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To cite this version:
Nabile Boussaid, Marco Caponigro, Thomas Chambrion. On the Ball-Marsden-Slemrod obstruction for bilinear control systems. 2019. hal-01537743v2

HAL Id: hal-01537743
https://hal.archives-ouvertes.fr/hal-01537743v2
Submitted on 13 Mar 2019

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On the Ball-Marsden-Slemrod obstruction for bilinear control systems

Nabille Boussaïd  
Laboratoire de Mathématiques de Besançon, UMR 6623  
Université de Bourgogne Franche-Comté, Besançon, France  
Nabile.Boussaid@univ-fcomte.fr

Marco Caponigro  
Équipe M2N  
Conservatoire National des Arts et Métiers, Paris, France  
Marco.Caponigro@cnam.fr

Thomas Chambrion  
Université de Lorraine, CNRS, Inria, IECL, Nancy, France  
Thomas.Chambrion@univ-lorraine.fr

Abstract—In this paper we present an extension to the case of $L^1$-controls of a famous result by Ball–Marsden–Slemrod on the obstruction to the controllability of bilinear control systems in infinite dimensional spaces.

I. INTRODUCTION

A. Bilinear control systems

Let $X$ be a Banach space, $A : D(A) 	o X$ a linear operator in $X$ with domain $D(A)$, $B : X 	o X$ a linear bounded operator and $\psi_0$ an element in $X$.

We consider a following bilinear control system on $X$

$$
\begin{align*}
\dot{\psi}(t) &= A\psi(t) + u(t)B\psi(t), \\
\psi(0) &= \psi_0,
\end{align*}
$$

where $u : [0, +\infty) \to \mathbb{R}$ is a scalar function representing the control.

Assumption 1: The pair $(A, B)$ of linear operators in $X$ satisfies

1) the operator $A$ generates a $C^0$-semigroup of linear bounded operators on $X$.

2) the operator $B$ is bounded.

Definition 1: Let $(A, B)$ satisfy Assumption 1 and let $T > 0$. A function $\psi : [0, T] \to X$ is a mild solution of (1) if for every $t$ in $[0, T]$,

$$
\psi(t) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A}B\psi(s)u(s)\,ds
$$

Equation (2) is often called Duhamel formula. Existence and uniqueness of mild solutions for equation (1) is given by the following result (see, for instance, Proposition 2.1 and Remark 2.7 in [BMS82]).

Proposition 1: Assume that $(A, B)$ satisfies Assumption 1. Then, for every $\psi_0$ in $X$, for every $u$ in $L^1_{\text{loc}}([0, +\infty), \mathbb{R})$, there exists a unique mild solution $t \mapsto \Upsilon_t^{u, \psi_0}$ to the Cauchy problem (1). Moreover, for every $\psi_0$ in $X$, the end-point mapping $\Upsilon_{\cdot, \psi_0} : [0, +\infty) \times L^1_{\text{loc}}([0, +\infty), \mathbb{R}) \to X$ is continuous.

Definition 2: Assume that $(A, B)$ satisfies Assumption 1 and let $\mathcal{U}$ be a subset of $L^1_{\text{loc}}([0, +\infty), \mathbb{R})$. For every $\psi_0$ in $X$, the attainable set from $\psi_0$ with controls in $\mathcal{U}$ is defined as

$$
\mathcal{A}(\psi_0, \mathcal{U}) = \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{U}} \{ \Upsilon_T^{u, \psi_0} \}.
$$

Our main result is the following property of the attainable set of system (1) with $L^1$ controls.

Theorem 2: Assume that $(A, B)$ satisfies Assumption 1. Then, for every $\psi_0$ in $X$, the attainable set $\mathcal{A}(\psi_0, L^1_{\text{loc}}([0, +\infty), \mathbb{R}))$ from $\psi_0$ with $L^1_{\text{loc}}$ controls is contained in a countable union of compacts subsets of $X$.

B. The Ball–Marsden–Slemrod obstruction

Our main result, Theorem 2 is an extension of the well-known Ball–Marsden–Slemrod obstruction to controllability (see also [ILT06]) which is as follows.

Theorem 3 (Theorem 3.6 in [BMS82]): Assume that $(A, B)$ satisfies Assumption 1. Then, for every $\psi_0$ in $X$, the attainable set $\mathcal{A}(\psi_0, \cup_{r>1} L^r_{\text{loc}}([0, +\infty), \mathbb{R}))$ from $\psi_0$ with $L^r_{\text{loc}}$ controls, $r > 1$, is contained in a countable union of compacts subsets of $X$.

A consequence of Theorem 3 to the framework of the conservative bilinear Schrödinger equation is given by Turinici.

Theorem 4 (Theorem 1 in [Tur00]): Assume that $(A, B)$ satisfies Assumption 1. Then, for every $\psi_0$ in $X$, the set $\cup_{r>1} \mathcal{A}(\psi_0, \cup_{r>1} L^r_{\text{loc}}([0, +\infty), \mathbb{R}))$ is contained in a countable union of compacts subsets of $X$.

Theorems 2 and 3 are basically empty in the case in which $X$ is finite dimensional, since, in this case, $X$ itself is a countable union of compact sets. On the other hand, when $X$ is infinite dimensional, these results represent a strong topological obstruction to the exact controllability. Indeed, compact subsets of an infinite dimensional Banach space have empty interiors and so is a countable union of closed subsets with empty interiors (as a consequence of Baire Theorem).

Whether the non-controllability result of Ball, Marsden, and Slemrod, Theorem 3 holds for $L^1$ control has been an open question for decades. Indeed, the proof of Theorem 3 does not apply directly to the $L^1$ case. To see what fails let
us briefly recall the method used in [BMS82] for the proof of Theorem 3. The first step is to write

\[ \mathcal{A}(\psi_0, t) = \bigcup_{T \geq r > 1} u \in L^T([0,T], \mathbb{R}) \{ \mathcal{Y}_{T,0}^n \} \]

Hence it is sufficient to prove that, for every \((l, m, k)\) in \(\mathbb{N}^3\), the set

\[ \mathcal{A}^{l,m,k} = \bigcup_{0 \leq i \leq l} \bigcup_{0 \leq i \leq u \leq k} \mathcal{Y}_{i,0}^{u} \]

has compact closure in \(X\). To this end, one considers a sequence \((\psi_n)_{n \in \mathbb{N}}\) in \(A^{l,m,k}\), associated with a sequence of times \((t_n)_{n \in \mathbb{N}}\) in \([0, t]\) and a sequence of controls \((u_n)_{n \in \mathbb{N}}\) in the ball of radius \(k\) of \(L^{1+1/m}(\mathbb{R})\). By compactness of \([0, t]\), each \(u_n\) is a finite sum of \(L^{1+1/m}(\mathbb{R})\) functions. The Banach space \(X\) is endowed with norm \(\| \cdot \|\). For every \(\psi_i\) in \(X\) and every \(r > 0\), \(B_X(\psi_i, r)\) denotes the ball of center \(\psi_i\) and of radius \(r\):

\[ B_X(\psi_i, r) = \{ \psi \in X \mid \| \psi - \psi_i \| < r \} \]

In the following, we shall make some assumptions on the space \(X\), the set \(A\), and the mapping \(A\). Notations

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\[ B_X(\psi_i, r) = \{ \psi \in X \mid \| \psi - \psi_i \| < r \} \]
Then, for every totally bounded subset $Y$ of $X$, the set $F([0, T] \times [0, T] \times Y)$ is totally bounded as well.

*Proof:* We claim that $G : (t, \psi) \mapsto e^{tA}\psi$ is jointly continuous in its two variables. Indeed, for every $\psi, \psi_0$ in $X$, for every $t, t_0 \geq 0$,

$$
\| e^{tA}\psi - e^{t_0A}\psi_0 \| \leq \| e^{tA}(\psi - \psi_0) \| + \| (e^{tA} - e^{t_0A})\psi_0 \|
$$

$$
\leq M e^{t_0} \| \psi - \psi_0 \| + \| (e^{tA} - e^{t_0A})\psi_0 \|.
$$

This last quantity tends to zero as $(t, \psi)$ tends to $(t_0, \psi_0)$. As a consequence, $F$ is continuous (as composition of continuous functions).

If $Y$ is totally bounded, the topological closure $\bar{Y}$ of $Y$ is compact (because the ambient space $X$ is complete). Hence $[0, T] \times [0, T] \times \bar{Y}$ is compact. By continuity, $F([0, T] \times [0, T] \times \bar{Y})$ is compact, hence is totally bounded. The set $F([0, T] \times [0, T] \times Y)$, which is contained in $F([0, T] \times [0, T] \times \bar{Y})$, is, therefore, totally bounded as well. ■

### C. Partition of unity in Banach spaces

**Definition 7:** Let $X$ be a Banach space. A family $(x_i)_{i \in I}$ of points of $X$ is locally finite if for every $x$ in $X$ and every $R > 0$, the cardinality of the set

$$
\left( \bigcup_{i \in I} \{x_i\} \right) \cap B_X(x, R)
$$

is finite.

**Definition 8:** Let $X$ be a Banach space, $Y$ be a subset of $X$, and $(O_i)_{i \in I}$ be an open cover of $Y$. A family $(\phi_i)_{i \in I}$ of continuous functions from $Y$ to $[0, 1]$ is called a partition of the unity of $Y$ adapted to the cover $(O_i)_{i \in I}$ if

(i) for every $i \in I$, $\phi_i(x) = 0$ for every $x \notin O_i$;

(ii) $\sum_{i \in I} \phi_i(x) = 1$ for every $x \in Y$.

**Proposition 10:** Let $X$ be a Banach space, $Y$ a subset of $X$, $\delta > 0$, $(x_j)_{j \in J}$ a locally finite family of points in $Y$ such that $Y \subset \bigcup_{j \in J} B_Y(x_j, \delta)$. Then, there exists $(\phi_j)_{j \in J}$ a partition of unity adapted to the open cover $(B(x_j, 2\delta))_{j \in J}$ of $Y$.

Moreover, if a family $(\phi_j)_{j \in J}$ is a partition of the unity adapted to the open cover $(B_X(x_j, 2\delta))_{j \in J}$, then for every $x \in Y$, $\| x - \sum_{j \in J} \phi_j(x) what(x) \| \leq 2\delta$.

**Proof:** We first prove the existence of a partition of the unity adapted to the open covering $(B_X(x_j, 2\delta))_{j \in J}$ of $Y$. To this end, we define, for every $j \in J$, the continuous functions $\varphi_j : X \to [0, 1]$ by

$$
\begin{align*}
\varphi_j(x) &= 1, & \text{if } \| x - x_j \| < \delta, \\
\varphi_j(x) &= 2 - \| x - x_j \|/\delta, & \text{if } \delta \leq \| x - x_j \| < 2\delta, \\
\varphi_j(x) &= 0, & \text{if } 2\delta \leq \| x - x_j \|.
\end{align*}
$$

Since the family $(x_j)_{j \in J}$ is locally finite, the sum $\sum_{j \in J} \varphi_j(x)$ converges for every $x$ in $Y$. Moreover, since $Y \subset \bigcup_{j \in J} B_X(x_j, \delta)$, the function $x \mapsto \sum_{j \in J} \varphi_j(x)$ does not vanish on $Y$. For every $j_0$ in $J$, we define $\phi_{\delta_0}$ by

$$
\phi_{\delta_0}(x) = \frac{1}{\sum_{j \in J} \varphi_j(x)} \varphi_{\delta_0}(x).
$$

and the family $(\phi_j)_{j \in J}$ is a partition of the unity adapted to the open cover $(B_X(x_j, 2\delta))_{j \in J}$ of $Y$.

We now prove the second point of Proposition 10. Let $(\phi_j)_{j \in J}$ be a partition of unity of $Y$ adapted to the cover $(B_X(x_j, 2\delta))_{j \in J}$. Then, for every $x \in Y$,

$$
\left\| \sum_{j \in J} \phi_j(x) - \phi_j(x) x_j \right\| \leq \sum_{j \in J} \| \phi_j(x) (x - x_j) \|
$$

$$\leq \sum_{j \in J} \| \phi_j(x) \| \| x - x_j \|.
$$

By construction, $\phi_j(x) = 0$ as soon as $\| x - x_j \| \geq 2\delta$. Hence,

$$
\left\| \sum_{j \in J} \phi_j(x) x_j \right\| \leq 2\delta \sum_{j \in J} \phi_j(x) \leq 2\delta,
$$

which concludes the proof. ■

### III. Dyson Expansion

#### A. The Dyson Operators

For every $u$ in $L^1_{loc}([0, +\infty), \mathbb{R})$, $p \in \mathbb{N}$, and $t \geq 0$ we define the linear bounded operator $W_p(t, u) : X \to X$ recursively by

$$
W_0(t, u)\psi = e^{(t-s)A}\psi
$$

$$
W_p(t, u)\psi = \int_0^t e^{(t-s)A}B W_{p-1}(s, u)\psi(u)ds,
$$

for $p \geq 1$, for every $\psi$ in $X$. We have the following estimate on the norm of the operator.

**Proposition 11:** For every $u$ in $L^1_{loc}([0, +\infty), \mathbb{R})$, $p \in \mathbb{N}$, and $t \geq 0$

$$
\|W_p(t, u)\| \leq \frac{Me^{ct}\|B\|p!\int_0^t |u(s)|ds^p}{p!}.
$$

**Proof:** We prove the result by induction on $p$ in $\mathbb{N}$. For $p = 0$ the result clearly follows from Proposition 5. Assume that the result holds for $p \geq 0$. Then, for every $\psi$ in $X$,

$$
\|W_{p+1}(t, u)\| \leq \int_0^t e^{(t-s)A} B W_{p}(s, u)\psi(u)ds,
$$

$$\leq M \int_0^t \|e^{(t-s)A}B\|p!\int_0^s |u(\tau)|d\tau u(s)ds,
$$

$$\leq M e^{ct}\|B\|p!\left(\int_0^t |u(\tau)|d\tau\right)^{p+1}/(p+1)!.
$$

The last inequality follows from Proposition 5. We conclude the proof by induction on $p$. ■
B. A compactness property

**Lemma 12:** For every $j$ in $\mathbb{N}$, $T \geq 0$ and $K \geq 0$, and $\psi$ in $X$ the set

$$\mathcal{W}_j^{T,K} = \{ W_j(t,u) \psi \mid 0 \leq t \leq T, \|u\|_{L^1} \leq K \},$$

is totally bounded.

**Proof:** We prove the result by induction on $j$ in $\mathbb{N}$. For $j = 0$, consider $\mathcal{W}_j^{T,K} = \{ e^{tA}\psi, 0 \leq t \leq T \}$ and let $(w_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{W}_j^{T,K}$. Then there exists a sequence $(t_n)_{n \in \mathbb{N}}$ such that $w_n = e^{t_n A} \psi$ for every $n$. Up to extraction $\lim_{n \to \infty} t_n = t \in [0,T]$ since $[0,T]$ is compact. By definition of $C_0$-semigroup, $\lim_{n \to \infty} e^{t_n A} \psi = e^{tA} \psi$. This proves that $\mathcal{W}_j^{T,K}$ is sequentially compact, hence compact and, in particular, totally bounded (Proposition 6).

Assume that, for $j \geq 0$, $\mathcal{W}_{j}^{T,K}$ is totally bounded. By Proposition 9, the set

$$\mathcal{Z}_j^{T,K} := \{ e^{(t-s)A}B\psi, \psi \in \mathcal{W}_j^{T,K}, 0 \leq s \leq t \leq T \} \subset F([0,T]^2 \times \mathcal{W}_j^{T,K})$$

is totally bounded as well.

Let $\varepsilon > 0$ be given and define $\delta = \frac{\varepsilon}{\|B\| K^2}$. Since $\mathcal{Z}_j^{T,K}$ is totally bounded, there exists a finite family $(x_i)_{1 \leq i \leq N_j}$ in $\mathcal{Z}_j^{T,K}$ such that

$$\mathcal{Z}_j^{T,K} \subset \bigcup_{i=1}^{N_j} B_X(x_i, \delta).$$

Let $(\phi_i)_{1 \leq i \leq N_j}$ be a partition of the unity adapted to the cover $\bigcup_{i=1}^{N_j} B(x_i, 2\delta)$ of $\mathcal{Z}_j^{T,K}$. Such a partition of the unity exists by Proposition 10, and moreover, for every $x$ in $\mathcal{Z}_j^{T,K}$, we have

$$\left\| x - \sum_{i=1}^{N_j} \phi_i(x) x_i \right\| \leq 2\delta. \quad (4)$$

Applying the inequality (4) with $x = e^{(t-s)A}Bw_j(s,u)\psi_0$, we get, for every $u$ in $L^1$ and every $(s,t)$ such that $0 \leq s \leq t \leq T$,

$$\left\| e^{(t-s)A}Bw_j(s,u)\psi_0 \right\| - \sum_{i=1}^{N_j} \phi_i(e^{(t-s)A}Bw_j(s,u)\psi_0)x_i \right\| \leq 2\delta.$$

Multiplying by $u(s)$ and integrating for $s$ in $[0,t]$, one gets for $\|u\|_{L^1} \leq K$

$$\left\| \int_0^t e^{(t-s)A}Bw_j(s,u)\psi_0u(s)ds - \sum_{i=1}^{N_j} \int_0^t \phi_i(e^{(t-s)A}Bw_j(s,u)\psi_0)u(s)dsx_i \right\| \leq 2\delta \|u\|_{L^1},$$

that is

$$\left\| \int_0^t e^{(t-s)A}Bw_j(s,u)\psi_0u(s)ds - \sum_{i=1}^{N_j} \int_0^t \phi_i(e^{(t-s)A}Bw_j(s,u)\psi_0)u(s)dsx_i \right\| \leq 2\delta K, \quad (5)$$

The set $\sum_{i=1}^{N_j} [0,K]x_i$ is compact by Proposition 7 and, hence, totally bounded. Then there exists a finite family $(y_i)_{1 \leq i \leq N'_j}$ such that

$$\sum_{i=1}^{N_j} [0,K]x_i \subset \bigcup_{i=1}^{N'_j} B_X(y_i, \delta). \quad (6)$$

From (5) and (6), one deduces that

$$\mathcal{W}_{j+1}^{T,K} \subset \bigcup_{i=1}^{N'_j} B_X(y_i, (2K+1)\delta) = \bigcup_{i=1}^{N'_j} B_X(y_i, \varepsilon).$$

This proves that $\mathcal{W}_{j+1}^{T,K}$ is totally bounded and concludes the proof.

**C. Convergence of the Dyson expansion**

**Proposition 13:** For every $u$ in $L^1([0,\infty), \mathbb{R})$, $t \geq 0$, and $\psi_0 \in X$

$$\left\| \mathcal{Y}_{t,0}^u \psi_0 \right\| \leq M e^{ct} \|\psi_0\| \exp \left( M e^{ct} \|B\| \int_0^t |u(s)|ds \right)$$

**Proof:** The proof follows the proof of [BMS82, Theorem 2.5]. By Duhamel formula (2) and Proposition 5,

$$\left\| \mathcal{Y}_{t,0}^u \psi_0 \right\| \leq \left\| e^{tA} \psi_0 \right\| + \int_0^t e^{(t-s)A}BY_{s,0}^u \psi_0 u(s)ds \leq M e^{ct} \|\psi_0\| + \int_0^t M e^{ct} \|B\| \|Y_{s,0}^u \psi_0\| u(s)ds,$$

and the conclusion follows by Gronwall lemma (see [BMS82, Lemma 2.6]).

**Proposition 14:** For every $u$ in $L^1([0,\infty), \mathbb{R})$, $p$ in $\mathbb{N}$, $t \geq 0$, and $\psi_0 \in X$

$$\lim_{p \to \infty} \int_0^t e^{(t-s)A}BW_p(s,t)\mathcal{Y}_{s,0}^u \psi_0 u(s)ds = 0.$$  

**Proof:** Consider

$$\left\| \int_0^t e^{(t-s)A}BW_p(s,t)\mathcal{Y}_{s,0}^u \psi_0 u(s)ds \right\| \leq \int_0^t \|e^{(s-t)A}\| \|B\| \|W_p(s,t)\| \|\mathcal{Y}_{s,0}^u \psi_0\| \|u(s)\|ds$$

and recall that $\|W_p(s,t)\|$ tends to zero as $p$ tends to infinity (Proposition 11).

**Proposition 15:** For every $u$ in $L^1_{loc}([0,\infty), \mathbb{R})$, $p$ in $\mathbb{N}$, $t \geq 0$, and $\psi_0 \in X$

$$\mathcal{Y}_{t,0}^u \psi_0 = \sum_{p=0}^{\infty} W_p(t,0)u \psi_0.$$
Hence, for every \( p \) and the result follow from Proposition 14 as \( p \) tends to \( \infty \), one gets 
\[
\sum_{j=0}^{N_e} W_j^{m,l} \subset \bigcup_{i=1}^{N_1} B_X (x_i, \varepsilon/2).
\] (8)

Gathering (7) and (8), one gets
\[
\sum_{j=0}^{N_e} W_j^{m,l} \subset \bigcup_{i=1}^{N_1} B_X (x_i, \varepsilon),
\]
which concludes the proof of Theorem 2.

V. ACKNOWLEDGMENTS

This work has been supported by the project DISQUO of the DEFI InFinInTI 2017 by CNRS and the QUACO project by ANR 17-CE40-0007-01.

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