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On the Ball–Marsden–Slemrod obstruction in bilinear control systems

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Abstract

This note presents an extension to the case of $L^1$ controls of a well-known obstruction to the controllability of bilinear control systems in infinite dimensional spaces.

1 Introduction

1.1 Bilinear control systems

Let $X$ be a Banach space, $A : D(A) \to X$ a linear operator in $X$ with domain $D(A)$, $B : X \to X$ a linear bounded operator and $\psi_0$ an element in $X$. We consider the following bilinear control system

\[
\begin{cases}
\dot{\psi}(t) = A\psi(t) + u(t)B\psi(t), \\
\psi(0) = \psi_0,
\end{cases}
\]

(1)

where $u : [0, +\infty) \to \mathbb{R}$ is a scalar function representing the control.

Assumption 1. The pair $(A, B)$ of linear operators in $X$ satisfies

1) the operator $A$ generates a $C^0$-semigroup of linear bounded operators on $X$.

2) the operator $B$ is bounded.
Definition 1. Assume that \((A, B)\) satisfies Assumption 1. Let \(T > 0\). A function \(\psi : [0, T] \rightarrow X\) is a mild solution of (1) if for every \(t \in [0, T]\),

\[
\psi(t) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A}B\psi(s)u(s)ds
\]  

Equation (2) is often called Duhamel formula.

Existence and uniqueness for equation (1) is given by the following result (see, for instance, Proposition 2.1 and Remark 2.7 in [BMS82]).

Proposition 1. Assume that \((A, B)\) satisfies Assumption 1. Then, for every \(\psi_0\) in \(X\), for every \(u\) in \(L^1_{\text{loc}}([0, +\infty), \mathbb{R})\), there exists a unique mild solution \(t \mapsto \Upsilon_{t, 0}\psi_0\) to the Cauchy problem (1). Moreover, for every \(\psi_0\) in \(X\), the end-point mapping \(\Upsilon_{\cdot, 0}\psi_0 : [0, +\infty) \times L^1_{\text{loc}}([0, +\infty), \mathbb{R}) \rightarrow X\) is continuous.

Definition 2. Assume that \((A, B)\) satisfies Assumption 1 and let \(U\) be a subset of \(L^1_{\text{loc}}([0, +\infty), \mathbb{R})\).

For every \(\psi_0\) in \(X\), the attainable set from \(\psi_0\) with controls in \(U\) is defined as

\[
\mathcal{A}(\psi_0, U) = \bigcup_{T \geq 0} \bigcup_{u \in U} \{\Upsilon_{T, 0}\psi_0\}.
\]

Our main result is the following property of the attainable set of system (1) with \(L^1\) controls.

Theorem 2. Assume that \((A, B)\) satisfies Assumption 1. Then, for every \(\psi_0\) in \(X\), the attainable set \(\mathcal{A}(\psi_0, L^1_{\text{loc}}([0, +\infty), \mathbb{R}))\) from \(\psi_0\) with \(L^1_{\text{loc}}\) controls is contained in a countable union of compacts subsets of \(X\).

1.2 The Ball–Marsden–Slemrod obstruction

Our main result Theorem 2 is an extension of the well-known Ball–Marsden–Slemrod obstruction to controllability.

Theorem 3 (Theorem 3.6 in [BMS82]). Assume that \((A, B)\) satisfies Assumption 1. Then, for every \(\psi_0\) in \(X\), the attainable set \(\mathcal{A}(\psi_0, \cup_{r>1} L^r_{\text{loc}}([0, +\infty), \mathbb{R}))\) from \(\psi_0\) with \(L^r_{\text{loc}}\) controls, \(r > 1\), is contained in a countable union of compacts subsets of \(X\).

A consequence of Theorem 3 to the framework of the conservative bilinear Schrödinger equation is given by Turinici.

Theorem 4 (Theorem 1 in [Tur00]). Assume that \((A, B)\) satisfies Assumption 1. Then, for every \(\psi_0\) in \(X\), the set \(\cup_{\alpha > 0} \mathcal{A}(\psi_0, \cup_{r>1} L^r_{\text{loc}}([0, +\infty), \mathbb{R}))\) is contained in a countable union of compacts subsets of \(X\).

Theorems 2 and 3 are basically empty in the case in which \(X\) is finite dimensional, indeed, in this case, \(X\) itself is a countable union of compact sets. On the other hand, when \(X\) is infinite dimensional, these results represent a strong topological obstruction to the exact controllability. Indeed, compact subsets of an infinite dimensional Banach space have empty interiors and so is a countable union of closed subsets with empty interiors (as a consequence of Baire Theorem).
Let us recall briefly the method used in [BMS82] for the proof of Theorem 3. The first step is to write

\[ A(\psi_0, \cup_{r>1} L_0^r([0, +\infty), R)) = \bigcup_{T \geq 0} \bigcup_{r>1} u \in L^r([0,T], R) \{ \Upsilon_{T,0}^u \psi_0 \} = \bigcup_{l \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \bigcup_{0 \leq t \leq l} \left( \bigcup_{\|u\|_{L_1+1/m} \leq k} \{ \Upsilon_{t,0}^u \psi_0 \} \right). \]

Hence it is sufficient to prove that, for every \((l, m, k)\) in \(\mathbb{N}^3\), the set

\[ A^{l,m,k} = \bigcup_{0 \leq t \leq l} \left( \bigcup_{\|u\|_{L_1+1/m} \leq k} \{ \Upsilon_{t,0}^u \psi_0 \} \right) \]

has compact closure in \(X\). To this end, one considers a sequence \((\psi_n)_{n \in \mathbb{N}}\) in \(A^{l,m,k}\), associated with a sequence of times \((t_n)_{n \in \mathbb{N}}\) in \([0, l]\) and a sequence of controls \((u_n)_{n \in \mathbb{N}}\) in the ball of radius \(k\) of \(L^{1+1/m}([0, +\infty), R)\). By compactness of \([0, l]\), up to extraction, one can assume that \((t_n)_{n \in \mathbb{N}}\) tends to \(t_\infty\) in \([0, l]\). By Banach–Alaoğlu–Bourbaki Theorem, the balls of \(L^{1+1/m}([0, +\infty), R)\) are weakly (sequentially) compact and, hence, up to extraction, one can assume that \((u_n)_{n \in \mathbb{N}}\) converges weakly in \(L^{1+1/m}([0, +\infty), R)\) to some \(u_\infty\). The hard point of the proof (Lemma 3.7 in [BMS82]) is then to show that \(\Upsilon_{t_n,0}^u \psi_0\) tends to \(\Upsilon_{t_\infty,0}^{u_\infty} \psi_0\) as \(n\) tends to infinity.

A crucial point in the proof of Theorem 3 given in [BMS82] is the fact that the closed balls of \(L^p\), \(p > 1\) are weakly sequentially compact. This is no longer true for the balls of \(L^1\), and this prevents a direct extension of the proof of Theorem 3 to the proof of Theorem 2.

### 1.3 Content

In this note we present a simple and short proof of Theorem 2. However, historical reasons have made different communities use incompatible terminologies and, in order to avoid ambiguities, we present in Section 2 a quick reminder of basic facts in Banach topologies. Section 3 gives a short introduction to the classical Dyson expansion (Section 3.1) and the proof of an instrumental compactness property (Section 3.2). We conclude in Section 4 with the proof of Theorem 2.

### 2 Basic facts on the topology in Banach spaces

#### 2.1 Notations

The Banach space \(X\) is endowed with norm \(\| \cdot \|\). For every \(\psi_c\) in \(X\) and every \(r > 0\), \(B_X(\psi_c, r)\) denotes the ball of center \(\psi_c\) and of radius \(r\):

\[ B_X(\psi_c, r) = \{ \psi \in X : \| \psi - \psi_c \| < r \}. \]

In the following, all we need to know about generators of \(C^0\)-semigroup is the classical result stated in Proposition 5 (see Chapter VII of [HP57]).

**Proposition 5.** Assume that \(A\) generates a \(C^0\)-semigroup. Then there exist \(M, \omega > 0\) such that \(\| e^{At} \| \leq Me^{\omega t} \) for every \(t \geq 0\).
2.2 Compact subset of Banach spaces

**Definition 3.** Let $X$ be a Banach space and $Y$ be a subset of $X$. A family $(O_i)_{i \in I}$ is an open cover of $Y$ if $O_i$ is open in $X$ for every $i$ in $I$ and $Y \subset \bigcup_{i \in I} O_i$.

**Definition 4.** Let $X$ be a Banach space. A subset $Y$ of $X$ is said to be compact if from any open cover of $Y$, it is possible to extract a finite cover of $Y$.

**Definition 5.** Let $X$ be a Banach space. A subset $Y$ of $X$ is said to be sequentially compact if from any sequence $(\psi_n)_{n \in \mathbb{N}}$ taking value in $Y$, it is possible to extract a subsequence $(\psi_{\phi(n)})_{n \in \mathbb{N}}$ converging in $Y$.

**Definition 6.** Let $X$ be a Banach space. A subset $Y$ of $X$ is said to be totally bounded if for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and a finite family $(x_i)_{1 \leq i \leq N}$ in $X$ such that

$$Y \subset \bigcup_{i=1}^{N} B_X(x_i, \varepsilon).$$

**Proposition 6.** Let $X$ be a Banach space. For every subset $Y$ of $X$, the following assertions are equivalent:

1. $Y$ is compact.
2. $Y$ is sequentially compact.
3. $Y$ is complete and totally bounded.
4. $Y$ is closed and totally bounded.

**Proposition 7.** Let $X$ be a Banach space, $N$ in $\mathbb{N}$ and $(Y_i)_{1 \leq i \leq N}$ a finite family of compact subsets of $X$. Then, the finite sum

$$\sum_{i=1}^{N} Y_i = \{ y_1 + y_2 + \ldots + y_N \mid y_i \in Y_i, i = 1, \ldots, N \}$$

is compact as well.

**Proposition 8.** Let $X$ be a Banach space, $N$ in $\mathbb{N}$ and $(Y_i)_{1 \leq i \leq N}$ a finite family of totally bounded subsets of $X$. Then, the finite sum

$$\sum_{i=1}^{N} Y_i = \{ y_1 + y_2 + \ldots + y_N \mid y_i \in Y_i, i = 1, \ldots, N \}$$

is totally bounded as well.

**Proposition 9.** Let $X$ be a Banach space, $T > 0$ and $(A, B)$ satisfies Assumption 1. Define the mapping

$$F : [0, T] \times [0, T] \times X \rightarrow X \quad (s, t, \psi) \mapsto e^{t-s}A B \psi$$

Then, for every totally bounded subset $Y$ of $X$, the set $F([0, T] \times [0, T] \times Y)$ is totally bounded as well.
Proof. We claim that \( G : (t, \psi) \mapsto e^{tA}\psi \) is jointly continuous in its two variables. Indeed, for every \( \psi, \psi_0 \) in \( X \), for every \( t, t_0 \geq 0 \),

\[
\|e^{tA}\psi - e^{t_0A}\psi_0\| \leq \|e^{tA}(\psi - \psi_0)\| + \|(e^{tA} - e^{t_0A})\psi_0\| \\
\leq Me^{\omega t}\|\psi - \psi_0\| + \|(e^{tA} - e^{t_0A})\psi_0\|.
\]

This last quantity tends to zero as \((t, \psi)\) tends to \((t_0, \psi_0)\). As a consequence, \( F \) is continuous (as composition of continuous functions).

If \( Y \) is totally bounded, the topological closure \( \bar{Y} \) of \( Y \) is compact (because the ambient space \( X \) is complete). Hence \([0, T] \times [0, T] \times \bar{Y} \) is compact. By continuity, \( F([0, T] \times [0, T] \times \bar{Y}) \) is compact, hence is totally bounded. The set \( F([0, T] \times [0, T] \times Y) \), which is contained in \( F([0, T] \times [0, T] \times \bar{Y}) \), is, therefore, totally bounded as well. \( \square \)

2.3 Partition of unity in Banach spaces

**Definition 7.** Let \( X \) a Banach space. A family \((x_i)_{i \in I}\) of points of \( X \) is locally finite if for every \( x \) in \( X \) and every \( R > 0 \), the cardinality of the set

\[
\left( \bigcup_{i \in I} \{x_i\} \right) \cap B_X(x, R)
\]

is finite.

**Definition 8.** Let \( X \) be a Banach space, \( Y \) a subset of \( X \), and \((O_i)_{i \in I}\) be an open cover of \( Y \). A family \((\phi_i)_{i \in I}\) of continuous functions from \( Y \) to \([0, 1]\) is called a partition of the unity of \( Y \) adapted to the cover \((O_i)_{i \in I}\) if

(i) for every \( i \in I \), \( \phi_i(x) = 0 \) for every \( x \notin O_i \);

(ii) \( \sum_{i \in I} \phi_i(x) = 1 \) for every \( x \in Y \).

**Proposition 10.** Let \( X \) be a Banach space, \( Y \) a subset of \( X \), \( \delta > 0 \), \((x_j)_{j \in J}\) a locally finite family of points in \( Y \) such that \( Y \subset \bigcup_{j \in J} B_X(x_j, \delta) \). Then, there exists \((\phi_j)_{j \in J}\) a partition of the unity adapted to the open cover \((B(x_j, 2\delta))_{j \in J}\) of \( Y \).

Moreover, if a family \((\phi_j)_{j \in J}\) is a partition of the unity adapted to the open cover \((B_X(x_j, 2\delta))_{j \in J}\), then for every \( x \) in \( Y \), \( \|x - \sum_{j \in J} \phi_j(x)\| \leq 2\delta \).

**Proof.** We first prove the existence of a partition of the unity adapted to the open cover \((B_X(x_j, 2\delta))_{j \in J}\) of \( Y \). To this end, we define, for every \( j \) in \( J \), the continuous functions \( \varphi_j : X \to [0, 1] \) by

\[
\begin{align*}
\varphi_j(x) &= 1, & \text{if } \|x - x_j\| < \delta, \\
\varphi_j(x) &= 2 - \|x - x_j\|/\delta, & \text{if } \delta \leq \|x - x_j\| < 2\delta, \\
\varphi_j(x) &= 0, & \text{if } 2\delta \leq \|x - x_j\|.
\end{align*}
\]

Since the family \((x_j)_{j \in J}\) is locally finite, the sum \( \sum_{j \in J} \varphi_j(x) \) converges for every \( x \) in \( Y \). Moreover, since \( Y \subset \bigcup_{j \in J} B(x_j, \delta) \), the function \( x \mapsto \sum_{j \in J} \varphi_j(x) \) does not vanish on \( Y \). For every \( j_0 \) in \( J \), we define \( \phi_{j_0} \) by

\[
\phi_{j_0}(x) = \varphi_{j_0}(x) \frac{1}{\sum_{j \in J} \varphi_j(x)}.
\]
and the family \((\phi_j)_{j \in J}\) is a partition of the unity adapted to the open cover \((B_X(x_j, 2\delta))_{j \in J}\) of \(Y\).

We now prove the second point of Proposition 10. Let \((\phi_j)_{j \in J}\) be a partition of unity of \(Y\) adapted to the cover \((B_X(x_j, 2\delta))_{j \in J}\). Then, for every \(x\) in \(Y\),

\[
\left\| x - \sum_{j \in J} \phi_j(x)x_j \right\| = \left\| \sum_{j \in J} \phi_j(x)x \right\| \quad = \quad \left\| \sum_{j \in J} \phi_j(x)(x - x_j) \right\| \quad \leq \quad \sum_{j \in J} \phi_j(x) \| x - x_j \|.
\]

By construction, \(\phi_j(x) = 0\) as soon as \(\| x - x_j \| \geq 2\delta\). Hence,

\[
\left\| x - \sum_{j \in J} \phi_j(x)x_j \right\| \leq 2\delta \sum_{j \in J} \phi_j(x) \leq 2\delta,
\]

which concludes the proof.

### 3 The Dyson expansion

#### 3.1 The Dyson Operators

For every \(u\) in \(L^1_{loc}([0, +\infty), \mathbb{R})\), \(p \in \mathbb{N}\), and \(t \geq 0\) we define the linear bounded operator \(W_p(t, u) : X \rightarrow X\) recursively by

\[
W_0(t, u)\psi = e^{(t-s)A}\psi
\]

\[
W_p(t, u)\psi = \int_0^t e^{(t-s)A}BW_{p-1}(s, u)\psi u(s)ds, \quad \text{for } p \geq 1,
\]

for every \(\psi\) in \(X\). We have the following estimate on the norm of the operator.

**Proposition 11.** For every \(u\) in \(L^1_{loc}([0, +\infty), \mathbb{R})\), \(p \in \mathbb{N}\), and \(t \geq 0\)

\[
\|W_p(t, u)\| \leq \frac{Me^{\omega t}\|B\|^p(\int_0^t |u(s)|ds)^p}{p!}.
\]

**Proof.** We prove the result by induction on \(p\) in \(\mathbb{N}\). For \(p = 0\) the result clearly follows from Proposition 5. Assume that the result holds for \(p \geq 0\). Then, for every \(\psi\) in \(X\),

\[
\|W_{p+1}(t, u)\psi\| \leq \int_0^t e^{(t-s)A}BW_p(s, u)\psi u(s)ds
\]

\[
\leq M \int_0^t e^{(t-s)\omega}\|B\|^p(\int_0^s |u(\tau)|d\tau)^p u(s)ds
\]

\[
\leq Me^{\omega t}\|B\|^{p+1}(\int_0^t |u(\tau)|d\tau)^{p+1}(p+1)!
\]

The last inequality follows from Proposition 5. We conclude the proof by induction on \(p\). \(\square\)
3.2 A compactness property

**Lemma 12.** For every \( j \) in \( \mathbb{N} \), \( T \geq 0 \) and \( K \geq 0 \), and \( \psi \) in \( X \) the set

\[
W_{j}^{T,K} = \{ W_{j}(t,u)\psi \mid 0 \leq t \leq T, \|u\|_{L^{1}} \leq K \},
\]

is totally bounded.

**Proof.** We prove the result by induction on \( j \) in \( \mathbb{N} \). For \( j = 0 \), consider \( W_{0}^{T,K} = \{ e^{tA}\psi, 0 \leq t \leq T \} \) and let \( (w_{n})_{n\in\mathbb{N}} \) be a sequence in \( W_{0}^{T,K} \). Then there exists a sequence \( (t_{n})_{n\in\mathbb{N}} \) such that \( w_{n} = e^{t_{n}A}\psi \) for every \( n \). Up to extraction \( \lim_{n\to\infty} t_{n} = t \in [0,T] \) since \([0,T]\) is compact. By definition of \( C^{0} \)-semigroup, \( \lim_{n\to\infty} e^{t_{n}A}\psi = e^{tA}\psi \). This proves that \( W_{0}^{T,K} \) is sequentially compact, hence compact and, in particular, totally bounded (Proposition 6).

Assume that, for \( j \geq 0 \), \( W_{j}^{T,K} \) is totally bounded. By Proposition 9, the set

\[
Z_{j}^{T,K} := \{ e^{(t-s)A}B\psi, \psi \in W_{j}^{T,K}, 0 \leq s \leq t \leq T \}
\]

is totally bounded as well.

Let \( \varepsilon > 0 \) be given and define \( \delta = \frac{\varepsilon}{2K+1} > 0 \). Since \( Z_{j}^{T,K} \) is totally bounded, there exists a finite family \( (x_{i})_{1 \leq i \leq N_{\delta}} \) in \( Z_{j}^{T,K} \) such that

\[
Z_{j}^{T,K} \subset \bigcup_{i=1}^{N_{\delta}} B_{X}(x_{i}, \delta).
\]

Let \( (\phi_{i})_{1 \leq i \leq N_{\delta}} \) be a partition of the unity adapted to the cover \( \cup_{i=1}^{N_{\delta}} B(x_{i}, 2\delta) \) of \( Z_{j}^{T,K} \). Such a partition of the unity exists by Proposition 10, and moreover, for every \( x \) in \( Z_{j}^{T,K} \), we have

\[
\left\| x - \sum_{i=1}^{N_{\delta}} \phi_{i}(x)x_{i} \right\| \leq 2\delta.
\]

Applying the inequality (4) with \( x = e^{(t-s)A}BW_{j}(s,u)\psi_{0} \), we get, for every \( u \) in \( L^{1} \) and every \( (s,t) \) such that \( 0 \leq s \leq t \),

\[
\left\| e^{(t-s)A}BW_{j}(s,u)\psi_{0} - \sum_{i=1}^{N_{\delta}} \phi_{i}(e^{(t-s)A}BW_{j}(s,u)\psi_{0})x_{i} \right\| \leq 2\delta.
\]

Multiplying by \( u(s) \) and integrating for \( s \) in \([0,t]\), one gets for \( \|u\|_{L^{1}} \leq K \)

\[
\left\| \int_{0}^{t} e^{(t-s)A}BW_{j}(s,u)\psi_{0}u(s)ds - \sum_{i=1}^{N_{\delta}} \int_{0}^{t} \phi_{i}(e^{(t-s)A}BW_{j}(s,u)\psi_{0))u(s)ds x_{i} \right\| \leq 2\delta K.
\]

The set \( \sum_{i=1}^{N_{\delta}} [0,K]x_{i} \) is compact by Proposition 7 and, hence, totally bounded. Then there exists a finite family \( (y_{i})_{1 \leq i \leq N_{\delta}'} \) such that

\[
\sum_{i=1}^{N_{\delta}} [0,K]x_{i} \subset \bigcup_{i=1}^{N_{\delta}'} B_{X}(y_{i}, \delta).
\]
From (5) and (6), one deduces that
\[
W^{T,K}_{j+1} \subset \bigcup_{i=1}^{N'_j} B_{X}(y_i, (2K+1)\delta) = \bigcup_{i=1}^{N'_j} B_{X}(y_i, \varepsilon).
\]
This proves that \(W^{T,K}_{j+1}\) is totally bounded and concludes the proof.

3.3 Convergence of the Dyson expansion

Proposition 13. For every \(u \in L^1([0, +\infty), \mathbb{R})\), \(t \geq 0\), and \(\psi_0 \in X\)
\[
\|\Upsilon_{t,0}^u \psi_0\| \leq M e^{\omega t} \|\psi_0\| \exp \left( M e^{\omega t} \|B\| \int_0^t |u(s)|ds \right)
\]

Proof. The proof follows the proof of [BMS82, Theorem 2.5]. By Duhamel formula (2) and Proposition 5,
\[
\|\Upsilon_{t,0}^u \psi_0\| = \left\| e^{tA} \psi_0 + \int_0^t e^{(t-s)A} B \Upsilon_{s,0}^u \psi_0 u(s)ds \right\| e^{t\omega} \psi_0 \\
\leq M e^{t\omega} \|\psi_0\| + \int_0^t M e^{\omega(t-s)} \|B\| \|\Upsilon_{s,0}^u \psi_0\| u(s)ds,
\]
and the conclusion follows by Gronwall lemma (see [BMS82, Lemma 2.6]).

Proposition 14. For every \(u \in L^1([0, +\infty), \mathbb{R})\), \(p \in \mathbb{N}\), \(t \geq 0\), and \(\psi_0 \in X\)
\[
\lim_{p \to \infty} \left\| \int_0^t e^{(t-s)A} BW_p(s,t) \Upsilon_{s,0}^u \psi_0 u(s)ds \right\| = 0.
\]

Proof. Consider
\[
\left\| \int_0^t e^{(t-s)A} BW_p(s,t) \Upsilon_{s,0}^u \psi_0 u(s)ds \right\| \leq \int_0^t \| e^{(t-s)A} \| \|B\| \|W_p(s,t)\| \|\Upsilon_{s,0}^u \psi_0\| u(s)|ds
\]
and recall that \(\|W_p(s,t)\|\) tends to zero as \(p\) tends to infinity (Proposition 11).

Proposition 15. For every \(u \in L^1_{loc}([0, +\infty), \mathbb{R})\), \(p \in \mathbb{N}\), \(t \geq 0\), and \(\psi_0 \in X\)
\[
\Upsilon_{t,0}^u \psi_0 = \sum_{p=0}^{\infty} W_p(t,0,u) \psi_0.
\]

Proof. Applying iteratively \(p\)-times Duhamel formula (2), one gets
\[
\Upsilon_{t,0}^u \psi_0 = e^{tA} \psi_0 + \int_0^t e^{(t-s)A} Bu(s) \Upsilon_{s,0}^u \psi_0 ds \\
= e^{tA} \psi_0 + \int_0^t e^{(t-s)A} Bu(s)e^{sA} \psi_0 ds + \int_0^t e^{(t-s)A} BW_1(s,t) \Upsilon_{s,0}^u \psi_0 u(s)ds \\
= \sum_{j=1}^{p} W_p(t,0) \psi_0 + \int_0^t e^{(t-s)A} BW_p(s,t) \Upsilon_{s,0}^u \psi_0 u(s)ds.
\]
Hence, for every \( p \geq 1 \),
\[
\Upsilon_{t,0}^u \psi_0 - \sum_{j=0}^p W_j(t,0) \psi_0 = \int_0^t e^{(t-s)A} B W_p(s,t) \Upsilon_{s,0}^u \psi_0 u(s) ds
\]
and the result follow from Proposition 14 as \( p \) tends to \( \infty \). \( \square \)

4 Proof of Theorem 2

We proceed now to the proof of Theorem 2. First of all, notice that, for every \( \psi_0 \) in \( X \),
\[
A(\psi_0, L^1_{\text{loc}}([0, +\infty), \mathbb{R})) = \bigcup_{l \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \{ \Upsilon_{t,0}^u \psi_0, \| u \|_{L^1} \leq l, 0 \leq t \leq m \},
\]
and it is enough to prove that, for every \( l \) and \( m \) in \( \mathbb{N} \), the set
\[
\{ \Upsilon_{t,0}^u \psi_0, \| u \|_{L^1} \leq l, 0 \leq t \leq m \}
\]
is totally bounded.

Let \( \varepsilon > 0 \). From the convergence of the Dyson expansion (Proposition 15) and the bound on the operators \( W_j \) (Proposition 11), there exists a integer \( N_\varepsilon \) such that
\[
\left\| \sum_{p \geq N_\varepsilon} W_p(t,u) \psi_0 \right\| \leq \frac{\varepsilon}{2}, \quad \text{(7)}
\]
for every \( t \) in \([0,m]\) and every \( u \) such that \( \| u \|_{L^1} \leq l \). For each \( j = 1, \ldots, N_\varepsilon \) the sets \( W_j^{m,l} \), defined by (3), are totally bounded (Lemma 12), hence their sum
\[
\sum_{j=0}^{N_\varepsilon} W_j^{m,l}
\]
is totally bounded as well (Proposition 8). Hence there exists a family \((x_i)_{1 \leq i \leq N_\varepsilon}\) of points of \( \sum_{j=0}^{N_\varepsilon} W_j^{m,l} \) such that
\[
\sum_{j=0}^{N_\varepsilon} W_j^{m,l} \subset \bigcup_{i=1}^{N_1} B_X \left( x_i, \frac{\varepsilon}{2} \right). \quad \text{(8)}
\]
Gathering (7) and (8), one gets
\[
\sum_{j=0}^{\infty} W_j^{m,l} \subset \bigcup_{i=1}^{N_1} B_X \left( x_i, \varepsilon \right),
\]
which concludes the proof of Theorem 2.

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References

