Spatial fuzzy consumer’s behavior: a multidimensional analysis
Claude Ponsard

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SPATIAL FUZZY CONSUMER'S BEHAVIOR

A multidimensional Analysis

Claude PONSARD

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Abstract.
This paper is devoted to a multidimensional analysis of the consumer's behavior when the decisionmaker is acting in a fuzzy space and pointing out an imprecise attitude.

At first, the process of decisionmaking is described with the help of three relationships between the set of goods which are supplied in several points, the set of their characteristics and the set of consumer's a priori possible behaviors. All these relations are fuzzy. The model applies the theory of fuzzy relations equations.

Then, the stages of the decision process are analyzed. Often fuzzy comportemental relations are like "black boxes". The mathematical solution of the model indicates in which conditions their valuations are possible.

The main interest of this method is to do without additive operations on subjective items and to use operators which are coherent with the fuzzy nature of the variables.

1. Introduction.
In Operational Research, especially in marketing and advertising studies, searchers have often to solve the following problem : what will be the behavior of a group of individuals in presence of a lot of consumption goods which are supplied in some scattered supply points and have
numerous and various characteristics?

The process of decisionmaking to be analyzed lays on subjective and qualitative relationships: some relations express the perception of real or fancied characteristics which are associated with goods and spatial structures of markets; other relations formulate the consumers' attitude with respect to these characteristics; at last, other relations express the revealed behaviors as to the goods such as they have been appreciated and valued.

The aim of this paper is to present a model which fully takes into account the subjective and qualitative nature of the relationships between man, space and goods. This model applies the theory of fuzzy relations equations. Initially, this theory was formulated by Sanchez (1974 and 1976). Then it was set out and developed by several authors (see especially Kaufmann 1977; Prevot 1977 and 1981; Yiganza 1982; Pedrycz 1983). Recently Zu-Wei (1980) introduced the necessary and sufficient conditions for the existence of solutions of fuzzy equations, without developing the proofs. They will be supplied in the Appendix next to this paper.

The following model is different, but complementary, of neoclassical fuzzy spatial models. In these ones, relationships between man, space and goods are expressed with fuzzy preferences relations which allow, under certain conditions, to define fuzzy utility functions (Ponsard 1979 and 1981 a). In this framework, consumer's partial equilibrium is the result of a fuzzy economic calculation: to maximize the fuzzy utility function with an elastic ressources constraint (Ponsard 1981 b). The same solution is available for the producer's equilibrium analysis, by means of suitable adaptations (Ponsard 1982 a). At last, in the particular case where the objective is precise and the constraint alone is fuzzy, the economic calculation is technically simpler (Ponsard 1982 b).
Now the aim of the present approach is to apply a multidimensional analysis to the exploration of the consumer's decisionmaking mechanism in a fuzzy context. At first, the complete decisionmaking process is described and formalized. Then, its different steps are analyzed in details. The conditions, which must be verified in order that the relations specifying each phase of the process be compatible with each other, are proved. They express the economic conditions for the fuzzy behavior coherence.

Remark. In order to avoid any ambiguity in the notation of mathematical terms, ordinary (non fuzzy) concepts are underlined, whereas fuzzy concepts are not.

2. The decisionmaking process.

The economic decisionmaker is an individual consumer or an homogeneous group of consumers. Its residential location is given. It enables to set out the delivered price system for the consumer. But consumption can take place either at the consumer's residence or at any other point. In this last case, the consumer's travelling expenses are incorporated in the delivered price.

Let \( \mathcal{G} = \{ g_p \} \), with \( p \in \mathcal{P} \), \( \mathcal{P} = [1, r] \) be a set of \( r \) goods indexed \( p \), and \( \mathcal{L} = \{ l_s \} \), with \( s \in \mathcal{S} \), \( \mathcal{S} = [1, \, t] \), be a set of \( t \) spots indexed \( s \) where the goods are supplied.

In order to express that a good can be supplied at several spots and a spot can supply several goods, we define a mapping, denoted by \( f \), from \( \mathcal{P} \times \mathcal{S} \) into \( \mathbb{N}^+ \) such that:

\[
\begin{align*}
\mathcal{P} \times \mathcal{S} & \quad \xrightarrow{f} \quad \mathbb{N}^+ \\
(p, s) & \quad \mapsto \quad f(p, s) = i, \; i \in I, \; I \subset \mathbb{N}^+ \\
\text{with} \; i & = \left[ (p - 1)S + 1, \, pS \right) \; \forall \; p \in \mathcal{P}
\end{align*}
\]
where $\bar{S} = \text{Card } S$

We put $f(r,t) = m$. Thus $\bar{I} = [1,m]$.

Thus, we define the set of located goods:

$$G \times L = X$$

with $X = \{ x_i \}, i \in \bar{I}$.

**Example.** The above mentioned formulation is complicated but it expresses a very simple relationship. It enables to indicate the nature of a given good and its supply spot with the help of one index only so that the symbols below are made easier.

Let four goods and three supply spots. Therefore $\bar{S} = 3$. We have (See Table 1).

Next, $X$ will be written in the form of a vector.

The located goods have numerous intrinsic characteristics. Also, a given good has many characteristics according to its supply spot.

Let $Y = \{ y_k \}, \text{ with } k \in K, K = [1,q]$, be the set of $q$ characteristics indexed $k$. For example, $k = 1$ means the aptitude of a located good to satisfy a given need; $k = 2$ designates the value which is associated with the state of contentment furnished by a consumption; $k = 3$ indicates the delivered price level such as the consumer appreciates it; $k = 4$ means the quality imputed to a located good; $k = 5$ designates the accessibility of goods depending on their supply spots location; and so on.

At last, let $Z = \{ z_j \}, \text{ with } j \in \bar{J}, \bar{J} = [1,n]$, the set of $n$ a priori possible behaviors that the consumer can have regarding the located goods and their characteristics. This set is not reduced to the purchasing behavior alone; it includes all the elements which characterize the spatial consumption behavior. For example, $j = 1$ designates the intention of purchasing some quantities of goods at some supply spots; $j = 2$ means the attachment to a trademark; $j = 3$ indicates the propensity to imitate other consumers' behavior; $j = 4$ designates the preference for a given supply spot; and so on.
Table 1

\[
\begin{array}{ccc}
\mathbf{g}_1 & \mathbf{g}_2 & \mathbf{g}_3 \\
\begin{array}{ccc}
(1,1) & (1,2) & (1,3) \\
(2,1) & (2,2) & (2,3) \\
(3,1) & (3,2) & (3,3) \\
(4,1) & (4,2) & (4,3) \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 \\
\begin{array}{ccc}
\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\
\mathbf{x}_4 & \mathbf{x}_5 & \mathbf{x}_6 \\
\mathbf{x}_7 & \mathbf{x}_8 & \mathbf{x}_9 \\
\mathbf{x}_{10} & \mathbf{x}_{11} & \mathbf{x}_{12} \\
\end{array}
\end{array}
\]

\[\mathbf{G} \times \mathbf{L} \rightarrow \mathbf{X}\]
The relationships between the sets $X$, $Y$ and $Z$ are like "black boxes", in the sense that they are very imprecise. First, the consumer has only exceptionally a complete information relating to his environment. Gener­ally, he knows imperfectly the space where he lives. Then, man is not a robot and the consumer has not a perfect aptitude to discriminate between goods and supply spots. He does not necessarily associate an exact valuation to a given characteristic. He does not always know if a good supplied at a given spot has a wanted characteristic. Accordingly, the decisionmaking is subject to a weakened rationality.

Therefore, to consider the relationships between $X$, $Y$ and $Z$ as fuzzy binary relations and to analyze them with the help of fuzzy relations equations theory is pertinent.

The decisionmaking process includes three stages: treatment of information by the consumer, valuation of characteristics and revealed behavior.

**Treatment of information.**

At first, a good $x_i$ has more or less a given characteristic depending on the consumer's appreciation and/or on the supply spot.

This treatment of data is dependent on the perception and the apprehen­sion of the available information relating to spatial environment and real or fancied characteristics of goods. It also depends on learning by doing phenomenon.

It means that $x_i$ is set in correspondence with $y_k$ at a given degree, de­noted by $\mu(x_i, y_k)$. We put that $\mu(x_i, y_k) \in [0,1]$. Thus, a fuzzy binary relation, denoted by $A$, between the elements of $X$ and $Y$ is defined. Relation $A$ is a fuzzy subset of $X \times Y$ such that:

$$A = \{(x_i, y_k), \mu_A : \forall x_i \in X, \forall y_k \in Y : \mu_A(x_i, y_k) \in [0,1]\}.$$

To simplify, we put: $\mu_A(x_i, y_k) = a_{ik}$. 
Thus \( a_{1k} \) means the valuation assigned to the located good \( x_1 \) for the characteristic \( y_k \) that it possesses more or less in the consumer's opinion.

**Valuation of characteristics**

A characteristic \( y_k \) is more or less interesting for the consumer. It has a relative value depending on the importance attributed to a state of satisfaction. In other words, the consumer's attitude with respect to goods and supply spots characteristics leads to the valuation of these characteristics.

Thus \( y_k \) is set in correspondence with \( z_j \) at a given degree, denoted by \( \mu (y_k, z_j) \), with \( \mu (y_k, z_j) \in [0,1] \). Thus a fuzzy binary relation, denoted by \( B \), between the elements of \( Y \) and \( Z \) is defined. Relation \( B \) is a fuzzy subset of \( Y \times Z \) such that:

\[
B = \left\{ (y_k, z_j), \mu_B ; \forall y_k \in Y, \forall z_j \in Z : \mu_B (y_k, z_j) \in [0,1] \right\}.
\]

We put: \( \mu_B (y_k, z_j) = b_{kj} \).

Thus \( b_{kj} \) means the valuation assigned to the characteristic \( y_k \) for the state of satisfaction \( z_j \) (a priori possible behavior) associated to it in the consumer's appreciation.

**Revealed behavior.**

At last, a located good \( x_i \) is more or less matter of a possible behavior.

Thus \( x_i \) is set in correspondence with \( z_j \) at the degree \( \mu (x_i, z_j) \), with \( \mu (x_i, z_j) \in [0,1] \). A fuzzy binary relation, denoted by \( C \), between the elements of \( X \) and \( Z \) is defined. Relation \( C \) is a fuzzy subset of \( X \times Z \) such that:

\[
C = \left\{ (x_i, z_j), \mu_C ; \forall x_i \in X, \forall z_j \in Z : \mu_C (x_i, z_j) \in [0,1] \right\}.
\]
We put: $V_C(x_i, z_j) = c_{ij}$.

Thus $c_{ij}$ is the valuation of the located good $x_i$ with respect to the consumer's possible behavior $z_j$. If we take again the above examples, it reveals the intention of buying a very great, great, mean, small or null quantity of good $x_i$, the degree of attachment to a trademark, the intensity of other consumers' imitation, the strength of the preference for a given supply spot; and so on.

The model of decisionmaking.

The complete model of decisionmaking is summarized on Figure 1.

Relation A expresses the step of information treatment, relation B the step of characteristics valuation and relation C represents the result (revealed behavior).

In usual consumption behavior models, especially in Expectancy-Value Theory (for a survey, see Martin 1976), $X$ is a vector of non located goods and $A$ a Boolean matrix: a good has or has not a given characteristic. Moreover $B$ and $C$ are matrices whose elements are measures, not valuations (fuzzy measures). At last, the authors put the relation: $C = A \times B$ which raises up numerous controversies concerning the additivity property and the interpretation of the product.

All these models lay on a particular specification. The following analysis requires different operators.

The composite fuzzy relation of $A$ and $B$ is the "max-min" composite relation, denoted by $B \circ A$, which is expressed, with the previous symbols:

$$V_{B \circ A}(x, z) = \bigvee_k [a_{ik} \land b_{kj}]$$

where $\lor$ and $\land$ designate the maximum and the minimum of the membership functions respectively. In this formulation, the "min" operator expresses the logical connective "and" (fuzzy intersection) for any couple of elements which are taken into $A$ and $B$. The "max" operator means the re-
result which is obtained for all the couples (fuzzy union).

Usual operators are particular cases of them. Indeed, in the ordinary (non fuzzy) case, we have:

\[ a_{ik} \in \{0,1\} \text{ and } b_{kj} \in \{0,1\} \]

and the composite relation of A and B is written:

\[ \mu_{B \circ A}(x, z) = \sum_k a_{ik} \cdot b_{kj} \]

where (.) indicates the Boolean multiplication and \( \sum \) the Boolean sum of the products.

Illustration (I).

Let three finite sets:

\[ X = \{ x_1, x_2, x_3, x_4 \} \] a set of four located goods

\[ Y = \{ y_1, y_2, y_3 \} \] a set of three characteristics

\[ Z = \{ z_1, z_2 \} \] a set of two a priori possible behaviors

and let \( A \subseteq X \times Y \) and \( B \subseteq Y \times Z \) be two fuzzy binary relations such that (see Table 2).

Then, \( B \circ A = C \) gives, by application of the max-min composite relation formula: (see Table 3).

3. Analysis of the decisionmaking process stages.

Now two interesting problems can be studied: in which conditions does equation \( B \circ A = C \) admit a solution B and a solution A? In other words, is it possible to make explicit the valuation of characteristics if A and C are known? In the same manner, is it possible to make explicit the step of information treatment if B and C are known?

The solution of the first problem will be explained thoroughly. The solution of the second question will be given without proof because it follows the same arguments. Moreover, in practice, relation B is the
Table 2

\[
A = \begin{array}{ccc}
    y_1 & y_2 & y_3 \\
    x_1 & 0.1 & 0.7 & 0.3 \\
    x_2 & 0.8 & 0.3 & 0.6 \\
    x_3 & 0.6 & 0.5 & 0.4 \\
    x_4 & 1 & 0.6 & 0.3 \\
\end{array}
\]

\[
B = \begin{array}{cc}
    y_1 & 0.7 & 0.5 \\
    y_2 & 0.2 & 0.4 \\
    y_3 & 0.2 & 0.5 \\
\end{array}
\]

Table 3

\[
C = \begin{array}{cc}
    z_1 & z_2 \\
    x_1 & 0.2 & 0.4 \\
    x_2 & 0.7 & 0.5 \\
    x_3 & 0.6 & 0.5 \\
    x_4 & 0.7 & 0.5 \\
\end{array}
\]

\[\text{and } C \subset \mathbb{X} \times \mathbb{Z}\]
Solution B of equation \( B \circ A = C \). General case.

The necessary and sufficient conditions for the existence of solutions \( B \) of equation \( B \circ A = C \) are set out by the following theorem.

Theorem. Equation \( B \circ A = C \) has a solution \( B = A' \odot C \) if and only if:

\[
(1) \forall a_{ik} \succ V c_{ij}, \forall k \in K, \forall j \in J, \forall i \in I
\]

\[
\text{i.e. } T = \{ k \in K | a_{ik} \succ V c_{ij} \} \neq \emptyset, \forall i \in I
\]

\[
(2) \forall k \in T \left( \frac{a_{k p} \alpha c_{p j}}{a_{k p} \alpha c_{p j}} \right) \succ V c_{ij}, \forall i \in I, \forall p \in I, \forall j \in J.
\]

where \( A' \) designates the transpose of \( A \)

and \( \alpha \) an operator such that \( c = a \alpha b \)

with \( c = 1 \) if \( a \leq b \)

\( c = b \) if \( a > b \).

The proof is tedious. It is stated in the Appendix (see Definition (1) and Theorem (3)).

The theorem demonstrates the existence of solutions, but not its uniqueness. In the Appendix, we shall prove that the above solution \( B \) is the greatest element in the set of possible solutions, i.e. the solution which outranks all the others. The non-uniqueness of the solution may appear as a deceptive result, but we must recall that the problem is put in a fuzzy context (for a complete mathematical statement, see Appendix, Theorems (1) to (5)).

Illustration (II).

We consider again illustration (I). Now \( A \) and \( C \) are given. We would compute \( B \).

First we verify if conditions (1) and (2) of the above theorem are satisfied.
(1) \( \forall k \ a_{ik} > \forall j \ c_{ij} \).

We have: \( a_{12} = 0.7 \) \( c_{12} = 0.4 \)
\( a_{21} = 0.8 \) \( c_{21} = 0.7 \)
\( a_{31} = 0.6 \) \( c_{31} = 0.6 \)
\( a_{41} = 1 \) \( c_{41} = 0.7 \)

Condition (1) is satisfied.

(2) \( \forall k \in T \left( \bigwedge_{p} \left( a_{kp} \land c_{pj} \right) \right) \forall j \ c_{ij} \)

with \( T = \left\{ k \in K \mid a_{ik} > \forall j \ c_{ij} \right\} \)

For \( i = 1 \), we have: \( T = \{ 2 \} \)

For \( i = 2, 3, 4 \), we have: \( T = \{ 1 \} \).

We have :
\( T = \{ 2 \} : (0.7 \land 0.4) \land (0.3 \land 0.5) \land (0.5 \land 0.5) \land (0.6 \land 0.5) = 0.4 \)
\( T = \{ 1 \} : (0.1 \land 0.2) \land (0.8 \land 0.7) \land (0.6 \land 0.6) \land (1 \land 0.7) = 0.7 \)

Condition (2) is fulfilled for any \( i \).

Solution \( B = A' \oplus C \). (see Table 4)

Matrix \( B \) of illustration (I) is found again.

**Particular case.**

The solution of the preceding problem can be improved in the particular case where matrix \( A \) contains one and only one element for a row which is equal to 1, all the others being null. This formulation fits the hypothesis in which the consumer, during the phase of information treatment, bounds his appreciation to a single characteristic of the located goods, for example the delivered price, and does not take into consideration the others. Or, it expresses that the consumer's data treatment is such that the most appreciated characteristic alone is chosen, all
Table 4

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\[
\begin{array}{cc}
0.2 & 0.4 \\
0.7 & 0.5 \\
0.6 & 0.5 \\
0.7 & 0.5 \\
\end{array}
\]

\[
A' = B
\]

\[
\begin{array}{cc}
y_1 & 0.7 & 0.5 \\
y_2 & 0.2 & 0.4 \\
y_3 & 0.2 & 0.5 \\
\end{array}
\]
the others being neglected, in the valuation of each located good. In such a context, the set of solutions B of equation $B \circ A = C$ has the structure of a complete lattice. In other words, there exist a greatest and a smallest element which bound the set of solutions. Indeed, we prove that the greatest element, denoted by $\hat{B}$, is equal to $A' \circ C$, as in the general case; the smallest element, denoted by $\hat{B}$, is equal to $A' \circ \bar{C}$ where the toplined terms designate the pseudocomplements and $A'$ the transpose of matrix A (see Appendix, Definitions (2) and (3) and Theorems (6) and (7)).

Illustration (III).
We consider again illustration (I). Now the matrix $A$ is such that the maximal elements of each row are equal to 1, all the others being null. Matrix $B$ is kept.

We have: (see Table 5)

Like in illustration (II), we verify that the necessary and sufficient conditions of a solution $B$ of equation $B \circ A = C$ are fulfilled.

We have:

(1) Maximal solution: $\hat{B} = A' \circ C$. (see Table 6)

(2) Minimal solution: $\hat{B} = A' \circ \bar{C}$. (see Table 7)

(3) In the two cases, we verify that matrix $C$ is found again. It suffices to compute $(A \circ \hat{B})$ and $(A \circ \hat{B})$ respectively.

Solution $A$ of equation $B \circ A = C$.
Now, if the phasis of data treatment is unknown, the equation must be solved into $A$, relations $B$ and $C$ being given.

Theorem: Equation $B \circ A = C$ has a solution $A = (B \circ C')'$ if and only if

$$\forall i \in I, \forall j \in J, \forall k \in K.$$

i.e. $T = \{ k \in K | b_{kj} \not\in \emptyset, \forall j \in J \}$
Table 5

\[
\begin{array}{cccc}
\text{x}_1 & \text{y}_1 & \text{y}_2 & \text{y}_3 \\
0 & 1 & 0 & \text{z}_1 \\
1 & 0 & 0 & \text{z}_2 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{array}
\]

Table 6

\[
\begin{array}{cccc}
\text{y}_1 & \text{y}_2 & \text{y}_3 \\
0.7 & 0.5 & \text{z}_1 \\
0.2 & 0.4 & \text{z}_2 \\
0.2 & 0.5 & \text{y}_3 \\
\end{array}
\]

Table 7

\[
\begin{array}{cccc}
\text{y}_1 & \text{y}_2 & \text{y}_3 \\
0.7 & 0.5 & \text{z}_1 \\
0.2 & 0.4 & \text{z}_2 \\
0 & 0 & \text{y}_3 \\
\end{array}
\]
\( \forall k \in T \left( \bigwedge_{\rho} (b_{kp} \alpha_c_{pj}) \right) \succ \bigvee_{i,j} c_{ij} \), \( \forall j \in J \), \( \forall p \in P \), \( \forall i \in I \).

where \( C' \) designates the transpose of \( C \) and \( (B \otimes C')' \) the transpose of \( (B \otimes C') \).

From this result, the analysis is analogous to the study which was stated for the valuation of the characteristics in the general case.

**Illustration (IV).**

Let the following equation :

(see Table 8)

After conditions (1) and (2) of the theorem have been verified, we have

(see Table 9)

4. Conclusion.

The consumer does not always know what he wants exactly. The fuzziness which distinguishes his behavior may foil empirical policies, for example marketing programming, advertising campaign or shops location choices. Also applied economics needs a theoretical model likely to formalize the fuzziness of behaviors and to analyse their effects. The particular case where behavior is not fuzzy is not opposite: it is simply an extremal case where membership functions are Boolean.

The conditions for the existence of solutions of composite fuzzy relations equations have a manifest economic significance. They express the conditions which must be fulfilled in order that behaviors reveal at least a weakened coherence. When they are not verified, the observed behaviors are incoherent and the model permits to know where the causes of this inconsistency are situated.

At last, the above model should be still more thoroughly examined. In the general case, the set of solutions has the structure of a sup-semilattice. The minimal solutions bounding inferiorly the set of solutions will require to be studied.
### Table 8

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The author would like to thank Mr. Prevot, associated with the Institute of Economic Mathematics, for fruitfull discussions he had with him and Mrs Penez for her assistance in translating this paper into English.
**Mathematical Appendix**

**Definition (1)**: A Brouwer's lattice is a lattice, denoted by \( L \), in which for any couple of given elements \( a \) and \( b \) the set of all elements \( x, x \in L \), such that \( a \wedge x \leq b \) contains a greatest element, denoted by \( a \alpha b \), called the relative pseudocomplement of \( a \) in \( b \). If \( L = [0,1] \) and if \( a, b \in L \), then:

- \( a \alpha b = 1 \) if \( a \leq b \)
- \( a \alpha b = b \) if \( a > b \)

The operator \( \alpha \) is neither commutative nor associative.

The relative pseudocomplement of \( a \) in \( b \) is unique.

**Definition (2)**: Let \( E \) be a set and \( L = [0,1] \) the set in which the elements of the subsets of \( E \) have their values.

Let \( A \) and \( B \) be two fuzzy subsets of \( E \). By definition, \( B \) is the pseudocomplement of \( A \) if:

\[ \forall x \in E : \mu_B (x) = 1 - \mu_A (x). \]

We note: \( B = \overline{A} \). Clearly, \( \overline{A} \) can be the pseudocomplement of \( A \) itself. We write then:

\[ \forall x \in E : \mu_{\overline{A}} (x) = 1 - \mu_A (x). \]

**Remark**. These two definitions must be carefully distinguished. In definition (1), \( L \) is endowed with the properties of a Brouwerian lattice, while in definition (2) \( L \) is viewed as a membership set.

**Theorem (1)**: Let \( A \subset X \times Y \) and \( C \subset X \times Z \). We have:

\[ (A' \circ C) \circ A \leq C \]

where \( A' \) is the transpose of the relation \( A \).

**Proof**.

We put \( W = (A' \circ C) \circ A \).
We have: \( \mu_w(x,z) = \bigvee_y \left( \mu_A(x,y) \land \mu_{A' \circ c}(y,z) \right) \)
by definition of the operation \( \circ \) (max-min composite fuzzy relation).

Let still:
\[
\mu_w(x,z) = \bigvee_y \left( \mu_A(x,y) \land \mu_{A}(x,y) \land \mu_{c}(x,z) \right)
\]

By definition of the operation \( \circ \), we have:
\[
\mu_w(x,z) = \bigvee_y \left( \mu_A(x,y) \land \mu_{A}(x,y) \land \mu_{c}(x,z) \right)
\]

Corollary. If \( B = A' \circ C \) is the solution of the equation \( B \circ A = C \), then 
\( (A' \circ C) \circ A = C \).

- trivial -

Theorem (2). Let \( A \subset X \times Y \) and \( B \subset Y \times Z \), we have: \( B \subset A' \circ C \).
The proof is similar to that of theorem (1).

Theorem (3). The equation \( B \circ A = C \) has a solution \( B = A' \circ C \) if and only if:

1. \( \forall k \exists a_{ik} \forall j c_{ij} , \forall k \in K , \forall j \in J , \forall i \in I \).

i.e. \( T = \{ k \in K \mid a_{ik} \forall j c_{ij} \} \neq \emptyset \), \( \forall i \in I \)

2. \( \forall k \in T \left( \land_p \left( a_{kp} \circ c_{pj} \right) \right) \forall j c_{ij} , \forall i \in I , \forall p \in I \), \( \forall j \in J \).

Proof.

Necessary conditions. In virtue of the corollary of theorem (1), if
\( B = A' \circ C \) is the solution of the equation \( B \circ A = C \), the equality 
\( (A' \circ C) \circ A = C \) is verified. We must prove that the conditions (1) and
(2) must be fulfilled in order that this equality be true. We have:

\[ \text{Let } B \circ A = (A' \otimes C) \circ A = \bigvee_{k} a_{ik} \land \left( \bigvee_{p} (a_{kp} \land c_{pj}) \right) \]

\[ = \bigvee_{k} a_{ik} \land (a_{k1} \land c_{i1}) \land \ldots \land (a_{ki} \land c_{ij}) \land \ldots \land (a_{km} \land c_{mn}) \]

\[ = \bigvee_{k} a_{ik} \land (a_{ik} \land c_{ij}) \land \left( \bigvee_{p} (a_{kp} \land c_{pj}) \right) \]

\[ = \bigvee_{k} a_{ik} \land \left( \bigvee_{p} (a_{kp} \land c_{pj}) \right) \]

because \( a_{ki} = a_{ik} \)

\[ = \left( \bigvee_{k} a_{ik} \right) \land \left( \bigvee_{p} (a_{kp} \land c_{pj}) \right) \]

because \( a \land (a \lor b) = a \land b \)

If condition (1) is not verified, i.e. if \( \exists i \in I \) / \( \bigvee_{k} a_{ik} < \bigvee_{j} c_{ij}, \forall k \in K, \forall j \in J \), there is no solution.

Indeed, let \( \bigvee_{j} c_{ij} = c_{i1} \lor \ldots \lor c_{ij} \lor \ldots \lor c_{in} = M \)

We have:

\[ \left( \bigvee_{k} a_{ik} \right) \land \left( \bigvee_{p} (a_{kp} \land c_{pj}) \right) \]

\[ = \left( \bigvee_{k} a_{ik} \right) \land M \land \left( \bigvee_{p} (a_{kp} \land c_{pj}) \right) \]

\[ = \left( \bigvee_{k} a_{ik} \right) \land \left( \bigvee_{p} (a_{kp} \land c_{pj}) \right) \land \bigvee_{k} a_{ik} \land c_{ij} \]

Consequently: \( B \circ A = (A' \otimes C) \circ A \neq C \). Condition (1) is necessary.

If condition (2) is not fulfilled, i.e. if

\[ \exists i \in I / \bigvee_{k \in T} \left( \bigvee_{p} (a_{kp} \land c_{pj}) \right) < \bigvee_{j} c_{ij}, \forall p \in I, \forall j \in J, \forall i \in I \]

there is no solution.
We have:

\[ B \circ A = (A' \alpha C) \circ A = \left[ \bigwedge_{k} \left( a_{ik} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) \right] \]

\[ \left[ \left\{ k \in \mathcal{T} \left( a_{ik} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) \right\} \right] \cup \left\{ k \in \mathcal{I} - \mathcal{T} \left( a_{ik} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) \right\} \]

where \( \mathcal{I} - \mathcal{T} = \{ k \in \mathcal{K} \mid a_{ik} < \bigvee_{j} c_{ij} \}, \forall i \in \mathcal{I} \)

But, \( \bigwedge_{k} \left( a_{ik} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) = \)

\[ = k \in \mathcal{T} \left( a_{ik} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) \]

\[ = k \in \mathcal{T} \left( a_{ik} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) \]

because \( a_{ik} = a_{ki} \)

\[ = k \in \mathcal{T} \left( a_{ik} \wedge c_{ij} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) \]

\[ = \left( k \in \mathcal{T} \left( a_{ik} \right) \right) \wedge \left( c_{ij} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) \]

By condition (1):

\[ k \in \mathcal{T} \left( a_{ik} \right) \wedge \left( c_{ij} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) \]

Hence:

\[ \forall k \in \mathcal{T}, \forall j \in \mathcal{J}, \forall i \in \mathcal{I} \]

We have:

\[ \forall i \in \mathcal{I}, \forall j \in \mathcal{J} \]

We have:

\[ \left( k \in \mathcal{T} \left( a_{ik} \right) \right) \wedge \left( c_{ij} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) = \]

\[ = \bigwedge_{j} \left( \bigwedge_{k} \left( a_{ik} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) \right) \forall j \in \mathcal{J}, \forall i \in \mathcal{I}, \forall p \in \mathcal{I} \]

\[ = \bigwedge_{j} \left( \bigwedge_{k} \left( a_{ik} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) \right) \]

\[ = \bigwedge_{k} \left( a_{ik} \wedge \left( \bigwedge_{p} (a_{kp} \alpha c_{pj}) \right) \right) < M \]
Finally, \( k \in \mathcal{I} \backslash \mathcal{T} \left( \bigwedge \left( a_{ik} \wedge \left( \bigwedge \left( a_{kp} \alpha c_{pj} \right) \right) \right) \right) < c_{ij} \).

In the same manner, we have:

\[
\begin{align*}
\forall k \in \mathcal{I} \backslash \mathcal{T} \left( a_{ik} \wedge \left( \bigwedge \left( a_{kp} \alpha c_{pj} \right) \right) \right) = \\
\left( \forall k \in \mathcal{I} \backslash \mathcal{T} \left( a_{ik} \wedge c_{ij} \right) \right) \wedge \left( \forall k \in \mathcal{I} \backslash \mathcal{T} \left( \bigwedge \left( a_{kp} \alpha c_{pj} \right) \right) \right)
\end{align*}
\]

We have:

\[
\forall k \in \mathcal{I} \backslash \mathcal{T} a_{ik} < c_{ij}, \quad \forall k \in \mathcal{I} \backslash \mathcal{T} a_{ik} < c_{ij}
\]

Hence:

\[
\forall k \in \mathcal{I} \backslash \mathcal{T} a_{ik} < c_{ij}, \quad \forall k \in \mathcal{I} \backslash \mathcal{T} a_{ik} < c_{ij}
\]

We have:

\[
\forall k \in \mathcal{I} \backslash \mathcal{T} a_{ik} < c_{ij}, \quad \forall k \in \mathcal{I} \backslash \mathcal{T} a_{ik} < c_{ij}
\]

Thus:

\[
\left( k \in \mathcal{I} \backslash \mathcal{T} a_{ik} \wedge c_{ij} \right) \wedge \left( k \in \mathcal{I} \backslash \mathcal{T} \left( \bigwedge \left( a_{kp} \alpha c_{pj} \right) \right) \right) \leq k \in \mathcal{I} \backslash \mathcal{T} a_{ik} < c_{ij}
\]

\[\forall k \in \mathcal{I} \backslash \mathcal{T} a_{ik} < c_{ij}, \quad \forall k \in \mathcal{I} \backslash \mathcal{T} a_{ik} < c_{ij}\]

Consequently: \( B \circ A = (A' \circ C) \circ A \neq C \). Condition (2) is necessary.

Sufficient conditions. We suppose that conditions (1) and (2) are fulfilled. We have:

\[
B \circ A = (A' \circ C) \circ A = \left[ \bigvee_{k \in \mathcal{K}} \left( \bigwedge \left( a_{ik} \wedge \left( \bigwedge \left( a_{kp} \alpha c_{pj} \right) \right) \right) \right) \right]
\]

\[
= \left[ \left\{ k \in \mathcal{I} \backslash \mathcal{T} \left( a_{ik} \wedge \left( \bigwedge \left( a_{kp} \alpha c_{pj} \right) \right) \right) \right\} \bigvee \left\{ k \in \mathcal{I} \backslash \mathcal{T} \left( a_{ik} \wedge \left( \bigwedge \left( a_{kp} \alpha c_{pj} \right) \right) \right) \right\} \right]
\]

\[\forall k \in \mathcal{K}, \quad \forall i \in \mathcal{I}, \quad \forall j \in \mathcal{J}, \quad \forall p \in \mathcal{I}\]

But:

\[
\forall k \in \mathcal{T} \left( a_{ik} \wedge \left( \bigwedge \left( a_{kp} \alpha c_{pj} \right) \right) \right) =
\]

\[
= \bigvee_{k \in \mathcal{T}} \left( a_{ik} \wedge (a_{1j} \alpha c_{ij}) \wedge \ldots \wedge (a_{kj} \alpha c_{ij}) \wedge \ldots (a_{km} \alpha c_{in}) \right)
\]

\[
= \bigvee_{k \in \mathcal{T}} \left( a_{ik} \wedge (a_{ik} \alpha c_{ij}) \wedge \left( \bigwedge \left( a_{kp} \alpha c_{pj} \right) \right) \right)
\]

because \( a_{ik} = a_{ki} \)

\[
= \bigvee_{k \in \mathcal{T}} \left( a_{ik} \wedge c_{ij} \wedge \left( \bigwedge \left( a_{kp} \alpha c_{pj} \right) \right) \right)
\]

because \( a \wedge (a \alpha b) = a \wedge b \)
\[
\left( k \in T \right) \left( a_{ik} \right) \wedge c_{ij} \leq \left( k \in T \right) \left( \left( a_{kp} \right) \wedge c_{pj} \right)
\]

By condition (1):
\[
\forall \; a \leq \forall \; c_{ij}, \; \forall \; k \in T, \; \forall \; i \in I, \; \forall \; j \in J
\]

We have:
\[
\forall \; k \in T \left( a_{ik} \right) \geq \forall \; c_{ij}, \; \forall \; i \in I, \; \forall \; j \in J
\]

\[
\forall \; k \in T \left( a_{ik} \right) \geq c_{i1} \wedge \cdots \wedge c_{ij} \wedge \cdots \wedge c_{in}
\]

By condition (2):
\[
\forall \; k \in T \left( \left( a_{kp} \right) \wedge c_{pj} \right) \geq \forall \; c_{ij}\]
\[
\forall \; i \in I, \; \forall \; j \in J, \; \forall \; p \in I
\]

We have:
\[
\forall \; k \in T \left( \left( a_{kp} \right) \wedge c_{pj} \right) \geq c_{i1} \wedge \cdots \wedge c_{ij} \wedge \cdots \wedge c_{in}
\]

Consequently:
\[
\left( k \in T \right) \left( a_{ik} \right) \wedge c_{ij} \leq \left( k \in T \right) \left( \left( a_{kp} \right) \wedge c_{pj} \right) = c_{ij}
\]
\[
\forall \; i \in I, \; \forall \; i \in I, \; \forall \; j \in J
\]

Therefore we have:
\[
B \circ A = \left[ c_{ij} \vee \left\{ k \in I - T \wedge \left( a_{ik} \wedge \left( a_{kp} \wedge c_{pj} \right) \right) \right\} \right]
\]

But:
\[
\forall \; k \in I - T \left( a_{ik} \wedge \left( \left( a_{kp} \wedge c_{pj} \right) \right) \right) =
\]
\[
= \left( k \in I - T \left( a_{ik} \right) \wedge c_{ij} \wedge \left( k \in I - T \left( \left( a_{kp} \wedge c_{pj} \right) \right) \right) \right) \leq c_{ij}
\]
\[
\forall \; i \in I, \; \forall \; j \in J, \; \forall \; p \in I
\]

Thus:
\[
\forall \; k \in I - T \left( a_{ik} \wedge \left( \left( a_{kp} \wedge c_{pj} \right) \right) \right) = c_{ij}
\]
\[
\forall \; i \in I, \; \forall \; p \in I, \; \forall \; j \in J
\]

Consequently:
\[
B \circ A = \left( A' \right) \circ C \circ A = [c_{ij}] = C.
\]

Conditions (1) and (2) are sufficient

\[Q.E.D.\]
Theorem (4). The equation $B \circ A = C$ has a solution $A = (B \ominus C')'$ if and only if:

1. $\bigvee_{k} b_{kj} > \bigvee_{i} c_{ij}, \forall i \in I, \forall j \in J, \forall k \in K$.

   i.e. $T = \left\{ k \in K \mid b_{kj} > \bigvee_{i} c_{ij} \right\} \neq \emptyset, \forall j \in J$

2. $\bigvee_{k} \left( \bigwedge_{p} \left( b_{kp} \alpha c_{pj} \right) \right) > \bigvee_{i} c_{ij}, \forall j \in J, \forall p \in J, \forall i \in I$.

The proof is similar to that of theorem (3).

Remark. The forms of theorems (3) and (4) are slightly different from Zu-Wei's formulations (1980) in which a third condition is put. In fact, the proofs can be derived from more concise statements.

Theorem (5). Let $S(B)$ be the set of fuzzy relations such that $B \circ A = C$. Then $(A' \ominus C)$ is the greatest element in $S(B)$.

Proof.

We put $B = A' \ominus C$ and we suppose that there exists $B \in S(B)$ such that $B \supset \nabla$.

As the max-min composite relation keeps the inclusion relation, we have $B \circ A \supset \nabla \circ A$.

But this relation is inconsistent with theorem (1) which states that:

$B \circ A \subseteq \nabla \circ A$

Therefore, $B$ is the greatest element in $S(B)$.

[Q.E.D.]

Definition (3): A binary relation $A \subseteq X \times Y$ is called an "elementary rowed relation" if and only if:

1. $\forall x \in X : (1) \exists ! y = y^*, y^* \in Y$, such that $\mu_{A}(x, y^*) = 1$

   where $\exists !$ is the quantifier of unique existence which means "it exists one and only one".

2. $\forall y \neq y^*, y \in Y : \mu_{A}(x, y) = 0.$
Therefore this relation is not fuzzy. The expression "elementary rowed relation" simply means "a single element for each row equal to 1, all the others being equal to zero". Each row is an elementary vector.

Theorem (6). Let $A \subseteq X \times Y$ an elementary rowed relation and $B \subseteq Y \times Z$ a fuzzy relation. We have:

$$B \circ A = B \circ A$$

Proof

We put : $C = B \circ A \subseteq X \times Z$. We have:

$$\forall (x, z) \in X \times Z : \mu_C(x, z) = \bigvee_y \left( \mu_A(x, y) \land \mu_B(y, z) \right)$$

By definition : $\mu_{B \circ A}(x, z) = \mu_C(x, z) = 1 - \mu_C(x, z)$.

In virtue of De Morgan theorems, we have:

$$\forall (x, z) \in X \times Z : \mu_C(x, z) = \bigvee_y \left( \mu_A(x, y) \lor \mu_B(y, z) \right)$$

$A$ being an elementary rowed relation, for $x = x_1$ given, there exists a unique element $y(x_1)$ such that:

$$\mu_A(x_1, y(x_1)) = 1 \quad \text{and} \quad \mu_A(x_1, y(x_1)) = 0$$

$$\forall y \neq y(x_1), \quad \mu_A(x, y) = 0 \quad \text{and} \quad \mu_A(x, y) = 1$$

Therefore, we have:

$$\mu_C(x, z) = \left( \mu_A(x_1, y(x_1)) \lor \mu_B(y(x_1), z) \right) \land \left( \mu_A(x, y) \lor \mu_B(y, z) \right)$$

$$= \left( 0 \lor \mu_B(y(x_1), z) \right) \land \left( 1 \lor \mu_B(y, z) \right)$$

$$= \mu_B(y(x_1), z)$$
On the other hand, \( \forall (x, z) \in X \times Z : \)

\[
\mu_B \circ_A (x, z) = \bigvee_y \left( \mu_A (x, y) \land \mu_B (y, z) \right)
\]

\[
= \left( \mu_A (x_1, y (x_1)) \land \mu_B (y (x_1), z) \right) \bigvee_y \left( \mu_A (x, y) \land \mu_B (y, z) \right)
\]

\[
y \neq y (x_1)
\]

\[
= 1 \land \mu_B \left( y (x_1), z \right) \bigvee_y \left( 0 \land \mu_B (y, z) \right)
\]

\[
y \neq y (x_1)
\]

\[
= \mu_B \left( y (x_1), z \right).
\]

\[\text{[Q.E.D.]}\]

**Theorem (7):** Let \( S(B) \) be the set of the solutions \( B \) of the equation \( B \circ_A C = C \) where \( A \) is an elementary rowed relation. Then:

1. \( S(B) \) has the structure of a complete Brouwerian lattice, the ordering relation of this lattice being the fuzzy inclusion relation.
2. \( B = A' \circ C \) is the greatest element of the lattice.
3. \( B = A' \circ C \) is the smallest element of the lattice.

**Proof.**

1. Let \( B_1 \in S(B) \) and \( B_2 \in S(B) \). It is easy to prove that the properties of a lattice are verified. Indeed, it suffices to point out that \( B_1 \) and \( B_2 \) fulfill the following properties: commutativity, associativity, idempotence, and absorption. These properties are dual for the fuzzy union and the fuzzy intersection.
2. \( B = A' \circ C \) by virtue of theorem (5).
3. \( B = A' \circ C \). Since \( A \) is an elementary rowed relation, in virtue of theorem (6) we have:

\[
\forall B \in S(B) : \overline{B} \circ A = \overline{C}
\]
Applying theorem (2) to $A$ and $B$, we have:

$$B \subset A' \otimes \overline{C}$$

or, by the definition of the pseudocomplement:

$$B \supseteq A' \otimes \overline{C}.$$ 

Thus, $A' \otimes \overline{C}$ being included into $B$ is the minimal solution and $\hat{B}$ is the least element of the lattice.

[Q.E.D.]
References


