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LARGE TIME BEHAVIOR OF SOLUTIONS OF LOCAL AND NONLOCAL NONDEGENERATE HAMILTON-JACOBI EQUATIONS WITH ORNSTEIN-UHLENBECK OPERATOR

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Abstract. We study the well-posedness of second order Hamilton-Jacobi equations with an Ornstein-Uhlenbeck operator in $\mathbb{R}^N$ and $\mathbb{R}^N \times [0, +\infty)$. As applications, we solve the associated ergodic problem associated to the stationary equation and obtain the large time behavior of the solutions of the evolution equation when it is nondegenerate. These results are some generalizations of the ones obtained by Fujita, Ishii & Loreti 2006 [18] by considering more general diffusion matrices or nonlocal operators of integro-differential type and general sublinear Hamiltonians. Our work uses as a key ingredient the a-priori Lipschitz estimates obtained in Chasseigne, Ley & Nguyen 2017 [10].

1. Introduction

The aims of this work are to study the existence and uniqueness of solutions of the equations

\begin{align}
\lambda u^\lambda - F(x, [u^\lambda]) + \langle b(x), Du^\lambda \rangle + H(x, Du^\lambda) &= f(x), \quad x \in \mathbb{R}^N, \ \lambda > 0, \\
\begin{cases}
\frac{\partial u}{\partial t} - F(x, [u]) + \langle b(x), Du \rangle + H(x, Du) &= f(x), \\
u(\cdot, 0) &= u_0(\cdot) \text{ in } \mathbb{R}^N
\end{cases} \quad (x, t) \in \mathbb{R}^N \times (0, +\infty)
\end{align}

and the large time behavior of solution $u(x, t)$ of (2), that is to prove that

\begin{align}
u(\cdot, t) + ct \to v(\cdot) \text{ locally uniformly in } \mathbb{R}^N \text{ as } t \to \infty,
\end{align}

where $(c, v) \in \mathbb{R} \times C(\mathbb{R}^N)$ is a solution of the associated ergodic problem

\begin{align}
c - F(x, [v]) + \langle b(x), Dv \rangle + H(x, Dv) &= f(x) \quad \text{in } \mathbb{R}^N.
\end{align}

Let us describe the main features of (1)-(2). The term $\langle b, D \rangle$ is an Ornstein-Uhlenbeck drift, i.e., there exists $\alpha > 0$ (the strength of the Ornstein-Uhlenbeck term) such that

\begin{align}
\langle b(x) - b(y), x - y \rangle &\geq \alpha |x - y|^2, \quad x, y \in \mathbb{R}^N,
\end{align}

the Hamiltonian $H$ is continuous and sublinear, i.e., there exists $C_H > 0$ such that

\begin{align}
|H(x, p)| &\leq C_H (1 + |p|), \quad x, p \in \mathbb{R}^N,
\end{align}

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and the operator $\mathcal{F}$ can be either \textit{local}
\begin{equation}
\mathcal{F}(x, [u]) = \text{tr}(A(x)D^2u) \quad \text{(classical diffusion)}
\end{equation}
where $A$ is a nonnegative symmetric matrix, or \textit{nonlocal}
\begin{equation}
\mathcal{F}(x, [u]) = \int_{\mathbb{R}^N} \{u(x + z) - u(x) - \langle Du(x), z \rangle \mathbb{I}_B(z)\} \nu(dz) \quad \text{(integro-differential)}
\end{equation}
and the datas $f, u_0$ satisfy
\begin{equation}
|g(x) - g(y)| \leq C_b(\phi_\mu(x) + \phi_\mu(y))|x - y|, \quad g = f \text{ or } g = u_0, \ x, y \in \mathbb{R}^N.
\end{equation}

Since we work on an unbounded domain and deal with unbounded solutions, we need to restrict them in some class
\begin{equation}
\mathcal{E}_\mu = \left\{ g : \mathbb{R}^N \to \mathbb{R} : \lim_{|x| \to +\infty} \frac{g(x)}{\phi_\mu(x)} = 0 \right\},
\end{equation}
where we choose
\begin{equation}
\phi_\mu(x) = e^{\mu \sqrt{1 + |x|^2}}, \quad \mu > 0.
\end{equation}

In the local case, the diffusion $A$ is anisotropic and we assume that $A = \sigma \sigma^T$ where $\sigma \in W^{1,\infty}(\mathbb{R}^N; \mathbb{M}_N)$, i.e.,
\begin{equation}
|\sigma(x)| \leq C_\sigma, \quad |\sigma(x) - \sigma(y)| \leq L_\sigma |x - y|, \quad x, y \in \mathbb{R}^N.
\end{equation}

In the nonlocal case, $\mathcal{F}$ has the form (8), where $\nu$ is a Lévy type measure, which is regular and nonnegative. In order that (8) is well-defined for our solutions in $\mathcal{E}_\mu$,
\begin{equation}
\mathcal{I}(x, \psi, D\psi) := \int_{\mathbb{R}^N} \{\psi(x + z) - \psi(x) - \langle D\psi(x), z \rangle \mathbb{I}_B(z)\} \nu(dz)
\end{equation}
has to be well-defined for any continuous $\psi \in \mathcal{E}_\mu$ which is $C^2$ in a neighborhood of $x$, which leads to assume that
\begin{equation}
\left\{ \begin{array}{l}
\text{There exists a constant } C_\nu^1 > 0 \text{ such that } \\
\int_B |z|^2 \nu(dz), \int_{B^c} \phi_\mu(z) \nu(dz) \leq C_\nu^1.
\end{array} \right.
\end{equation}

An important example of $\nu$ is the \textit{tempered $\beta$-stable law}
\begin{equation}
\nu(dz) = \frac{e^{-\mu|z|}}{|z|^{N+\beta}} dz,
\end{equation}
where $\beta \in (0, 2)$ is the \textit{order} of the integro-differential operator. Notice that, in the bounded framework when $\mu$ can be taken equal to 0, up to a normalizing constant, $-\mathcal{I} = (-\Delta)^{\beta/2}$ is the fractional Laplacian of order $\beta$, see [15] and [25] and references therein for further explanations about the integro-differential operator with Ornstein-Uhlenbeck drift.

Most of the results in this work are based on the Lipschitz estimates on the solutions of (1) and (2) obtained in [10], i.e.,
\begin{equation}
|u^\lambda(x) - u^\lambda(y)|, \ |u(x, t) - u(y, t)| \leq C (\phi_\mu(x) + \phi_\mu(y))|x - y|, \quad x, y \in \mathbb{R}^N,
\end{equation}
where $C$ is independent of $\lambda > 0$, $t \in [0, T)$, $T > 0$. The uniformity of these estimates with respect to $\lambda, t$ is a crucial point for the applications, i.e., to be able to solve the ergodic problem (4) and to prove the large time behavior (3). They are established for both degenerate and nondegenerate equations. Let us recall that equations (1), (2) are called nondegenerate in [10] when

\[ A(x) \geq \rho Id, \quad \text{for some } \rho > 0, \]

in the local case, which is the classical assumption of ellipticity. In the nonlocal one, we work with Lévy measures $\nu$ satisfying (14) and

\[
\left\{ \begin{array}{l}
0 < \eta < 1 \text{ and } C_\nu^2 > 0 \text{ such that, for all } \gamma > 0, \\
\int_{C_{\nu,\gamma}(a)} |z|^2 \nu(dz) \geq C_\nu^2 \eta^{N+\frac{1}{2}} \gamma^{2-\beta},
\end{array} \right.
\]

where $C_{\nu,\gamma}(a) := \{ z \in B_\gamma : (1 - \eta)|z||a| \leq |\langle a, z \rangle| \}$. We say that the nonlocal equation is nondegenerate when the order $\beta$ belongs to the interval $(1, 2)$, since in this case, (18) gives a kind of ellipticity. This assumption, which holds true for the typical example (15), was introduced in [6] and allows to adapt Ishii-Lions’ method to nonlocal integro-differential equation. We refer to [10] for details and comments.

When working in the whole space with unbounded solutions, the Ornstein-Uhlenbeck operator has a very important role which gives some compactness properties of solutions. From a PDE point of view, this property translates into a supersolution property for the growth function $\phi_\mu$ (see [10, Lemma 2.1]), that is, there exist $C, K > 0$ such that

\[
\mathcal{L}[\phi_\mu](x) := -\mathcal{F}(x, [\phi_\mu]) + \langle b(x), D\phi_\mu(x) \rangle - C|D\phi_\mu(x)| \geq \phi_\mu(x) - K, \quad x \in \mathbb{R}^N.
\]

This property is the crucial tool used in [18, 10] to obtain (16), the existence, uniqueness as well as the long time behavior of solutions through the strong maximum principle.

As far as the long time behavior is concerned, there have been many results obtained for second order equations. But most of them are investigated in periodic settings. We refer to [5, 18, 17, 19, 6, 7, 9, 22, 23] and the references therein. There are few results in the unbounded settings, essentially the work of Fujita, Ishii & Loreti 2006 [18]. In [18], they concern with the local equation with a pure Laplacian diffusion,

\[
\frac{\partial u}{\partial t} - \Delta u + \alpha \langle x, Du \rangle + H(Du) = f(x), \quad (x, t) \in \mathbb{R}^N \times (0, \infty).
\]

The same results are investigated with the datas $f, u_0$ and the solutions belonging to the class (10) with

\[
\phi_\mu(x) = e^{\mu|x|^2},
\]

and $0 < \mu < \alpha$, which seems to be the optimal growth condition when thinking to the classical heat equation. But in [18], $H$ is Lipschitz continuous and independent of $x$, thus the well-posedness of the equations is a straightforward consequence in the classical viscosity solutions theory (by comparison and Perron’s method). Moreover, when considering the diffusion $-\Delta$, the proofs are simpler since it is possible to work with smooth solutions thanks to Schauder theory.

We now describe briefly our main results. We prove the existence of continuous viscosity solutions satisfying (16) for nondegenerate equations (1) and (2). This result holds
true in the case of both the general classical diffusion (7) and the nonlocal diffusion (8) and with Hamiltonians $H(x, p)$ which are merely sublinear. When $H$ satisfies in addition
\[ |H(x, p) - H(x, q)| \leq L_H |p - q|, \text{ for all } x, p, q \in \mathbb{R}^N, \]
then the solution is moreover unique. To prove these results we cannot use directly the classical viscosity solution machinery since comparison principle between discontinuous viscosity solutions does not necessarily hold without the classical structure assumptions, see [20, 13, 3] for instance. The idea, which was already used in [5, 22] for instance, is to use a uniformly continuous truncation both for the Hamiltonian and the datas $f, u_0$. It is then possible to build some continuous viscosity solutions to the approximate problems and to pass to the limit thanks to the estimates (16). As a by-product, we obtain $1/2$-H"older estimates in time for the solutions of the evolution problem (2). Since our approach uses (16) as a crucial tool, it is mostly useful for nondegenerate equations for which (16) holds by [10].

However, for possibly degenerate equations, we obtain the same kind of results but using this time classical techniques requiring a stronger assumption on the Hamiltonian,
\[
\begin{cases}
|H(x, p) - H(y, p)| \leq L_H |x - y|(1 + |p|), & x, y, p, q \in \mathbb{R}^N, \\
|H(x, p) - H(x, q)| \leq L_H |p - q|, \\
|H(x, 0)| \leq L_H.
\end{cases}
\]

The other main result in our work is to obtain the convergence (3). The proof is more classical and follows the arguments of [18] but some adaptations are needed in presence of a nonlocal operator and due to the fact that we work with nonsmooth solutions instead of classical ones.

The paper is organized as follows. In Section 2, we study the well-posedness of the equations (1) and (2) at first in the nondegenerate case and then in the general case with stronger assumtions on the Hamiltonian. Section 3 is devoted to the ergodic problem (4) and to the proof of the convergence (3). Some technical results are collected in Section 4.

Notations. In the whole paper, $S_N$ denotes the set of symmetric matrices of size $N$ equipped with the norm $|A| = (\sum_{1 \leq i, j \leq N} a_{ij}^2)^{1/2}$, $B(x, \delta)$ is the open ball of center $x$ and radius $\delta > 0$ (written $B_\delta$ if $x = 0$) and $B^c(x, \delta) = \mathbb{R}^N \setminus B(x, \delta)$.

Let $T \in (0, \infty)$, we write $Q_T = \mathbb{R}^N \times (0, T)$ and $Q = Q_\infty$, we introduce
\[
\mathcal{E}_\mu^+(\mathbb{R}^N) = \{ v : \mathbb{R}^N \to \mathbb{R} : \limsup_{|x| \to +\infty} \frac{v(x)}{\phi_\mu(x)} \leq 0 \},
\]
\[
\mathcal{E}_\mu^+(\overline{Q}_T) = \{ v : \overline{Q}_T \to \mathbb{R} : \limsup_{|x| \to +\infty} \sup_{0 \leq t < T} \frac{v(x, t)}{\phi_\mu(x)} \leq 0 \},
\]
\[
\mathcal{E}_\mu^- := -\mathcal{E}_\mu^+ \text{ and } \mathcal{E}_\mu := \mathcal{E}_\mu^+ \cap \mathcal{E}_\mu^- \text{, where } \phi_\mu \text{ is defined by (11). Notice that } v \in \mathcal{E}_\mu(\mathbb{R}^N) \text{ if and only if for all } \epsilon > 0, \text{ there exists } M(\epsilon) > 0 \text{ such that}
\]
\[ |v(x)| \leq \epsilon \phi(x) + M(\epsilon) \text{ for all } x \in \mathbb{R}^N. \]

In the whole article, we deal with viscosity solutions of (1), (2). Classical references in the local case are [13, 21, 16] and for nonlocal integro-differential equations, we refer the reader to [8, 1, 10].
2. Well-posedness and regularity of the stationary and evolution problems

In two first parts of this Section, we build continuous solutions for (1)-(2) when supposing that the Hamiltonian is sublinear, i.e., (6) holds without further assumption, and that the equation is non-degenerate in the sense explained in the introduction. The proofs in this case are strongly based on the a priori Lipschitz estimates obtained in [10], which hold thanks to the nondegeneracy of the equation together with the effect of Ornstein-Ulhenbeck term. The last part is devoted to build solutions using the classical theory of viscosity solutions for possible degenerate equations. Some additional assumptions on $H$ and on the strength of the Ornstein-Ulhenbeck term are then needed (but we do not use the Lipschitz estimates (16)).

Throughout this Section, we write $\phi$ for $\phi_\mu$ defined by (11).

2.1. Well-posedness of the stationary problem. We start with a comparison principle for solutions of (1) satisfying (16).

Proposition 2.1. Suppose that (5), (20), $f \in C(\mathbb{R}^N)$ and either (12) or (14) hold. Let $u \in USC(\mathbb{R}^N) \cap \mathcal{E}_\mu^+(\mathbb{R}^N)$ and $v \in LSC(\mathbb{R}^N) \cap \mathcal{E}_\mu^-(\mathbb{R}^N)$ be a viscosity sub and supersolution of (1), respectively. Assume that either $u$ or $v$ satisfies (16). Then $u \leq v$ in $\mathbb{R}^N$.

Proof of Proposition 2.1. We argue by contradiction assuming that $u(z) - v(z) \geq 2\eta > 0$ for some $z \in \mathbb{R}^N$. We consider

$$
\Psi(x, y) = u(x) - v(y) - \frac{|x - y|^2}{2\epsilon^2} - \beta(\phi(x) + \phi(y)),
$$

where $\epsilon, \beta$ are positive parameters. For small $\beta$ we have $\Psi(z, z) \geq \eta$. Since $u \in \mathcal{E}_\mu^+(\mathbb{R}^N)$, $v \in \mathcal{E}_\mu^-(\mathbb{R}^N)$, $\Psi$ attains a maximum at $(\bar{x}, \bar{y}) \in B(0, R_\beta) \times B(0, R_\beta)$, where $R_\beta$ does not depend on $\epsilon$. It follows that $u(x) - v(y) - \beta(\phi(x) + \phi(y))$ is bounded in $B(0, R_\beta) \times B(0, R_\beta)$, so the following classical properties (see [3]) hold up to some subsequence,

$$
|\bar{x} - \bar{y}|^2 \to 0, \quad \bar{x}, \bar{y} \to \bar{x} \in B(0, R_\beta) \text{ as } \epsilon \to 0, \ \beta \text{ is fixed},
$$

Assuming that $v$ for instance satisfies (16), since $\Psi(\bar{x}, \bar{x}) \leq \Psi(\bar{x}, \bar{y})$, we have

$$
\frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} \leq v(\bar{x}) - v(\bar{y}) + \beta(\phi(\bar{x}) - \phi(\bar{y}))
\leq C|\bar{x} - \bar{y}|(\phi(\bar{x}) + \phi(\bar{y})) + \beta\mu|\bar{x} - \bar{y}|(\phi(\bar{x}) + \phi(\bar{y})),
$$

using that

$$
|\phi(\bar{x}) - \phi(\bar{y})| \leq \mu(\phi(\bar{x}) + \phi(\bar{y}))|\bar{x} - \bar{y}|.
$$

This implies that $p_\epsilon := \frac{\bar{x} - \bar{y}}{\epsilon^2}$ remains bounded when $\epsilon \to 0$ and, up to some subsequence, $p_\epsilon \to \hat{p}$, for some $\hat{p} \in \mathbb{R}^N$. 

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We write the viscosity inequalities at \((\bar{x}, \bar{y})\) using [13, Theorem 3.2] in the local case and [8, Corollary 1] in the nonlocal one. In the local case, for every \(\rho > 0\), there exist \((p_\epsilon + \beta D\phi(\bar{x}), X) \in J^{2+} u(\bar{x}), (p_\epsilon - \beta D\phi(\bar{y}), Y) \in J^{2-} v(\bar{y})\) such that

\[
\begin{pmatrix}
X & O \\
O & -Y
\end{pmatrix} \leq A + \rho A^2, \quad \text{where } A = \frac{2}{\epsilon^2} \begin{pmatrix}
I & -I \\
-I & I
\end{pmatrix} + \beta \begin{pmatrix}
D^2\phi(\bar{x}) & 0 \\
0 & D^2\phi(\bar{y})
\end{pmatrix}
\]

and \(\rho A^2 = O(\rho)\) (\(\rho\) will be sent to 0 first). It follows

\[
\lambda(u(\bar{x}) - v(\bar{y})) - (F(\bar{x}, [u]) - F(\bar{y}, [v])) + \langle b(\bar{x}) - b(\bar{y}), p_\epsilon \rangle + \beta\langle b(\bar{x}), D\phi(\bar{x}) \rangle + \beta\langle b(\bar{y}), D\phi(\bar{y}) \rangle + H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) - H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y}))
\]

\[
\leq f(\bar{x}) - f(\bar{y}),
\]

where \(F(\bar{x}, [u]) = \text{tr}(A(\bar{x}) X)\) and \(F(\bar{y}, [u]) = \text{tr}(A(\bar{y}) Y)\) in the local case and \(F(\bar{x}, [u]) = \mathcal{I}(\bar{x}, u, p_\epsilon + \beta D\phi(\bar{x}))\) and \(F(\bar{y}, [u]) = \mathcal{I}(\bar{y}, u, p_\epsilon - \beta D\phi(\bar{y}))\) in the nonlocal one.

We estimate the \(F\)-terms by using the results of [10] for the test function \(|\bar{x} - \bar{y}|^2 + \beta(\phi(x) + \phi(y))\). When \(F\) is the local operator defined by (7), applying [10, Lemma 2.2], we obtain

\[
\text{tr}(A(\bar{x}) X - A(\bar{y}) Y) \leq L_2 \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} + \beta\text{tr}(A(\bar{x}) D^2\phi(\bar{x})) + \beta\text{tr}(A(\bar{y}) D^2\phi(\bar{y})) + O(\rho).
\]

When \(F\) is the nonlocal operator defined by (8), applying [10, Proposition 2.1], we get

\[
\mathcal{I}(\bar{x}, u, p_\epsilon + \beta D\phi(\bar{x})) - \mathcal{I}(\bar{y}, v, p_\epsilon - \beta D\phi(\bar{y})) \leq \beta\mathcal{I}(\bar{x}, \phi, D\phi) + \beta\mathcal{I}(\bar{y}, \phi, D\phi).
\]

Therefore, in any case we have

\[
F \leq \beta F(\bar{x}, \phi) + \beta F(\bar{y}, \phi) + L_2 \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} + O(\rho).
\]

Since \(\Psi(\bar{x}, \bar{y}) \geq \Psi(z, z) \geq \eta\), we have \(u(\bar{x}) - v(\bar{y}) \geq \eta\). Using (5), taking into account (26) and sending \(\rho \to 0\), inequality (25) leads to

\[
\lambda\eta - \beta\mathcal{F}(\bar{x}, [\phi]) - \beta\mathcal{F}(\bar{y}, [\phi]) - L_2 \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} + \alpha \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} + \beta\langle b(\bar{x}), D\phi(\bar{x}) \rangle + \beta\langle b(\bar{y}), D\phi(\bar{y}) \rangle + H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) - H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y})) \leq f(\bar{x}) - f(\bar{y}).
\]

Now sending \(\epsilon\) to 0, using (23) and since \(f \in C(\mathbb{R}^N)\) we obtain

\[
\lambda\eta - 2\beta\mathcal{F}(\bar{x}, [\phi]) + 2\beta\langle b(\bar{x}), D\phi(\bar{x}) \rangle + H(\bar{x}, \hat{p} + \beta D\phi(\bar{x})) - H(\bar{y}, \hat{p} - \beta D\phi(\bar{x})) \leq 0.
\]

Since \(H(x, p)\) is lipschitz in \(p\) uniformly in \(x\), i.e., (20) holds, we get

\[
\lambda\eta - 2\beta\mathcal{F}(\bar{x}, [\phi]) + 2\beta\langle b(\bar{x}), D\phi(\bar{x}) \rangle - 2\beta L_H |D\phi(\bar{x})| \leq 0.
\]

From (19), there exists a constant \(K(L_H, F) > 0\) such that

\[-\mathcal{F}(x, [\phi]) + \langle b(x), D\phi(x) \rangle - L_H |D\phi(x)| \geq \phi(x) - K \quad \forall x \in \mathbb{R}^N.
\]

Therefore, we have

\[
\lambda\eta + 2\beta\phi(\bar{x}) - 2\beta K \leq 0.
\]

Since \(\phi > 0\), sending \(\beta\) to 0, we get a contradiction. \(\Box\)
Theorem 2.1. Suppose that (5), (6) and that \( f \in C(\mathbb{R}^N) \cap E_\mu(\mathbb{R}^N) \) satisfying (9). Assume either (12)-(17) or (14)-(18) with \( \beta \in (1,2) \) holds. For all \( \lambda \in (0,1) \), there exists a continuous viscosity solution \( u^\lambda \) of (1) such that

\[
|u^\lambda(x) - u^\lambda(y)| \leq C(\phi_\mu(x) + \phi_\mu(y))|x - y|, \quad x, y \in \mathbb{R}^N,
\]

where \( C > 0 \) is a constant independent of \( \lambda \). In addition, if (20) holds then the solution is unique in \( C(\mathbb{R}^N) \cap E_\mu(\mathbb{R}^N) \).

Proof of Theorem 2.1.

1. Construction of a continuous viscosity solution to a truncated equation. In order to recover the classical framework of viscosity solutions, we first truncate the data on the equations.

Recall that \( \phi(x) = e^{\mu \sqrt{|x|^2 + 1}} \) and \( f \in E_\mu(\mathbb{R}^N) \). By (22), for every \( m \geq 1 \), there exists \( C(m) > 0 \) such that

\[
f(x) \geq -\frac{1}{2m} \phi(x) - C(m).
\]

Therefore, there exists \( R_m > 0 \) such that

\[
f(x) + \frac{1}{m} \phi(x) \geq m, \quad \text{for} \quad |x| \geq R_m.
\]

We then define

\[
f_m(x) = \min\{f(x) + \frac{1}{m} \phi(x), m\}.
\]

The function \( f_m \) is bounded by some constant \( C_m \), still satisfies (9) with the constant \( C_f + \frac{\mu}{m} \) and \( f_m \to f \) locally uniformly in \( \mathbb{R}^N \). Moreover, \( f_m \) is Lipschitz continuous in \( \mathbb{R}^N \) with

\[
|f_m(x) - f_m(y)| \leq \left( C_f + \frac{\mu}{m} \sup_{B(0,R_m)} 2\phi \right) |x - y| =: L_m |x - y|
\]

Indeed, from (29), if \( x, y \notin B(0, R_m) \), then \( f_m(x) = f_m(y) = m \) and the property is true. If \( x, y \in B(0, R_m) \), then, from (29), (9) and (24),

\[
|f_m(x) - f_m(y)| \leq |f(x) - f(y)| + \frac{1}{m} \phi(x) - \phi(y)\]

\[
\leq C_f |x - y| (\phi(x) + \phi(y)) + \frac{\mu}{m} |x - y| (\phi(x) + \phi(y)) \leq L_m |x - y|.
\]

Finally, if \( x \in B(0, R_m) \) and \( y \notin B(0, R_m) \), then, for \( \{z\} = [x,y] \cap \partial B(0,R_m) \), we have

\[
|f_m(x) - f_m(y)| = |f_m(x) - f_m(z)|\]

and we conclude by the previous computation since \( |x - z| \leq |x - y| \).

Let \( n \geq 1 \), we now truncate the Hamiltonian by defining an Hamiltonian \( H_{mn} \) such that

\[
H_{mn}(x,p) = \begin{cases} 
H_m(x,p) & \text{if} \ |p| \leq n \\
H_m(x,n\frac{p}{|p|}) & \text{if} \ |p| \geq n,
\end{cases}
\]

with \( H_m(x,p) = \begin{cases} 
H(x,p) & \text{if} \ |x| \leq m \\
H(m\frac{x}{|x|},p) & \text{if} \ |x| \geq m.
\end{cases}
\]

It is easy to verify that \( H_{mn} \in BUC(\mathbb{R}^N \times \mathbb{R}^N) \) with a modulus of continuity depending on \( m, n \) and satisfies (6) with the same constant \( C_H \). Indeed,
for $|p| \geq n$,

$$|H_m(x, p)| = |H_m(x, n\frac{p}{|p|})| = \begin{cases} H(x, n\frac{p}{|p|}), |x| \leq m \\ H(m\frac{p}{|p|}, n\frac{p}{|p|}), |x| \geq m \end{cases} \leq C_H(1 + n) \leq C_H(1 + |p|),$$

for $|p| \leq n$,

$$|H_m(x, p)| = |H_m(x, p)| = \begin{cases} H(x, p), |x| \leq m \\ H(m\frac{p}{|p|}, p), |x| \geq m \end{cases} \leq C_H(1 + |p|).$$

Obviously, $H_m$ converges locally uniformly in $\mathbb{R}^N \times \mathbb{R}^N$ to $H_m$ when $n \to +\infty$ and $H_m$ converges locally uniformly in $\mathbb{R}^N \times \mathbb{R}^N$ to $H$ when $m \to +\infty$.

We then consider the new equation

$$\lambda u - \mathcal{F}(x, [u]) + \langle b(x), D_u \rangle + H_m(x, Du) = f_m(x) \quad \text{in } \mathbb{R}^N. \tag{32}$$

Classical strong comparison principle holds for bounded discontinuous viscosity sub and supersolutions (see Theorem 4.1 in the Appendix). Noticing that $\lambda u_{\lambda,mn}(x) = \pm \lambda^{-1}(C_m + C_H)$ are respectively a super and a subsolution of (32), we obtain by means of Perron’s method, the existence and uniqueness of a continuous viscosity solution $u_{\lambda,mn}$ of (32) such that $|\lambda u_{\lambda,mn}| \leq \tilde{C}_m := C_m + C_H$ independent of $n$. We refer to classical references [13] for the details.

2. Convergence of the solution of the approximate equation to a continuous solution of (1). Recall that $H_m$ satisfies (6) with constants $C_H$ independent of $m, n$. Moreover, from (30), we have $f_m$ is $L_m$-lipschitz. Since either (12)-(17) or (14)-(18) with $\beta \in (1, 2)$ holds, then applying the a priori Lipschitz estimates [10, Theorem 2.1] for bounded solutions $u_{\lambda,mn}$ we obtain that $u_{\lambda,mn}$ is $K_m -$lipschitz continuous, i.e.,

$$|u_{\lambda,mn}(x) - u_{\lambda,mn}(y)| \leq K_m|x - y| \quad \text{for all } x, y \in \mathbb{R}^N. \tag{33}$$

Therefore, the family $(u_{\lambda,mn})_{n \geq 1}$ is uniformly equicontinuous in $\mathbb{R}^N$. By Ascoli Theorem, it follows that, up to some subsequence,

$$u_{\lambda,mn} \to u_{\lambda,m} \quad \text{as } n \to +\infty \text{ locally uniformly in } \mathbb{R}^N.$$  

By stability ([1, 8, 13]), $u_{\lambda,m}$ is a continuous viscosity solution of (32) with $H_m$ in place of $H_m$ and still satisfies (33) and $|\lambda u_{\lambda,m}| \leq \tilde{C}_m$.

Similarly $H_m$ (respectively $f_m$) satisfies (6) (respectively (9)) with constants $C_H$ and $C_f + \mu$ independent of $m \geq 1$. Applying [10, Theorem 2.1] again, we obtain that $u_{\lambda,m}$ satisfies (28) with $C$ independent of $\lambda, m$. To apply Ascoli Theorem when sending $m \to \infty$, we need some local $L^\infty$ bound for $u_{\lambda,m}$ independent of $m$. It is the purpose of the following Lemma.

Lemma 2.1. For every $\epsilon \in (0, 1)$, there exists $C(\epsilon) > 0$ independent of $m$ and $\lambda$ such that

$$|\lambda u_{\lambda,m}(x)| \leq \epsilon \phi(x) + C(\epsilon). \tag{34}$$

In particular, for all $R > 0$, there exists a constant $C_R > 0$ independent of $m$ and $\lambda \in (0, 1)$ such that

$$|\lambda u_{\lambda,m}(x)| \leq C_R, \quad \text{for all } x \in B(0, R).$$
**Proof of Lemma 2.1.** Let $\epsilon \in (0, 1)$, $y \in \mathbb{R}^N$ such that
\[ u_{\lambda,m}(y) - \epsilon \phi(y) = \max_{\mathbb{R}^N} \{ u_{\lambda,m}(x) - \epsilon \phi(x) \}. \]
Since $u_{\lambda,m}$ is a viscosity solution of (32), at the maximum point, we have
\[ \lambda u_{\lambda,m}(y) - \mathcal{F}(y, [\epsilon \phi]) + \langle b(y), \epsilon D\phi(y) \rangle + H_m(y, \epsilon D\phi(y)) \leq f_m(y). \]
Recall that $H_m$ satisfies (6) with $C_H$ independent of $m$. Hence using (19), we get
\[ \lambda u_{\lambda,m}(y) \leq f_m(y) - \epsilon \phi(y) + \epsilon K + C_H. \]
Let $m \geq \frac{2}{\epsilon}$. Since $f \in \mathcal{E}_\mu(\mathbb{R}^N)$, by (22), there exists $M(\frac{\epsilon}{2}) > 0$ such that
\[ f(y) \leq \frac{\epsilon}{2} \phi(y) + M(\frac{\epsilon}{2}). \]
Hence, from (35) and by the definition of $f_m$ we obtain
\[ \lambda u_{\lambda,m}(y) \leq f(y) + \frac{1}{m} \phi(y) - \epsilon \phi(y) + \epsilon K + C_H \leq M(\frac{\epsilon}{2}) + \epsilon K + C_H. \]
Moreover, since $y$ is a maximum point of $u_{\lambda,m} - \epsilon \phi$, we have, for all $\lambda \in (0, 1)$, and $x \in B(0, R)$, $R > 0$,
\[ \lambda u_{\lambda,m}(x) \leq \lambda \epsilon \phi(x) + \lambda u_{\lambda,m}(y) - \lambda \epsilon \phi(y) \leq \epsilon \phi(x) + M(\frac{\epsilon}{2}) + \epsilon K + C_H \leq C_R, \]
where $C_R = \max_{B(0, R)} \{ \epsilon \phi(x) + M(\frac{\epsilon}{2}) + \epsilon K + C_H \}$ independent of $m$ and $\lambda$.

The proof for the opposite inequality is the same by considering $\min_{\mathbb{R}^N} \{ u_{\lambda,m}(x) + \epsilon \phi(x) \}$. \hfill \Box

Now we can apply Ascoli Theorem to get, up to some subsequence, $u_{\lambda,m} \to u_\lambda$ as $m \to \infty$ locally uniformly in $\mathbb{R}^N$ and $u_\lambda$ is a continuous viscosity solution of (1) satisfying (28).

It remains to prove that $u_\lambda \in \mathcal{E}_\mu(\mathbb{R}^N)$. By (34), since $u_{\lambda,m} \to u_\lambda$ as $m \to \infty$, we get
\[ |\lambda u_\lambda(x)| \leq \epsilon \phi(x) + C(\epsilon), \text{ for all } x \in \mathbb{R}^N. \]
This holds for any $\epsilon > 0$, it means that $u_\lambda \in \mathcal{E}_\mu(\mathbb{R}^N)$.

We conclude to the existence of a continuous solution $u^\lambda$ of (1) belonging to the class (27)-(28).

3. **Uniqueness of the solution of (1) in $C(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$.** Under the additional assumption (20), it is a direct consequence of comparison principle (see Proposition 2.1). \hfill \Box

### 2.2. Well-posedness of the evolution problem.

We recall that $Q_T = \mathbb{R}^N \times (0, T)$ and $Q = Q_\infty$.

**Proposition 2.2.** Suppose that (5), (20) and either (12) or (14) hold. Let $u \in USC(\overline{Q}_T) \cap \mathcal{E}_\mu^+(\overline{Q}_T)$ and $v \in LSC(\overline{Q}_T) \cap \mathcal{E}_\mu^-(\overline{Q}_T)$ be a viscosity sub and supersolution of (2) with $u(\cdot, 0) = u_0(\cdot)$, $f = f_1 \in C(\mathbb{R}^N)$ and $v(\cdot, 0) = v_0(\cdot)$, $f = f_2 \in C(\mathbb{R}^N)$, respectively. Assume either $u(\cdot, t)$ or $v(\cdot, t)$ satisfies (16) and $\sup_{\mathbb{R}^N} \{ u_0(x) - v_0(x) \}, |(f_1 - f_2)^+|_\infty < +\infty$. Then, for all $(x, t) \in \overline{Q}_T$,
\[ u(x, t) - v(x, t) \leq \sup_{\mathbb{R}^N} \{ u_0(y) - v_0(y) \} + t|(f_1 - f_2)^+|_\infty. \]
Proof of Proposition 2.2. Set $U = \sup_{x \in \mathbb{R}^N} \{u_0(x) - v_0(x)\} < +\infty$, we are going to prove that for all $\rho > 0$,

$$u(x, t) - v(x, t) - \frac{\rho}{T - t} - |(f_1 - f_2)^+|_\infty t - U \leq 0 \quad \text{for all } (x, t) \in \overline{Q}_T.$$

We argue by contradiction assuming that there exists some $(z, s) \in \overline{Q}_T$ such that

$$u(z, s) - v(z, s) - \frac{\rho}{T - s} - |(f_1 - f_2)^+|_\infty s - U \geq 2\gamma > 0.$$

Let $\epsilon, \beta > 0$, we consider

$$\Psi(x, y, t) = u(x, t) - v(y, t) - \frac{|x - y|^2}{\epsilon^2} - \frac{\rho}{T - t} - t|(f_1 - f_2)^+|_\infty - U - \beta(\phi(x) + \phi(y)),$$

and set

$$\varphi(x, y, t) = \frac{|x - y|^2}{\epsilon^2} + \frac{\rho}{T - t} + t|(f_1 - f_2)^+|_\infty + U + \beta(\phi(x) + \phi(y)).$$

For small $\beta$ we have $\Psi(z, z, s) \geq \gamma$. Since $u \in \mathcal{E}_\mu^+(\overline{Q}_T)$ and $v \in \mathcal{E}_\mu^-(\overline{Q}_T)$, the maximum of $\Psi$ is achieved at some point $(\tilde{x}, \tilde{y}, \tilde{t}) \in B(0, R_\beta) \times B(0, R_\beta) \times [0, T)$, where $R_\beta$ does not depend on $\epsilon$. It follows that $u(x, t) - v(y, t) - \frac{\rho}{T - t} - t|(f_1 - f_2)^+|_\infty - U - \beta(\phi(x) + \phi(y))$ is bounded in $B(0, R_\beta) \times B(0, R_\beta) \times [0, T)$, so the following classical properties hold (see [3]) up to some subsequence,

$$|\overline{x} - \overline{y}|^2 \epsilon^2 \rightarrow 0, \quad \overline{x}, \overline{y} \rightarrow \hat{x} \in B(0, R_\beta) \text{ as } \epsilon \rightarrow 0, \beta \text{ is fixed}.$$

Assuming that $v(\cdot, \tilde{t})$ satisfies (16), since $\Psi(\overline{x}, \overline{y}, \tilde{t}) \leq \Psi(\overline{x}, \overline{y}, \tilde{t})$ then we have

$$\frac{|\overline{x} - \overline{y}|^2}{\epsilon^2} \leq v(\overline{x}, \tilde{t}) - v(\overline{y}, \tilde{t}) + \beta(\phi(\overline{x}) - \phi(\overline{y}))$$

$$\leq C|\overline{x} - \overline{y}|(\phi(\overline{x}) + \phi(\overline{y}))+ \beta \mu|\overline{x} - \overline{y}|(\phi(\overline{x}) + \phi(\overline{y})).$$

This implies that $p_\epsilon := 2 \frac{\overline{x} - \overline{y}}{\epsilon^2}$ remains bounded when $\epsilon \rightarrow 0$ and, up to some subsequence, $p_\epsilon \rightarrow \hat{p}$, for some $\hat{p} \in \mathbb{R}^N$.

If $\tilde{t} = 0$, since $\Psi(\overline{x}, \overline{y}, 0) \geq \Psi(z, z, s) \geq \gamma$, we have

$$\gamma \leq u(\overline{x}, 0) - v(\overline{y}, 0) - \frac{|\overline{x} - \overline{y}|^2}{\epsilon^2} - U - \frac{\rho}{T} - \beta(\phi(\overline{x}) + \phi(\overline{y}))$$

so

$$\gamma \leq \limsup_{\epsilon \rightarrow 0} \{u(\overline{x}, 0) - v(\overline{y}, 0) - U - \frac{\rho}{T} - \beta(\phi(\overline{x}) + \phi(\overline{y}))\} \leq -\frac{\rho}{T} - 2\beta \phi(\hat{x}),$$

which is a contradiction.

Therefore $\tilde{t} > 0$ and we can write the viscosity inequalities at $(\overline{x}, \overline{y}, \tilde{t})$ using [13, Theorem 8.3] in the local case and [8, Corollary 2] in the nonlocal one. From [13, Theorem 8.3], for any $\zeta > 0$, there exist $r, s \in \mathbb{R}$ and $X, Y \in \mathcal{S}^N$ such that $(r, D_x \varphi, X) \in \overline{P}^{2,+} u(\overline{x}, \tilde{t}), (s, -D_y \varphi, Y) \in \overline{P}^{2,-} v(\overline{y}, \tilde{t}),$

$$r - s = \varphi(\overline{x}, \overline{y}, \tilde{t}) = \frac{\rho}{(T - \tilde{t})^2} + |(f_1 - f_2)^+|_\infty \geq \frac{\rho}{T^2} + |(f_1 - f_2)^+|_\infty.$$
and $D_x \phi(x, y, t) = p_\epsilon + \beta D\phi(x)$, 
$-D_y \phi(x, y, t) = p_\epsilon - \beta D\phi(y)$,

$$\begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq A + \zeta A^2,$$

where $A = D^2 \phi(x, y, t)$

and $\zeta A^2 = O(\zeta)$ ($\zeta$ will be sent to 0 first).

Writing the viscosity inequalities at $(\bar{x}, y, \bar{t})$, we have

$$\sqrt{\frac{0}{T^2}} + |(f_1 - f_2)^+|_\infty - (\mathcal{F}(\bar{x}, [u]) - \mathcal{F}(\bar{y}, [v])) + \langle b(\bar{x}) - b(\bar{y}), \frac{2(\bar{x} - \bar{y})}{\epsilon^2} \rangle$$

$$+ \beta \langle b(\bar{x}), D\phi(\bar{x}) \rangle + \beta \langle b(\bar{y}), D\phi(\bar{y}) \rangle + H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) - H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y}))$$

$$\leq f_1(\bar{x}) - f_2(\bar{y}),$$

where $\mathcal{F}(\bar{x}, [u]) = \text{tr}(A(\bar{x})X)$, $\mathcal{F}(\bar{y}, [u]) = \text{tr}(A(\bar{y})Y)$ in the local case and $\mathcal{F}(\bar{x}, [u]) = \mathcal{I}(\bar{x}, u, p_\epsilon + \beta D\phi(\bar{x}))$, $\mathcal{F}(\bar{y}, [u]) = \mathcal{I}(\bar{y}, u, p_\epsilon - \beta D\phi(\bar{y}))$ in the nonlocal one.

When $\mathcal{F}$ is the local operator and (12) holds, applying [10, Lemma 2.2] for the test function (36), we obtain

$$\text{tr}(A(\bar{x})X - A(\bar{y})Y) \leq \frac{L_2^2|\bar{x} - \bar{y}|^2}{\epsilon^2} + \beta \text{tr}(A(\bar{x})D^2\phi(\bar{x})) + \beta \text{tr}(A(\bar{y})D^2\phi(\bar{y})) + O(\zeta).$$

When $\mathcal{F}$ is the nonlocal operator defined by (8), applying [10, Proposition 2.1] with the test function (36), we get

$$\mathcal{I}(\bar{x}, u, p_\epsilon + \beta D\phi(\bar{x})) - \mathcal{I}(\bar{y}, v, p_\epsilon - \beta D\phi(\bar{y})) \leq \beta \mathcal{I}(\bar{x}, \phi, D\phi) + \beta \mathcal{I}(\bar{y}, \phi, D\phi).$$

Therefore, in both cases we obtain

$$\mathcal{F} \leq \beta \mathcal{F}(\bar{x}, [\phi]) + \beta \mathcal{F}(\bar{y}, [\phi]) + \frac{L_2^2|\bar{x} - \bar{y}|^2}{\epsilon^2} + O(\zeta).$$

Using (5) to estimate the $b$-terms in (38), taking into account (39) and sending $\zeta \to 0$, we obtain

$$\sqrt{\frac{0}{T^2}} + |(f_1 - f_2)^+|_\infty - \frac{L_2^2|\bar{x} - \bar{y}|^2}{\epsilon^2} + \frac{2}{\epsilon^2} \sigma|\bar{x} - \bar{y}|^2 + \beta(-\mathcal{F}(\bar{x}, [\phi]) - \mathcal{F}(\bar{y}, [\phi])$$

$$+ \langle b(\bar{x}), D\phi(\bar{x}) \rangle + \langle b(\bar{y}), D\phi(\bar{y}) \rangle) + H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) - H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y}))$$

$$\leq f_1(\bar{x}) - f_2(\bar{y}).$$

Now sending $\epsilon \to 0$, using (37) and since $f_1, f_2 \in C(\mathbb{R}^N)$, we get

$$\sqrt{\frac{0}{T^2}} + |(f_1 - f_2)^+|_\infty + 2\beta(-\mathcal{F}(\hat{x}, [\phi]) + \langle b(\hat{x}), D\phi(\hat{x}) \rangle)$$

$$+ H(\hat{x}, \hat{p} + \beta D\phi(\hat{x})) - H(\hat{x}, \hat{p} - \beta D\phi(\hat{x}))$$

$$\leq f_1(\hat{x}) - f_2(\hat{x}),$$

for some $\hat{x} \in B(0, R_\beta)$.

Since $H(x, \cdot)$ satisfies (20) then,

$$\sqrt{\frac{0}{T^2}} - 2\beta \mathcal{F}(\hat{x}, [\phi]) + 2\beta \langle b(\hat{x}), D\phi(\hat{x}) \rangle - 2\beta L_H|D\phi(\hat{x})| \leq 0.$$

From (19), there exists a constant $K(L_H, \mathcal{F}) > 0$ such that

$$-\mathcal{F}(\hat{x}, [\phi]) + \langle b(\hat{x}), D\phi(\hat{x}) \rangle - L_H|D\phi(\hat{x})| \geq \phi(\hat{x}) - K.$$

Therefore, we obtain

$$\sqrt{\frac{0}{T^2}} + 2\beta \phi(\hat{x}) - 2\beta K \leq 0.$$
Since $\phi > 0$, sending $\beta \to 0$ we get a contradiction.  

We have proved that, for all $(x, t) \in \overline{Q}_T$,

$$u(x, t) - v(x, t) - \frac{D}{T-t} - t (f_1 - f_2) + |\infty| - \max_{\mathbb{R}^N} \{u(y, 0) - v(y, 0)\} \leq 0.$$  

Sending $\varrho \to 0$ we get the conclusion.  

\textbf{Theorem 2.2.} Suppose (5), (6) and that $f, u_0 \in \mathcal{E}_\mu(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ satisfy (9) with constant $C_f, C_0$. Assume either (12)-(17) or (14)-(18) with $\beta \in (1, 2)$ hold. Then, there exists a continuous viscosity solution $u$ of (2) such that

\begin{align}
(40) & \quad u \in \mathcal{E}_\mu(\bar{Q}), \\
(41) & \quad |u(x, t) - u(y, t)| \leq C|x - y|((\phi(x) + \phi(y)), \quad x, y \in \mathbb{R}^N, \quad t \in [0, T),
\end{align}

where $C > 0$ is a constant independent of $T$. In addition, if (20) holds then the solution is unique in $C(\bar{Q}) \cap \mathcal{E}_\mu(\bar{Q})$.

\textbf{Proof of Theorem 2.2.} We only give a sketch of proof since it is similar with the proof of Theorem 2.1.

1. Construction of a continuous viscosity solution to a truncated equation. Let $m \geq 1$, we first truncate the initial data as we did for $f_m$ in the proof of Theorem 2.1 by considering

$$u_{0m}(x) = \min\{u_0(x) + \frac{1}{m^\nu}(x, m)\}.$$  

Since $u_0 \in \mathcal{E}_\mu(\mathbb{R}^N)$, we get

\begin{align}
(43) & \quad |u_{0m}(x)| \leq C_m, \\
(44) & \quad |u_{0m}(x) - u_{0m}(y)| \leq L_m|x - y|.
\end{align}

Moreover, $u_{0m}$ still satisfies (9) with the constant $C_0 + \mu$ and $u_{0m} \to u_0$ locally uniformly in $\mathbb{R}^N$.

We then introduce the truncated evolution problem (2) with $H_{mn}$ (respectively $f_m$) defined by (31) (respectively (29)) for $m, n \geq 1$ and with the initial data defined by (42). The classical comparison principle (see Theorem 4.2) holds for bounded discontinuous viscosity sub and supersolutions of

$$u_t - \mathcal{F}(x, [u]) + \langle b(x), Du \rangle + H_{mn}(x, Du) = f_m(x) \quad \text{in} \quad Q_T,$$

with the initial data $u_{mn}(x, 0) = u_{0m}(x)$.

Notice that $u_{mn}^- = \pm(C_m + (C_m + C_H)\mu)\mu$ are respectively a super and a subsolution of (45) satisfying the initial conditions

$$u_{mn}^+(x, 0) = -C_m \leq u_{0m}(x) \leq C_m \leq u_{mn}^-(x, 0).$$

Then by means of Perron’s method, we obtain the existence and uniqueness of a bounded continuous viscosity solution $u_{mn}$ of (45) such that $|u_{mn}| \leq \tilde{C}_{mT}$ independent of $n$. We refer to classical references [13] for the details.

2. Convergence of the solution of the truncated equation to a continuous solution of (2). Recall that $H_{mn}$ satisfies (6) with constant $C_H$ independent of $m, n$. Moreover, from (30) and (44) we have $f_m$ and $u_{0m}$ are $L_m$-lipschitz. Since either (12)-(17) or (14)-(18) with $\beta \in (1, 2)$ hold, then applying [10, Theorem 3.1] for bounded solution $u_{mn}$, we obtain that $u_{mn}$ is $K_m$-lipschitz continuous, i.e.,

$$|u_{mn}(x, t) - u_{mn}(y, t)| \leq K_m|x - y| \quad \text{for all} \quad x, y \in \mathbb{R}^N, t \in [0, T).$$
where $K_m$ is independent of $T$. Therefore, the family $(u_{mn})_{n \geq 1}$ is uniformly equicontinuous and bounded in $\overline{Q}$. It follows that, up to some subsequence,
\[ u_{mn} \to u_m \quad \text{as} \quad n \to +\infty \quad \text{locally uniformly in} \quad \overline{Q}. \]
By stability [1, 8, 13], $u_m$ is a viscosity solution of (32) with $H_m$ in place of $H_{mn}$.

Similarly $H_m$ (respectively $f_m$) satisfies (6) (respectively (9)) with constants $C_H$ and $C_f + \mu$, $u_{om}$ satisfies (9) with constant $C_0 + \mu$ independent of $m$. By applying [10, Theorem 3.1] again, we obtain that $u_m$ satisfies (41) with $C$ independent of $m$ and $T$.

To apply Ascoli Theorem sending $m \to \infty$, we need some local bound for $u_m$. Therefore we need to use following Lemma the proof of which is given in the Appendix.

**Lemma 2.2.** Let $T > 0$. For all $\epsilon \in (0, 1)$, there exists $M(\epsilon) > 0$ such that
\[ |u_m(x, t)| \leq \epsilon \phi(x) + M(\epsilon)(1 + |x| + t) \quad \text{for all} \quad (x, t) \in \overline{Q}_T, \]
where $M(\epsilon)$ is independent of $m$ and $T$. In particular, for all $R > 0$, there exists a constant $C_{RT} > 0$ independent of $m$ such that
\[ |u_m(x, t)| \leq C_{RT}, \quad \text{for all} \quad x \in B(0, R), \quad t \in [0, T), \]
and $u_m \in \mathcal{E}_\mu(\overline{Q}_T)$.

From Lemma 2.2, the family $(u_m)_{m \geq 1}$ is uniformly equicontinuous and bounded on compact subsets of $\overline{Q}_T$. By Ascoli Theorem, it follows that, up to some subsequence, $u_m \to u_T$ as $m \to +\infty$ locally uniformly in $\overline{Q}_T$. By stability, $u_T$ is a continuous viscosity solution of (2) in $\overline{Q}_T$. Notice that $u_T$ still satisfies (41) with $C$ independent of $T$ and (46). It is now easy to use a diagonal process to build a solution $u$ of (2) in $Q$ which also satisfies (41) and (46). In particular $u$ is in $\mathcal{E}_\mu(\overline{Q})$ for all $T > 0$ so is in $\mathcal{E}_\mu(\overline{Q}_T)$. It ends the proof of existence.

3. **Uniqueness of the solution of (2) in the class $C(\overline{Q}) \cap \mathcal{E}_\mu(\overline{Q})$.** It is a direct consequence of the comparison principle (see Proposition 2.2) if we assume in addition (20) holds. □

2.3. **Regularity results with respect to time for the evolution problem.** The next lemma gives some time regularity estimates of a solution for which the space regularity is known. This is well known for local equations but does not seem to be written for nonlocal ones. We provide a general statement and a proof for the nonlocal case by adapting the arguments of [4, Lemma 9.1].

**Lemma 2.3.** Let $R > 0$, $0 \leq t_0 < T$, $x_0 \in \mathbb{R}^N$, set $\Omega_{x_0, t_0, R, T} := B(x_0, R) \times (t_0, T)$ and consider a viscosity solution $u \in C(\overline{\Omega}_{x_0, t_0, R+1, T}) \cap \mathcal{E}_\mu(\overline{Q})$ of
\[ u_t - \mathcal{F}(x, [u]) + \langle b(x), Du \rangle + H(x, Du) = f(x), \quad (x, t) \in \Omega_{x_0, t_0, R+1, T}, \]
where $b, H$ are continuous and $\mathcal{F}$ satisfies either (12) (local case) or (14) (nonlocal case). If
\[ |u(y, t_0) - u(x, t_0)| \leq m(|y - x|) \quad \text{for} \quad x, y \in \overline{B}(x_0, R + 1), \]
for some modulus of continuity $m$, then there exists a modulus of continuity $\tilde{m}$ depending only on $m, |u|_{L^\infty(\Omega_{x_0, t_0, R+1, T})}, b, H, \mu$ and $\sigma$ or $\nu$ such that
\[ |u(x, t) - u(x, t_0)| \leq \tilde{m}(|t - t_0|) \quad \text{for} \quad x \in B(x_0, \frac{R}{2}), \quad t \in [t_0, T]. \]
If \( m(r) = Lr \), then \( \tilde{m}(r) = \tilde{L}\sqrt{r} \), where \( \tilde{L} \) depends on \( L, |u|_{L^\infty(\Omega_{x_0,t_0,R+1,T})}, b, H, \mu \) and \( \sigma \) or \( \nu \).

**Remark 2.1.** Notice that in our framework, (48) holds true for \( m(r) = Lr \), see (16).

**Proof of Lemma 2.3.** We fix \( \eta > 0 \) and we want to find some constants \( C, K > 0 \) depending only on \( m, |u|_{L^\infty(\Omega_{x_0,t_0,R+1,T})}, b, H, \sigma \) or \( \nu \) and \( \mu \) such that, for any \( x \in B(x_0, R/2) \) and every \((y,t) \in \overline{B}_{x_0,t_0, R+1,T} \), we have

\[
(50) \quad -\eta - C|y - x|^2 - K(t - t_0) \leq u(y, t) - u(x, t_0) \leq \eta + C|y - x|^2 + K(t - t_0).
\]

We prove only the second inequality, the first one being proved in a similar way. Let us fix \( x \in B(x_0, R/2) \) and consider \((y,t)\) as the running variable in the following.

At first, if we choose

\[
(51) \quad C > \frac{8|u|_{L^\infty(\Omega_{x_0,t_0,R+1,T})}}{R^2},
\]

the desired inequality is fulfilled on \((\overline{B}(x_0, R + 1) \setminus B(x_0, R)) \times [t_0, T] \) for every \( \eta, K > 0 \). Indeed, \( |y - x| > R/2 \) in this region. Notice that \( C \) is chosen independent of \( x \in B(x_0, R/2) \).

Next, we want to ensure that the inequality holds on \( \overline{B}(x_0, R + 1) \times \{t_0\} \). We argue by contradiction assuming that there exists \( \eta > 0 \) such that, for every \( C > 0 \), there exists \( y_C \in \overline{B}(x_0, R + 1) \) such that

\[
(52) \quad u(y_C, t_0) - u(x, t_0) > \eta + C|y_C - x|^2.
\]

It follows that

\[
|y_C - x| \leq \sqrt{\frac{2|u|_{L^\infty(\Omega_{x_0,t_0,R+1,T})}}{C}}.
\]

Thus \( |y_C - x| \to 0 \) as \( C \to +\infty \). Coming back to (52) and using (48), we infer

\[
m(|y_C - x|) \geq u(y_C, t_0) - u(x, t_0) \geq \eta.
\]

We obtain a contradiction if \( C \) is large enough since the left-hand side tends to 0 as \( C \to +\infty \). Notice that the choice of \( C \) to obtain the inequality on \( \overline{B}(x_0, R + 1) \times \{t_0\} \) depends only on \( \eta, |u|_{L^\infty(\Omega_{x_0,t_0,R+1,T})} \) and \( m \).

Therefore, by choosing \( C \) large enough, the desired inequality holds on

\[
(53) \quad (\overline{B}(x_0, R + 1) \setminus B(x_0, R)) \times [t_0, T] \cup (\overline{B}(x_0, R + 1) \times \{t_0\}).
\]

We then consider

\[
(54) \quad \max_{\overline{B}_{x_0,t_0, R+1,T}} \{u - \chi\} \quad \text{where} \quad \chi(y,t) := u(x, t_0) + \eta + C|y - x|^2 + K(t - t_0).
\]

If the maximum is nonpositive, then the desired inequality holds. Otherwise, the maximum is positive and, from (53), is achieved at an interior point \((\bar{y}, \bar{t})\) in \( \Omega_{x_0,t_0,R,T} \). We can write the viscosity inequality for the subsolution \( u \) at this point using the smooth
test-function $\chi$. Since $(\bar{y}, \bar{t})$ is a maximum point of $u - \chi$ in $B(x_0, R) \times [t_0, T]$, we obtain (see [8, Definition 2])
\begin{equation}
K - \int_B (\chi(\bar{y} + z, \bar{t}) - \chi(\bar{y}, \bar{t}) - \langle D\chi(\bar{y}, \bar{t}), z \rangle)\nu(dz)
- \int_{B^c} (u(\bar{y} + z, \bar{t}) - u(\bar{y}, \bar{t}))\nu(dz) + \langle b(\bar{y}), D\chi(\bar{y}, \bar{t}) \rangle + H(\bar{y}, D\chi(\bar{y}, \bar{t})) \leq f(\bar{y}).
\end{equation}
We estimate the terms in the inequality using that $D\chi(y, t) = 2C(y-x)$, $D^2\chi(y, t) = 2CI$ and $|\bar{y} - x| \leq 2R$. We have

\begin{equation}
|\langle b(\bar{y}), D\chi(\bar{y}, \bar{t}) \rangle + H(\bar{y}, D\chi(\bar{y}, \bar{t})) - f(\bar{y})|
\leq \max_{y \in B(x_0, R)} \{|b(y)| |D\chi|_{L^\infty(\Omega_{x_0, t_0, 0, R, T})} + |f(y)| + \max_{|\xi| \leq |D\chi|_{L^\infty(\Omega_{x_0, t_0, 0, R, T})}} |H(y, \xi)|\}
\leq \max_{y \in B(x_0, R)} \{4CR|b(y)| + |f(y)| + \max_{|\xi| \leq 4CR} |H(y, \xi)|\},
\end{equation}
and, using (14),

\begin{equation}
\left|\int_B (\chi(\bar{y} + z, \bar{t}) - \chi(\bar{y}, \bar{t}) - \langle D\chi(\bar{y}, \bar{t}), z \rangle)\nu(dz)\right|
\leq \frac{1}{2} |D^2\chi|_{L^\infty(\Omega_{x_0, t_0, 0, R, T})} \int_B |z|^2 \nu(dz) \leq CC^1 \nu.
\end{equation}
Since $u \in E\mu(\bar{Q}) \subset E\mu(Q_T)$, by (22) for $\epsilon = 1$, we have

\begin{equation}
|u(y, t)| \leq \phi_\mu(y) + M(1) = \phi_\mu(y) + M_T \quad \text{for all } y \in B(x_0, R), t \in [t_0, T]
\end{equation}
for some constant $M_T$ depending on $T$. It follows, using (14) again,

\begin{equation}
\left|\int_{B^c} (u(\bar{y} + z, \bar{t}) - u(\bar{y}, \bar{t}))\nu(dz)\right|
\leq \int_{B^c} (\phi_\mu(\bar{y} + z) + \phi_\mu(\bar{y}) + 2M_T)\nu(dz)
\leq 2\left(\max_{B(x_0, R)} \phi_\mu + M_T\right)C^1 \nu.
\end{equation}
It follows that, if $K > 0$ is chosen such that
\begin{equation}
K > \max_{y \in B(x_0, R)} \left\{4CR|b(y)| + |f(y)| + \max_{|\xi| \leq 4CR} |H(y, \xi)| + (C + 2M_T + 2\phi_\mu(y))C^1 \nu\right\},
\end{equation}
then $\chi$ is a strict supersolution of (47) in $\Omega_{x_0, t_0, 0, R, T}$ and (55) does not hold. Therefore, (54) is nonpositive and the desired inequality holds. Notice that $K$ depends on $x_0, t_0, R, T$, the data $s$ and $\eta, m, |u|_{L^\infty(\Omega_{x_0, t_0, 0, R, T})}$ through the constant $C$.

By (50), we obtain that for every $\eta > 0$,
\begin{equation}
|u(x, t) - u(x, t_0)| \leq \eta + K(\eta)(t - t_0) \quad \text{for every } x \in B(x_0, \frac{R}{2}), t \in [t_0, T],
\end{equation}
where we emphasize the dependence of $K$ with respect to $\eta$. It is standard that by optimizing this estimate with respect to $\eta$ we obtain a modulus of continuity, but let us do it for the sake of clarity. In order to solve $\eta = K(\eta)|t - t_0|$, we define $g : (0, \infty) \rightarrow (0, \infty)$ as the inverse function of $s \mapsto s/K(s)$. Notice that since $\eta \mapsto K(\eta)$ can be chosen as continuous, decreasing and such that $K(\eta) \rightarrow \infty$ as $\eta \rightarrow 0$, the function $g$ is continuous on $(0, +\infty)$, increasing and such that $g(0^+) = 0$ (in other words, $g$ is a modulus of continuity).
Now, choosing the specific value of \( \eta := g(|t - t_0|) \) yields

\[
|u(x, t) - u(x, t_0)| \leq 2g(|t - t_0|) \quad \text{for every } x \in B(x_0, \frac{R}{2}), \ t \in [t_0, T],
\]

and this yields (49) with \( \tilde{m} := 2g \) which is also modulus of continuity.

Now, assume that \( m(r) = Lr \). Looking at the above proof, we notice on the one side that, since

\[
|u(x, t_0) - u(y, t_0)| \leq L|x - y| \leq \eta + \frac{L^2}{4\eta}|x - y|^2,
\]

the desired inequality (50) holds on \( B(x_0, R + 1) \times \{t_0\} \) if \( C \geq \frac{L^2}{4\eta} \). Therefore (50) holds providing \( C \) satisfies the latter inequality and (51). On the other side, we see that

\[
|D\chi(\tilde{y}, \tilde{t})| \leq |Du|_{L^\infty(\Omega_{x_0, t_0, R+1, T})} \leq L
\]

coming back to (56), we see that it is enough to choose \( K \) such that

\[
K > A_1C + A_2 + 2M_TC^1_\nu,
\]

where \( A_1, A_2 \) depends only on the datas, \( x_0, R \) and \( L \). Choosing \( C \) and \( K \) as above, (57) then reads, for every \( \eta > 0 \) and \( x \in B(x_0, \frac{R}{2}), t \in [t_0, T], \)

\[
|u(x, t) - u(x, t_0)| \leq \eta + \left( A_1\left( \frac{8|u|_{L^\infty(\Omega_{x_0, t_0, R+1, T})}}{R^2} + \frac{L^2}{4\eta} \right) + A_2 + 2M_TC^1_\nu \right)|t - t_0|.
\]

Minimizing the right-hand side with respect to \( \eta > 0 \), we get the conclusion. \( \square \)

2.4. Well-posedness of the stationary and evolution equation by using classical techniques. In this section, we investigate the results for degenerate equations. In this case, some additional assumptions on \( H \) and on the strength of the Ornstein-Uhlenbeck operator are needed. Even if [10] provides also a priori Lipschitz estimates in this framework, these estimates are not needed to prove comparison and build continuous solutions. The additional assumptions on \( H \) allows us to use here the classical machinery of viscosity solutions. It is the strategy we choose in this section.

2.4.1. Results for the stationary problem.

**Theorem 2.3.** Let \( u \in USC(\mathbb{R}^N) \cap \mathcal{E}^+_\mu(\mathbb{R}^N) \) and \( v \in LSC(\mathbb{R}^N) \cap \mathcal{E}^-_\mu(\mathbb{R}^N) \) be a viscosity sub and supersolution of (1), respectively. Suppose that (5) (9), (21) and either (12) or (14) hold. Then there is a unique solution \( u^\lambda \in C(\mathbb{R}^N) \cap \mathcal{E}^\lambda_\mu(\mathbb{R}^N) \) of (1).

**Corollary 2.1.** Under the assumptions of Theorem 2.3 with

\[
\begin{align*}
|H(x,p) - H(y,p)| &\leq C_H|x - y|(1 + |p|), \quad x, y, p, q \in \mathbb{R}^N \\
|H(x,p) - H(x,q)| &\leq C_H|p - q|(1 + |x|), \\
|H(x,0)| &\leq C_H(1 + |x|)
\end{align*}
\]

in place of (21). Then for any \( \alpha > 2C_H \), there is a unique solution \( u^\lambda \in C(\mathbb{R}^N) \cap \mathcal{E}^\lambda_\mu(\mathbb{R}^N) \) of (1).
Remark 2.2. In the above two results, we do not assume anymore that the equation is nondegenerate but we use stronger assumptions on $H$, which are the classical assumptions required in viscosity solutions (see [13, 3] and references therein). In the Corollary, when dealing with the weaker assumption (58), we need an additional condition on the strength $\alpha$ of the Ornstein-Uhlenbeck operator in order to guarantee that (19) holds. Actually, (19) is a crucial tool to build a viscosity sub and supersolution for the equation (see [18]).

Proof of Theorem 2.3. The proof of existence of a solution is easily adapted thanks to [18, Theorem 4.2] and the comparison result. In this proof we only focus to prove comparison principle and then the uniqueness is a consequence of this result.

Recall from (19), there is a constant $K > 0$ such that

$$-F(x, |\phi|) + \langle b(x), D\phi(x) \rangle - L_H|D\phi(x)| \geq \phi(x) - K \quad \text{in } \mathbb{R}^N.$$  

(59)

Fix such a constant $K$ and, for $\epsilon \in (0, 1)$, we define the functions $u_\epsilon, v_\epsilon$ on $\mathbb{R}^N$ as following

$$u_\epsilon(x) = u(x) - \epsilon(\phi(x) + \lambda^{-1}K), \quad v_\epsilon(x) = v(x) + \epsilon(\phi(x) + \lambda^{-1}K)$$

Observe that $u_\epsilon, v_\epsilon \in USC(\mathbb{R}^N)$ and using (59), we easily verify that $u_\epsilon$ and $v_\epsilon$ are respectively a viscosity subsolution and a viscosity supersolution of (1).

Now we prove that $u_\epsilon \leq v_\epsilon$ in $\mathbb{R}^N$. We argue by contradiction assuming that $u_\epsilon(z) - v_\epsilon(z) \geq 2\delta > 0$ for some $z \in \mathbb{R}^N$.

We consider,

$$\Psi(x, y) = \sup_{\mathbb{R}^N \times \mathbb{R}^N} \{u_\epsilon(x) - v_\epsilon(y) - \frac{|x-y|^2}{2\eta^2} - \beta(|x|^2 + |y|^2)\},$$

where $\eta, \beta$ are small parameters. For small $\beta$ we have $\Psi(z, z) \geq \delta$. Since $u \in \mathcal{E}_\mu^+(\mathbb{R}^N), v \in \mathcal{E}_\mu^-(\mathbb{R}^N)$, we suppose that $\Psi$ attains its maximum at some points $(x, y) \in B(0, R_\beta) \times B(0, R_\beta)$, where $R_\beta$ does not depend on $\eta$. The following classical properties hold (see [3]) up to some subsequence,

$$\frac{|x-y|^2}{2\eta^2} \rightarrow 0 \quad \text{as } \eta \rightarrow 0 \quad \text{and} \quad \beta |x|, \beta |y| \rightarrow 0 \quad \text{as } \beta \rightarrow 0.$$  

(61)

Setting $p_\eta = \frac{x-y}{\eta}$. Applying [13, Theorem 3.2] in the local case and [8, Corollary 1] in the nonlocal one to learn that, for any $\varrho > 0$, there exist $X, Y \in \mathcal{S}^N$ such that $(p_\eta + 2\beta x, X) \in \mathcal{J}^{2-}u_\epsilon(x), (p_\eta - 2\beta y, Y) \in \mathcal{J}^{2-}v_\epsilon(y)$, and

$$\left(\begin{array}{cc} X & O \\ O & -Y \end{array}\right) \leq E + \varrho E^2,$$

where $E = \frac{1}{\eta^2} \left(\begin{array}{cc} I & -I \\ -I & I \end{array}\right) + 2\beta \left(\begin{array}{cc} I & 0 \\ 0 & I \end{array}\right)$

and $\varrho E^2 = O(\varrho)$ ($\varrho$ will be sent to 0 first).

Writing the viscosity inequalities at $(x, y)$, we obtain

$$\lambda(u_\epsilon(x) - v_\epsilon(y)) - (\mathcal{F}(x, [u]) - \mathcal{F}(y, [v])) + \langle b(x) - b(y), p_\eta \rangle + 2\beta(b(x), x) + 2\beta(b(y), y) + H(x, p_\eta + 2\beta x) - H(y, p_\eta - 2\beta y) \leq f(x) - f(y),$$

where $\mathcal{F}(x, [u]) = tr(A(x)X)$ and $\mathcal{F}(y, [u]) = tr(A(y)Y)$ in the local case and $\mathcal{F}(x, [u]) = \mathcal{I}(x, u, p_\eta + 2\beta x)$ and $\mathcal{F}(y, [u]) = \mathcal{I}(y, u, p_\eta - 2\beta y)$ in the nonlocal one.
When $\mathcal{F}$ is the local operator defined by (7) and (12) holds only, by [10, Lemma 2.2], we easily get

$$\text{tr}(A(x)X - A(y)Y) \leq L_\sigma^2 \frac{|x - y|^2}{\eta^2} + 2\beta\text{tr}(A(x)) + 2\beta\text{tr}(A(y)) + O(\varrho)$$

$$\leq L_\sigma^2 \frac{|x - y|^2}{\eta^2} + 4\beta|\sigma|^2 + O(\varrho).$$

When $\mathcal{F}$ is the nonlocal operator defined by (8), apply [10, Proposition 2.1] for the classical test function, we get

$$\mathcal{I}(x, u, p_n + 2\beta x) - \mathcal{I}(y, v, p_n - 2\beta y) \leq \beta\mathcal{I}(x, \beta|x|^2, 2x) + \beta\mathcal{I}(y, |y|^2, 2y).$$

Moreover, from (14) we have

$$\mathcal{I}(x, |x|^2, 2x) = \int_{B^c} (|x + z|^2 - |x|^2)\nu(dz) + \int_B (|x + z|^2 - |x|^2 - (2x, z))\nu(dz)$$

$$\leq 2|x| \int_{B^c} |z|\nu(dz) + 2 \int_B |z|^2\nu(dz)$$

$$\leq 2C_\nu^1(|x| + 1).$$

Hence, we obtain

$$\mathcal{I}(x, u, p_n + 2\beta x) - \mathcal{I}(y, v, p_n - 2\beta y) \leq 2\beta C_\nu^1(|x| + |y| + 2).$$

Therefore, in any cases we obtain

$$\mathcal{F} \leq \beta C(\mathcal{F})(|x| + |y| + 2) + L_\sigma^2 \frac{|x - y|^2}{\eta^2} + O(\varrho).$$

From (5), we have

$$\langle b(x) - b(y), p_n \rangle + 2\beta\langle b(x) - b(0), x \rangle + 2\beta\langle b(y) - b(0), y \rangle + 2\beta\langle b(0), x + y \rangle$$

$$\geq \alpha\frac{|x - y|^2}{\eta^2} + 2\beta\alpha(|x|^2 + |y|^2) - 2\beta|b(0)||x| + |y|).$$

To estimate $H$-terms, using (21), we have

$$|H(x, p_n + 2\beta x) - H(y, p_n - 2\beta y)|$$

$$\leq L_H|x - y| + \frac{|x - y|^2}{2\eta^2} + 2\beta L_H|x - y||x| + 2\beta L_H(|x| + |y|).$$

Since $\delta \leq \Psi(z, z) \leq u_\varphi(x) - v_\varphi(y)$, combining (63), (64), (65) and sending $\varrho \to 0$, we have

$$\lambda\delta + \alpha\frac{|x - y|^2}{\eta^2} + 2\beta\alpha(|x|^2 + |y|^2)$$

$$\leq \beta C(\mathcal{F})(|x| + |y| + 2) + L_\sigma^2 \frac{|x - y|^2}{\eta^2} + L_H|x - y| + L_H \frac{|x - y|^2}{2\eta^2}$$

$$+ 2\beta L_H|x - y||x| + 2\beta L_H(|x| + |y|) + f(x) - f(y).$$

Since $f \in C(\mathbb{R}^N)$, sending $\eta$ to 0 then $\beta$ to 0 and using (61) we get a contradiction.
We have proved that \( u_\epsilon \leq v_\epsilon \) for all \( \epsilon \in (0, 1) \), i.e.,
\[
u(x) - v(x) - 2\epsilon(\phi(x) + \lambda^{-1}K) \leq 0 \quad x \in \mathbb{R}^N, \; \epsilon \in (0, 1).
\]
Sending \( \epsilon \) to 0 we conclude that \( u \leq v \) in \( \mathbb{R}^N \).

**Proof of Corollary 2.1.** From [10, Lemma 2.1], for any \( \alpha > 2C_H \), there exists a constant \( K > 0 \) such that
\[-\mathcal{F}(x, [\phi]) + b(x, D\phi(x)) - C_H|D\phi(x)|(1 + |x|) \geq \phi(x) - K \quad \text{in} \; \mathbb{R}^N.
\]
Then we easily verify that \( u_\epsilon \) and \( v_\epsilon \) defined by (60) are respectively a viscosity sub-solution and a viscosity supersolution of (1).

Hence, we do the same arguments as in the proof of Theorem 2.3 with the estimate of \( H \) in (65) replaced by
\[
H(x, p_\eta + 2\beta x) - H(y, p_\eta - 2\beta y)
\leq C_H|x - y|(1 + |p_\eta| + 2\beta|x|) + 2\beta C_H(|x| + |y|)(1 + |x|),
\]
where (58) is used. Therefore, instead of (66), we have
\[
\lambda \delta + \alpha \frac{|x - y|^2}{\eta^2} + 2\beta \alpha(|x|^2 + |y|^2)
\leq 4\beta C(\mathcal{F}) + L^2 \frac{|x - y|^2}{\eta^2} + C_H|x - y|(1 + |p_\eta| + 2\beta|x|)
+ 2\beta C_H(|x| + |y|)(1 + |x|) + f(x) - f(y).
\]

Since \( f \in C(\mathbb{R}^N) \) and \( \alpha > 2C_H \), sending \( \eta \) to 0 and then \( \beta \) to 0 we reach contradiction and get a conclusion as in Theorem 2.3.

The existence and uniqueness of a solution are established basing on [18] and the above comparison result with an additional assumption on the strength of the Ornstein-Uhlenbeck operator \( \alpha > 2C_H \).

\[ \square \]

2.4.2. Results for the evolution problem.

**Theorem 2.4.** Let \( u \in USCU(\overline{Q}_T) \cap E^+_{\mu}(\overline{Q}_T) \) and \( v \in LSC(\overline{Q}_T) \cap E^-(\overline{Q}_T) \) be a viscosity sub and supersolution of (2), respectively. Suppose that (5), (9), (21) and either (12) or (14) hold. Assume that \( u(x,0) \leq v(x,0) \) for all \( x \in \mathbb{R}^N \), then there is a unique solution \( u \in C(\overline{Q}_T) \cap E_{\mu}(\overline{Q}_T) \) of (2).

**Corollary 2.2.** Under the assumptions of Theorem 2.4 with (58) in place of (21). Then for any \( \alpha > 2C_H \), there is a unique solution \( u \in C(\overline{Q}_T) \cap E_{\mu}(\overline{Q}_T) \) of (2).

**Remark 2.3.** The same comments as in Remark 2.2 hold.

**Proof of Theorem 2.4.** We first recall that there exists a constant \( K > 0 \) so that (19) holds. Fix any \( \epsilon > 0 \) and define the function \( u_\epsilon \in USCU(\overline{Q}_T) \) and \( v_\epsilon \in LSC(\overline{Q}_T) \) by
\[
u_\epsilon(x,t) = u(x,t) - \epsilon\phi(x) - \epsilon Kt, \quad v_\epsilon(x,t) = v(x,t) + \epsilon\phi(x) + \epsilon Kt.
\]
Observe that \( u_\epsilon - v_\epsilon \in USCU(\overline{Q}_T) \) and that \( u_\epsilon \) and \( v_\epsilon \) are, respectively, a viscosity subsolution and a viscosity supersolution of (2) in \( Q_T \).

Now we apply Proposition 2.2 for \( u_\epsilon \) and \( v_\epsilon \) with \( f_1 = f_2 = f \) we get that
\[
u_\epsilon(x,t) - v_\epsilon(x,t) \leq \sup_{\mathbb{R}^N} \{u(y,0) - v(y,0) - 2\epsilon \phi(y)\} \leq 0 \quad \text{for any} \; \epsilon > 0.
\]
Sending $\epsilon \to 0$ allows us to conclude that $u \leq v$ on $\overline{Q}_T$.

The proof of existence is done thanks to Perron’s method as [18, Theorem 2.2], the uniqueness of solution is a direct consequence of the above comparison result. □

**Proof of Corollary 2.2.** The proof of this Corollary is an adaptation of the one of Corollary 2.1 using the same extension to the parabolic case as explained in the proof of Theorem 2.4. More precisely, we suppose in addition that $\alpha > 2C_H$, where $\alpha$ comes from (5) and we use (58) instead of (21). □

3. APPLICATION TO ERGODIC PROBLEM AND LONG TIME BEHAVIOR OF SOLUTIONS

In this Section we will use some uniform estimates (16) obtained by [10] to solve (4) and then study the convergence (3). The idea comes back to the seminal work of Lions-Papanicolaou-Varadhan [24]. Let $u^\lambda$ be a solution of (1) satisfying (16) with constant independent of $\lambda$, we consider $w^\lambda(x) = u^\lambda(x) - u^\lambda(0)$ and aim at sending $\lambda$ to 0. The family $(w^\lambda)_{\lambda \in (0,1)}$ still satisfies (16). It is locally bounded since, by (16), we have $|w^\lambda(x)| \leq C (\phi_\mu(x) + \phi_\mu(0)) |x|$ so, in this unbounded case, $w^\lambda$ does not belong anymore to $E_{\mu}(\mathbb{R}^N)$ but to a slightly bigger class. We therefore need to take a safety margin for the growth condition in the nonlocal case.

This create an additional difficulty in the nonlocal case. More precisely, from now on, we fix $\overline{\mu} > \mu > 0$ and we assume that

$$
\text{(67)} \quad \text{the measure } \nu \text{ in (13) satisfies (14) with } \overline{\mu}.
$$

Notice that the nonlocal operator $I$ given by (13) is well-defined for all function in $E_\gamma$, $\gamma \leq \overline{\mu}$.

3.1. Application to ergodic problem.

**Theorem 3.1.** Under the assumptions of Theorem 2.1 (assuming in addition (67) in the nonlocal case), there exists a solution $(c,v) \in \mathbb{R} \times C(\mathbb{R}^N)$ of (4) such that

$$
\text{(68)} \quad v \in \bigcap_{\mu < \gamma < \overline{\mu}} E_\gamma(\mathbb{R}^N).
$$

**Remark 3.1.** The same result holds for degenerate equations under the assumptions of Corollary 2.1 with an additional condition on the strength of the Ornstein-Uhlenbeck operator (see [10, Theorem 2.2]). Since our goal is to prove the long time behavior of solutions of the evolution equation (2), we choose to state the result only for non-degenerate equations.

**Proof of Theorem 3.1.** Let $u^\lambda \in C(\mathbb{R}^N) \cap E_\mu(\mathbb{R}^N)$, $\lambda \in (0,1)$, be a solution of (1) given by Theorem 2.1. Define $w^\lambda, z^\lambda \in C(\mathbb{R}^N)$ by $w^\lambda(x) := u^\lambda(x) - u^\lambda(0)$ and $z^\lambda(x) := \lambda u^\lambda(x)$, respectively. Then in view of (28) and Lemma 2.1, there are constant $C, C(1) > 0$ independent of $\lambda$ such that, for all $x, y \in \mathbb{R}^N$,

$$
|z^\lambda(0)| \leq \phi_\mu(0) + C(1),
$$

$$
|z^\lambda(x) - z^\lambda(0)| = |\lambda u^\lambda(x) - \lambda u^\lambda(0)| \leq C|x| (\phi_\mu(x) + \phi_\mu(0)),
$$

$$
|w^\lambda(x)| \leq C|x| (\phi_\mu(x) + \phi_\mu(0)),
$$

$$
|w^\lambda(x) - w^\lambda(y)| \leq C|x - y| (\phi_\mu(x) + \phi_\mu(y)).
$$
Therefore, \( \{w^\lambda\}_{\lambda \in (0,1)} \) is a uniformly bounded and equi-continuous family on any balls of \( \mathbb{R}^N \). By Ascoli’s theorem, up to subsequences, we obtain
\[
z^\lambda \to c, \quad w^\lambda \to v, \quad \text{locally uniformly in } \mathbb{R}^N \text{ as } \lambda \to 0,
\]
for some \( c \in \mathbb{R} \) and \( v \in C(\mathbb{R}^N) \). By the stability of viscosity solutions (see \([1, 8, 13]\)), we find that \( v \) satisfies (4) in the viscosity sense. Let \( \mu < \gamma < \bar{\mu} \). Since
\[
\lim_{|x| \to \infty} \frac{|x| \phi_\mu(x)}{\phi_\gamma(x)} = 0,
\]
we see from (69) that \( v \in \mathcal{E}_\gamma(\mathbb{R}^N) \). \( \square \)

To prove the uniqueness of the ergodic constant and the solution up to additive constants in (4), we need to linearize the equation in order to apply the strong maximum principle. To do that, we need to assume that (20) and (21) hold.

**Theorem 3.2.** Under the assumptions of Theorem 2.1 (assuming in addition (67) in the nonlocal case), let \( (c, v_1), (d, v_2) \in \mathbb{R} \times (C(\mathbb{R}^N) \cap \mathcal{E}_\gamma(\mathbb{R}^N)) \) with \( \mu < \gamma < \bar{\mu} \) be respectively a subsolution and a supersolution of (4).

(i) If (20) holds, then \( c \leq d \);

(ii) If (21) holds and \( c = d \), then there is a constant \( C \in \mathbb{R} \) such that \( v_1 - v_2 = C \) in \( \mathbb{R}^N \).

**Proof of Theorem 3.2.**

(i) We argue by contradiction assuming that \( c > d \) and choose \( \epsilon > 0 \) small enough so that

\[
2\epsilon K \gamma < c - d,
\]
where \( K = K_\gamma \) appearing in (19). Since \( (c, v_1), (d, v_2) \) are sub- and supersolutions of (4), we can easily verify that \( \bar{v}_1(x, t) = v_1(x) - \epsilon \phi_\gamma(x) + ct \) is viscosity subsolution of
\[
v_t - \mathcal{F}(x, [v]) + \langle b(x), Dv(x, t) \rangle + H(x, Dv(x, t)) = f_1(x) \quad \text{in } Q_T = \mathbb{R}^N \times (0, T)
\]
and \( \bar{v}_2(x, t) = v_2(x) + \epsilon \phi_\gamma(x) + dt \) is viscosity supersolution of
\[
v_t - \mathcal{F}(x, [v]) + \langle b(x), Dv(x, t) \rangle + H(x, Dv(x, t)) = f_2(x) \quad \text{in } Q_T,
\]
where \( f_1(x) = f(x) - \epsilon \phi_\gamma(x) + \epsilon K \gamma, f_2(x) = f(x) + \epsilon \phi_\gamma(x) - \epsilon K \gamma \). Since (20) holds, we can apply Proposition 2.2 for \( \bar{v}_1 \) and \( \bar{v}_2 \) to obtain that, for all \( (x, t) \in Q_T \),
\[
v_1(x) - v_2(x) - 2\epsilon \phi_\gamma(x) + (c - d)t \leq \sup_{\mathbb{R}^N} \{v_1(y) - v_2(y) - 2\epsilon \phi_\gamma(y)\} + t|2\epsilon K \gamma - 2\epsilon \phi_\gamma|_\infty.
\]
Taking \( x \) as close as we want to where the sup is achieved, this implies that
\[
(c - d)t \leq 2\epsilon K \gamma t, \quad \text{for all } t > 0,
\]
which is a contradiction. Thus \( c \leq d \).

(ii) For the proof of the second statement, we use the following Lemma, the proof of which is classical and given in the Appendix:

**Lemma 3.1.** Under the assumptions of Theorem 2.1 (assuming in addition (67) in the nonlocal case), suppose that (21) holds. Let \( (c, v_1), (c, v_2) \in \mathbb{R} \times (C(\mathbb{R}^N) \cap \mathcal{E}_\gamma(\mathbb{R}^N)) \) be a viscosity sub and supersolution of (4) respectively, with \( \mu < \gamma < \bar{\mu} \). Then \( \omega = v_1 - v_2 \) is a continuous viscosity subsolution of
\[
-\mathcal{F}(x, [\omega]) + \langle b(x), D\omega(x) \rangle - L_H|D\omega| = 0.
\]
Now let \( \epsilon > 0 \). Since \( \omega = v_1 - v_2 \in \mathcal{E}_\gamma(\mathbb{R}^N) \), \( \omega - \epsilon \phi_\gamma \) attains a maximum at some \( x_\epsilon \in \mathbb{R}^N \). From Lemma 3.1, using \( \epsilon \phi_\gamma \) as a test function for \( w \) we have
\[
-F(x_\epsilon, [\epsilon \phi_\gamma]) + \langle b(x), D\epsilon \phi_\gamma(x) \rangle - L_H|D\epsilon \phi_\gamma(x)| \leq 0.
\]
(71)
Recall from (19) that there is a constant \( K_\gamma > 0 \) such that
\[
-F(x, [\phi_\gamma]) + \langle b(x), D\phi_\gamma(x) \rangle - L_H|D\phi_\gamma(x)| \geq \phi_\gamma(x) - K_\gamma \quad \text{for } x \in \mathbb{R}^N.
\]
Therefore, there is a constant \( R_\gamma > 0 \) independent of \( \epsilon \) such that, for \( x \in \mathbb{R}^N \backslash B(0, R_\gamma) \),
\[
-F(x, [\epsilon \phi_\gamma]) + \langle b(x), D\epsilon \phi_\gamma(x) \rangle - L_H|D\epsilon \phi_\gamma(x)| \geq \epsilon(\phi_\gamma(x) - K_\gamma) > 0.
\]
From (71) and (72) we deduce that \( \omega - \epsilon \phi_\gamma \) can only attain a maximum at \( x_\epsilon \in B(0, R_\gamma) \). It means that
\[
\sup_{\mathbb{R}^N}(\omega - \epsilon \phi_\gamma) = \max_{B(0, R_\gamma)}(\omega - \epsilon \phi_\gamma).
\]
Hence, easy to see that
\[
\sup_{\mathbb{R}^N} \omega = \max_{B(0, R_\gamma)} \omega
\]
Therefore, we have
\[
\sup_{\mathbb{R}^N} \omega = \max_{B(0, r)} \omega \quad \text{for any } r > R_\gamma.
\]
Since (70) is nondegenerate, by applying the strong maximum principle to \( \omega \) in \( B(0, r) \) (see [2, 14] in the local case and [12, 11] in the nonlocal one), we conclude that \( \omega \) is a constant function on \( B(0, r) \) with arbitrary \( r > R_\gamma \), which guarantees that \( \omega(x) = \omega(0) \) for all \( x \in \mathbb{R}^N \).

3.2. Application to long time behavior of solutions. We study the long time behavior of solutions of (2) in the non-degenerate case.

**Theorem 3.3.** Let \( \mu > 0 \). Suppose (5), (6), (21) and that \( f, u_0 \in \mathcal{E}_\mu(\mathbb{R}^N) \cap C(\mathbb{R}^N) \) satisfying (9). Assume either (12)-(17) (local case) or (14)-(18)-(67) with \( \beta \in (1, 2) \) and \( \overline{\mu} > \mu \) (nonlocal case). Let \( u \in \mathcal{E}_\mu(\overline{Q}) \cap C(\overline{Q}) \) be the unique solution of (2) and \( (c, v) \in \mathbb{R} \times (C(\mathbb{R}^N) \cap \mathcal{E}_\gamma(\mathbb{R}^N)) \) a solution of (4) for some \( \mu < \gamma < \overline{\mu} \). Then there is a constant \( a \in \mathbb{R} \) such that
\[
\lim_{t \to \infty} \max_{B(0, R)} |u(x, t) - (ct + v(x) + a)| = 0 \quad \text{for all } R > 0.
\]
(73)
Notice that, under our assumptions, Theorems 2.1, 2.2, 3.1 and 3.2 hold.

Before giving the proof, let us state some preliminaries. The key ingredient is the Lipschitz estimates (16) obtained in [10]. Then, the proof of Theorem 3.3 is quite close to the one of [18, Theorem 5.1]. We follow its lines but there are changes, first because the equation may be nonlocal, and second because we do not work with \( C^2 \)-smooth solutions.

At first, up to replace \( f(x) \) by \( f(x) - c \) (which still satisfies (9)) and the solution \( u(x, t) \) by \( u(x, t) - ct \), we may assume, without loss of generality, that \( c = 0 \).

In what follows we introduce the function
\[
w(x, t) = u(x, t) - v(x) \quad \text{on } \overline{Q}.
\]
(74)
Since \( c = 0 \), \( v \) is a viscosity solution of
\[
-F(x, [v]) + \langle b(x), Dv(x) \rangle + H(x, Dv(x)) = f(x) \quad \text{in } \mathbb{R}^N
\]
(75)
and $u$ is the viscosity solution of
\[ u_t - \mathcal{F}(x, u) + \langle b(x), Du(x, t) \rangle + H(x, Du(x, t)) = f(x) \quad \text{in } Q, \]
then, by Lemma 3.1 (actually we use the parabolic version of this Lemma, which is obtained by straightforward adaptations in its proof), $w$ is a viscosity subsolution of
\[
\mathcal{P}[w](x, t) := w_t - \mathcal{F}(x, [w]) + \langle b(x), Dw \rangle - L_H |Dw| = 0 \quad \text{in } Q.
\]
Thanks to (19) (with $\gamma$ instead of $\mu$), there exists $K = K(\gamma, L_H)$ such that
\[-\mathcal{F}(x, [\phi_{\gamma}]) + \langle b(x), D\phi_{\gamma}(x) \rangle - L_H |D\phi_{\gamma}(x)| \geq \phi_{\gamma}(x) - K \quad \text{in } \mathbb{R}^N.
\]
Therefore
\[
\varphi(x, t) := (\phi_{\gamma}(x) - K)e^{-t}
\]
is a smooth supersolution of
\[
\mathcal{P}[\varphi](x, t) \geq 0 \quad \text{in } Q.
\]

We divide the proof of Theorem 3.3 into several lemmas. The following lemma gives some boundedness of $w$ with respect to $t$ (recall that $c = 0$).

**Lemma 3.2.** Under the assumptions of Theorem 3.3, for every $0 < \epsilon < 1$, there exists $C(\epsilon) > 0$ such that
\[
|w(x, t)| \leq \epsilon\phi_{\gamma}(x) + C(\epsilon), \quad (x, t) \in \overline{Q},
\]

**Proof of Lemma 3.2.** The proof is an easy application of the maximum principle. By Theorem 2.2, since $u_0 \in \mathcal{E}_\mu(\mathbb{R}^N)$, $u \in \mathcal{E}_\mu(\mathbb{R}^N \times [0, T])$ for every $T > 0$. Recalling that $v \in \mathcal{E}_\gamma(\mathbb{R}^N)$, we obtain that $w(\cdot, 0) \in \mathcal{E}_\gamma(\mathbb{R}^N)$ and $w \in \mathcal{E}_\gamma(\mathbb{R}^N \times [0, T])$ for every $T > 0$. We fix $T > 0$. It follows from (22) that
\[ |w(x, 0)| = |u_0(x) - v(x)| \leq \epsilon\phi_{\gamma}(x) + M(\epsilon) \leq \epsilon\varphi(x, 0) + \epsilon K + M(\epsilon) =: \epsilon\varphi(x, 0) + C(\epsilon).
\]
Notice that $C(\epsilon)$ is independent of $T$. Moreover, $w(x, t) - \epsilon\varphi(x, t) \to -\infty$ as $|x| \to +\infty$ uniformly with respect to $t \in [0, T]$. Therefore, there exists $R = R(T, \epsilon, \gamma) > 0$ such that $w(x, t) \leq \epsilon\varphi(x, t)$ in $\left(\mathbb{R}^N \setminus B(0, R)\right) \times [0, T] = B^c(0, R) \times [0, T]$. Finally,
\[ w(x, t) \leq \epsilon\varphi(x, t) + C(\epsilon) \quad \text{in } (B(0, R) \times \{0\}) \cup (B^c(0, R) \times [0, T]).
\]
Therefore, if, for some $\eta > 0$,
\[ M(\eta) := \max_{\mathbb{R}^N \times [0, T]} \{w(x, t) - \epsilon\varphi(x, t) - C(\epsilon) - \frac{\eta}{T - t}\} > 0,
\]
then the maximum is achieved at an interior point $(\bar{x}, \bar{t})$ of the parabolic cylinder $B(0, R) \times [0, T]$. Using that $w$ is a viscosity subsolution of (76), we obtain $\mathcal{P}[\epsilon\varphi + C(\epsilon) + \frac{\eta}{T - t}](\bar{x}, \bar{t}) \leq 0$, which is a contradiction with (78) since $\mathcal{P}(C(\epsilon) + \frac{\eta}{T - t}) > 0$. It follows that
\[ w(x, t) - \epsilon\varphi(x, t) - C(\epsilon) - \frac{\eta}{T - t} \leq M(\eta) \leq 0 \quad \text{for all } x \in \mathbb{R}^N, t \in [0, T], \eta > 0.
\]
Sending $\eta$ to 0 and recalling that $C(\epsilon)$ does not depend on $T$, we get (79). \hfill \square

**Lemma 3.3.** Under the assumptions of Theorem 3.3, for every $R > 0$, there exists $L_R > 0$ (independent of $t$) such that
\[
|u(x, t) - u(x, s)| \leq L_R \sqrt{|t - s|} \quad \text{for all } x \in B(0, R), t, s \in [0, +\infty).
\]
Proof of Lemma 3.3. It is a direct consequence of Lemma 2.3. Indeed, take \( x_0 = 0, \ t_0 = 0 \) and \( \Omega_{0,0,2R+1,T} = B(0, 2R + 1) \times (0, T) \) in Lemma 2.3. By Lemma 3.2, \( |u|_{L^\infty(\Omega_{0,0,2R+1,T})} \) depends only on \( R \) (but not on \( T \)). Notice also that \( M_T \) which appears in the proof of Lemma 2.3 can be chosen independent of \( T \) thanks to Lemma 3.2. The conclusion follows. Indeed, Lemma 3.2 implies the following time-independent bound \( |u(x,t)| \leq |v(x)| + \epsilon \phi_\gamma(x) + C(\epsilon) \).

\[ (81) \]

Lemma 3.4. Under the assumptions of Theorem 3.3, the sets \( \{u(\cdot, t) : t \geq 0\} \) and \( \{u(\cdot, \cdot + t) : t \geq 0\} \) are precompact in \( C(\mathbb{R}^N) \) and \( C(\overline{Q}) \), respectively.

Proof of Lemma 3.4. The proof is a straightforward consequence of the boundedness and equicontinuity of both families on bounded subsets of \( \mathbb{R}^N \). These properties follow from Theorem 2.2 and Lemmas 3.2 and 3.3. □

We introduce the half-relaxed limits (see [13, 8])

\[ \overline{u}(x) = \limsup_{t \to \infty} u(x,t), \quad \underline{u}(x) = \liminf_{t \to \infty} u(x,t). \]

Lemma 3.5. Under the assumptions of Theorem 3.3, there exist a solution \( v \in C(\mathbb{R}^N) \cap \mathcal{E}_\gamma(\mathbb{R}^N) \) of (75) satisfying (16) and \( \overline{C}, \underline{C} \in \mathbb{R} \) such that

\[ (81) \]

Proof of Lemma 3.5. By Theorem 2.2 and Lemma 3.2, we obtain easily that \( \overline{u} \) and \( \underline{u} \) are well-defined, belong to \( \mathcal{E}_\gamma(\mathbb{R}^N) \), satisfy the Lipschitz estimates (16). By classical stability results ([13, 8]), \( \overline{u} \) is a viscosity subsolution and \( \underline{u} \) a viscosity supersolution of (75). By Theorem 3.1 (under our assumptions which leads to \( c = 0 \)), there exists a solution \( (0, v) \) of (4). The existence of \( \overline{C}, \underline{C} \) such that (81) holds follows directly from Theorem 3.2 (ii). □

Now, we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. To prove the convergence (73), in view of Lemma 3.4, it is sufficient to prove that \( \overline{C} = \underline{C} \) in Lemma 3.5. Since \( \overline{u} \geq \underline{u} \), we have \( \overline{C} \leq \underline{C} \) and it remains to establish \( \overline{C} \geq \underline{C} \).

We claim that there exists \( u_\infty \geq \underline{u} \) in the \( \omega \)-limit set

\[ \Omega(u) = \{ \omega \in C(\overline{Q}) : \text{there exits } t_j \to +\infty \text{ such that } u(\cdot, \cdot + t_j) \to \omega \text{ in } C(\overline{Q}) \} \]

such that

\[ (82) \]

Indeed, by (16) for \( u \), we have

\[ (83) \]

hence, there exists \( t_j \to +\infty \) such that \( u(0, t_j) \to \underline{u}(0) \). Therefore, using Lemma 3.4 again, there exists a subsequence (still denoted \( (t_j) \) and \( u_\infty \in C(\overline{Q}) \) such that \( u(\cdot, \cdot + t_j - 1) \to u_\infty \) in \( C(\overline{Q}) \). It is clear that \( \underline{u} \leq u_\infty \in \Omega(u) \) and, since \( u(0,1 + t_j - 1) = u(0,t_j) \to \underline{u}(0) \), we get (82) and the claim is proved.

Now, we prove that there is a sequence \( s_j \to +\infty \) such that

\[ (84) \]

as \( j \to \infty \).
From the previous claim, the function $\zeta \in C(\overline{Q})$ defined by $\zeta(x, t) = u(x) - u_\infty(x, t)$ attains a maximum over $\overline{Q}$ at the point $(0, 1)$. Moreover, $u$ is a viscosity solution (so subsolution) of (75) and, by stability, $u_\infty$ is a viscosity solution (so supersolution) of (2).

Thanks to Lemma 3.1, we get $\zeta$ is a viscosity subsolution of (76). By applying the strong maximum principle to $\zeta$ (adapting the proof of Theorem 3.2 to the case of parabolic equations), we find that $\zeta$ is constant in $\overline{Q}$. Since $\zeta(0, 1) = 0$, we obtain $u(x) = u_\infty(x, t)$ for all $(x, t) \in \overline{Q}$. But, by the definition of $\Omega(u)$, there is a sequence $s_j \to +\infty$ such that $u(\cdot, \cdot + s_j) \to u_\infty$ in $C(\overline{Q})$. This shows (84).

For $j \in \mathbb{N}$ and $\epsilon > 0$, define

$$w_j(x, t) := u(x, t + s_j) - v(x) + \underline{C} = w(x, t + s_j) + \underline{C} = u(x, t + s_j) - u(x),$$

where $s_j$ is defined in (84) and we used (74) and (81) for the last two equalities. By Lemma 3.2,

$$w_j(x, 0) = w(x, s_j) + \underline{C} \leq \frac{\epsilon}{2} \phi_\gamma(x) + C(\frac{\epsilon}{2}) + \underline{C},$$

hence there exists $R = R_\epsilon > 0$ large enough such that

$$w_j(x, 0) \leq \epsilon(\phi_\gamma(x) - K) = \epsilon \varphi(x, 0) \text{ for } x \in \mathbb{R}^N \setminus B(0, R_\epsilon),$$

where $\varphi$ is defined in (77). For $x$ in the compact subset $\overline{B}(0, R_\epsilon)$, up to fix $j$ big enough, by (84), we infer

$$w_j(x, 0) = u(x, s_j) - u(x) \leq \epsilon \leq \epsilon \varphi(x, 0) + (K + 1)\epsilon.$$

Therefore

$$w_j(x, 0) \leq \epsilon \varphi(x, 0) + (K + 1)\epsilon \text{ for } x \in \mathbb{R}^N.$$

Since $w_j \in \mathcal{E}_\gamma(\overline{Q})$ is a subsolution and $\epsilon \varphi \in \mathcal{E}_\gamma(\overline{Q})$ is a supersolution of (76) in $\overline{Q}_T$ for any $T > 0$, by the comparison principle of Proposition 2.2 in $\mathcal{E}_\gamma(\overline{Q}_T)$, we obtain

$$w_j(x, t) - \epsilon \varphi(x, t) \leq \sup_{\mathbb{R}^N} \{w_j(\cdot, 0) - \epsilon \varphi(\cdot, 0)\} \leq (K + 1)\epsilon \text{ for } (x, t) \in \overline{Q}_T.$$

Since this bound does not depend on $T > 0$, the previous inequality holds in $\overline{Q}$ and it follows that

$$u(x, t + s_j) \leq v(x) - \underline{C} + \epsilon(\phi_\gamma(x) - K)e^{-t} + (K + 1)\epsilon$$

and therefore, using Lemma 3.5,

$$\limsup_{t \to \infty} u(x, t + s_j) = \underline{u}(x) = v(x) - \overline{C} \leq v(x) - \underline{C} + (K + 1)\epsilon.$$

Sending $\epsilon$ to 0, we get the desired inequality $\overline{C} \geq \underline{C}$. It ends the proof. \qed

4. Appendix

**Theorem 4.1.** Let $u \in USC(\mathbb{R}^N)$ and $v \in LSC(\mathbb{R}^N)$ be bounded viscosity sub and supersolution of (32), respectively. Assume that $f \in BUC(\mathbb{R}^N)$, $H \in BUC(\mathbb{R}^N \times \mathbb{R}^N)$, (5) and either (12) or (14) hold. Then $u \leq v$ in $\mathbb{R}^N$. 

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Lemma 2.3], for any \( \epsilon \) bounded case. Since \( H \in BUC(\mathbb{R}^N \times \mathbb{R}^N) \), instead of using (65) we get

\[
|H(x, p_\eta + 2\beta x) - H(y, p_\eta - 2\beta y)| \leq \omega(|x - y| + 2\beta|x + y|).
\]

Moreover, since \( f \in BUC(\mathbb{R}^N) \), then (66) is replaced by

\[
\lambda \delta + \alpha \frac{|x - y|^2}{\eta^2} + 2\beta\alpha(|x|^2 + |y|^2)
\leq \beta C(F)(|x| + |y| + 2) + L_\sigma \frac{|x - y|^2}{\eta^2} + \omega_H(|x - y| + 2\beta|x + y|) + \omega_f(|x - y|).
\]

Sending \( \eta \) to 0 then \( \beta \) to 0 and using (61) we get a contradiction. \( \square \)

Theorem 4.2. Let \( u \in USC(\overline{Q}_T) \) and \( v \in LSC(\overline{Q}_T) \) be bounded viscosity sub and supersolution of (45), respectively. Assume that \( f \in BUC(\mathbb{R}^N) \), \( H \in BUC(\mathbb{R}^N \times \mathbb{R}^N) \), (5), either (12) or (14) hold and \( u(x, 0) \leq v(x, 0) \), \( x \in \mathbb{R}^N \). Then \( u \leq v \) in \( \overline{Q}_T \).

The proof is easily adapted by the one of Theorem 4.1.

Proof of Lemma 2.2. The main idea of the proof is followed by the one of [18, Theorem 2.2].

By the definition of \( u_{0m} \) in (42), we have \( u_{0m} \) is still in \( \mathcal{E}_\mu(\mathbb{R}^N) \cap C(\mathbb{R}^N) \). By [18, Lemma 2.3], for any \( \epsilon \in (0, 1) \) there is a constant \( M(\epsilon) > 0 \) such that for all \( x, y \in \mathbb{R}^N \),

\[
|u_{0m}(x) - u_{0m}(y)| \leq \epsilon(\phi(x) - \phi(y)) + M(\epsilon)|x - y|.
\]

Fix \( \{M(\epsilon)|\epsilon \in (0, 1)\} \). Let \( y \in \mathbb{R}^N, \epsilon \in (0, 1), A > 0 \) and set \( \langle x \rangle = \sqrt{|x|^2 + 1} \) and

\[
g(x, t) = u_{0m}(y) + \epsilon(\phi(x) + \phi(y)) + M(\epsilon)\langle x - y \rangle + At \quad \text{for } (x, t) \in \overline{Q}_T.
\]

We compute that for \( (x, t) \in Q_T \), set \( q = \frac{x - y}{(x - y)} \),

\[
Dg(x, t) = \epsilon D\phi(x) + M(\epsilon)q; \quad D^2g(x, t) = \epsilon D^2\phi(x) + \frac{M(\epsilon)}{\langle x - y \rangle} [I - q \otimes q].
\]

When \( F \) is the local operator defined by (7), from (85), we have

\[
\mathcal{F}(x, [g]) = \epsilon \text{tr}(A(x)D^2\phi(x)) + \frac{M(\epsilon)}{\langle x - y \rangle} [\text{tr}(A(x)) - \text{tr}(A(x)q \otimes q)]
\leq \epsilon \text{tr}(A(x)D^2\phi(x)) + M(\epsilon)|\sigma|^2.
\]
When $F$ is the nonlocal operator defined by (8), from (85), we have

$$F(x,[g]) = \epsilon \int_{\mathbb{R}^N} (\phi(x+z) - \phi(x) - \langle D\phi(x), z \rangle B(z)) \nu(dz)$$

$$+ M(\epsilon) \int_{\mathbb{R}^N} \langle x + z - y \rangle - \langle x - y \rangle - \langle D((x-y)), z \rangle B(z) \nu(dz)$$

$$= \epsilon I(x, \phi, D\phi) + M(\epsilon) \int_{B^c} \int_0^1 \frac{\langle x - y + sz, z \rangle}{\sqrt{|x - y + sz|^2 + 1}} ds \nu(dz)$$

$$+ M(\epsilon) \int_B \int_0^1 (1-s) \left( |z|^2 - \frac{\langle x - y + sz, z \rangle}{\sqrt{|x - y + sz|^2 + 1}} \right)^2 ds \nu(dz)$$

$$\leq \epsilon I(x, \phi, D\phi) + M(\epsilon) \left( \int_{B^c} |z| \nu(dz) + \int_B |z|^2 \nu(dz) \right)$$

$$\leq \epsilon I(x, \phi, D\phi) + M(\epsilon) C_{\nu}.$$  

The last inequality is obtained thanks to (14). Therefore, from (86) and (87) we get

$$F(x,[g]) \leq \epsilon F(x,[\phi]) + M(\epsilon) C(F),$$

where $C(F)$ is the constant depending either on $\sigma$ or on $\nu$.

From (5) and (85), we have

$$\langle b(x), Dg \rangle = \epsilon \langle b(x), D\phi(x) \rangle + \frac{M(\epsilon)}{\langle x-y \rangle} \langle (b(x) - b(y), x - y) + (b(y), x - y) \rangle$$

$$\geq \epsilon \langle b(x), D\phi(x) \rangle - M(\epsilon) |b(y)|.$$  

And from (6) and (85) we have

$$H(x, Dg(x,t)) \geq -C_H - \epsilon C_H |D\phi(x)| - M(\epsilon) C_H.$$  

Then from (88), (89) and (90), we obtain

$$g_t - F(x,[g]) + \langle b(x), Dg(x,t) \rangle + H(x, Dg(x,t))$$

$$\geq A + \epsilon (-F(x,[\phi]) + \langle b(x), D\phi \rangle - C_H |D\phi|) - M(\epsilon) [C(F) + |b(y)| + C_H] - C_H.$$  

We choose a constant $K > 0$ so that (19) holds. Since $f_m \in \mathcal{E}_\mu(\mathbb{R}^N)$, by (22), for each $\epsilon \in (0,1)$ we may choose a constant $C(\epsilon) > 0$ so that

$$|f_m(x)| \leq \epsilon \phi(x) + C(\epsilon) \text{ for } x \in \mathbb{R}^N, \ \forall m \geq 1.$$  

For each $y \in \mathbb{R}^N$ and $\epsilon \in (0,1)$ we set

$$A(y, \epsilon) = M(\epsilon) [C(F) + |b(y)| + C_H] + C_H + \epsilon K + C(\epsilon),$$

and define the functions $\psi^+ \in C^\infty(Q_T)$, parametrized by $y, \epsilon$, by

$$\psi^+(x,t; y, \epsilon) = u_{0m}(y) + \epsilon (\phi(x) + \phi(y)) + M(\epsilon) (x - y) + A(y, \epsilon) t.$$  

Observe that, for any $y \in \mathbb{R}^N$ and $\epsilon \in (0,1)$, the function $h(x,t) := \psi^+(x,t; y, \epsilon)$ satisfies

$$h_t - F(x,[h]) + \langle b(x), Dh(x,t) \rangle + H(x, Dh(x,t))$$

$$\geq A(y, \epsilon) + \epsilon (\phi(x) - K) - M(\epsilon) [C(F) + |b(y)| + C_H] - C_H$$

$$\geq f_m(x) \text{ for } (x,t) \in Q_T.$$  

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That is, $h$ is classical (and hence viscosity) supersolution of (45). Observe also that
\[ h(x, 0) \geq u_{0m}(y) + \epsilon(\phi(x) + \phi(y)) + M(\epsilon)x - y \geq u_0(x) \text{ for } (x, t) \in \overline{Q}_T. \]
Similarly we define the function
\[ k(x, t) := \psi^-(x, t; y, \epsilon) = u_{0m}(y) - \epsilon(\phi(x) + \phi(y)) - M(\epsilon)x - y - A(y, \epsilon)t, \]
and then observe as before that $k$ is a viscosity subsolution of (45) and for $(x, t) \in \overline{Q}_T$,
\[ u_{0m}(x) \geq u_{0m}(y) - \epsilon(\phi(x) + \phi(y)) - M(\epsilon)x - y = k(x, 0). \]
By comparison principle (see Theorem 4.2) we obtain, for any $y \in \mathbb{R}^N$,
\[ k(x, t) \leq u_{nn}(x, t) \leq h(x, t). \]
Hence, for $(x, t) \in \overline{Q}_T$, $\epsilon \in (0, 1)$ we have
\[ |u_{nn}(x, t)| \leq \epsilon \phi(x) + |u_{0m}(0)| + \epsilon \phi(0) + M(\epsilon)x + A(0, \epsilon)t. \]
Since $|u_{0m}(0)| \leq |u_0(0) + \phi(0)|$, $m \geq 1$. Then we get
\[ |u_{nn}(x, t)| \leq \epsilon \phi(x) + E(\epsilon)(1 + |x| + t), \quad (x, t) \in \overline{Q}_T, \]
where $E(\epsilon) = \max\{|\epsilon \phi(0) + M(\epsilon)| + |u_0(0) + \phi(0)|; A(0, \epsilon)|$ independent of $m, n$ and $T$.
Moreover, let $R > 0$, we have
\[ |u_{nn}(x, t)| \leq C_{RT}, \quad (x, t) \in B(0, R) \times [0, T), \]
where $C_{RT} = \sup_{B(0,R) \times [0,T]} \{ \epsilon(\phi(x) + E(\epsilon)(1 + |x| + t) \}$ independent of $m$ and $n$.
On the other hand, for $(x, t) \in \overline{Q}_T$, $\epsilon \in (0, 1)$ we have
\[ \lim_{|x| \to \infty} \sup_{0 \leq t < T} \frac{|u_{nn}(x, t)|}{\phi(x)} \leq \epsilon + \lim_{|x| \to \infty} \sup_{0 \leq t < T} \frac{|u_0(0)| + (\epsilon + 1)\phi(0) + M(\epsilon)x + A(0, \epsilon)t}{\phi(x)}, \]
which guarantees that $u_{nn} \in \mathcal{E}_{\mu}(\overline{Q}_T)$. Since $u_{nn} \to u_m$ as $n \to \infty$ locally uniformly in $\overline{Q}_T$. Then we get
\[ u_m \in \mathcal{E}_{\mu}(\overline{Q}_T), \quad |u_m(x, t)| \leq \epsilon \phi(x) + E(\epsilon)(1 + |x| + t), \quad (x, t) \in \overline{Q}_T, \]
\[ |u_m(x, t)| \leq C_{RT}, \quad (x, t) \in B(0, R) \times [0, T), \]
where $E(\epsilon)$ is independent of $m$, $T$ and $C_{RT}$ is independent of $m$. \hfill \Box

**Proof of Lemma 3.1**. We divide the proof in several steps.

**Step 1. Viscosity inequalities for $v_1$ and $v_2$.** This step is classical in viscosity theory. Let $\varphi \in C^2(\mathbb{R}^N)$ and $\bar{x} \in \mathbb{R}^N$ be a local maximum point of $\omega - \varphi$. We can assume that this maximum is strict in the same ball $\overline{B}(\bar{x}, R)$ for some $R > 0$. Let $\Theta(x, y) = \varphi(x) + \frac{|x-y|^2}{\epsilon^2}$ and consider
\[ M_\epsilon := \max_{x, y \in \overline{B}(\bar{x}, R)} \{ v_1(x) - v_2(y) - \Theta(x, y) \}. \]
This maximum is achieved at a point $(x_\epsilon, y_\epsilon)$ and, since the maximum is strict, we know [3] that
\[ (x_\epsilon, y_\epsilon) \to \bar{x}, \quad \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2} \to 0 \quad \text{as} \quad \epsilon \to 0 \quad M_\epsilon = v_1(x_\epsilon) - v_2(y_\epsilon) - \Theta(x_\epsilon, y_\epsilon) \to v_1(\bar{x}) - v_2(\bar{x}) - \varphi(\bar{x}) = \omega(\bar{x}) - \varphi(\bar{x}). \]
Now, we apply [13, Theorem 3.2] in the local case and [8, Corollary 1] in the nonlocal one to learn that, for any \( \varrho > 0 \), there exist \( X, Y \in S^N \) such that \((D_x \Theta(x_e, y_e), X) \in \mathcal{T}^+v_1(x_e), (-D_y \Theta(x_e, y_e), Y) \in \mathcal{T}^-v_2(y_e)\) and
\[
\begin{pmatrix}
X \\
O \\
-\varrho M^2
\end{pmatrix} \leq M + \varrho M^2
\]
where \( M = D^2 \Theta(x_e, y_e) = \begin{pmatrix}
D^2 \varphi(x_e) + 2I/\epsilon^2 \\
-2I/\epsilon^2 \\
2I/\epsilon^2
\end{pmatrix} \) and \( \varrho M^2 = O(\varrho) \) (\( \varrho \) will be sent to 0 first). Setting \( p_e = 2x_e-\epsilon y_e \), we have
\[
(92) \quad D_x \Theta(x_e, y_e) = p_e + D \varphi(x_e), \quad D_y \Theta(x_e, y_e) = -p_e.
\]

Writing the viscosity inequalities for \( v_1 \) and \( v_2 \) at \((x_e, y_e)\) we have
\[
\begin{align*}
\text{Step 2. Estimate of } S := & \quad \mathcal{F}(x_e, [v_1]) - \mathcal{F}(y_e, [v_2]).
\end{align*}
\]

\textit{Step 2.1.} When \( \mathcal{F} \) is the local operator defined by (7). Using (12), we easily get
\[
(93) \quad S \leq \text{tr}(\sigma(x_e)\sigma(x_e)^T D^2 \varphi(x_e)) + \frac{2L^2}{\epsilon^2} |x_e - y_e|^2 \to \text{tr}(\sigma(\bar{x})\sigma(\bar{x})^T D^2 \varphi(\bar{x})), \quad \text{as } \epsilon \to 0.
\]

\textit{Step 2.2.} When \( \mathcal{F} \) is the nonlocal operator defined by (8). For each \( \delta > 0 \), we have
\[
S = I[B_\delta](x_e, \Theta, D_x \Theta) + I[B_\delta](x_e, v_1, D_x \Theta) - I[B_\delta](y_e, \Theta, -D_y \Theta) - I[B_\delta](y_e, v_2, -D_y \Theta).
\]

From (92), we first estimate
\[
(95) \quad I_1 := I[B_\delta](x_e, \Theta, D_x \Theta) - I[B_\delta](y_e, \Theta, -D_y \Theta)
\]
\[
= \int_{B_\delta} \{ \varphi(x_e + z) - \varphi(x_e) + \frac{|x - y + z|^2 - |x - y|^2}{\epsilon^2} - \langle D \varphi(x_e), z \rangle \} v(dz)
\]
\[
= I[B_\delta](x_e, \varphi, D \varphi) + \frac{1}{\epsilon^2} o_\delta(1).
\]

On the other hand, at the maximum point \((x_e, y_e)\) we have
\[
v_1(x_e + z) - v_2(y_e + z) - (v_1(x_e) - v_2(y_e)) \leq \varphi(x_e + z) - \varphi(x_e),
\]
for each \( z \in B \). Hence, for each \( 0 < \delta < \kappa < 1 \), using this inequality we obtain
\[
(96) \quad I_2 := I[B_\delta](x_e, v_1, D_x \Theta) - I[B_\delta](y_e, v_2, -D_y \Theta) \leq J^\kappa + I[B_\kappa \setminus B_\delta](x_e, \varphi, D \varphi),
\]
where
\[
J^\kappa = \int_{B^\kappa_\delta} \{ v_1(x_e + z) - v_2(y_e + z) - (v_1(x_e) - v_2(y_e)) - \langle D \varphi(x_e), z \rangle \text{I}_B(z) \} v(dz).
\]

Therefore from (95) and (96), we conclude that for all \( 0 < \delta < \kappa < 1 \)
\[
(97) \quad S = I_1 + I_2 \leq J^\kappa + I[B_\kappa](x_e, \varphi, D \varphi) + \frac{1}{\epsilon^2} o_\delta(1).
\]
Since \(v_i, v_2 \in \mathcal{E}_\gamma(\mathbb{R}^N)\), there exists \(C > 0\) such that \(|v_i(x)| \leq C\phi_\gamma(x)\), \(\forall i = 1, 2, x \in \mathbb{R}^N\). Let \(\gamma < \bar{\gamma}\), thanks to (67) we have \(\int_{B_\epsilon} \phi_\gamma(z) \nu(dz) < +\infty\). Hence, applying Dominated convergence Theorem and using (91), we get, for each \(k > 0\) fixed,
\[
\limsup_{\epsilon \to 0} J^k \leq \mathcal{I}[B^\epsilon_k](\bar{x}, \omega, D\varphi).
\]
Therefore, letting \(\delta \to 0\) and then \(\epsilon \to 0\) in (97), using (91) we obtain
\[
(98) \quad \limsup_{\epsilon \to 0} \mathcal{S} \leq \mathcal{I}(\bar{x}, \omega, D\varphi).
\]

Step 3. Estimate of \(\mathcal{B} := \langle b(x), D_x \Theta \rangle - \langle b(y), -D_y \Theta \rangle\). From (5) and (92) we have
\[
(99) \quad \mathcal{B} = \langle b(x), p_\epsilon + D\varphi(x) \rangle - \langle b(y), p_\epsilon \rangle \geq 2\alpha \frac{|x - y|^2}{\epsilon^2} + \langle b(x), D\varphi(x) \rangle.
\]

Step 4. Estimate of \(\mathcal{H} := H(x, D_x \Theta) - H(y, -D_y \Theta)\). From (21) and (92) we have
\[
(100) \quad \mathcal{H} \geq -L_H |D\varphi(x)| - L_H |x - y| - 2L_H \frac{|x - y|^2}{\epsilon^2}.
\]

Step 5. Estimate of \(\mathcal{F} := f(y) - f(x)\). Since \(f \in C(\mathbb{R}^N)\), hence we have
\[
(101) \quad \mathcal{F} \leq o_\epsilon(1).
\]

Step 6. Conclusion. Combining (94), (98), (99), (100), (101) to (93) and sending \(\epsilon \to 0\), we obtain
\[
-\mathcal{F}(\bar{x}, [\omega]) + \langle b(\bar{x}), D\varphi(\bar{x}) \rangle - L_H |D\varphi(\bar{x})| \leq 0,
\]
which means exactly that \(\omega\) is a subsolution of (70). \(\square\)

References

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