On the Ball–Marsden–Slemrod obstruction in bilinear control systems

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Abstract

This note presents an extension to the case of L^1 controls of a well-known obstruction to the controllability of bilinear control systems in infinite dimensional spaces.

1 Introduction

1.1 Bilinear control systems

Let X be a Banach space, $A : D(A) \to X$ a linear operator in X with domain $D(A)$, $B : X \to X$ a linear bounded operator and ψ_0 an element in X. We consider the following bilinear control system

$$\begin{cases} \dot{\psi}(t) & = A\psi(t) + u(t)B\psi(t), \\ \psi(0) & = \psi_0, \end{cases} \tag{1}$$

where $u : [0, +\infty) \to \mathbf{R}$ is a scalar function representing the control.

Assumption 1. *The pair (A, B) of linear operators in X satisfies*

1) *the operator A generates a C^0-semigroup of linear bounded operators on X.*

2) *the operator B is bounded.*

Definition 1. *Assume that* (A, B) *satisfies Assumption 1. Let* $T > 0$*. A function* $\psi : [0, T] \to X$ *is a* mild solution *of (1) if for every* t *in* $[0, T]$*,*

$$\psi(t) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A} B\psi(s)u(s)\mathrm{d}s \tag{2}$$

Equation (2) is often called Duhamel formula.

Existence and uniqueness for equation (1) is given by the following result (see, for instance, Proposition 2.1 and Remark 2.7 in [BMS82]).

Proposition 1. *Assume that* (A, B) *satisfies Assumption 1. Then, for every* ψ_0 *in* X*, for every* u *in* $L^1_{loc}([0, +\infty), \mathbf{R})$*, there exists a unique mild solution* $t \mapsto \Upsilon^u_{t,0}\psi_0$ *to the Cauchy problem (1). Moreover, for every* ψ_0 *in* X*, the end-point mapping* $\Upsilon_{\cdot,0}\psi_0 : [0, +\infty) \times L^1_{loc}([0, +\infty), \mathbf{R}) \to X$ *is continuous.*

Definition 2. *Assume that* (A, B) *satisfies Assumption 1 and let* $\mathcal{U}$ *be a subset of* $L^1_{loc}([0, +\infty), \mathbf{R})$*. For every* ψ_0 *in* X*, the* attainable set from ψ_0 with controls in $\mathcal{U}$ *is defined as*

$$\mathcal{A}(\psi_0, \mathcal{U}) = \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{U}} \{\Upsilon^u_{T,0}\psi_0\}.$$

Our main result is the following property of the attainable set of system (1) with L^1 controls.

Theorem 2. *Assume that* (A, B) *satisfies Assumption 1. Then, for every* ψ_0 *in* X*, the attainable set* $\mathcal{A}(\psi_0, L^1_{loc}([0, +\infty), \mathbf{R}))$ *from* ψ_0 *with* L^1_{loc} *controls is contained in a countable union of compacts subsets of* X*.*

1.2 The Ball–Marsden–Slemrod obstruction

Our main result Theorem 2 is an extension of the well-known Ball–Marsden–Slemrod obstruction to controllability.

Theorem 3 (Theorem 3.6 in [BMS82])**.** *Assume that* (A, B) *satisfies Assumption 1. Then, for every* ψ_0 *in* X*, the attainable set* $\mathcal{A}(\psi_0, \cup_{r>1} L^r_{loc}([0, +\infty), \mathbf{R}))$ *from* ψ_0 *with* L^r_{loc} *controls,* $r > 1$*, is contained in a countable union of compacts subsets of* X*.*

A consequence of Theorem 3 to the framework of the conservative bilinear Schrödinger equation is given by Turinici.

Theorem 4 (Theorem 1 in [Tur00])**.** *Assume that* (A, B) *satisfies Assumption 1. Then, for every* ψ_0 *in* X*, the set* $\cup_{\alpha>0}\alpha\mathcal{A}(\psi_0, \cup_{r>1} L^r_{loc}([0, +\infty), \mathbf{R}))$ *is contained in a countable union of compacts subsets of* X*.*

Theorems 2 and 3 are basically empty in the case in which X is finite dimensional, indeed, in this case, X itself is a countable union of compact sets. On the other hand, when X is infinite dimensional, these results represent a strong topological obstruction to the exact controllability. Indeed, compact subsets of an infinite dimensional Banach space have empty interiors and so is a countable union of closed subsets with empty interiors (as a consequence of Baire Theorem).

Let us recall briefly the method used in [BMS82] for the proof of Theorem 3. The first step is to write

$$\mathcal{A}(\psi_0, \cup_{r>1} L^r_{loc}([0,+\infty), \mathbf{R})) = \bigcup_{T\geq 0} \bigcup_{r>1} \bigcup_{u\in L^r([0,T],\mathbf{R})} \{\Upsilon^u_{T,0}\psi_0\}$$
$$= \bigcup_{l\in\mathbf{N}} \bigcup_{m\in\mathbf{N}} \bigcup_{k\in\mathbf{N}} \bigcup_{0\leq t\leq l} \left(\bigcup_{\|u\|_{L^{1+1/m}}\leq k} \{\Upsilon^u_{t,0}\psi_0\} \right).$$

Hence it is sufficient to prove that, for every (l,m,k) in $\mathbf{N}^3$, the set

$$\mathcal{A}^{l,m,k} = \bigcup_{0\leq t\leq l} \left(\bigcup_{\|u\|_{L^{1+1/m}}\leq k} \{\Upsilon^u_{t,0}\psi_0\} \right)$$

has compact closure in X. To this end, one considers a sequence $(\psi_n)_{n\in\mathbf{N}}$ in $\mathcal{A}^{l,m,k}$, associated with a sequence of times $(t_n)_{n\in\mathbf{N}}$ in $[0,l]$ and a sequence of controls $(u_n)_{n\in\mathbf{N}}$ in the ball of radius k of $L^{1+1/m}([0,+\infty),\mathbf{R})$. By compactness of $[0,l]$, up to extraction, one can assume that $(t_n)_{n\in\mathbf{N}}$ tends to t_∞ in $[0,l]$. By Banach–Alaoglu–Bourbaki Theorem, the balls of $L^{1+1/m}([0,+\infty),\mathbf{R})$ are weakly (sequentially) compact and, hence, up to extraction, one can assume that $(u_n)_{n\in\mathbf{N}}$ converges weakly in $L^{1+1/m}([0,+\infty),\mathbf{R})$ to some u_∞. The hard point of the proof (Lemma 3.7 in [BMS82]) is then to show that $\Upsilon^{u_n}_{t_n,0}\psi_0$ tends to $\Upsilon^{u_\infty}_{t_\infty,0}$ as n tends to infinity.

A crucial point in the proof of Theorem 3 given in [BMS82] is the fact that the closed balls of L^p, $p>1$ are weakly sequentially compact. This is no longer true for the balls of L^1, and this prevents a direct extension of the proof of Theorem 3 to the proof of Theorem 2.

1.3 Content

In this note we present a simple and short proof of Theorem 2. However, historical reasons have made different communities use incompatible terminologies and, in order to avoid ambiguities, we present in Section 2 a quick reminder of basic facts in Banach topologies. Section 3 gives a short introduction to the classical Dyson expansion (Section 3.1) and the proof of an instrumental compactness property (Section 3.2). We conclude in Section 4 with the proof of Theorem 2.

2 Basic facts on the topology in Banach spaces

2.1 Notations

The Banach space X is endowed with norm $\|\cdot\|$. For every ψ_c in X and every $r>0$, $B_X(\psi_c,r)$ denotes the ball of center ψ_c and of radius r:

$$B_X(\psi_c,r) = \{\psi \in X | \|\psi - \psi_c\| < r\}.$$

In the following, all we need to know about generators of C^0-semigroup is the classical result stated in Proposition 5 (see Chapter VII of [HP57]).

Proposition 5. *Assume that A generates a C^0-semigroup. Then there exist $M,\omega>0$ such that $\|e^{At}\| \leq Me^{\omega t}$ for every $t\geq 0$.*

2.2 Compact subset of Banach spaces

Definition 3. *Let X be a Banach space and Y be a subset of X. A family $(O_i)_{i\in I}$ is an open cover of Y if O_i is open in X for every i in I and $Y \subset \cup_{i\in I} O_i$.*

Definition 4. *Let X be a Banach space. A subset Y of X is said to be* compact *if from any open cover of Y, it is possible to extract a finite cover of Y.*

Definition 5. *Let X be a Banach space. A subset Y of X is said to be* sequentially compact *if from any sequence $(\psi_n)_{n\in\mathbf{N}}$ taking value in Y , it is possible to extract a subsequence $(\psi_{\phi(n)})_{n\in\mathbf{N}}$ converging in Y.*

Definition 6. *Let X be a Banach space. A subset Y of X is said to be* totally bounded *if for every $\varepsilon > 0$, there exist $N \in \mathbf{N}$ and a finite family $(x_i)_{1\leq i\leq N}$ in X such that*

$$Y \subset \bigcup_{i=1}^{N} B_X(x_i, \varepsilon).$$

Proposition 6. *Let X be a Banach space. For every subset Y of X, the following assertions are equivalent:*

1. *Y is compact.*
2. *Y is sequentially compact.*
3. *Y is complete and totally bounded.*
4. *Y is closed and totally bounded.*

Proposition 7. *Let X be a Banach space, N in $\mathbf{N}$ and $(Y_i)_{1\leq i\leq N}$ a finite family of compact subsets of X. Then, the finite sum*

$$\sum_{i=1}^{N} Y_i = \{y_1 + y_2 + \ldots + y_N \mid y_i \in Y_i, i = 1, \ldots, N\}$$

is compact as well.

Proposition 8. *Let X be a Banach space, N in $\mathbf{N}$ and $(Y_i)_{1\leq i\leq N}$ a finite family of totally bounded subsets of X. Then, the finite sum*

$$\sum_{i=1}^{N} Y_i = \{y_1 + y_2 + \ldots + y_N \mid y_i \in Y_i, i = 1, \ldots, N\}$$

is totally bounded as well.

Proposition 9. *Let X be a Banach space, $T > 0$ and (A, B) satisfies Assumption 1. Define the mapping*

$$\begin{array}{rccc} F: & [0,T]\times[0,T]\times X & \to & X \\ & (s,t,\psi) & \mapsto & e^{|t-s|A}B\psi \end{array}$$

Then, for every totally bounded subset Y of X, the set $F([0,T]\times[0,T]\times Y)$ is totally bounded as well.

Proof. We claim that $G : (t, \psi) \mapsto e^{tA}\psi$ is jointly continuous in its two variables. Indeed, for every ψ, ψ_0 in X, for every $t, t_0 \geq 0$,

$$\begin{aligned} \|e^{tA}\psi - e^{t_0 A}\psi_0\| &\leq \|e^{tA}(\psi - \psi_0)\| + \|(e^{tA} - e^{t_0 A})\psi_0\| \\ &\leq Me^{\omega t}\|\psi - \psi_0\| + \|(e^{tA} - e^{t_0 A})\psi_0\|. \end{aligned}$$

This last quantity tends to zero as (t, ψ) tends to (t_0, ψ_0). As a consequence, F is continuous (as composition of continuous functions).

If Y is totally bounded, the topological closure $\bar{Y}$ of Y is compact (because the ambient space X is complete). Hence $[0,T] \times [0,T] \times \bar{Y}$ is compact. By continuity, $F([0,T] \times [0,T] \times \bar{Y})$ is compact, hence is totally bounded. The set $F([0,T] \times [0,T] \times Y)$, which is contained in $F([0,T] \times [0,T] \times \bar{Y})$, is, therefore, totally bounded as well. □

2.3 Partition of unity in Banach spaces

Definition 7. *Let X a Banach space. A family $(x_i)_{i \in I}$ of points of X is* locally finite *if for every x in X and every $R > 0$, the cardinality of the set*

$$\left(\bigcup_{i \in I} \{x_i\} \right) \cap B_X(x, R)$$

is finite.

Definition 8. *Let X be a Banach space, Y be a subset of X, and $(O_i)_{i \in I}$ be an open cover of Y. A family $(\phi_i)_{i \in I}$ of continuous functions from Y to $[0,1]$ is called a* partition of the unity of Y adapted to the cover $(O_i)_{i \in I}$ *if*

(i) for every $i \in I$, $\phi_i(x) = 0$ for every $x \notin O_i$;

(ii) $\sum_{i \in I} \phi_i(x) = 1$ for every $x \in Y$.

Proposition 10. *Let X be a Banach space, Y a subset of X, $\delta > 0$, $(x_j)_{j \in J}$ a locally finite family of points in Y such that $Y \subset \cup_{j \in J} B_X(x_j, \delta)$. Then, there exists $(\phi_j)_{j \in J}$ a partition of the unity adapted to the open cover $(B(x_j, 2\delta))_{j \in J}$ of Y.*

Moreover, if a family $(\phi_j)_{j \in J}$ is a partition of the unity adapted to the open cover $(B_X(x_j, 2\delta))_{j \in J}$, then for every x in Y, $\|x - \sum_{j \in J} \phi_j(x) x_j\| \leq 2\delta$.

Proof. We first prove the existence of a partition of the unity adapted to the open cover $(B_X(x_j, 2\delta))_{j \in J}$ of Y. To this end, we define, for every j in J, the continuous functions $\varphi_j : X \to [0,1]$ by

$$\begin{cases} \varphi_j(x) = 1, & \text{if } \|x - x_j\| < \delta, \\ \phi_j(x) = 2 - \|x - x_j\|/\delta, & \text{if } \delta \leq \|x - x_j\| < 2\delta, \\ \varphi_j(x) = 0, & \text{if } 2\delta \leq \|x - x_j\|. \end{cases}$$

Since the family $(x_j)_{j \in J}$ is locally finite, the sum $\sum_{j \in J} \varphi_j(x)$ converges for every x in Y. Moreover, since $Y \subset \cup_{j \in J} B(x_j, \delta)$, the function $x \mapsto \sum_{j \in J} \varphi_j(x)$ does not vanish on Y. For every j_0 in J, we define ϕ_{j_0} by

$$\phi_{j_0}(x) = \varphi_{j_0}(x) \frac{1}{\sum_{j \in J} \varphi_j(x)},$$

and the family $(\phi_j)_{j\in J}$ is a partition of the unity adapted to the open cover $(B_X(x_j,2\delta))_{j\in J}$ of Y.

We now prove the second point of Proposition 10. Let $(\phi_j)_{j\in J}$ be a partition of unity of Y adapted to the cover $(B_X(x_j,2\delta))_{j\in J}$. Then, for every x in Y,

$$\begin{aligned}\left\|x-\sum_{j\in J}\phi_j(x)x_j\right\| &= \left\|\sum_{j\in J}\phi_j(x)x-\sum_{j\in J}\phi_j(x)x_j\right\| \\ &= \left\|\sum_{j\in J}\phi_j(x)(x-x_j)\right\| \\ &\leq \sum_{j\in J}\phi_j(x)\left\|x-x_j\right\|.\end{aligned}$$

By construction, $\phi_j(x)=0$ as soon as $\|x-x_j\|\geq 2\delta$. Hence,

$$\left\|x-\sum_{j\in J}\phi_j(x)x_j\right\| \leq 2\delta\sum_{j\in J}\phi_j(x)\leq 2\delta,$$

which concludes the proof. □

3 The Dyson expansion

3.1 The Dyson Operators

For every u in $L^1_{loc}([0,+\infty),\mathbf{R})$, $p\in\mathbf{N}$, and $t\geq 0$ we define the linear bounded operator $W_p(t,u): X\to X$ recursively by

$$\begin{aligned}&W_0(t,u)\psi = e^{(t-s)A}\psi\\ &W_p(t,u)\psi=\int_0^t e^{(t-s)A}BW_{p-1}(s,u)\psi u(s)\mathrm{d}s, \qquad \text{for } p\geq 1,\end{aligned}$$

for every ψ in X. We have the following estimate on the norm of the operator.

Proposition 11. *For every u in $L^1_{loc}([0,+\infty),\mathbf{R})$, $p\in\mathbf{N}$, and $t\geq 0$*

$$\|W_p(t,u)\|\leq \frac{Me^{\omega t}\|B\|^p(\int_0^t|u(s)|\mathrm{d}s)^p}{p!}.$$

Proof. We prove the result by induction on p in $\mathbf{N}$. For $p=0$ the result clearly follows from Proposition 5. Assume that the result holds for $p\geq 0$. Then, for every ψ in X,

$$\begin{aligned}\|W_{p+1}(t,u)\psi\| &\leq \int_0^t e^{(t-s)A}BW_p(s,u)\psi u(s)\mathrm{d}s\\ &\leq M\int_0^t e^{(t-s)\omega}\|B\|\frac{e^{\omega s}\|B\|^p(\int_0^s|u(\tau)|\mathrm{d}\tau)^p}{p!}u(s)\mathrm{d}s\\ &\leq Me^{\omega t}\|B\|^{p+1}\frac{(\int_0^t|u(\tau)|\mathrm{d}\tau)^{p+1}}{(p+1)!}.\end{aligned}$$

The last inequality follows from Proposition 5. We conclude the proof by induction on p. □

3.2 A compactness property

Lemma 12. *For every j in $\mathbf{N}$, $T \geq 0$ and $K \geq 0$, and ψ in X the set*

$$\mathcal{W}_j^{T,K} = \{W_j(t,u)\psi \mid 0 \leq t \leq T, \|u\|_{L^1} \leq K\}, \tag{3}$$

is totally bounded

Proof. We prove the result by induction on j in $\mathbf{N}$. For $j = 0$, consider $\mathcal{W}_0^{T,K} = \{e^{tA}\psi, 0 \leq t \leq T\}$ and let $(w_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{W}_0^{T,K}$. Then there exists a sequence $(t_n)_{n\in\mathbf{N}}$ such that $w_n = e^{t_nA}\psi$ for every n. Up to extraction $\lim_{n\to\infty} t_n = t \in [0,T]$ since $[0,T]$ is compact. By definition of C^0-semigroup, $\lim_{n\to\infty} e^{t_nA}\psi = e^{tA}\psi$. This proves that $\mathcal{W}_0^{T,K}$ is sequentially compact, hence compact and, in particular, totally bounded (Proposition 6).

Assume that, for $j \geq 0$, $\mathcal{W}_j^{T,K}$ is totally bounded. By Proposition 9, the set

$$\begin{aligned} Z_j^{T,K} &:= \{e^{(t-s)A}B\psi, \psi \in W_j^{T,K}, 0 \leq s \leq t \leq T\} \\ &\subset F([0,T]^2 \times \mathcal{W}_j^{T,K}) \end{aligned}$$

is totally bounded as well.

Let $\varepsilon > 0$ be given and define $\delta = \frac{\varepsilon}{2K+1} > 0$. Since $Z_j^{T,K}$ is totally bounded, there exists a finite family $(x_i)_{1\leq i\leq N_\delta}$ in $Z_j^{T,K}$ such that

$$Z_j^{T,K} \subset \bigcup_{i=1}^{N_\delta} B_X(x_i, \delta).$$

Let $(\phi_i)_{\leq i \leq N_\delta}$ be a partition of the unity adapted to the cover $\cup_{i=1}^{N_\delta} B(x_i, 2\delta)$ of $Z_j^{T,K}$. Such a partition of the unity exists by Proposition 10, and moreover, for every x in $Z_j^{T,K}$, we have

$$\left\| x - \sum_{i=1}^{N_\delta} \phi_i(x)x_i \right\| \leq 2\delta. \tag{4}$$

Applying the inequality (4) with $x = e^{(t-s)A}BW_j(s,u)\psi_0$, we get, for every u in L^1 and every (s,t) such that $0 \leq s \leq t$,

$$\left\| e^{(t-s)A}BW_j(s,u)\psi_0 - \sum_{i=1}^{N_\delta} \phi_i(e^{(t-s)A}BW_j(s,u)\psi_0)x_i \right\| \leq 2\delta.$$

Multiplying by $u(s)$ and integrating for s in $[0,t]$, one gets for $\|u\|_{L^1} \leq K$

$$\left\| \int_0^t e^{(t-s)A}BW_j(s,u)\psi_0 u(s)\mathrm{d}s - \sum_{i=1}^{N_\delta} \int_0^t \phi_i(e^{(t-s)A}BW_j(s,u)\psi_0)u(s)\mathrm{d}s\, x_i \right\| \leq 2\delta K. \tag{5}$$

The set $\sum_{i=1}^{N_\delta}[0,K]x_i$ is compact by Proposition 7 and, hence, totally bounded. Then there exists a finite family $(y_i)_{1\leq i\leq N'_\delta}$ such that

$$\sum_{i=1}^{N_\delta}[0,K]x_i \subset \bigcup_{i=1}^{N'_\delta} B_X(y_i, \delta). \tag{6}$$

From (5) and (6), one deduces that

$$\mathcal{W}_{j+1}^{T,K} \subset \bigcup_{i=1}^{N'_\delta} B_X(y_i,(2K+1)\delta) = \bigcup_{i=1}^{N'_\delta} B_X(y_i,\varepsilon).$$

This proves that $\mathcal{W}_{j+1}^{T,K}$ is totally bounded and concludes the proof. □

3.3 Convergence of the Dyson expansion

Proposition 13. *For every u in $L^1([0,+\infty),\mathbf{R})$, $t \geq 0$, and $\psi_0 \in X$*

$$\left\|\Upsilon_{t,0}^u \psi_0\right\| \leq M e^{\omega t}\|\psi_0\| \exp\left(M e^{\omega t}\|B\| \int_0^t |u(s)|\mathrm{d}s\right)$$

Proof. The proof follows the proof of [BMS82, Theorem 2.5]. By Duhamel formula (2) and Proposition 5,

$$\begin{aligned}\left\|\Upsilon_{t,0}^u \psi_0\right\| &= \left\|e^{tA}\psi_0 + \int_0^t e^{(t-s)A} B \Upsilon_{s,0}^u \psi_0 u(s)\mathrm{d}s\right\| e^{t\omega}\psi_0 \\ &\leq M e^{t\omega}\|\psi_0\| + \int_0^t M e^{\omega(t-s)}\|B\| \left\|\Upsilon_{s,0}^u\psi_0\right\| u(s)\mathrm{d}s,\end{aligned}$$

and the conclusion follows by Gronwall lemma (see [BMS82, Lemma 2.6]). □

Proposition 14. *For every u in $L^1([0,+\infty),\mathbf{R})$, p in $\mathbf{N}$, $t \geq 0$, and ψ_0 in X*

$$\lim_{p\to\infty}\left\|\int_0^t e^{(t-s)A} B W_p(s,t)\Upsilon_{s,0}^u \psi_0 u(s)\mathrm{d}s\right\| = 0.$$

Proof. Consider

$$\left\|\int_0^t e^{(t-s)A} B W_p(s,t)\Upsilon_{s,0}^u \psi_0 u(s)\mathrm{d}s\right\| \leq \int_0^t \|e^{(t-s)A}\|\|B\|\|W_p(s,t)|\Upsilon_{s,0}^u\psi_0\||u(s)|\mathrm{d}s$$

and recall that $\|W_p(s,t)\|$ tends to zero as p tends to infinity (Proposition 11). □

Proposition 15. *For every u in $L^1_{loc}([0,+\infty),\mathbf{R})$, p in $\mathbf{N}$, $t \geq 0$, and ψ_0 in X*

$$\Upsilon_{t,0}^u\psi_0 = \sum_{p=0}^{\infty} W_p(t,0,u)\psi_0.$$

Proof. Applying iteratively p-times Duhamel formula (2), one gets

$$\begin{aligned}\Upsilon_{t,0}^u\psi_0 &= e^{tA}\psi_0 + \int_0^t e^{(t-s)A} B u(s)\Upsilon_{s,0}^u\psi_0\mathrm{d}s \\ &= e^{tA}\psi_0 + \int_0^t e^{(t-s)A} B u(s) e^{sA}\psi_0\mathrm{d}s + \int_0^t e^{(t-s)A} B W_1(s,t)\Upsilon_{s,0}^u\psi_0 u(s)u(s)\mathrm{d}s \\ &= \sum_{j=1}^{p} W_p(t,0)\psi_0 + \int_0^t e^{(t-s)A} B W_p(s,t)\Upsilon_{s,0}^u\psi_0 u(s)\mathrm{d}s.\end{aligned}$$

Hence, for every $p \geq 1$,

$$\Upsilon^u_{t,0}\psi_0 - \sum_{j=0}^{p} W_j(t,0)\psi_0 = \int_0^t e^{(t-s)A} B W_p(s,t) \Upsilon^u_{s,0}\psi_0 u(s) \mathrm{d}s$$

and the result follow from Proposition 14 as p tends to ∞. $\square$

4 Proof of Theorem 2

We proceed now to the proof of Theorem 2. First of all, notice that, for every ψ_0 in X,

$$\begin{aligned}\mathcal{A}(\psi_0, L^1_{loc}([0,+\infty), \mathbf{R}) \\ = \bigcup_{l\in\mathbf{N}} \bigcup_{m\in\mathbf{N}} \{\Upsilon^u_{t,0}\psi_0, \|u\|_{L^1} \leq l, 0 \leq t \leq m\},\end{aligned}$$

and it is enough to prove that, for every l and m in $\mathbf{N}$, the set

$$\{\Upsilon^u_{t,0}\psi_0, \|u\|_{L^1} \leq l, 0 \leq t \leq m\}$$

is totally bounded.

Let $\varepsilon > 0$. From the convergence of the Dyson expansion (Proposition 15) and the bound on the operators W_j (Proposition 11), there exists a integer N_ε such that

$$\left\| \sum_{p\geq N_\varepsilon} W_p(t,u)\psi_0 \right\| \leq \frac{\varepsilon}{2}, \tag{7}$$

for every t in $[0,m]$ and every u such that $\|u\|_{L^1} \leq l$. For each $j = 1, \ldots, N_\varepsilon$ the sets $\mathcal{W}^{m,l}_j$, defined by (3), are totally bounded (Lemma 12), hence their sum

$$\sum_{j=0}^{N_\varepsilon} \mathcal{W}^{m,l}_j$$

is totally bounded as well (Proposition 8). Hence there exists a family $(x_i)_{1\leq i\leq N_1}$ of points of $\sum_{j=0}^{N_\varepsilon} \mathcal{W}^{m,l}_j$ such that

$$\sum_{j=0}^{N_\varepsilon} \mathcal{W}^{m,l}_j \subset \bigcup_{i=1}^{N_1} B_X\left(x_i, \frac{\varepsilon}{2}\right). \tag{8}$$

Gathering (7) and (8), one gets

$$\sum_{j=0}^{\infty} \mathcal{W}^{m,l}_j \subset \bigcup_{i=1}^{N_1} B_X\left(x_i, \varepsilon\right),$$

which concludes the proof of Theorem 2.

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