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Robust entanglement with three-dimensional nonreciprocal photonic topological insulators

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We investigate spontaneous and pumped entanglement of two-level systems in the vicinity of a photonic topological insulator interface, which supports a nonreciprocal (unidirectional), scattering-immune, and topologically protected surface-plasmon polariton in the band gap of the bulk material. To this end, we derive a master equation for qubit interactions in a general three-dimensional, nonreciprocal, inhomogeneous, and lossy environment. The environment is represented exactly, via the photonic Green’s function. The resulting entanglement is shown to be extremely robust to defects occurring in the material system, such that strong entanglement is maintained even if the interface exhibits electrically large and geometrically sharp discontinuities. Alternatively, depending on the initial excitation state, using a nonreciprocal environment allows two qubits to remain unentangled even for very close spacing. The topological nature of the material is manifest in the insensitivity of the entanglement to variations in the material parameters that preserve the gap Chern number.

Our formulation and results should be useful for both fundamental investigations of quantum dynamics in nonreciprocal environments and technological applications related to entanglement in two-level systems.

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I. INTRODUCTION

Entanglement as a quantum resource is important for a range of emerging applications, including quantum computing [1] and quantum cryptography [2]. A main obstacle to the development of entanglement-based systems is decoherence associated with the unavoidable coupling between a quantum system and the degrees of freedom of the surrounding environment [3]. However, reservoir engineering methods have changed the idea of trying to minimize coupling to the environment to one of modifying the properties of the environment in order to achieve a desired state. These methods include using dissipative dynamics [4–9], recently extended to systems out of thermal equilibrium [10–14], as well as, e.g., exploiting the effect of measurements and feedback to achieve a desired final state [15,16].

Another emerging resource for reservoir engineering is the use of nonreciprocal environments [17]. In particular, there has been considerable investigation of quantum spin networks in chiral waveguides [18–24]. The previous work on spin dynamics in quantum chiral environments has focused on one-dimensional (1D) waveguide models. Here, we investigate two-level (spin) qubit interactions mediated by unidirectional surface-plasmon polaritons (SPPs) at the interface of a photonic topological insulator (PTI) and a topological-trivial material.

PTIs represent a broad class of materials that are attracting wide interest for both fundamental and applied reasons [25–28]. Perhaps their most celebrated aspect is their ability to support SPPs that are unidirectional, propagate in the bulk band gap, and are topologically protected from backscattering at discontinuities [29–36]. PTIs can be broadly divided into two classes: (i) those with broken time-reversal symmetry, which are photonic analogs of quantum Hall insulators [photonic quantum Hall effect (PQHE)], and (ii) those that are time-reversal invariant but have broken inversion symmetry, which are photonic analogs of electronic topological insulators or quantum spin Hall insulators [photonic quantum spin Hall Effect (PQSHE)]. Although as a specific example we consider PTIs of the PQHE type, the formulation presented here is general.

In this paper, we develop a master equation (ME) for three-dimensional (3D), nonreciprocal, inhomogeneous, and lossy environments, based on the macroscopic canonical quantization scheme described in [37–39], extended to nonreciprocal media [40]. In Sec. II A we present the master equation (derived in Appendix A), and in Sec. II B we provide the equations for concurrence as a measure of entanglement. In Sec. III we consider the topological aspect of concurrence for a PQHE-type PTI system consisting of a plasma continuum. Then, qubit entanglement dynamics are examined for several waveguiding systems. We focus on the aspects unique to the topological and nonreciprocal environment, such as the preservation of entanglement in the presence of large defects. Three appendices present a derivation of the master equation and discussion of various approximations, a comparison with previous 1D chiral MEs and discussion of the 1D, two-dimensional (2D), and 3D Green’s functions, and a derivation of the unidirectional concurrence.

II. THEORETICAL MODEL

In this section, we first present a general ME valid for both reciprocal and nonreciprocal, inhomogeneous, and lossy environments. This form is valid for 3D, 2D, and 1D systems since it is expressed in terms of the electromagnetic Green’s function. Then, we present concurrence expressions for the
The resulting unidirectional SPP provides a strongly nonreciprocal environment for qubit entanglement.

unidirectional case. The physical system we will consider is that of two qubits at the interface of a PTI and another (eventually topologically trivial) medium, as depicted in Fig. 1, although the development is completely general.

A. Master equation for general 3D nonreciprocal environments

We consider qubits with transition frequency $\omega_0$ interacting through a general nonreciprocal environment. For a derivation in the reciprocal case, see [41].

The classical electric field satisfies

$$\mathbf{E}(\mathbf{r},\omega) = \mathbf{J}_{\text{ext}}(\mathbf{r},\omega),$$

where $c$ is the vacuum speed of light, $\mu(\mathbf{r},\omega)$ and $\epsilon(\mathbf{r},\omega)$ are the material permeability and permittivity, and $\mathbf{J}_{\text{ext}}(\mathbf{r},\omega)$ is the noise current. In this paper, we suppose that the medium is nonmagnetic, $\mu(\mathbf{r},\omega) = 1$, where $1$ is the unit dyad, but that the permittivity is a tensorial quantity. By defining the noise current in terms of polarization as $\mathbf{J} = -i\omega \mathbf{P}$, which is associated with material absorption by the fluctuation-dissipation theorem, the electric-field Green’s tensor is the solution of

$$\mathbf{G}(\mathbf{r},\mathbf{r}’,\omega) = i\delta(\mathbf{r} - \mathbf{r}’),$$

and the electric field is $\mathbf{E}(\mathbf{r},\omega) = (\omega^2/c^2) \int d\mathbf{r}’ \mathbf{G}(\mathbf{r},\mathbf{r}’,\omega) \cdot \mathbf{P}$. Following the standard macroscopic canonical quantization [37–39], the noise polarization can be expressed in terms of the bosonic field annihilation operator as [40]

$$\hat{\mathbf{P}}_i(\mathbf{r},\omega) = -i\sqrt{\hbar \epsilon_0 / \pi} \mathbf{T}(\mathbf{r},\omega) \cdot \hat{\mathbf{f}}(\mathbf{r},\omega),$$

where

$$\mathbf{T}(\mathbf{r},\omega) \cdot \mathbf{T}^\dagger(\mathbf{r},\omega) = \frac{1}{2i} [\mathbf{e}(\mathbf{r},\omega) - \mathbf{e}^\dagger(\mathbf{r},\omega)],$$

and, for the special case of a symmetric permittivity tensor (e.g., a reciprocal medium), $\mathbf{T}(\mathbf{r},\omega) = \sqrt{\text{Im}(\mathbf{e}(\mathbf{r},\omega))}$. The bosonic field operators $\hat{\mathbf{f}}(\mathbf{r},\omega)$ obey the commutation relations $[\hat{f}_i(\mathbf{r},\omega), \hat{f}_j(\mathbf{r}',\omega')] = \delta_{ij} \delta(\mathbf{r} - \mathbf{r}') \delta(\omega - \omega')$ and $[\hat{f}_i(\mathbf{r},\omega), \hat{f}_j(\mathbf{r}',\omega')] = 0$. The noise polarization operator generates the electric-field operator

$$\hat{\mathbf{E}}(\mathbf{r},\omega) = \int d\mathbf{r}' \mathbf{G}(\mathbf{r},\mathbf{r}',\omega) \cdot \mathbf{T}(\mathbf{r},\omega) \cdot \hat{\mathbf{f}}(\mathbf{r}',\omega),$$

where $\mathbf{G}(\mathbf{r},\omega)$ is the classical electric-field Green’s tensor. Using this formulation, we arrive at the master equation (see Appendix A for details)

$$\partial_t \rho_s(t) = -\frac{i}{\hbar} [\hat{H}_s + V^{AF}, \rho_s(t)] + \mathcal{L} \rho(t),$$

where

$$\mathcal{L} \rho_s(t) = \sum_{i} \frac{\Gamma_{ij}(\omega_0)}{2} \left[ (2\sigma_i \rho_s(t) \sigma_j^\dagger - \sigma_j \rho_s(t) \sigma_i^\dagger \sigma_j^\dagger) - \frac{1}{2} \left[ \sigma_i \rho_s(t) \sigma_j^\dagger + \sigma_j \rho_s(t) \sigma_i^\dagger \right] \right]$$

$$+ \sum_{i,j} \left[ g_{ij}(\omega_0) \left[ \sigma_j \rho_s(t) \sigma_i^\dagger - i \sigma_j^\dagger \right] + \left[ i \sigma_i \rho_s(t) \sigma_j^\dagger \right] \right].$$

Equation (7) is applicable to both reciprocal and nonreciprocal environments and an arbitrary number of qubits. In Eq. (7), $\mathcal{L}$ is the Lindblad superoperator for the general nonreciprocal medium, involving the dissipative decay rate, $\Gamma_{ij}(\omega_0)$, and the coherent coupling term, $g_{ij}(\omega_0)$, in terms of the electromagnetic Green’s dyadic evaluated at the qubit transition frequency $\omega_0$;

$$\Gamma_{ij}(\omega_0) = \frac{2\omega_0^2}{\epsilon_0 \hbar c^2} \sum_{a,b=x,y,z} \sum_{\alpha} d_{a\alpha} \text{Im}[G_{a\alpha}(\mathbf{r}_j,\mathbf{r}_j,\omega_0)] d_{b\alpha}^\dagger,$$

$$g_{ij}(\omega_0) = \frac{\omega_0^2}{\epsilon_0 \hbar c^2} \sum_{a,b=x,y,z} \sum_{\alpha} d_{a\alpha} \text{Re}[G_{a\alpha}(\mathbf{r}_j,\mathbf{r}_j,\omega_0)] d_{b\alpha}^\dagger.$$

The Hamiltonian of the decoupled qubits is

$$\hat{H}_s = \sum_i \hbar \Delta_0 \sigma_i^\dagger \sigma_i,$$

where $\Delta_0 = \omega_0 - \omega_l - \delta_i$, with $\delta_i = g_{ii}$, being the Lamb shift and $\omega_l$ the laser frequency of an external source. The Lamb shift for optical emitters is in general on the order of a few GHz, therefore the effect of the Lamb shift for optical frequencies is small ($\omega_l \sim 10^{15}$ Hz, $\delta_i \sim 10^6$ Hz), and can be ignored, or assumed to be accounted for in the definition of the transition frequency $\omega_0$. In Eq. (6), the term

$$V^{AF} = -\hbar (\Omega e^{-i\Delta t} \sigma_1^\dagger + \Omega^* e^{i\Delta t} \sigma_1),$$

represents the external coherent drive applied to each qubit at laser frequency $\omega_l$. Due to its large amplitude we treat the drive field as a $c$ number where $\delta_i = d_i$. $\Omega$ is a Rabi frequency and $\Delta_0 = \omega_0 - \omega_l$ is the detuning parameter.

For the reciprocal case where $\Gamma_{ij} = \Gamma_{ji}$ and $g_{ij} = g_{ji}$ it can be shown that Eq. (7) is the well-known reciprocal (bidirectional) master equation [42,43]. In the reciprocal case, some terms associated with $g_{ij} = g_{ji}$ cancel each other out and are eliminated from the dissipative term. For example,
\(\sigma_i \rho_s(t) \sigma^j, i \neq j,\) appears in the nonreciprocal case but is absent in the reciprocal case.

For a system of two qubits, Eq. (7) can be written in the simple form
\[
\mathcal{L}_\rho(t) = \sum_{j=1,2} \Gamma_{ij} (2 \sigma_j \rho_s \sigma^j - \rho_s \sigma_j \sigma^j - \sigma_j \sigma^j \rho_s)
+ \left( \frac{\Gamma_1}{2} + ig_{21} \right) (\sigma_2 \rho_s \sigma^1 - \rho_s \sigma_1 \sigma^2)
+ \left( \frac{\Gamma_1}{2} - ig_{21} \right) (\sigma_1 \rho_s \sigma^2 - \rho_s \sigma_2 \sigma^1)
+ \left( \frac{\Gamma_1}{2} + ig_{12} \right) (\sigma_2 \rho_s \sigma^1 - \rho_s \sigma_1 \sigma^2)
+ \left( \frac{\Gamma_1}{2} - ig_{12} \right) (\sigma_1 \rho_s \sigma^2 - \rho_s \sigma_2 \sigma^1).
\]

Eq. (12), is the presence of the sinusoidal term in Eq. (13). When \(g_{12}\) is strong enough, this sinusoidal term causes oscillations in the transient concurrence related to photons being recycled between the two qubits, with a period that corresponds to the round trip time of the coupled qubits through the reciprocal medium (Rabi oscillations). For the unidirectional case Eq. (12) Rabi oscillations cannot occur.

It was shown in [43] that for qubits coupled to an infinite reciprocal waveguide system, the positions of \(\Gamma_{ij}\) maxima or minima correspond to positions of \(g_{ij}\) minima or maxima (for finite waveguides, see [45]). Thus, in general, coherent and dissipative regimes become dominant at different separations between emitters. It was further shown in [43] that for an infinite reciprocal plasmonic waveguide the best entanglement was obtained when \(\Gamma_{ij}\) was large and \(g_{ij}\) was small (forming the dissipative regime), which forces a restriction on the positioning of the qubits in the reciprocal case. However, in the unidirectional case the qubit positioning is unimportant, as detailed in [18], and the qubits can be anywhere in the coherent or dissipative regimes, which is a practical advantage of these unidirectional systems. In three dimensions it is also true that being in the coherent or dissipative regime makes no difference for a unidirectional system, as evidenced by Eq. (12), where it is seen that large concurrence arises merely from the magnitude of the Green’s function being large.

### III. NUMERICAL RESULTS

A unidirectional SPP can be provided by the interface between a PTI and a topologically trivial material. When operated in a common band gap of the two materials (or if the trivial medium is opaque), the SPP is unidirectional, topologically protected from backscattering, and diffraction immune, providing an ideal implementation of a strongly nonreciprocal system for qubit interactions. Although here we implement a PTI as a PQHE using a continuum plasma [34–36], many other implementations of PTIs are possible, of both PQHE and PQSHE types, and qualitatively would behave in a similar manner.

#### A. Continuum photonic topological insulator realization of a nonreciprocal SPP environment

We assume a magnetized plasma having the permittivity tensor
\[
\varepsilon = \begin{bmatrix}
\varepsilon_{11} & i\varepsilon_{12} & 0 \\
-i\varepsilon_{12} & \varepsilon_{11} & 0 \\
0 & 0 & \varepsilon_{33}
\end{bmatrix}
\]
where
\[
\varepsilon_{11} = 1 + \frac{\omega_p^2}{\omega_c^2} \frac{(v - i\omega)^2 + \omega_c^2}{(v - i\omega)^2 + \omega_c^2},
\]
\[
\varepsilon_{12} = \frac{\omega_p^2}{\omega_c^2 (v - i\omega)^2 + \omega_c^2},
\]
\[
\varepsilon_{33} = 1 + \frac{\omega_p^2}{\omega_c^2 (v - i\omega)^2 + \omega_c^2},
\]
and where \(\omega_c = (q_e/m_e)B_c\) is the cyclotron frequency, \(B_c\) is the applied bias field, \(\omega_p^2 = N_e q_e^2 / \varepsilon_0 m_e\) is the squared plasma frequency, \(N_e\) is the free-electron density and \(q_e\) and \(m_e\) are the electron charge and mass, respectively, and \(\nu\)
placed in the plasma region, very close to the interface at

medium the gap Chern number is $C_{\text{gap}}$

the magnetized plasma and a topologically trivial (simple)

are not considered further in this paper. The Chern numbers

Chern number of the bulk TE mode is trivial, and so TE modes

in Fig. 3).

SPP that crosses the band gap (band structure is shown later,

the presence of one nonreciprocal, backscattering-immune TM

are indicated by dashed red and solid blue, respectively.

Figure 2(b) shows the excited unidirectional SPP at the

interface of the magnetized plasma and the opaque medium,

demonstrating the unidirectional nature of the SPP, and Fig. 2(c)

driven concurrence and pump strength. The concurrence is

stop the pump strength.

Figure 2(d) shows the steady-state concurrence for a wide

range of laser intensities (a laser pump can be applied to the

qubits via, e.g., a fiber penetrating into the material). It can be

seen that the laser intensity cannot be too large, otherwise

the qubits will interact mostly with the laser. Ideally, the

pump should be strong enough to keep the system interacting,

but weak enough for the qubit interaction to dominate the

dynamics. It is clear from Fig. 2(d) that unequal pumping

leads to larger steady-state concurrence.

C. Topological aspect of entanglement

In this section we briefly show the topological aspect of

entanglement in a PTI system. Figure 3 shows the reciprocal

bulk bands (solid blue) for the biased plasma, and the

unidirectional gap-crossing SPP (dashed red) dispersion for a

biased-plasma–opaque-medium interface, for different values

of bias. For $\omega_e > 0$ the gap Chern number is $-1$ [34,35],

indicating the presence of one nonreciprocal, backscattering-immune TM

SPP that crosses the band gap (band structure is shown later, in

Fig. 3).

B. Entanglement evaluation in different environments

We first consider the behavior of the concurrence for

qubits in several different environments, and establish that

the best entanglement occurs for a PTI–opaque-medium interface.

Figure 2(a) shows a comparison of concurrence between four

different cases of two qubits interacting (1) through vacuum,

(2) at the interface of a gold half space and vacuum, (3) at

the interface of a magnetized plasma and vacuum, and (4) at

the interface of a magnetized plasma and an opaque medium.

Here and in the following, the Green’s function is calculated

numerically [46], and the qubits are considered to be placed

very close to the interface, at $y = \lambda_0/60$ in the plasma region.

However, we numerically found that all plots and conclusions

remain unchanged when the qubits are at a distance at least a

height $< \lambda_0/30$ in the plasma region.

The system of qubits was initially prepared in state $|4\rangle =

|e_1\rangle \otimes |g_2\rangle$, such that the left qubit is initially in the excited

state while the right qubit is in the ground state.

From Fig. 2(a), it is clear that the biased-plasma–opaque

medium interface is far superior to the other cases. The poor

performance of the gold-vacuum interface is somewhat

surprising, since a strong SPP can be excited and the

separation is chosen to be in the dissipative regime, which

is a best-case scenario for the reciprocal case. Making the

gold lossless does not significantly improve the concurrence

(results not shown). The problem is primarily due to the lack

of lateral confinement of the SPP, since in [43,45], where

various waveguide geometries are used to confine the mode,

concurrence values are obtained much better than vacuum

values. The unidirectional case has much higher $C$ even for

the flat interface, since energy is focused in one direction.

Furthermore, for the biased-plasma case, one bulk region is

opaque and the other has a band gap, and so no radiation goes

into either bulk medium, whereas for a gold-vacuum interface

energy can flow into the vacuum. Thus, in the following, we

focus on the magnetized-plasma–opaque-medium geometry.

Figure 2(b) shows the excited unidirectional SPP at the

interface of the magnetized plasma and the opaque medium,

demonstrating the unidirectional nature of the SPP, and Fig. 2(c)

drives that a PTI–opaque-medium interface is far superior to the other cases.

The Chern numbers are very insensitive to the bias as long as the topology does

not change.
not change; however, when the gap closes and reopens the concurrence vanishes.

D. Preserving entanglement in the presence of large defects

Perhaps the most important aspect of using PTIs for entanglement is the possibility of robust SPPs, topologically immune to backscattering (and immune to diffraction if operated in the bulk band gap) in the presence of any arbitrary large obstacle or defect. To examine this, we compare two cases: (1) the interface between an opaque medium and a biased plasma and (2) the interface between the same opaque medium and an unbiased plasma.

In the nonreciprocal case, this unidirectional and scattering-immune SPP provides the ability to preserve the entangled state of two qubits in plasmonic systems even in the presence of very nonideal interfaces. Figure 5 shows the transient concurrence for the cases of biased or unbiased plasmas with flat and defected interfaces. Although for the flat interface the biased plasma provides better concurrence than the reciprocal (unbiased) case, this could be perhaps altered by adjustment of the two material half-space properties. However, the point is that in the presence of a defect, as shown in the right panel, the reciprocal SPP suffers from a strong reflection at the defect, as expected, whereas the nonreciprocal SPP (biased plasma) detours around the defect, leading to the same concurrence as without the defect.

E. Finite-width waveguide

The previous results were for an infinitely wide interface. In this section we examine the effect of lateral confinement of the SPP [35]. Figure 6(a) shows the finite-width waveguide geometry. In order to efficiently confine the SPP along the propagation axis, the plasma is extended past the interface to form partially extended sidewalls. Only partial side walls are needed to prevent radiation in space, since the SPP is confined to the vicinity of the interface.

Lateral confinement of the unidirectional SPP improves both the transient and steady-state (pumped) concurrence. Figure 6(b) shows the transient and steady-state concurrence of two qubits initially prepared in state $|4\rangle = |e_1, g_2\rangle$. In comparison to Fig. 2(a), it can be seen that lateral confinement increases both the maximum transient concurrence and the steady-state concurrence. Figure 6(c) shows the dynamics of the qubits under external pumping, where $\rho_{11}, \rho_{22}, \rho_{33},$ and $\rho_{44}$ are the probabilities of finding both qubits in the ground state, both qubits in the excited state, the first qubit in the ground state, and the second qubit in the excited state, and vice versa, respectively. Figure 6(d) shows the steady-state concurrence for a wide range of pump values. The behavior is similar to the case of the infinite interface, Fig. 2(d), except that the range of pump...
values that result in large steady-state concurrence is extended, and the maximum achievable steady-state concurrence is larger in the case of the finite-width waveguide.

In Fig. 7, qubit concurrence is shown for a finite-width waveguide having a defect which spans the entire waveguide width. It can be seen that the concurrence is minimally affected by the defect. Although not shown, as with Fig. 5, in the reciprocal (unbiased) case the defect eliminates the concurrence.

FIG. 6. (a) Finite-width waveguide formed by an opaque medium and biased plasma. (b) Transient and driven concurrence for two qubits interacting through the finite-width waveguide. For the biased plasma, \( \omega_p/\omega_0 = 0.95 \) and \( \omega_s/\omega_0 = 0.21 \), and for the opaque medium \( \varepsilon = -2 \). (c) Dynamics of the qubits under external pumping. (d) Steady-state concurrence for different pump values. Waveguide width is 1.8 \( \mu m \) (1.2\( \lambda_0 \)), qubit separation is 2.4 \( \mu m \) (1.6\( \lambda_0 \)), and \( \omega_0/2\pi = 200 \) THz.

FIG. 7. Transient concurrence of two qubits interacting in a finite-width waveguide [see Fig. 6(a)] consisting of an opaque medium \( (\varepsilon = -2) \) and a biased plasma \( (\omega_p/\omega_0 = 0.95, \omega_s/\omega_0 = 0.21) \). The defect contour length is of the order of a free-space wavelength, and spans the width of the waveguide, \( W = 1.8 \mu m \) (1.2\( \lambda_0 \)). Qubit spacing for the flat interface is 2.4 \( \mu m \) (1.6\( \lambda_0 \)), and for the interface with defect the line-of-sight spacing is 2.4 \( \mu m \). The system of qubits is initially prepared in the state \( |4\rangle = |e_1, g_2\rangle \) and \( \omega_0/2\pi = 200 \) THz.

F. Effect of different initial-state preparations

An interesting behavior of the concurrence arising from having a unidirectional SPP is that, e.g., if the medium supports only a right going SPP, then the initially excited qubit should be the left qubit, otherwise the qubits remain unentangled, as shown in Fig. 8(a) for the unpumped case. Figure 8(b) shows the dynamics of the qubits for this unpumped case. It can be seen that \( \rho_{33} \), which is the probability of finding the right qubit in the excited state and the left qubit in the ground state, starts from 1 and then drops rapidly. However, \( \rho_{24} \), which is the probability of finding the excitation in the left qubit with the right qubit in the ground state, is always zero, meaning that the excitation lost from the right qubit never gets captured by the left qubit. This behavior is particular to a unidirectional environment, and allows for keeping two qubits disentangled at any qubit separation, even if one of them carries an excitation.

However, by applying an external pump we can achieve nonzero concurrence, as also depicted in Fig. 8(a).

The pump is turned on at \( t = 0 \), and instead of immediately becoming nonzero, the concurrence remains zero for a period of time, then starts rising as a sudden birth in concurrence and reaches a nonzero steady-state value. This delayed sudden birth is quite different from the pumped reciprocal case.

It is also possible to consider different initial states which can give other possible unidirectional SPP assisted dynamical evolutions. Figure 9 shows the case of the initial state being the maximally entangled Bell state \( |\Psi_{Bell}\rangle = (|1\rangle + |2\rangle)/\sqrt{2} \).

We consider that the qubits are interacting through the finite-width waveguide depicted in Fig. 6(a). Figure 9(a) shows the time evolution of the concurrence for both pumped and nonpumped cases. In contrast to the previous cases, the concurrence starts from one due to the maximum degree of entanglement of the initial Bell state. For the nonpumped case the concurrence diminishes in time as the system becomes disentangled, resulting in a sudden death of entanglement. It remains zero for a period of time, then the entanglement experiences a rebirth before decaying exponentially at long times. For the externally pumped case, the concurrence exponentially decays but the qubits do not become completely disentangled. Figure 9(b) shows the dynamics of the qubits.
We have derived a master equation for qubit dynamics in a general three-dimensional, nonreciprocal, inhomogeneous, and lossy environment. Spontaneous and pumped entanglement were investigated for two qubits in the vicinity of a photonic topological insulator surface, which supports a nonreciprocal (unidirectional), scattering-immune surface-plasmon polariton in the band gap of the bulk material. We have illustrated the topological nature of the entanglement, and it was shown that large defects in the interface do not impact entanglement for the PTI case, whereas a defect has considerable effect for the reciprocal case. Several initial qubit states were considered, as well as the influence of pump intensity and material loss. Particularities arising from the unidirectional nature of the qubit communication were highlighted.

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**APPENDIX A: MASTER EQUATION DERIVATION AND APPROXIMATIONS**

Starting with Eqs. (1)–(5), we note that the nonreciprocal Green’s tensor has the following useful property [40]:

$$2i \frac{\hbar^2}{c^2} \int d^3r' G(r, r', \omega) \cdot \mathbf{T}(r'', \omega) \cdot T^\dagger(r'', \omega)G^\dagger(r', r'', \omega) = G(r, r', \omega) - G^\dagger(r, r', \omega).$$  \hspace{1cm} (A1)

Under the dipole approximation, the governing Hamiltonian of a system of qubits (two-level atoms) interacting with the surrounding environment can be written as

$$H = \int d^3r \int_0^\infty d\omega d\omega_0 \hat{\mathcal{F}}(\mathbf{r}, \omega) \hat{\mathbf{f}}(\mathbf{r}, \omega) + \sum_i \hbar \omega_i \hat{\sigma}_i^+ \hat{\sigma}_i - \sum_i \int_0^\infty d\omega \hat{d}_i \cdot \mathbf{E}(\mathbf{r}, \omega) + \text{H.c.},$$  \hspace{1cm} (A2)

where the right side can be decomposed into the reservoir Hamiltonian $H_R$ (first term), the qubit Hamiltonian $H_i$ (second term), and the interaction Hamiltonian $H_{ir}$ (third term). We can modify the total Hamiltonian to include the coherent drive (external laser pump) Hamiltonian $V^{AF}$, given later [Eq. (10)]. We transform to a frame rotating with the laser frequency $\omega_l [H \rightarrow \hat{U}(t)H \hat{U}^\dagger(t), \hat{U}(t) = e^{-i\omega_l t} \sum_i \hat{\sigma}_i^+ \hat{\sigma}_i]$ and write the total density matrix of the qubit system and reservoir according to the Schrödinger equation $\dot{\hat{\rho}} = -i[H, \hat{\rho}]/\hbar$, then we transform to the interaction picture $[\hat{O}_I = \hat{U}(t)H \hat{U}^\dagger(t), \hat{U}(t) = e^{-i[H_I + H_R, \hat{\rho}_T]/\hbar}]$ where $\dot{\hat{\rho}}_{T,I} = -i[H_I, \hat{\rho}_{T,I}]$ with $H_I = H_{ir,I}$. We integrate to find

$$\rho_{T,I} = \rho_I(0)R_0 + \frac{-i}{\hbar} \int_0^t dt' [H_I(t'), \rho_{T,I}(t')]$$  \hspace{1cm} (A3)

where $R_0$ is the initial reservoir density matrix. In the interaction picture, by considering $\Gamma_{ii} \ll \omega$ for optical frequencies we make the rotating wave approximation (RWA) in $H_I$ and drop the rapidly varying counter-rotating terms proportional

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**IV. CONCLUSIONS**

for the pumped case. The population probabilities $\rho_{11}$ and $\rho_{22}$ start from 0.5 due to the Bell state preparation. An interesting behavior in the qubit dynamics is the unequal steady-state values $\rho_{33}$ and $\rho_{44}$ under pumping with equal intensities $|\Omega_1| = |\Omega_2|$ (in the reciprocal case, $\rho_{33} = \rho_{44}$).

### G. Lossy biased plasma

In a lossy medium the SPP loses power as it propagates along the interface, resulting in weaker qubit entanglement. In order to study the effect of loss, we suppose the qubits are interacting through an infinite interface as considered in Fig. 2, but for three different collision frequencies: $\nu = 0$ and $\nu/2\pi = 270$ and 500 MHz. Qubits are initially prepared in the state $|4\rangle = |e_1\rangle \otimes |g_2\rangle$. Figure 10, left panel, shows the transient concurrence. Increasing the collision frequency reduces the concurrence, and for collision frequencies greater than 500 MHz loss dominates the system and an entangled steady-state is not achievable for this relatively wide qubit separation of $1.6\lambda_0$.

The right panel of Fig. 10 shows the steady-state concurrence of the pumped system versus pumping intensity. In comparison to the lossless case [Fig. 2(d)], the range of pump intensities that give nonzero steady-state concurrence has decreased, and the maximum achievable concurrence value is diminished compared to the lossless case.

**FIG. 10.** Left panel: Transient concurrence of two qubits interacting through an infinite interface between a biased plasma ($\omega_p/\omega_0 = 0.95$ and $\omega_0/\omega_0 = 0.21$) and an opaque medium ($\varepsilon = -2$) for different values of the collision frequency. Right panel: Steady-state concurrence for different pump values in the lossy case. Qubit separation is $2.4 \mu m (1.6\lambda_0)$ and $\omega_p/2\pi = 200$ THz.

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to $\sigma_i^\dagger(t')\hat{f}^\dagger(\hat{r}',\omega)e^{i(\omega_0-\omega)t'}$ and its Hermitian conjugate. The interaction Hamiltonian in the interaction picture reduces to

$$H_I(t) = -\sum_i \left( \int_0^\infty d\omega \sigma_i^\dagger(t)d_j e^{i(\omega_0-\omega)t} + \text{H.c.} \right),$$

(A4)

To find the system density matrix we insert Eq. (A3) into the interaction picture Schrödinger equation and trace over the reservoir:

$$\partial_t \rho_I = \text{Tr}_R \left\{ \frac{-i}{\hbar} [H_I, \rho_I(0)R_0] \right\}$$

$$-\frac{1}{\hbar^2} \int_0^\infty dt' \text{Tr}_R \{ [H_I(t'), [H_I(t'), \rho_{T,I}(t')]] \}. \quad (A5)$$

Aside from the rotating wave approximation, we apply a number of other approximations to the density matrix to simplify this further (see Appendix A). We first take the mean initial system reservoir coupling to be zero such that $\text{Tr}_R \{ \frac{\hbar}{2\pi} [H_I, \rho_I(0)R_0] \} = 0$. Then we apply the Born approximation, which states that the reservoir will be largely unaffected by its interaction by the system. Next, we assume that the evolution of the density matrix only depends on its current state (Born-Markov approximation). The Born-Markov approximation comes from the assumption that the reservoir relaxation time is much faster than the relaxation time of the system, and so the memory effect of the reservoir can be ignored. Lastly, we make a second Markov approximation, extending the upper limit of the time integral to infinity to produce a fully Markovian equation. With these simplifications we have

$$\partial_t \rho_I = -\frac{1}{\hbar^2} \int_0^\infty dt' \text{Tr}_R \{ [H_I(t'), [H_I(t'), \rho_I(t)]R_0] \}. \quad (A6)$$

We suppose that the atomic transition frequency of the qubits is $\omega_0$. Then, for the first term in Eq. (A6) we have

$$\text{Tr}_R \{ [H_I(t), H_I(t - t') \rho_I(t)R_0] \}$$

$$\sum_{i,j} d_{ai}d_{bj} \int_0^\infty d\omega e^{i(\omega_0-\omega)t'} \sigma_i^\dagger \sigma_j \text{Tr}_R \{ [\hat{f}^\dagger(\hat{r}',\omega)e^{i(\omega_0-\omega)t'}\rho_{T,I}R_0] \}$$

$$\times \langle \hat{E}_\alpha(\hat{r},\omega)\hat{E}_\beta^\dagger(\hat{r},\omega)R_0 \rangle \quad (A7)$$

where

$$\sigma_i = |g_i\rangle\langle e_i|, \sigma_i^\dagger = |e_i\rangle\langle g_i| \quad \text{(A8)}$$

are the atomic lowering and raising operators describing energy-level transitions for each qubit, and where it is supposed that one of the qubits is polarized along $\alpha$ and the other one is polarized along $\beta$. Considering Eq. (5) for the nonreciprocal Green’s tensor and $\text{Tr}_R \{ [\hat{r}(\omega)\hat{f}^\dagger(\hat{r}',\omega)R_0] \} = (\hbar(\omega) + 1)\delta(\hat{r} - \hat{r}')\delta(\omega - \omega')$ with zero thermal photon occupation $\hbar(\omega) = 0$, it can be easily shown that

$$\text{Tr}_R \{ E_{\alpha}(\hat{r},\omega)E_{\beta}^\dagger(\hat{r},\omega)R_0 \}$$

$$= \frac{\hbar}{\pi \epsilon_0 c^4} \int d^3r G_{\alpha\beta}(\hat{r},\hat{r},\omega)$$

$$\times \left[ \epsilon_{\alpha\beta}(\hat{r},\omega) - \epsilon_{\alpha\beta}(\hat{r}',\omega) \right]$$

$$\frac{\omega^2}{2i} \text{Re} G_{\alpha\beta}(\hat{r},\hat{r},\omega)$$

$$= \frac{\hbar}{2\pi \epsilon_0} \frac{\omega^2}{c^4} \{ G_{\alpha,\beta}(\hat{r},\hat{r},\omega) - G_{\beta,\alpha}(\hat{r},\hat{r},\omega) \}. \quad (A9)$$

Thus, we have

$$\text{Tr}_R \{ [H_I(t), H_I(t - t') \rho_I(t)R_0] \}$$

$$= \frac{\hbar}{2\pi \epsilon_0 c^2} \sum_{i,j} \sigma_i^\dagger \sigma_j \rho_I(t)$$

$$\times \int_0^\infty [da_i G_{\alpha\beta}(\hat{r},\hat{r},\omega)d_{bj} - da_j G_{\alpha\beta}^*(\hat{r},\hat{r},\omega)d_{ai}]$$

$$\times \omega^2 d\omega e^{i(\omega_0-\omega)t'}. \quad (A10)$$

Following the same procedure for the second term in Eq. (A6),

$$\text{Tr}_R \{ [H_I(t - t') \rho_I(t)R_0 H_I(t)] \}$$

$$= \frac{\hbar}{2\pi \epsilon_0 c^2} \sum_{i,j} \sigma_i \rho_I(t) \sigma_j^\dagger$$

$$\times \int_0^\infty [da_i G_{\alpha\beta}(\hat{r},\hat{r},\omega)d_{bj} - da_j G_{\alpha\beta}^*(\hat{r},\hat{r},\omega)d_{ai}]$$

$$\times \omega^2 d\omega e^{i(\omega_0-\omega)t'}. \quad (A11)$$

Replacing Eqs. (A10) and (A11) in Eq. (A6) and performing the time integral over $t'$ gives the evolution of the density matrix in the interaction picture, where we have used the Kramers-Kronig relation

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Re} G_{\alpha\beta}(\omega)}{\omega - \omega_0} d\omega = -\pi \text{Im} G_{\alpha\beta},$$

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Im} G_{\alpha\beta}(\omega)}{\omega - \omega_0} d\omega = \pi \text{Re} G_{\alpha\beta}. \quad (A12)$$

Transforming back to the Schrödinger picture, we obtain the master equation for the two-level system dynamics Eqs. (6) and (7).

Next, we briefly discuss the approximations used in the derivation of the master equations (6) and (7).

The first approximation made in the derivation is the RWA where in the interaction picture we drop the rapidly varying counter-rotating terms in $H_I$. This approximation is valid for $\Gamma_{ij} \ll \omega_0$. The qubit transition frequency is $\omega_0/2\pi = 200$ THz, and we assume a dipole moment $\alpha \approx 60$ D. For an interface made of lossless biased plasma, $\Gamma_{ii}/2\pi \sim 450$ MHz, and, for the lossy biased plasma with $\nu/2\pi = 500$ MHz, $\Gamma_{ii}/2\pi \sim 2$ GHz. For the non-biased-plasma–opaque-medium interface (interface supporting reciprocal SPP) $\Gamma_{ii}/2\pi \sim 75$ MHz. In all cases the condition for the validity of the RWA is strongly met.

We also applied the Born-Markov approximation, which comes from the assumption that the reservoir relaxation time, $\tau_R$, is much faster than the relaxation time of the qubit system $\tau_S \approx 1/\Gamma_{ii}$. This allows for the expansion of the exact equation of motion for the density matrix up to second order, and makes the quantum master equation local in time. For a nonreciprocal
The 1D chiral theory is based on the notion of right and left, a 1D chiral reservoir presented in [18,21] (see also [17,20]).

The bath relaxation time can be estimated by looking at the decay time of the correlation function, \( \tau_R \). In the systems considered here the bulk modes are discontinuous at \( x_i \), whereas the interface SPP is strongly nonreciprocal (unidirectional). Thus, to compare with the 1D chiral ME it is sensible to consider the SPP (nonreciprocal) contribution.

As defined in Eq. (B2), \( \Gamma_{ij,SPP} \) is discontinuous at \( x_i = x_j \) in the nonreciprocal case, i.e., \( \Gamma_{ij,SPP} = 2\beta_{ij}\Gamma_{11} |x_i - x_j| \), whereas at \( x_i = x_j \), \( \Gamma_{ij,SPP} = (\beta_{12} + \beta_{21})\Gamma_{11} \). As we show below, the SPP contribution in the considered PTI system is indeed discontinuous at \( x_i = x_j \). However, the exact \( \Gamma_{ij} \), which contains both the SPP and radiation continuum, is continuous at the source point even in the nonreciprocal case. As another example of this, a 3D analytical Green’s function for a nonreciprocal bulk medium is provided in [49] [see their Eq. (117)], where \( \gamma_{ij} \) is also seen to be continuous.

Equating Eq. (11) in the 1D case [i.e., using Eq. (B2)] and Eq. (B1) term by term, the two Lindblad superoperators will be equal if

\[
\mathcal{L}_R \rho(t) = \sum_{j=1,2} \gamma_j (2\sigma_j \rho \sigma_j^\dagger - \rho \sigma_j^\dagger \sigma_j - \sigma_j \rho \sigma_j^\dagger)
+ \gamma_R e^{i k_s (x_j - x_i)}(\sigma_j \rho \sigma_j^\dagger - \sigma_j^\dagger \sigma_j \rho \sigma_j^\dagger) + \gamma_L e^{-i k_s (x_j - x_i)}(\sigma_j \rho \sigma_j^\dagger - \sigma_j^\dagger \sigma_j \rho \sigma_j^\dagger),
\]

where \( k_{LR} = \omega_0/v_{L,R} \), with \( v_e \) being the group velocity of the guided photons.

If we assume now a plasmonic environment, the total emission of the source can be divided into several decay channels: \( \Gamma_{11} = \Gamma_r + \Gamma_{SPP} \), where \( \Gamma_r \) represents free-space radiation, \( \Gamma_{SPP} \) represents losses in the material (quenching), and \( \Gamma_{SPP} \) represents excitation of SPPs. Material absorption and radiation do not contribute to strong qubit-qubit interactions, and therefore we are interested in systems with strong decay through the plasmon channel, \( \Gamma_{SPP} \), where the fraction of all emissions that are coupled to plasmons is expressed by

\[
\beta_{ij} = \beta_{ij,SPP}/\Gamma_{11}, \quad i \neq j.
\]

Assuming a plasmonic environment with a preferred propagation axis, here taken as \( x_i \), in order to connect our formulation with previous 1D chiral formulations [18,21] we introduce a particular 1D plasmonic version of Eq. (8):

\[
\gamma_{ij} = \gamma_{ij,SPP} = \gamma_{ij} \Gamma_{11} e^{-k_s (|x_i - x_j|)} \sin[k_s (x_i - x_j)],
\]

\[
\gamma_{ij} = \gamma_{ij,SPP} = \left\{ \begin{aligned}
\beta_{12} + \beta_{21} \Gamma_{11}, & \quad i = j,
\frac{1}{2} \beta_{ij} \Gamma_{11} e^{-k_s (|x_i - x_j|)} \cos[k_s (x_i - x_j)], & \quad i \neq j.
\end{aligned} \right.
\]

The Green’s function consists of homogeneous (vacuum) and scattered terms, and \( \tau_R \) will be dominated by the slower scattered field contribution [for the vacuum term, \( \tau_R (T) = \hbar / \pi k_B T \) [47], so that \( \tau_R (300 K) \sim 10 \) fs]. For the scattered part of the Green’s function for an interface made of nonbiased plasma and an opaque medium (interface supporting reciprocal SPP), using the Green’s function in [48], \( \tau_R \sim 10^{-11} \) s for \( \nu = 500 \) and 270 MHz, whereas \( \tau_R = 1/\Gamma_{11} \sim 10^{-5} \) s, so that we can ignore the reservoir relaxation time.

**APPENDIX B: COMPARISON WITH PREVIOUS 1D CHIRAL THEORY AND WITH 2D AND 3D GREEN’S FUNCTIONS**

Here we discuss the relation between the general ME we derived in terms of the exact electromagnetic Green’s function, resulting, for two qubits, in the Lindblad Eq. (11), and the 1D phenomenological ME for two-level systems coupled to a 1D chiral reservoir presented in [18,21] (see also [17,20]).

The 1D chiral theory is based on the notion of right and left, defining couplings \( \gamma_{R,L} \), whereas the theory presented here is based on qubit interactions \( \Gamma_{ij} \); note that \( \Gamma_{ij} \) plays the role of a \( \Gamma_{\text{right}} \) if \( x_i > x_j \), but plays the role of a \( \Gamma_{\text{left}} \) if \( x_i < x_j \). To facilitate the comparison with the 1D chiral theory we will assume two qubits with positions \( x_1 \) and \( x_2 \), with \( x_3 > x_1 \). In [18,21] phenomenological quantities \( \gamma_{R,L}, \gamma_{11} \) for \( i = 1,2 \) are utilized, and setting \( \gamma_{11} = \gamma_{22} = \gamma_R = \gamma_L = \gamma_{12} = \gamma_{21} \), the 1D chiral Lindblad superoperator is

\[
\mathcal{L}_R \rho(t) = \sum_{j=1,2} \gamma_j (2\sigma_j \rho \sigma_j^\dagger - \rho \sigma_j^\dagger \sigma_j - \sigma_j \rho \sigma_j^\dagger) + \gamma_R e^{i k_s (x_j - x_i)}(\sigma_j \rho \sigma_j^\dagger - \sigma_j^\dagger \sigma_j \rho \sigma_j^\dagger).
\]
then Eqs. (B3)–(B5) are satisfied and Eq. (11) becomes strictly equal to Eq. (B1). It is worth stressing that physically the two formulations still differ, since Eq. (B7) is not exact (phase velocity and group velocity are different quantities). Nonetheless it is interesting to try to connect the phenomenological parameters in the model Eq. (B1) to the corresponding ones in Eq. (11), which are obtained in terms of the Green’s function, and hence can be computed for arbitrary environments.

Using the rates defined in Eq. (B2), Eq. (12) reduces to

$$ C^{3D}(t) = 2β_311Γ_{11} e^{-β_0^2(x^2 + y^2)} e^{-Γ_{11} t}, $$

(B8)

which is distance independent in the lossless case, as noted in [18] [using Eq. (B6), Eq. (B8) is the same as Eq. (6) in [18]].

1. Discontinuity of the SPP and 2D Green’s function

Here we show that for the strongly nonreciprocal (unidirectional) case, and for a general nonreciprocal case, near the source point the SPP contribution to the Green’s function is discontinuous. We also show that for nonreciprocal systems, $Γ_{21} > Γ_{11}$ can occur.

To avoid analytical complications of the general 3D case, we first assume a simple 2D model of a $z$-directed and $z$-invariant magnetic current source located at $x = 0, y = d$ inside a biased-plasma half space, adjacent to an opaque half space occupying $y < 0$, as depicted in Fig. 11(a). The resulting magnetic field in the plasma is [50,51]

$$ H_z^{inc} = \frac{A_0}{2\pi} \int_{-∞}^{+∞} \frac{1}{2γ_p} R_0 e^{−γ_p(y+d)+iγ_p d} dk_x, $$

(B9)

where $A_0 = iωε_0ε_{eff}I_m$, where $I_m$ is the magnetic current (set to unity) and $R_0$ accounts for the interface,

$$ R_0 = \frac{γ_p}{γ_m} + \frac{iε_m k_x}{γ_m} + \frac{ε_m k_x^2}{γ_m} $$

(B10)

where $γ_p = \sqrt{k_x^2 + ε_m k_x^2}$, $γ_m = \sqrt{k_x^2 + ε_m k_x^2}$, and $ε_m$ is the permittivity of the metal (opaque medium). The field in the absence of the interface is

$$ H_z^{inc} = \frac{A_0}{2\pi} \int_{-∞}^{+∞} \frac{1}{2γ_p} e^{−γ_p|y−d|+iγ_p d} dk_x $$

$$ = \frac{A_0}{−4i} H_{0}^{(1)}(k_0 \sqrt{ε_{eff}ρ}) $$

(B11)

where $H_{0}^{(1)}$ is the Hankel function of the first kind and order zero and $ρ = \sqrt{x^2 + (y − d)^2}$. The source-point singularity is contained in $\text{Im}(H_{0}^{(1)})$, and $Γ \sim \text{Im}(G_{xy}) \sim \text{Re}(E_{z}) \sim \text{Re}(H_{z})$.

The interface reflection coefficient $R_0$, leading to the scattered field, contains pole singularities at the SPP wave numbers (e.g., the denominator of $R_0$ is the SPP dispersion equation). For $|ε_m| \rightarrow ∞$ (perfect conductor), there is one pole at $k_{spp,±} = ±k_0\sqrt{γ_1}$ for $ω_c \geq 0$. For $|ε_m|$ finite the dispersion equation must be solved numerically, and the plasma may be strongly nonreciprocal, supporting a unidirectional SPP (operating in the bulk band gap), nonreciprocal supporting SPPs traveling in opposite directions with unequal wave numbers (operating above the bulk band gap), or, in the unbiased (no band gap) case, reciprocal.

Complex-plane analysis of the magnetic field leads to its evaluation as the sum of a branch cut integral (continuous spectrum) and a discrete residue (SPP) contribution, the latter being

$$ H_{rs}^{inc} = θ(−x)A_0 \text{Res}^{(−)}e^{−γ_p^{(−)}(y+d)+iγ_p^{(−)} d} dk_x $$

$$ + θ(x)A_0 \text{Res}^{(+)}e^{−γ_p^{(+)}(y+d)+iγ_p^{(+) } d} dk_x $$

(B12)

where $\text{Res}^{(±)}$ is the residue of $R_0$ evaluated at $k_x = k_{spp}^{(±)}$, and $γ_p^{(±)} = \sqrt{(k_{spp}^{(±)})^2 − ϵ_{eff}k_0^2}$, where $k_x^{(±)}$ is the SPP pole for $k_x \geq 0$ (forward propagating or backward propagating), and where $θ(x)$ is the Heaviside step function. In the strongly nonreciprocal (unidirectional) case, only one pole is present, leading to only one term in Eq. (B12).

Figure 11(b) shows the magnetic field in the bulk band gap for $ω_c > 0$ obtained by numerical evaluation of the Sommerfeld integral (B9), and by assuming only the residue component Eq. (B12) (since we operate in the bulk band gap and the gap Chern number is $−1$, then there is one unidirectional SPP). The opaque medium is topologically trivial, and is an unbiased plasma having $ε = −2$. As shown in the inset to Fig. 11(c), the residue accurately approximates the field except very close to the source, where the real
FIG. 12. (a) Magnetic field $H_x(x)$ at the interface of an $\varepsilon = -0.47$ half space and a magnetized plasma having $\omega_p/\omega_0 = 0.82$ and $\omega_0/\omega_0 = 0.17$ where $\omega_0/2\pi = 230$ THz. The magnetic line source is located $\lambda_0/10$ above the interface in the plasma region, and the field is evaluated at $(x,y = \lambda_0/10, z = 0)$. (b) Field behavior in the vicinity of the source showing the discontinuity of the residue component.

part of the residue ($\propto \Gamma_{	ext{SP}}$) has an unphysical discontinuity, indicated by the two black dots. In this case, the radiation continuum compensates for the discontinuity of the residue, such that the real part of the full Sommerfeld integral ($\propto \Gamma$) is continuous, and the SPP peak is pushed away from the source point.

As a result of the importance of the radiation continuum near the source, at some points $H_x(x = 0) < H_x(x > 0)$, so that $\Gamma_21$ exceeds $\Gamma_{11}$. Figure 11(d) shows the unbiased (reciprocal) case for the full Sommerfeld integral, where the field peak occurs at $x = 0$ and $\Gamma_21 < \Gamma_{11}$ at all points. In general, there is a quadrature relationship between the dissipative and coherent rates.

Figure 12 shows the magnetic field at a frequency outside the band gap, where we have two SPPs propagating in opposite directions with unequal wave numbers. As with the unidirectional case, the residue shows a discontinuity at the source point.

2. 3D Green’s function

Considering now the 3D case of an electric dipole source at the interface, Fig. 13 shows the dissipative decay and coherent rates of Eq. (8) computed using the finite element method (COMSOL, [46]). In this case, it is impossible to separate the discrete and continuum contributions to the fields. Figures 13(a) and 13(b) show the rates for qubits at the interface as a function of qubit separation for two frequencies within the band gap, and Fig. 13(c) shows the rates normalized by $\Gamma_{11}$ for a fixed separation as a function of height above the interface. It can be seen that, as predicted by the previous analytical 2D model, it occurs that $\Gamma$ is nearly discontinuous at the source point (the discontinuity of the discrete spectrum is softened by the radiation continuum), and that $\Gamma_21 > \Gamma_{11}$ at some points. The coherent rate becomes unbounded at the source due to the well-known divergence of the real part of the Green’s function.

APPENDIX C: CONCURRENCE IN THE UNIDIRECTIONAL CASE

In this section we derive the concurrence for a unidirectional system.

FIG. 13. (a) Dissipative decay (solid blue) and coherent (dashed red) rates at the interface of a biased plasma ($\omega_p/\omega_0 = 0.95, \omega_0/\omega_0 = 0.21$) and an opaque medium ($\epsilon = -2$) at $\omega_0/2\pi = 207$ THz. The same as panel (a) but for 207 THz. The black circle demonstrates the point dipole source, and the dipole moment is $d = 60$ D. (c) The normalized rates as a function of the height of the two qubits above the interface for a fixed separation of 2.1 $\mu$m.

Suppose that the system of qubits is communicating through a strongly nonreciprocal environment, so that the communication is strictly unidirectional, such as occurs for SPPs at PTI interfaces. Assuming that $G(r_1,r_2)$ and $G(r_2,r_1)$ are the dyadic Green’s function propagators along two opposite directions, the unidirectionality assumption leads to, e.g., $G(r_1,r_2) = 0$ ($\Gamma_{12} = g_{12} = 0$) and $G(r_2,r_1) \neq 0$.

Under this unidirectionality assumption, the 3D Lindblad superoperator (7) reduces to

$$\frac{\partial \rho_s(t)}{\partial t} = -\frac{i}{\hbar} [H_s + V^A F, \rho_s(t)] + \frac{\Gamma_{11}}{2} \left[ 2 \sigma_1 \rho_s(t) \sigma_1^\dagger - \sigma_1^\dagger \sigma_1 \rho_s(t) - \rho_s(t) \sigma_1 \sigma_1^\dagger \right]$$

$$+ \frac{\Gamma_{21}}{2} \left[ 2 \sigma_2 \rho_s(t) \sigma_2^\dagger - \sigma_2^\dagger \sigma_2 \rho_s(t) - \rho_s(t) \sigma_2 \sigma_2^\dagger \right]$$

$$+ \frac{\Gamma_{21}}{2} \left[ g_{21} \left[ \sigma_2 \rho_s(t) \sigma_1^\dagger - \rho_s(t) \sigma_1 \sigma_2^\dagger \right] \right]$$

$$+ \frac{\Gamma_{21}}{2} \left[ g_{21} \left[ \sigma_1 \rho_s(t) \sigma_2^\dagger - \rho_s(t) \sigma_2 \sigma_1^\dagger \right] \right]$$

where it has been assumed that $\Gamma_{11} = \Gamma_{22}$.

Defining the basis

$$|1\rangle = |g_1\rangle \otimes |g_2\rangle = |g_1, g_2\rangle, |2\rangle = |e_1\rangle \otimes |e_2\rangle = |e_1, e_2\rangle, |3\rangle = |g_1\rangle \otimes |e_2\rangle = |g_1, e_2\rangle, |4\rangle = |e_1\rangle \otimes |g_2\rangle = |e_1, g_2\rangle$$

and considering the system of qubits to be initially prepared in the state $|4\rangle = |e_1\rangle \otimes |g_2\rangle$, it can be shown that for the nonpumped case the nonzero components of the density matrix

$$\rho_{11} \rightarrow \rho_{11} + \frac{\Gamma_{11}}{2}$$

$$\rho_{22} \rightarrow \rho_{22} + \frac{\Gamma_{22}}{2}$$

$$\rho_{12} \rightarrow \rho_{12} + \frac{\Gamma_{12}}{2}$$

$$\rho_{21} \rightarrow \rho_{21} + \frac{\Gamma_{21}}{2}$$

and the concurrence of the system is given by

$$C = \max \left\{ 0, \sqrt{\rho_{12} \rho_{21} - \rho_{11} \rho_{22}} \right\}$$

where $\rho_{12} = \rho_{21}^\dagger$.
in Eq. (C1) are \( \rho = \rho_3 \) 
\[
\begin{align*}
\dot{\rho}_{11} &= \Gamma_{11}(\rho_{33} + \rho_{44}) + \gamma \rho_{34} + \gamma^* \rho_{43}, \\
\dot{\rho}_{33} &= -\Gamma_{11} \rho_{33} - \gamma \rho_{34} - \gamma^* \rho_{43}, \\
\dot{\rho}_{34} &= -\Gamma_{11} \rho_{34} - \gamma^* \rho_{44}, \\
\dot{\rho}_{43} &= -\Gamma_{11} \rho_{43} - \gamma \rho_{44}, \\
\dot{\rho}_{44} &= -\Gamma_{11} \rho_{44}
\end{align*}
\]  
(C3)

where \( \gamma = \Gamma_{21}/2 + ig_{21} \). For all times the density matrix is block diagonal. Concurrence for arbitrary materials can be calculated as [44]

\[
C = \max(0, \sqrt{u_1} - \sqrt{u_2} - \sqrt{u_3} - \sqrt{u_4}) .
\]  
(C4)

where \( u_i \) are arranged in descending order of the eigenvalues of the matrix \( \rho(t) \rho^\dagger(t) \), where \( \rho(t) = \sigma_y \otimes \sigma_z, \rho^\dagger(t) \sigma_y \otimes \sigma_z \) is the spin-flip density matrix with \( \sigma_i \) being the Pauli matrix.

We have

\[
\rho(t) \rho^\dagger(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x & y & 0 \\ 0 & z & x & 0 \end{bmatrix} \Rightarrow u_1 = x + \sqrt{x^2}, \quad u_2 = x - \sqrt{x^2}, \quad u_3 = 0, \quad u_4 = 0
\]  
(C5)

such that \( x = |\rho_{34}|^2 + |\rho_{33}|^2, y = 2|\rho_{34}|^2 \), and \( z = 2|\rho_{43}|^2, \) and

\[
\begin{align*}
\rho_{43}(t) &= e^{-\Gamma_{11} t}, \\
\rho_{34}(t) &= -\gamma \rho_{34} e^{-\Gamma_{11} t}, \\
\rho_{33}(t) &= -\gamma^* \rho_{33} e^{-\Gamma_{11} t}, \\
\rho_{11}(t) &= 1 - e^{-\Gamma_{11} t} - |\gamma|^2 t^2 e^{-\Gamma_{11} t},
\end{align*}
\]  
(C6)

which leads to Eq. (12).


