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Abstract—A method is described for representing a structural component by means of its vibration modes. The modes used to describe the component may have the connection points to the remainder of the structure free, or fixed, or some points free and some points fixed. The modes may either be calculated or experimentally measured. Statically determined deflection influence coefficients may be used to improve the accuracy of the representation.

The advantages claimed for the method derive from the generality of the conditions under which the component modes are calculated (or measured). Thus the boundary conditions may be selected to optimize accuracy or, in the event that the modes have already been obtained, the method permits the *available* data to be used. Examples are presented that illustrate use of the method, and the significance of the improvements derived from static calculations.

INTRODUCTION

COMPONENT mode synthesis is a method of solving structural dynamic problems in which the degrees of freedom consist, largely or entirely, of the uncoupled vibration modes that exist when the structure is divided into several separate parts. The method owes its current popularity [1, 2, 3, 4] to the economics of computerized finite element structural analysis. It is a matter of common experience that computer time in a vibration analysis increases more than linearly with an increase in the number of degrees of freedom (about as the square or cube, depending on the eigenvalue extraction method and the type of structure). Thus, if a structure is divided into N parts, the N separate component mode analyses consume less time than the analysis of the whole structure, and the difference in time will, it is hoped, exceed the time required to *synthesize* the overall solution from the component modes. A net saving in time results if this is true, but at the cost of a loss in accuracy due to the retention of only a fraction of the component modes in the synthesis.

Component mode synthesis need not be formulated in a highly sophisticated manner in order to be useful. Vibration analysts, and more particularly vibration test engineers, often think of appendages and equipment packages as spring-mass combinations in which the spring constants are calculated from the masses and the frequencies of the lower vibration modes. *Electrical* engineers, however, were the first to develop a systematic approach to the representation of a system in terms of its components. The impetus was provided by the need to design networks with specified properties as seen at their terminals; as a result, electrical engineers learned to think of groups of circuit components as 'black boxes' that could be described in terms of their 'impedances.' An important early (1924) development was Foster's Reactance Theorem [5] in which the impedance of a general two-terminal reactive network was synthesized in terms of electrical components derived from the vibration modes (zeros and poles) of the circuit. Two of Foster's canonical forms are shown in Fig. 1. Those readers who are familiar with the mobility analogy [6] will recognize the second form as the electrical analog of a parallel collection of sprung masses, each of which represents a 'clamped' vibration mode of the subsystem. The first

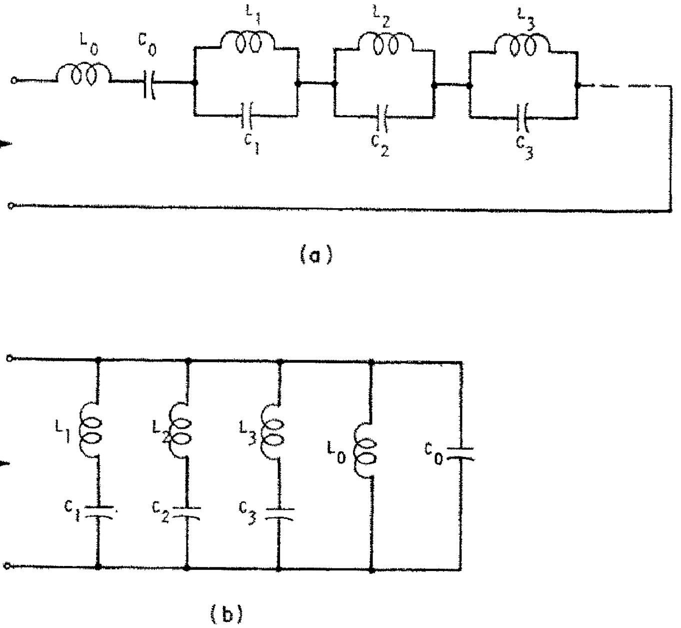


FIG. 1. Foster's Canonical forms for purely reactive networks. (a) Series combination of parallel-resonant circuits. (b) Parallel combination of series-resonant circuits.

form, which is a bit harder for mechanical engineers to interpret, is a series combination of circuits each of which represents a 'free' vibration mode of the subsystem.

The author's attention was first drawn to component mode synthesis by the need for an economic representation of substructures in analog computer solutions of structural dynamic problems. 'Economy' in this sense meant the representation of a substructure using only a minimum number of electrical elements. Foster's Reactance Theorem provided a solution for a substructure with a single connection point (terminal pair), but a generalization was required for substructures with two or more connection points. The viewpoint was taken that the connection points of the substructure could be either clamped or free (or some clamped and some free) when the vibration modes were evaluated. The properties of the substructure models were represented exclusively in terms of properties that could be obtained from vibration tests. This feature, which is a product of an electrical engineer's 'black box' thinking, contrasts sharply with more recently developed methods [1, 2, 3] which require knowledge of the detailed properties (stiffness and mass matrices) of the substructure.

The author's early work [7, 8] on component mode synthesis was expressed in terms of electrical analogies and so remained inaccessible to many mechanical engineers. More recently it has been restated in purely mathematical terms as part of the documentation for the NASTRAN structural analysis program [9]; the latter exposition is, however, limited to the more elementary aspects of the theory. The purpose of the present paper is to present a complete exposition of the author's hybrid method of component mode synthesis, including both published and unpublished results.

REPRESENTATIONS IN TERMS OF FREE VIBRATION MODES

When part of a structure is described by vibration modes, it must first be ascertained how the degrees of freedom at which it is connected to the remainder of the structure were supported when the vibration modes were measured (or computed). Three cases are distinguished:

(1) All connection coordinates free.
(2) All connection coordinates restrained.
(3) Some connection coordinates free and some restrained.

The first condition is usually employed in vibration tests or analyses of large parts, such as an airplane fuselage. The second condition is usually employed in vibration tests or analyses of small parts, such as a horizontal stabilizer. Case 3 (some connection coordinates free and some restrained) contains the first two as subcases. Generality with respect to the boundary conditions for the vibration modes is important for two reasons: first, if test data or the results of a previous analysis are used, the analyst may have no options left regarding the selection of boundary conditions; second, better results will be obtained if the boundary conditions for the substructure modes resemble the true conditions of the composite structure, e.g., a free boundary condition is better than a restrained boundary condition for a point on a heavy structure where a light appendage will be attached.

Case 1 will be discussed first, because it is the simplest. The required data are the vibration mode frequencies, ω_i, the mode shapes or eigenvectors, $\{\phi_i\}$, and the modal masses, m_i. The eigenvectors need not be normalized in any particular manner. Let the degrees of freedom at the points of connection to the remainder of the structure be designated by the vector $\{\mathbf{u}_c\}$. Then the motions of these points are related to the modal coordinates $\{\xi_i\}$ of the substructure by

$$\{\mathbf{u}_c\}=[\phi_{ci}]\{\xi_i\} \tag{1}$$

which may be regarded as an equation of constraint between dependent coordinates $\{\mathbf{u}_c\}$ and independent coordinates $\{\xi_i\}$. The columns of $[\phi_{ci}]$ are the eigenvectors, $\{\phi_i\}$, abbreviated to include only the degrees of freedom at connection points, $\{\mathbf{u}_c\}$. Usually only a finite number of eigenvectors are included in $[\phi_{ci}]$ and this approximation produces an idealized model for the substructure that is too stiff. Specification of the substructure is completed by calculating the generalized mass, m_i, stiffness, k_i, and damping, b_i, associated with each modal coordinate, ξ_i, as follows.

$$m_i=\{\phi_i\}^T[M_p]\{\phi_i\} \tag{2}$$

$$k_i=\omega_i^2 m_i \tag{3}$$

$$b_i=g_i m_i \omega_i \tag{4}$$

where g_i is a damping factor for the ith mode and $[M_p]$ is the mass matrix of the substructure. Reference [7] describes a method for calculating m_i from experimental measurements at the connection points. This matter need not detain us, however, if we assume that the detailed properties of the substructure are known, so that m_i can be evaluated from equation (2).

The equation of motion for the generalized coordinate, ξ_i, is

$$(m_i p^2 + b_i p + k_i)\xi_i = \{\phi_{ci}\}^T \{\mathbf{f}_c\} = f_i \tag{5}$$

where $\{\mathbf{f}_c\}$ is the vector of forces applied to the substructure at the connection points, $\{\phi_{ci}\}$ is the eigenvector, $\{\phi_i\}$, abbreviated to include only the degrees of freedom at connection points, and $p = \mathrm{d}/\mathrm{d}t$.

Equations (1) and (5) are the equations of state of the idealized model of the substructure and equations (2), (3) and (4) define its parameters. The way in which they may be incorporated into a digital computer program for structural analysis depends on the general provisions of the program. The NASTRAN program [9] accepts equations of constraint (equation (1)) as input data, permits both generalized scalar coordinates (ξ_i) and physical displacement components ($\mathbf{u}_c$) to be specified as degrees of freedom, and includes lumped masses, springs, and dampers (m_i, k_i and b_i) as structural elements. Thus NASTRAN includes all of the necessary provisions for the user to describe the structural model of the component to the program. Figure 2 is a schematic representation of the model. All of the input data (connections, element properties, and coefficients of constraint) are inserted via data cards. The relationship between forces of constraint,

$$\{\mathbf{f}_i\} = [\phi_{ci}]^T \{\mathbf{f}_c\} \tag{6}$$

is automatically satisfied by internal procedures.

It is not hard to imagine how the procedure could be completely automated so that, for example, the results of a vibration analysis of the substructure could be posted directly to the analysis of the composite structure. A potential difficulty which should be pointed out occurs because the degrees of freedom of the boundary connection points, $\{\mathbf{u}_c\}$, are removed from the list of independent degrees of freedom by equation (1). Thus, if additional modally synthesized substructures are connected to the same points, an over-constraint will result unless the number of modal coordinates, $\{\xi_i\}$, in each additional substructure equals or exceeds the number of degrees of freedom in $\{\mathbf{u}_c\}$. This difficulty can be avoided by employing some of the procedures that will be described later.

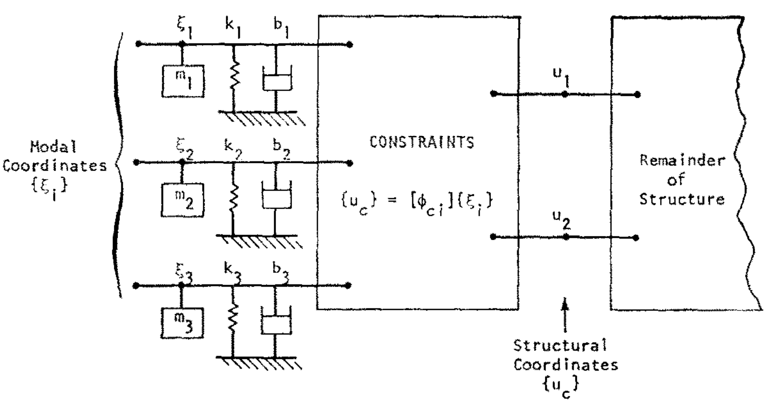


FIG. 2. Representation of part of a structure by its vibration modes when all connection points are free while the modes are calculated.

THE HYBRID REPRESENTATION

The derivation of an idealized model for the case when some, or all, of the connection points are restrained during measurement (or calculation) of the substructure modes, is considerably more involved than the derivation for the case of free vibration modes which has just been described. A general solution, that was first published in [7], is developed below. The objective is to derive a set of constraint relationships between the modal coordinates and the degrees of freedom at connection points (both free and restrained). The modal mass, damping, and stiffness properties will, as in the case of free vibration modes, be simulated by scalar structural elements.

Let the degrees of freedom of the substructure be partitioned into $\{u_a\}$, degrees of freedom that are free in the substructure modes, and, $\{u_b\}$, degrees of freedom that are restrained in the substructure modes. The equations of motion for the substructure (without damping) can then be written as

$$\left\{\begin{array}{c} f_a \\ \hdashline f_b \end{array}\right\} = \left[\begin{array}{c:c} K_{aa}+M_{aa}p^2 & K_{ab} \\ \hdashline K_{ab}^T & K_{bb} \end{array}\right] \left\{\begin{array}{c} u_a \\ \hdashline u_b \end{array}\right\} \tag{7}$$

$\{f_a\}$ and $\{f_b\}$ are forces applied to the substructure. The mass of the substructure is assumed to be concentrated at the free coordinates, $\{u_a\}$, which include all coordinates not restrained in the substructure modes. Any substructure mass on the restrained coordinates, $\{u_b\}$, should be lumped into the remainder of the structure because the masses on restrained coordinates produce no effect during the vibration modes of the substructure. They may, accordingly, be ignored in the modal representation of the substructure. The stiffness matrix is partitioned in equation (7) according to free and restrained coordinates. Note that $\{u_a\}$ contains the free connection coordinates as a subset.

The substructure mode shapes are described by a modal transformation between the free coordinates, $\{u_a\}$, and modal coordinates, $\{\xi_i\}$.

$$\{u_a\} = [\phi_{ai}]\{\xi_i\} \tag{8}$$

The corresponding generalized forces on the modal coordinates are

$$\{f_i\} = [\phi_{ai}]^T\{f_a\} \tag{9}$$

By virtue of the orthogonality property of vibration modes

$$[\phi_{ai}]^T[K_{aa}+p^2M_{aa}][\phi_{ai}] = [k_i+m_ip^2] \tag{10}$$

where $[k_i]$ and $[m_i]$ are diagonal matrices of the modal coefficients computed by equations (2) and (3). Thus, using, equations (8), (9) and (10) to transform equation (7),

$$\left\{\begin{array}{c} f_i \\ \hdashline f_b \end{array}\right\} = \left[\begin{array}{c:c} k_i+m_ip^2 & \phi_{ai}^T K_{ab} \\ \hdashline K_{ab}^T\phi_{ai} & K_{bb} \end{array}\right] \left\{\begin{array}{c} \xi_i \\ \hdashline u_b \end{array}\right\} \tag{11}$$

It is convenient to separate the inertia forces from equation (11) so that, defining

$$\{\bar{f}_i\} = \{f_i\} - [m_i]\{p^2\xi_i\} \tag{12}$$

$$\begin{Bmatrix} \bar{f}_i \\ \hline f_b \end{Bmatrix} = \left[\begin{array}{c:c} k_i & \phi_{ai}{}^T K_{ab} \\ \hdashline K_{ab}{}^T \phi_{ai} & K_{bb} \end{array}\right] \begin{Bmatrix} \xi_i \\ \hline u_b \end{Bmatrix} \tag{13}$$

Equation (13) is a stiffness equation in standard form. A form that leads more directly to a useful physical model is obtained by placing ξ_i on the left-hand side. Thus,

$$\begin{Bmatrix} \xi_i \\ \hline f_b \end{Bmatrix} = \left[\begin{array}{c:c} k_i^{-1} & \psi_{ib} \\ \hdashline -\psi_{ib}{}^T & \bar{K}_{bb} \end{array}\right] \begin{Bmatrix} \bar{f}_i \\ \hline u_b \end{Bmatrix} \tag{14}$$

where,

$$[\psi_{ib}] = -[k_i]^{-1}[\phi_{ai}]^T[K_{ab}] \tag{15}$$

and

$$[\bar{K}_{bb}] = [K_{bb}] - [\psi_{ib}]^T[k_i][\psi_{ib}] \tag{16}$$

If the set of restrained points, $\{u_b\}$, is nonredundant, the matrix $[\bar{K}_{bb}]$ is null. This condition is assumed in the present discussion. It will be relaxed in the next section. The matrix, $[\psi_{ib}]$, is calculated from properties of the vibration modes as follows. During a vibration mode, $\{u_b\} = 0$, and the vector of forces acting on the constraints is, from equations (13) and (14),

$$\{F_b\} = -\{f_b\} = [\psi_{ib}]^T\{\bar{f}_i\} = [\psi_{ib}]^T[k_i]\{\xi_i\} \tag{17}$$

Define $[K_{bi}]$ to be the matrix of forces on the constraints due to unit values of the modal coordinates while the substructure is vibrating in its normal modes, i.e., a matrix of *eigen-reactions*.

$$\{F_b\} = [K_{bi}]\{\xi_i\} \tag{18}$$

Then, comparing equations (17) and (18),

$$[\psi_{ib}] = [k_i]^{-1}[K_{bi}]^T \tag{19}$$

or, in other words, $[\psi_{ib}]$ is equal to $[K_{bi}]^T$ with each row divided by the appropriate element of $[k_i]$. $[\psi_{ib}]$ may also be used to define an auxiliary set of modal coordinates

$$\{\zeta_i\} = [\psi_{ib}]\{u_b\} \tag{20}$$

Then, from the top half of equation (14)

$$\{\bar{f}_i\} = [k_i]\{\xi_i - \zeta_i\} \tag{21}$$

The free connection coordinates $\{u_c\}$ are a subset of $\{u_a\}$. The relation between $\{u_c\}$ and the modal coordinates $\{\xi_i\}$ is

$$\{u_c\} = [\phi_{ci}]\{\xi_i\} \tag{22}$$

where $[\phi_{ci}]$ is the appropriate partition of $[\phi_{ai}]$.

Equations (12), (20), (21) and (22) provide a complete description of the substructure. They are used to construct the idealized model of the substructure, shown in Fig. 3. The

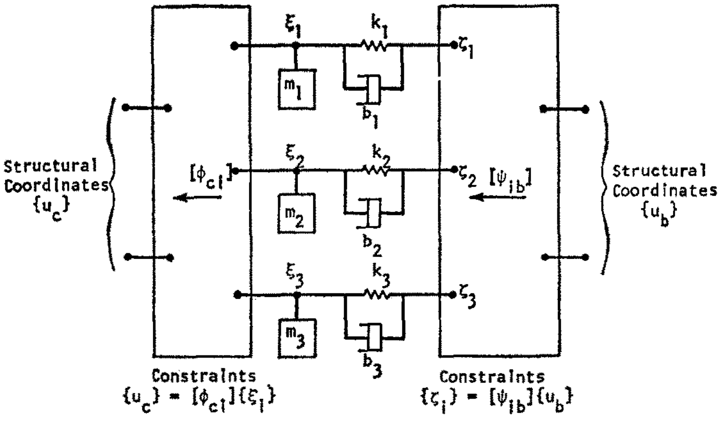


FIG. 3. Representation of part of a structure by its vibration modes in the general case when some connection points are free and some are rigidly constrained.

modal dampers, b_i, are placed across the modal springs, k_i, if they simulate structural damping. If they simulate damping due to the viscosity of a surrounding fluid environment, they should be placed between the modal coordinates and ground. The user may also, if he desires, retain some nonconnection coordinates in the model in order to record motions at other points in the substructure. This is done by constructing constraints from additional rows of equation (8). Equation (20) expresses a new set of constraint equations between the auxiliary modal coordinates and the degrees of freedom that are restrained in substructure modes.

If the NASTRAN computer program is used, the input data for the substructure consist of the coefficients of the equations of constraint, $[\phi_{ci}]$ and $[\psi_{ib}]$, the values of the scalar elements, m_i, b_i and k_i, and their connections. Again, the procedure could easily be automated to the extent of replacing the data cards by card images obtained from a modal analysis of the substructure, but this has not been done in NASTRAN.

The techniques discussed above provide the capability for complete dynamic partitioning of a structure, since all of its parts, rather than a few, may be represented by their respective vibration modes. The general case diagrammed in Fig. 3 is particularly useful for this purpose. Consider, for example, the missile structure shown in Fig. 4. The missile is physically partitioned with support conditions for the calculation of uncoupled vibration modes as shown in the figure. The first partition, (a), is unsupported while the others are cantilevered. The lumped element model for the composite system consists of parts with the form of Fig. 3 connected in tandem. The directions of the arrows in the constraint blocks point from independent coordinates toward dependent coordinates. It is evident from the form of the lumped element model that the independent degrees of freedom consist of the modal coordinates, $\{\xi_a\}$, $\{\xi_b\}$, $\{\xi_c\}$ etc., and that the displacement sets $\{u_{a,b}\}$, $\{\zeta_b\}$, $\{u_{b,c}\}$ etc., are all constrained. Furthermore, the overconstraint problem mentioned earlier has been avoided. The dynamic equations, when written by the stiffness method, are banded with bandwidth equal to the number of modal coordinates in three successive partitions.

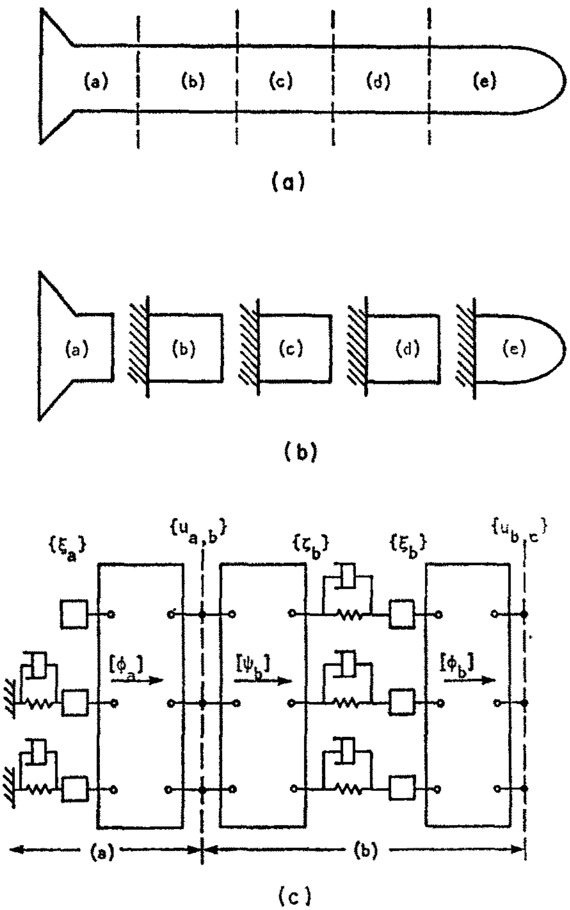


FIG. 4. Dynamic partitioning of missile structure. (a) Missile structure, unpartitioned. (b) Support conditions for partitions while calculating substructure vibration modes. (c) Portion of composite model.

THE USE OF STATICALLY DERIVED MATRICES TO IMPROVE THE REPRESENTATION

The substructure model developed in the preceding section was derived exclusively from modal properties which may be obtained either from analysis or from vibration test. Usually only those modes that occur below a user-selected maximum frequency are used so that, in general, the substructure model is approximate to the extent that the effects of its higher modes are omitted. In addition, the stiffness matrix, $[\bar{K}_{bb}]$, due to the presence of redundant supports, was ignored.

The substructure model may be improved by including additional elements derived from static approximations to the effects of the higher modes, and by accounting for the stiffness matrix due to the redundant supports. The derivation follows.

Let the restrained coordinates, $\{u_b\}$, be partitioned into a set of determinate reaction points, $\{u_r\}$, and a set of redundant reaction points, $\{u_s\}$.

$$\{u_b\} = \left\{\frac{u_s}{u_r}\right\}. \tag{23}$$

Note that, in general, $\{u_s\}$ may be null and that $\{u_r\}$ may be deficient. In order to make static deflection measurements, however, it is necessary that $\{u_r\}$ be sufficient to prevent rigid body motions.

The quantities that are determined statically are expressed by the following matrix equation, which is valid only at frequencies that are low compared to the lowest mode of the subsystem.

$$\left\{\begin{array}{c} u_c \\ \hdashline u_s \\ \hdashline f_r \end{array}\right\} = \left[\begin{array}{c:c:c} Z_{cc} & Z_{cs} & S_{cr} \\ \hdashline Z_{cs} & Z_{ss} & S_{sr} \\ \hdashline -S_{cr}^T & -S_{sr}^T & M_{rr}p^2 \end{array}\right] \left\{\begin{array}{c} f_c \\ \hdashline f_s \\ \hdashline u_r \end{array}\right\} \tag{24}$$

In equation (24), the submatrix

$$[Z] = \left[\begin{array}{c:c} Z_{cc} & Z_{cs} \\ \hdashline Z_{cs}^T & Z_{ss} \end{array}\right] \tag{25}$$

is the matrix of deflection influence coefficients at the free coordinates and at the redundant support coordinates when the determinate reaction coordinates are rigidly restrained. The submatrices $[S_{cr}]$ and $[S_{sr}]$ express the rigid body property of the substructure, namely they give the motions at the free and redundant support coordinates due to motions at the determinate supports. They are determined by the positions of the connection points and by the directions of the components of motion at the connection points. The submatrix $[M_{rr}]$ expresses the rigid mass property as seen at the determinate reaction points.

It may not always be practical to supply the submatrices in equation (24). $[S_{cr}]$ and $[S_{sr}]$ can, at least, always be calculated from the coordinate geometry. $[M_{rr}]$ should also be relatively easy to obtain. The submatrix components of the deflection influence coefficients have varying degrees of importance. The $[Z_{cc}]$ matrix will be approximated by the modal part of the representation, so that it may not be important if a large number of modes is included. The $[Z_{ss}]$ matrix and part of the $[Z_{cs}]$ matrix will not be approximated by the modal representation, and they may be important if some of the redundant supports are far away from the determinate supports.

For the purpose of deriving the improved model of the substructure, it is convenient to interchange the positions of $\{u_s\}$ and $\{f_s\}$ in equation (24). The result is

$$\left\{\begin{array}{c} u_c \\ \hdashline f_s \\ \hdashline f_r \end{array}\right\} = \left[\begin{array}{c:c:c} \bar{Z}_{cc} & \bar{S}_{cs} & \bar{S}_{cr} \\ \hdashline -\bar{S}_{cr}^T & \bar{K}_{ss} & \bar{K}_{sr} \\ \hdashline -\bar{S}_{cr}^T & \bar{K}_{sr}^T & \bar{K}_{rr}+M_{rr}p^2 \end{array}\right] \left\{\begin{array}{c} f_c \\ \hdashline u_s \\ \hdashline u_r \end{array}\right\} \tag{26}$$

where,

$$\left.\begin{aligned}
&[\bar{Z}_{cc}]=[Z_{cc}]-[Z_{cs}][\bar{Z}_{ss}]^{-1}[Z_{cs}]^T\\
&[\bar{S}_{cs}]=[Z_{cs}][Z_{ss}]^{-1}\\
&[\bar{S}_{cr}]=[S_{cr}]-[Z_{cs}][Z_{ss}]^{-1}[S_{sr}]\\
&[\bar{K}_{ss}]=[Z_{ss}]^{-1}\\
&[\bar{K}_{sr}]=-[Z_{ss}]^{-1}[S_{sr}]\\
&[\bar{K}_{rr}]=[S_{sr}]^T[Z_{ss}]^{-1}[S_{sr}]
\end{aligned}\right\} \tag{27}$$

Equation (26) may be consolidated as follows:

$$\begin{Bmatrix} u_c \\ \hline f_b \end{Bmatrix}=\left[\begin{array}{c:c} \bar{Z}_{cc} & S_{cb} \\ \hdashline -S_{cb}^T & \bar{K}_{bb}+M_{bb}p^2 \end{array}\right]\begin{Bmatrix} f_c \\ \hline u_b \end{Bmatrix} \tag{28}$$

where,

$$[S_{cb}]=[\bar{S}_{cs}\,\vdots\,\bar{S}_{cr}] \tag{29}$$

$$[\bar{K}_{bb}]=\left[\begin{array}{c:c} \bar{K}_{ss} & \bar{K}_{sr} \\ \hdashline \bar{K}_{sr}^T & \bar{K}_{rr} \end{array}\right] \tag{30}$$

and

$$[M_{bb}]=\left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & M_{rr} \end{array}\right] \tag{31}$$

It may be noted that $[\bar{K}_{bb}]$ in equation (28) is identically equal to $[\bar{K}_{bb}]$ in equation (14). In some cases $[\bar{Z}_{cc}]$, the deflection influence coefficient matrix with *all* support points restrained, may be more readily available than $[Z_{cc}]$. This option should be provided to the user.

The improved substructure representation is obtained by adding terms to the modal representation, so that it gives results that approach equation (28) at frequencies that are low compared to the lowest mode. To this end, the equations of the modal representation will be cast in the form of equation (28). The relevant equations of the modal representation are, by consolidation of equations (8), (9), (12), (17), (20), and (21),

$$\begin{Bmatrix} \xi_i \\ \hline f_b^{(m)} \end{Bmatrix}=\left[\begin{array}{c:c} k_i^{-1} & \psi_{ib} \\ \hdashline -\psi_{ib}^T & 0 \end{array}\right]\begin{Bmatrix} f_i-m_ip^2\xi_i \\ \hline u_b \end{Bmatrix} \tag{32}$$

$$\{f_i\}=[\phi_{ci}]^T\{f_c\} \tag{33}$$

and

$$\{u_c^{(m)}\}=[\phi_{ci}]\{\xi_i\} \tag{34}$$

Upon combining equations (32), (33) and (34), we obtain

$$\begin{Bmatrix} u_c^{(m)} \\ \hdashline f_b^{(m)} \end{Bmatrix} = \left[\begin{array}{c:c} \phi_{ci} k_i^{-1} \phi_{ci}^T & \phi_{ci}\psi_{ib} \\ \hdashline -\psi_{ib}^T \phi_{ci}^T & 0 \end{array}\right] \begin{Bmatrix} f_c \\ \hdashline u_b \end{Bmatrix} + \left[\begin{array}{c} -\phi_{ci} k_i^{-1} m_i \\ \hdashline \psi_{ib}^T m_i \end{array}\right] \{p^2 \xi_i\}. \tag{35}$$

The first term in equation (35) is in the correct form but the second term is not. If the frequency is low, however, it may be assumed that

$$[k_i]^{-1}[m_i]\{p^2\xi_i\} \ll [\psi_{ib}]\{u_b\} \tag{36}$$

and that, using this result in equation (32),

$$\{\xi_i\} \simeq [\psi_{ib}]\{u_b\}. \tag{37}$$

Thus, employing these approximations, equation (35) may be written as

$$\begin{Bmatrix} u_c^{(m)} \\ \hdashline f_b^{(m)} \end{Bmatrix} = \left[\begin{array}{c:c} Z_{cc}^{(m)} & S_{cb}^{(m)} \\ \hdashline -S_{cb}^{(m)T} & M_{bb}^{(m)} p^2 \end{array}\right] \begin{Bmatrix} f_c \\ \hdashline u_b \end{Bmatrix} \tag{38}$$

where,

$$\left.\begin{aligned} [Z_{cc}^{(m)}] &= [\phi_{ci}][k_i]^{-1}[\phi_{ci}]^T \\ [S_{cb}^{(m)}] &= [\phi_{ci}][\psi_{ib}] \\ [M_{bb}^{(m)}] &= [\psi_{ib}]^T[m_i][\psi_{ib}] \end{aligned}\right\} \tag{39}$$

Let the vector

$$\begin{Bmatrix} u_c \\ \hdashline f_b \end{Bmatrix}$$

consist of a part due to the modal representation, designated by superscript (m), and a 'residual' part, designated by a superscript (r).

$$\begin{Bmatrix} u_c \\ \hdashline f_b \end{Bmatrix} = \begin{Bmatrix} u_c^{(m)} \\ \hdashline f_b^{(m)} \end{Bmatrix} + \begin{Bmatrix} u_c^{(r)} \\ \hdashline f_b^{(r)} \end{Bmatrix} \tag{40}$$

By comparison of equations (28) and (38)

$$\begin{Bmatrix} u_c^{(r)} \\ \hdashline f_b^{(r)} \end{Bmatrix} = \left[\begin{array}{c:c} Z_{cc}^{(r)} & S_{cb}^{(r)} \\ \hdashline -S_{cb}^{(r)T} & K_{bb} + M_{bb}^{(r)} p^2 \end{array}\right] \begin{Bmatrix} f_c \\ \hdashline u_b \end{Bmatrix} \tag{41}$$

where,

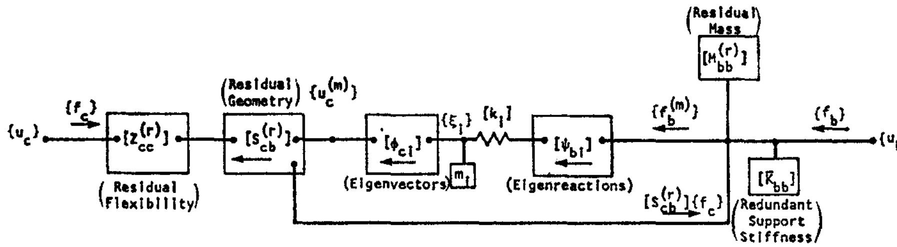
$$[Z_{cc}^{(r)}] = [Z_{cc}] - [Z_{cc}^{(m)}], \text{ the residual flexibility matrix} \tag{42}$$

$$[S_{cb}^{(r)}] = [S_{cb}] - [S_{cb}^{(m)}], \text{ the residual geometry matrix} \tag{43}$$

$$[M_{bb}^{(r)}] = [M_{bb}] - [M_{bb}^{(m)}], \text{ the residual mass matrix}. \tag{44}$$

If the support points are determinate, all of the residual matrices will tend to zero as the number of modes is increased. Furthermore $[Z_{cc}^{(r)}]$ is positive definite and $[M_{bb}^{(r)}]$ will be positive definite if there are no redundant reaction points. The complete improved representation of the substructure is shown in Fig. 5. It consists of stiffness, mass, and flexibility matrices, and of constraint matrices ($[S_{cb}^{(r)}]$, $[\phi_{ci}]$, and $[\psi_{bi}]$). It is, therefore, a subsystem that may be included in a dynamic analysis.

The complete representation may be simplified if all of the connection coordinates are free, or if all of the connection coordinates are restrained. These simplified forms are shown in Figs. 6 and 7.



$$\{u_c\} = [Z_{cc}^{(r)}]\{f_c\} + [S_{cb}^{(r)}]\{u_b\} + \{u_c^{(m)}\} \qquad \{f_b\} = [\bar{K}_{bb} + M_{bb}^{(r)}p^2]\{u_b\} - [S_{cb}^{(r)}]^T\{f_c\} + \{f_b^{(m)}\}$$

FIG. 5. Improved representation of a substructure for the general case.

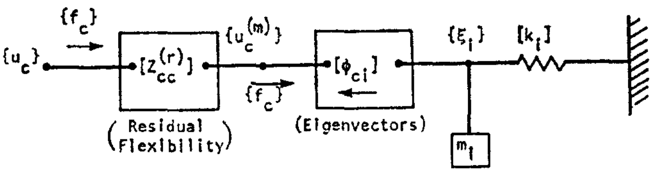


FIG. 6. Improved representation for case in which all connection coordinates are free in modal calculation.

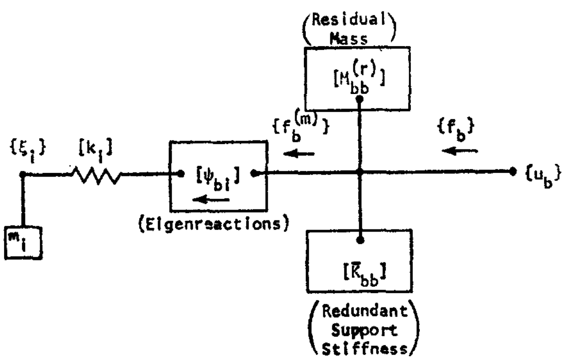


FIG. 7. Improved representation for case in which all connection coordinates are constrained in modal calculation.

The simplified form shown in Fig. 6 has been extensively studied in [10]. Some of the results are described in the next section. It is shown in [10] that the residual flexibility matrix $[Z_{cc}^{(r)}]$ is important for the solution of aeroelastic problems (unless very many modes are used), and that it may be calculated even if the substructure is capable of rigid body motion. The following procedure is used when rigid body substructure modes are present:

(1) Calculate the matrix of deflection influence coefficients $[Z_{mm}]$ of the substructure at all points $\{u_m\}$ where masses are concentrated, with *any* set of determinate supports.

(2) Calculate

$$[B_{mm}]=[\phi_{m0}][\phi_{m0}]^T[M_{mm}] \tag{45}$$

where $[M_{mm}]$ is the complete mass matrix and $[\phi_{m0}]$ is the matrix of rigid body modes (one column for each rigid body mode).

(3) The elements of $[\bar{Z}_{cc}]$ are in the appropriate rows and columns of

$$[\bar{Z}_{mm}]=[I-B_{mm}][Z_{mm}][I-B_{mm}]^T \tag{46}$$

Note that only (c) rows of $[B_{mm}]$ need to be calculated. The residual flexibility matrix is then calculated from equation (42).

The derivation of equation (46) given in [10] is quite lengthy and will not be repeated. A somewhat shorter derivation appears in [11]. The effects of the presence of rigid body modes in the general case (i.e., rigid body modes that are not removed by the constrained connection coordinates) have not been explored.

An independent derivation which is equivalent to the simplified model shown in Fig. 7, including use of the term 'residual mass,' has recently appeared [12]. It can also be shown that 'residual mass' is implicitly contained in the Hurty method [1]. The author's original paper [7] includes the residual mass concept directly, where it is called 'the mass approximation to higher modes.' The improved representation of a substructure for the general case of modes computed with hybrid boundary conditions, Fig. 5, has not previously been published.

The discussion of this section will be concluded by displaying the complete equations of motion of the substructure for the general case as they would appear in a finite element computer program that uses the matrix displacement method. The independent degrees of freedom are the connection coordinates $\{u_b\}$ and $\{u_c\}$ and the generalized modal coordinates $\{\xi_i\}$. The complete equations of motion for the substructure are

$$\begin{Bmatrix} 0 \\ \hdashline f_c \\ \hdashline f_b \end{Bmatrix} = \left[\begin{array}{c:c:c} m_i p^2 + B_{ii}p + K_{ii} & K_{ic} & K_{ib} + B_{ib}p \\ \hdashline K_{ic}^T & K_{cc} & K_{cb} \\ \hdashline K_{ib}^T + B_{ib}^T p & K_{cb}^T & M_{bb}^{(r)}p^2 + B_{bb}p + K_{bb} \end{array}\right] \begin{Bmatrix} \xi_i \\ \hdashline u_c \\ \hdashline u_b \end{Bmatrix} \tag{47}$$

where,

$[m_i]$ = modal mass matrix
$[M_{bb}^{(r)}]$ = residual mass matrix

and, by examination of preceding results,

$$\left.\begin{aligned}
&[K_{ii}]=[k_i]+[\phi_{ci}]^T[Z_{cc}^{(r)}]^{-1}[\phi_{ci}]\\
&[K_{ic}]=-[\phi_{ci}]^T[Z_{cc}^{(r)}]^{-1}\\
&[K_{ib}]=-[k_i][\psi_{ib}]+[\phi_{ci}]^T[Z_{cc}^{(r)}]^{-1}[S_{cb}^{(r)}]\\
&[K_{cc}]=[Z_{cc}^{(r)}]^{-1}\\
&[K_{cb}]=-[Z_{cc}^{(r)}]^{-1}[S_{cb}^{(r)}]\\
&[K_{bb}]=[\bar{K}_{bb}]+[\psi_{ib}]^T[k_i][\psi_{ib}]+[S_{cb}^{(r)}]^T[Z_{cc}^{(r)}]^{-1}[S_{cb}^{(r)}]\\
&[B_{ii}]=[k_ig_i]\\
&[B_{ib}]=-[k_ig_i][\psi_{ib}]\\
&[B_{bb}]=[\psi_{ib}]^T[k_ig_i][\psi_{ib}]\,.
\end{aligned}\right\}\quad(48)$$

It is possible, particularly if the input data have been derived from physical tests, that a further transformation may be needed to rotate the directions of $\{u_c\}$ and $\{u_b\}$ so as to be compatible with the directions used to describe the motions of the adjoining structures. Another possibility is that, if many modes are used, the residual flexibility matrix may be very small, leading to ill-conditioning of the resulting equations. If this condition is suspected, or if the static deflection influence coefficients, $[Z_{cc}]$, are not supplied, the free connection coordinates, $\{u_c\}$, should be considered to be dependent coordinates and eliminated by means of equation (1).

EXAMPLES OF THE USE OF RESIDUAL MASS AND FLEXIBILITY MATRICES

The first example to be considered is a uniform torsion bar that is built in at one end. The problem is to calculate the displacement, θ_0, at the free end due to a sinusoidal torque, T, imposed at radian frequency, ω. The problem is sufficiently simple that the results may be expressed in closed form and evaluated exactly. The differential equation and boundary conditions are

$$\left.\begin{aligned}
&JG\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}+I\omega^2\theta=0\\
&\theta=0 \quad \text{at} \quad x=0\\
&T=JG\frac{\mathrm{d}\theta}{\mathrm{d}x} \quad \text{at} \quad x=l
\end{aligned}\right\}\quad(49)$$

The exact solution of the problem is

$$\frac{\theta_0}{T}=\frac{l}{JG}\left[\frac{\tan\left(\frac{\pi}{2}\frac{\omega}{\omega_1}\right)}{\frac{\pi}{2}\frac{\omega}{\omega_1}}\right] \tag{50}$$

where,

$$\omega_1=\frac{\pi}{2l}\sqrt{\frac{JG}{I}} \tag{51}$$

is the lowest cantilever mode frequency.

The cantilever vibration modes of the torsion bar are:

$$\left.\begin{aligned} &\phi_k(x)=\sin\left(\frac{2k-1}{2}\frac{\pi x}{l}\right), \qquad k=1, 2, 3, \ldots \\ &\text{with frequencies} \\ &\omega_k=\frac{2k-1}{2}\frac{\pi}{l}\sqrt{\frac{JG}{I}} \end{aligned}\right\} \tag{52}$$

If the solution is synthesized from cantilever modes, the rotation, θ_0, will be a 'free' coordinate in the terminology developed in previous discussion. The result of the modal synthesis is

$$\frac{\theta_0}{T}=\frac{l}{JG}\frac{2}{\pi^2}\sum_{k=1}^{M}\left(\frac{2}{2k-1}\right)^2\left(\frac{1}{1-\left(\frac{\omega}{\omega_k}\right)^2}\right) \tag{53}$$

where M is the highest mode included in the synthesis. The residual flexibility, $Z^{(r)}$, is the difference between the results produced by equations (50) and (53) for $\omega=0$,

$$Z^{(r)}=\frac{l}{JG}\left(1-\frac{2}{\pi^2}\sum_{k=1}^{M}\left(\frac{2}{2k-1}\right)^2\right) \tag{54}$$

Adding equation (54) to equation (53), the improved result is

$$\frac{\theta_0}{T}=\frac{l}{JG}\left[1+\frac{2}{\pi^2}\sum_{k=1}^{M}\left(\frac{2}{2k-1}\right)^2\frac{\left(\frac{\omega}{\omega_k}\right)^2}{1-\left(\frac{\omega}{\omega_k}\right)^2}\right] \tag{55}$$

TABLE 1.

Results for cantilever torsion problem

$$\frac{\theta_0}{T} \cdot \frac{JG}{l}$$

Number of modes (M)	$\frac{\omega}{\omega_1}=0$		$\frac{\omega}{\omega_1}=0{\cdot}5$		$\frac{\omega}{\omega_1}=1{\cdot}5$	
	Without $Z^{(r)}$	With $Z^{(r)}$	Without $Z^{(r)}$	With $Z^{(r)}$	Without $Z^{(r)}$	With $Z^{(r)}$
0	0	1·0000	0	1·00000	0	1·00000
1	0·81057	1·0000	1·08076	1·27019	−0·64846	−0·45903
2	0·90063	1·0000	1·17339	1·27276	−0·52838	−0·42901
3	0·93305	1·0000	1·20614	1·27309	−0·49275	−0·42580
4	0·94959	1·0000	1·22276	1·27317	−0·47541	−0·42500
5	0·95969	1·0000	1·23289	1·27320	−0·46502	−0·42471
Exact solution	1·0000		1·27323		−0·42441	

Equations (53) and (55) are compared in Table 1 for three values of ω/ω_1 and different values of M. The improvement provided by residual flexibility is quite apparent.

As a second example, let the torsion bar be free at the end $x=0$, and construct a modal synthesis based on modes in which the rotation at $x=l$ is rigidly constrained to zero. The exact solution for forced vibration at $x=l$ is

$$\frac{T}{\theta_0} = -Il\omega^2 \left[\frac{\tan\left(\frac{\pi}{2}\frac{\omega}{\omega_1}\right)}{\frac{\pi}{2}\frac{\omega}{\omega_1}} \right] \tag{56}$$

where, again, ω_1 is the lowest cantilever mode frequency.

The modal synthesis for the second example is also based on cantilever modes, but this time the end $x=l$ is restrained. The relationship between the torque, T, and the modal parameters is, from inspection of Fig. 3,

$$T = -\theta_0 \cdot \sum_{j=1}^{M} (\psi_j)^2 \left[\frac{m_j \omega^2}{1-\left(\frac{\omega}{\omega_j}\right)^2} \right]. \tag{57}$$

The force of reaction in each mode is, assuming unit amplitude at the free end

$$F_j = JG\frac{d\theta}{dx}\bigg|_{x=l} = -JG\left(\frac{2k-1}{2}\right)\frac{\pi}{l} \tag{58}$$

so that, from equation (19)

$$\psi_j = -\frac{JG\left(\frac{2j-1}{2}\right)\frac{\pi}{l}}{k_j} \tag{59}$$

where the modal stiffness

$$k_j = \omega_j^2 m_j = \left(\frac{(2j-1}{2}\frac{\pi}{l}\sqrt{\frac{JG}{I}}\right)^2 \cdot \frac{Il}{2} = \frac{\pi^2 JG}{2l}\left(\frac{2j-1}{2}\right)^2 \tag{60}$$

Thus,

$$\psi_j = -\frac{2}{\pi}\left(\frac{2}{2j-1}\right) \tag{61}$$

and, by substitution into equation (57),

$$\frac{T}{\theta_0} = -Il\omega^2 \cdot \frac{2}{\pi^2}\sum_{j=1}^{M}\left(\frac{2}{2j-1}\right)^2 \frac{2}{1-\left(\frac{\omega}{\omega_j}\right)^2} \tag{62}$$

The residual mass is the difference between the total inertia of the rod, Il, and the result given by equation (62) for $\omega/\omega_j = 0$:

$$M^{(r)} = Il\left[1 - \frac{2}{\pi^2}\sum_{j=1}^{M}\left(\frac{2}{2j-1}\right)^2\right] \tag{63}$$

The improved solution, obtained by adding $-M^{(r)}\omega^2$ to equation (62) is:

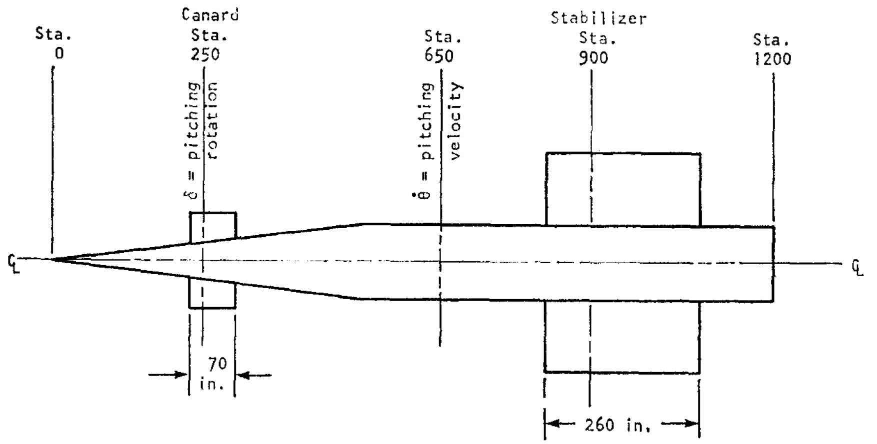
$$\frac{T}{\theta_0} = -Il\omega^2\left[1 + \frac{2}{\pi^2}\sum_{j=1}^{M}\left(\frac{2}{2j-1}\right)^2 \frac{\left(\frac{\omega}{\omega_j}\right)^2}{1-\left(\frac{\omega}{\omega_j}\right)^2}\right] \tag{64}$$

Equations (62) and (64) are identical in form to equations (53) and (55), respectively, so that the comparison of accuracy shown in Table 1 also applies to the second example. In general, residual mass has the same importance for a synthesis using restrained modes that residual flexibility has for a synthesis using free modes.

Some results from [10] will be used as a final example to illustrate the importance of residual flexibility in aeroelastic analysis. The example is the missile configuration shown in Fig. 8 which consists of a slender body and two elastically connected control surfaces. The problem is to compute the pitching velocity of the missile in response to sinusoidal oscillations of the forward control surface.

The configuration was initially analyzed on an analog computer by a lumped element model including aerodynamic influence coefficients for the control surfaces. The results of the initial analysis are shown in Fig. 9a. The vibration modes of the lumped element model were computed and used as a basis for a modal synthesis. (The synthesis includes two rigid body modes, as many as five elastic modes, and the same aerodynamic influence coefficients that were used in the initial analysis.) In addition, residual flexibility was computed using equations (46) and (42).

Typical results, illustrating the effects of including residual flexibility, are shown in Figs. 9b, c, d and e. The resonance near 0·7 Hz is the response of the short period pitching mode of the vehicle. It is seen that the synthesis without residual flexibility misses the peak response of the short period mode by about 5 db (a factor of 1·78) and that the inclusion of as many as five elastic modes does not improve the accuracy. The addition of residual flexibility, on the other hand, reduces the error in the short period mode to about one db, even when only one elastic mode is used, and significantly improves the accuracy of the response of the elastic modes.



Notes: The body is flexible

The aerodynamic surfaces have an elastic connection

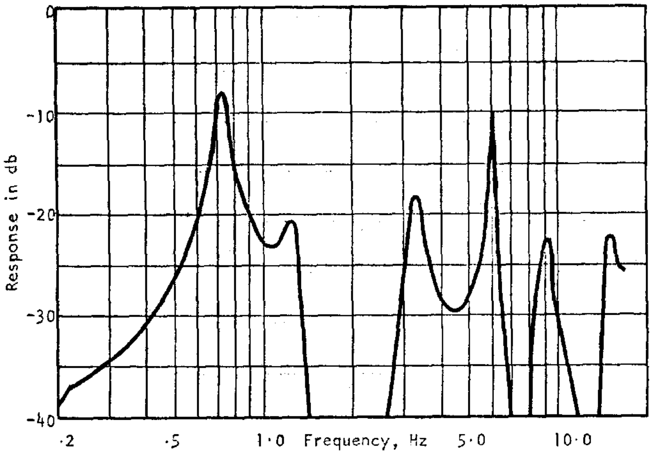
Uncoupled frequencies of cantilevered surfaces:

Canard bending 25 cps
Canard pitch 250 cps
Stabilizer bending 10 cps
Stabilizer pitch 20 cps

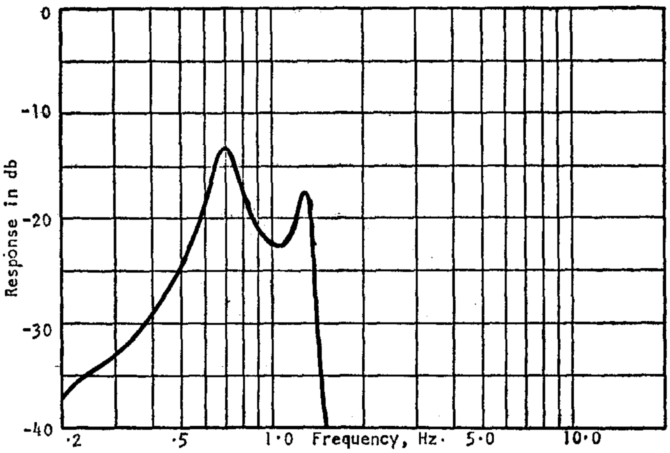
Coupled frequencies of entire system:

1.42 cps
3.42 cps
5.87 cps
9.17 cps
10.12 cps

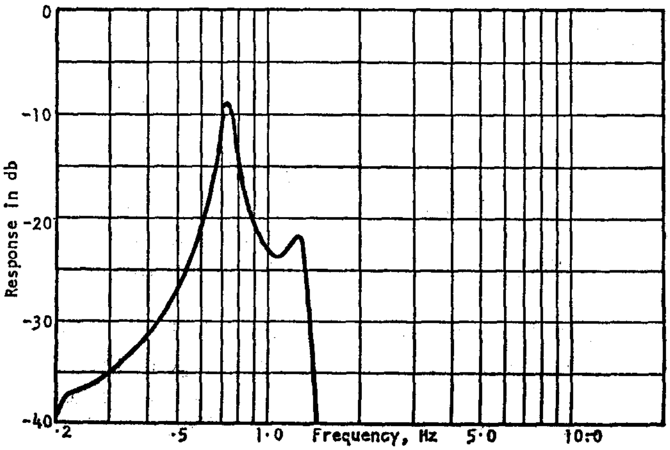
FIG. 8. Missile configuration, taken from [10].


0
-10
-20
-30
-40
Response in db
·2
·5
1·0
Frequency, Hz
5·0
10·0

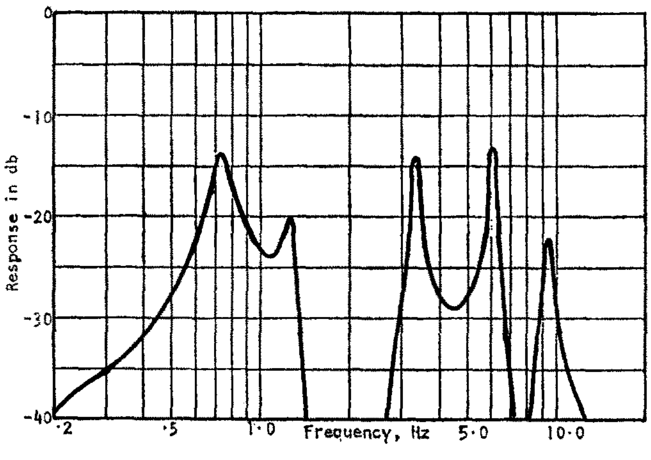
(a)


0
-10
-20
-30
-40
Response in db
·2
·5
1·0
Frequency, Hz·
5·0
10·0

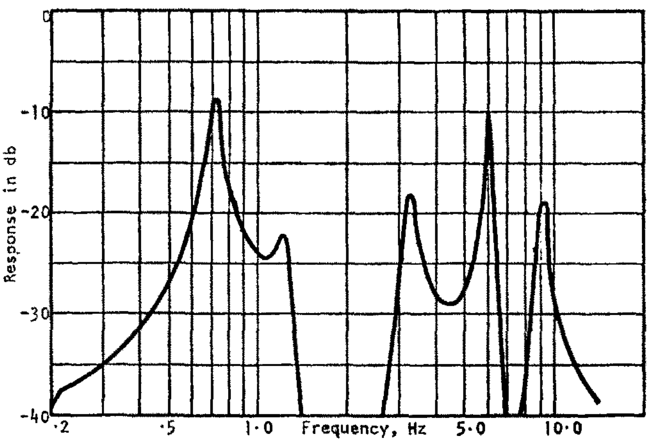
(b)


0
-10
-20
-30
-40
Response in db
·2
·5
1·0
Frequency, Hz
5·0
10·0

(c)



(d)



(e)

FIG. 9. Pitching response to control surface oscillation; $\dot{\theta}/\delta$. (a) Initial analysis of lumped element model; (b) modal synthesis, one elastic mode, *no* residual flexibility; (c) modal synthesis, one elastic mode, *with* residual flexibility; (d) five elastic modes, no residual flexibility; (e) five elastic modes, with residual flexibility.

It is not difficult to discover why residual flexibility is so important in this example. The uncoupled mode data listed in Fig. 8 show that three of the four important control surface modes are beyond the frequency range of the modes included in the synthesis, so that control surface flexilibilities are largely ignored. They are, however, included in the residual flexibility.

An important lesson to be learned from this example is that it is frequently not sufficient to include only the modes whose frequencies lie in the range of the exciting forces. Either enough modes should be included to account for all important flexibilities, or residual flexibility should be added to approximate the effects of omitted higher modes. The same remarks apply to residual mass in a synthesis using restrained modes, and to both residual mass and residual flexibility in a synthesis using hybrid modes.

REFERENCES

[1] W. C. HURTY, *Dynamic Analysis of Systems by Component Mode Synthesis.* JPL TR 32-530 (1964).

[2] W. C. HURTY, Dynamic analysis of structural systems using component modes. *AIAA J.* 3, 678–685 (1965).

[3] G. M. L. GLADWELL, Branch mode analysis of vibrating systems. *J. Sound Vibration* 1, 41–59 (1964).

[4] R. R. CRAIG and M. C. BAMPTON, Coupling of substructures for dynamic analyses. *AIAA Jl.* 6, 1313 (1968).

[5] R. N. FOSTER, A reactance theorem. *Bell Syst. Tech. J.* 3, 259, April (1924).

[6] F. A. FIRESTONE, The mobility method of computing the vibration of linear mechanical and acoustic systems: mechanical-electrical analogies. *J. Appl. Phys.* 9, 373–387 (1938).

[7] R. H. MACNEAL, *Vibrations of Composite Systems.* Office of Scientific Research Report OSR TN-55-120, October (1954).

[8] R. H. MACNEAL, *Electric Circuit Analogies for Elastic Structures*, 240–245. John Wiley, New York (1962).

[9] R. H. MACNEAL (Ed.), *NASTRAN Theoretical Manual*, NASA SP-221, Section 14.1, September (1970).

[10] R. G. SCHWENDLER and R. H. MACNEAL, *Optimum Structural Representation in Aeroelastic Analyses.* Aero. Systems Div., Air Force Systems Command Report No. ASD-TR-61-680, March (1962).

[11] B. F. PEARCE, *Topics on Flexible Airplane Dynamics*, ASD-TDR-63-334, Part I, July (1963).

[12] R. M. BAMFORD, B. K. WADE and W. H. GAYMAN, *Equivalent Spring-Mass System for Normal Modes.* Jet Propulsion Laboratory Technical Memo 33-380, February 15 (1971).