Ordering Garside groups

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We introduce a structure on a Garside group that we call Dehornoy structure and we show that an iteration of such a structure leads to a left-order on the group. We define two conditions on a Garside group G and we show that, if G satisfies these two conditions, then G has a Dehornoy structure. Then we show that the Artin groups of type A and of type $I_2(m)$, $m \geq 4$, satisfy these conditions, and therefore have Dehornoy structures. As indicated by the terminology, one of the orders obtained by this method on the Artin groups of type A coincides with the Dehornoy order.

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1 Introduction

A group G is said to be *left-orderable* if there exists a total order $<$ on G invariant by left-multiplication. Recall that a subset P of G is a *subsemigroup* if $\alpha\beta \in P$ for all $\alpha, \beta \in P$. It is easily checked that a left-order $<$ on G is determined by a subsemigroup P such that $G = P \sqcup P^{-1} \sqcup \{1\}$: we have $\alpha < \beta$ if and only if $\alpha^{-1}\beta \in P$. In this case the subsemigroup P is called the *positive cone* of $<$.

The first explicit left-order on the braid group $\mathcal{B}_n$ was determined by Dehornoy [4]. The fact that $\mathcal{B}_n$ is left-orderable is important, but, furthermore, the Dehornoy order is interesting by itself, and there is a extensive literature on it. We refer to Dehornoy–Dynnikov–Rolfsen–Wiest [8] for a complete report on left-orders on braid groups and on the Dehornoy order in particular. The definition of the Dehornoy order is based on the following construction.

Let G be a group and let $S = \{s_1, s_2, \dots, s_n\}$ be a finite ordered generating set for G. Let $i \in \{1, 2, \dots, n\}$. We say that $\alpha \in G$ is *s_i-positive* (resp. *s_i-negative*) if α is written in the form $\alpha = \alpha_0 s_i \alpha_1 \cdots s_i \alpha_m$ (resp. $\alpha = \alpha_0 s_i^{-1} \alpha_1 \cdots s_i^{-1} \alpha_m$) with $m \geq 1$

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and $\alpha_0, \alpha_1, \dots, \alpha_m \in \langle s_{i+1}, \dots, s_n \rangle$. For each $i \in \{1, 2, \dots, n\}$ we denote by P_i^+ (resp. P_i^-) the set of s_i-positive elements (resp. s_i-negative elements) of G. The key point in the definition of the Dehornoy order is the following.

Theorem 1.1 (Dehornoy [4]) *Let $G = \mathcal{B}_{n+1}$ be the braid group on $n+1$ strands and let $S = \{s_1, s_2, \dots, s_n\}$ be its standard generating set. For each $i \in \{1, 2, \dots, n\}$ we have the disjoint union $\langle s_i, s_{i+1}, \dots, s_n \rangle = P_i^+ \sqcup P_i^- \sqcup \langle s_{i+1}, \dots, s_n \rangle$.*

Let $G = \mathcal{B}_{n+1}$ be the braid group on $n+1$ strands. Set $P_D = P_1^+ \sqcup P_2^+ \sqcup \cdots \sqcup P_n^+$. Then, by Theorem 1.1, P_D is the positive cone for a left-order $<_D$ on G. This is the *Dehornoy order*.

A careful reader will notice that Theorem 1.1 leads to more than one left-order on $\mathcal{B}_{n+1}$. Indeed, if $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \{+, -\}^n$, then $P^\epsilon = P_1^{\epsilon_1} \sqcup P_2^{\epsilon_2} \sqcup \cdots \sqcup P_n^{\epsilon_n}$ is a positive cone for a left-order on $\mathcal{B}_{n+1}$. The case $\epsilon = (+, -, +, \dots)$ is particularly interesting because, by Dubrovina–Dubrovin [11], in this case P^ϵ determines an isolated left-order in the space of left-orders on $\mathcal{B}_{n+1}$.

Our goal in the present paper is to extend the Dehornoy order to some Garside groups.

A first approach would consist on keeping the same definition, as follows. Let G be a group and let $S = \{s_1, s_2, \dots, s_n\}$ be a finite ordered generating set for G. Again, we denote by P_i^+ (resp. P_i^-) the set of s_i-positive elements (resp. s_i-negative elements) of G. Then we say that S determines a *Dehornoy structure* (*in Ito's sense*) if, for each $i \in \{1, \dots, n\}$, we have the disjoint union $\langle s_i, s_{i+1}, \dots, s_n \rangle = P_i^+ \sqcup P_i^- \sqcup \langle s_{i+1}, \dots, s_n \rangle$. In this case, as for the braid group, for each $\epsilon \in \{+, -\}^n$ the set $P^\epsilon = P_1^{\epsilon_1} \sqcup P_2^{\epsilon_2} \sqcup \cdots \sqcup P_n^{\epsilon_n}$ is the positive cone for a left-order on G. This approach was used by Ito [16] to construct isolated left-orders in the space of left-orders of some groups.

In the present paper we will consider another approach of the Dehornoy order in terms of Garside groups (see Dehornoy [6], Fromentin [13], Fromentin–Paris [14]), and our definition of Dehornoy structure will be different from that in Ito's sense given above.

In Section 2 we recall some basic and preliminary definitions and results on Garside groups. We refer to Dehornoy et al. [7] for a full account on the theory. In Section 3 we give our (new) definition of Dehornoy structure and show how such a structure leads to a left-order on the group (see Proposition 3.1). Then we define two conditions on a Garside group, that we call Condition A and Condition B, and show that a Garside group which satisfies these two conditions has a Dehornoy structure (see Theorem 3.2).

The aim of the rest of the paper is to apply Theorem 3.2 to the Artin groups of type A, that is, the braid groups, and the Artin groups of dihedral type. In Section 4 we prove that a braid group with its standard Garside structure satisfies Condition A and Condition B (see Theorem 4.1), and therefore has a Dehornoy structure in the sense of the definition of Section 3 (see Corollary 4.2). We also prove that the left-orders on the group induced by this structure are the same as the left-orders induced by Theorem 1.1 (see Proposition 4.4), as expected. Section 5 and Section 6 are dedicated to the Artin groups of dihedral type. There is a difference between the even case, treated in Section 5, and the odd case, treated in Section 6. The latter case requires much more calculations. In both cases we show that such a group satisfies Condition A and Condition B, and therefore admits a Dehornoy structure. Then we show that the left-orders obtained from this Dehornoy structure can also be obtained via an embedding of the group in a braid group defined by Crisp [3].

2 Preliminaries

Let G be a group and let M be a submonoid of G such that $M \cap M^{-1} = \{1\}$. Then we have two partial orders $\le_R$ and $\le_L$ on G defined by $\alpha \le_R \beta$ if $\beta\alpha^{-1} \in M$, and $\alpha \le_L \beta$ if $\alpha^{-1}\beta \in M$. For each $a \in M$ we set $\mathrm{Div}_R(a) = \{b \in M \mid b \le_R a\}$ and $\mathrm{Div}_L(a) = \{b \in M \mid b \le_L a\}$. We say that $a \in M$ is *balanced* if $\mathrm{Div}_R(a) = \mathrm{Div}_L(a)$. In that case we set $\mathrm{Div}(a) = \mathrm{Div}_R(a) = \mathrm{Div}_L(a)$. We say that M is *Noetherian* if for each element $a \in M$ there is an integer $n \ge 1$ such that a cannot be written as a product of more than n non-trivial elements.

Definition Let G be a group, let M be a submonoid of G such that $M \cap M^{-1} = \{1\}$, and let Δ be a balanced element of M. We say that G is a *Garside group* with *Garside structure* (G, M, Δ) if:

(a) M is Noetherian;

(b) $\mathrm{Div}(\Delta)$ is finite, it generates M as a monoid, and it generates G as a group;

(c) $(G, \le_R)$ is a lattice.

Let (G, M, Δ) be a Garside structure on G. Then Δ is called the *Garside element* and the elements of $\mathrm{Div}(\Delta)$ are called the *simple elements* (of the Garside structure). The lattice operations of $(G, \le_R)$ are denoted by $\wedge_R$ and $\vee_R$. The ordered set $(G, \le_L)$ is also a lattice and its lattice operations are denoted by $\wedge_L$ and $\vee_L$.

Now take a Garside group G with Garside structure (G, M, Δ) and set $\mathcal{S} = \mathrm{Div}(\Delta) \setminus \{1\}$. The word length of an element $\alpha \in G$ with respect to $\mathcal{S}$ is denoted by $\lg(\alpha) = \lg_{\mathcal{S}}(\alpha)$. The right *greedy normal form* of an element $a \in M$ is the unique expression

$a = u_p \cdots u_2 u_1$ of a over $\mathcal{S}$ satisfying $(u_p \cdots u_i) \wedge_R \Delta = u_i$ for all $i \in \{1, \dots, p\}$. We define the left *greedy normal form* of an element of M in a similar way. The following two theorems contain several key results of the theory of Garside groups.

Theorem 2.1 (Dehornoy–Paris [9], Dehornoy [5]) *(1) Let $a \in M$ and let $a = u_p \cdots u_2 u_1$ be the greedy normal form of a. Then $\lg(a) = p$.*

(2) Let $\alpha \in G$. There exists a unique pair $(a, b) \in M \times M$ such that $\alpha = ab^{-1}$ and $a \wedge_R b = 1$. In that case we have $\lg(\alpha) = \lg(a) + \lg(b)$.

The expression of α given in Theorem 2.1 (2) is called the (right) *orthogonal form* of α. The *left orthogonal form* of an element of G is defined in a similar way.

We say that an element $a \in M$ is *unmovable* if $\Delta \not\leq_R a$ or, equivalently, if $\Delta \not\leq_L a$.

Theorem 2.2 (Dehornoy–Paris [9], Dehornoy [5]) *Let $\alpha \in G$. There exists a unique pair $(a, k) \in M \times \mathbb{Z}$ such that a is unmovable and $\alpha = a\Delta^k$.*

The expression of α given above is called the (right) Δ*-form* of α. We define the *left* Δ*-form* of an element of G in a similar way.

Definition Let δ be a balanced element of M. Denote by G_δ (resp. M_δ) the subgroup of G (resp. the submonoid of M) generated by $\mathrm{Div}(\delta)$. We say that $(G_\delta, M_\delta, \delta)$ is a *parabolic substructure* of (G, M, Δ) if δ is balanced and $\mathrm{Div}(\delta) = \mathrm{Div}(\Delta) \cap M_\delta$. In that case G_δ is called a *parabolic subgroup* of G and M_δ is called a *parabolic submonoid* of M.

Remark Let H be a parabolic subgroup of G. Then there exists a unique parabolic substructure $(G_\delta, M_\delta, \delta)$ of (G, M, Δ) such that $H = G_\delta$. Indeed, the above element δ should be the greatest element in $H \cap \mathrm{Div}(\Delta)$ for the order relation $\leq_R$, hence δ is entirely determined by H. Similarly, if N is a parabolic submonoid of M, then there exists a unique parabolic substructure $(G_\delta, M_\delta, \delta)$ such that $N = M_\delta$, where δ is the greatest element of $\mathrm{Div}(\Delta) \cap N$ for the order relation $\leq_R$. So, we can speak of a parabolic subgroup or of a parabolic submonoid without necessarily specifying the corresponding element δ or the triple $(G_\delta, M_\delta, \delta)$.

Theorem 2.3 (Godelle [15]) *Let (H, N, δ) be a parabolic substructure of (G, M, Δ).*

(1) H is a Garside group with Garside structure (H, N, δ).

(2) *Let $a \in N$ and let $a = u_p \cdots u_2 u_1$ be the greedy normal form of a with respect to (G, M, Δ). Then $u_i \in \mathrm{Div}(\delta)$ for all $i \in \{1, 2, \dots, p\}$ and $a = u_p \cdots u_2 u_1$ is the greedy normal form of a with respect to (H, N, δ).*

(3) *Let $\alpha, \beta \in H$ and $\gamma \in G$ such that $\alpha \le_R \gamma \le_R \beta$. Then $\gamma \in H$.*

(4) *Let $\alpha, \beta \in H$. Then $\alpha \wedge_R \beta, \alpha \vee_R \beta \in H$.*

(5) *Let $\alpha \in H$ and let $\alpha = ab^{-1}$ be the orthogonal form of α with respect to (G, M, Δ). Then $a, b \in N$ and $\alpha = ab^{-1}$ is the orthogonal form of α with respect to (H, N, δ).*

Example Let S be a finite set. A *Coxeter matrix* over S is a square matrix $M = (m_{s,t})_{s,t \in S}$ indexed by the elements of S with coefficients in $\mathbb{N} \cup \{\infty\}$ such that $m_{s,s} = 1$ for all $s \in S$ and $m_{s,t} = m_{t,s} \ge 2$ for all $s, t \in S$, $s \neq t$. If s, t are two letters and m is an integer ≥ 2 we denote by $\Pi(s, t, m)$ the word $sts \cdots$ of length m. In other words $\Pi(s, t, m) = (st)^{\frac{m}{2}}$ if m is even and $\Pi(s, t, m) = (st)^{\frac{m-1}{2}} s$ if m is odd. The *Artin group* associated with M is the group $A = A_M$ defined by the presentation

$$A = \langle S \mid \Pi(s, t, m_{s,t}) = \Pi(t, s, m_{s,t}) \text{ for } s, t \in S,\ s \neq t \text{ and } m_{s,t} \neq \infty \rangle \,.$$

The *Coxeter group* associated with M is the quotient $W = W_M$ of A by the relations $s^2 = 1$, $s \in S$. We say that A is of *spherical type* if W is finite. The braid groups are the star examples of Artin groups of spherical type.

We denote by A^+ the monoid having the following monoid presentation.

$$A^+ = \langle S \mid \Pi(s, t, m_{s,t}) = \Pi(t, s, m_{s,t}) \text{ for } s, t \in S,\ s \neq t \text{ and } m_{s,t} \neq \infty \rangle^+ \,.$$

By Paris [17] the natural homomorphism $A^+ \to A$ is injective. So, we can consider A^+ as a submonoid of A. It is easily checked that $A^+ \cap (A^+)^{-1} = \{1\}$, hence we can consider the order relations $\le_R$ and $\le_L$ on A. Suppose that A is of spherical type. Then, by Brieskorn–Saito [1] and Deligne [10], for all $\alpha, \beta \in A$ the elements $\alpha \wedge_R \beta$ and $\alpha \vee_R \beta$ exist, and (A, A^+, Δ) is a Garside structure, where $\Delta = \vee_R S$. Let X be a subset of S and let A_X be the subgroup of A generated by X. Then, again by Brieskorn–Saito [1] and Deligne [10], A_X is a parabolic subgroup of A and it is an Artin group of spherical type.

The triple (G, M, Δ) denotes again an arbitrary Garside structure on a group G. Besides the greedy normal forms, we will use some other normal forms of the elements of M defined from a pair (N_2, N_1) of parabolic submonoids of M. Their definition is based on the following.

Proposition 2.4 (Dehornoy [6]) *Let N be a parabolic submonoid of M. For each $a \in M$ there exists a unique $b \in N$ such that $\{c \in N \mid c \le_R a\} = \{c \in N \mid c \le_R b\}$.*

The element b of Proposition 2.4 is called the (right) *N-tail* of a and is denoted by $b = \tau_N(a) = \tau_{N,R}(a)$. We define in a similar way the left *N-tail* of a, denoted by $\tau_{N,L}(a)$.

Now, assume that N_1 and N_2 are two parabolic submonoids of M such that $N_2 \cup N_1$ generates M. Then each nontrivial element $a \in M$ is uniquely written in the form $a = a_p \cdots a_2 a_1$ where $a_p \neq 1$, $a_i = \tau_{N_1}(a_p \cdots a_i)$ if i is odd, and $a_i = \tau_{N_2}(a_p \cdots a_i)$ if i is even. This expression is called the (right) *alternating form* of a with respect to (N_2, N_1). Note that we may have $a_1 = 1$, but $a_i \neq 1$ for all $i \in \{2, \dots, p\}$. The number p is called the *(N_2, N_1)-breadth* of a and is denoted by $p = \mathrm{bh}(a) = \mathrm{bh}_{N_2,N_1}(a)$. By extension we set $\mathrm{bh}(1) = 1$ so that $a \in N_1 \Leftrightarrow \mathrm{bh}(a) = 1$.

Now, consider the standard Garside structure $(\mathcal{B}_{n+1}, \mathcal{B}_{n+1}^+, \Delta)$ on the braid group $\mathcal{B}_{n+1}$. Let $S = \{s_1, s_2, \dots, s_n\}$ be the standard generating system of $\mathcal{B}_{n+1}$, N_1 be the submonoid of $\mathcal{B}_{n+1}^+$ generated by $\{s_2, \dots, s_n\}$, and N_2 be the submonoid generated by $\{s_1, \dots, s_{n-1}\}$. Then N_1 and N_2 are parabolic submonoids of $\mathcal{B}_{n+1}^+$ and they are both isomorphic to $\mathcal{B}_n^+$. Observe that $N_1 \cup N_2$ generates $\mathcal{B}_{n+1}^+$, hence we can consider alternating forms with respect to (N_2, N_1). The definitions of the next section are inspired by the following.

Theorem 2.5 (Fromentin–Paris [14]) *Let $a \in \mathcal{B}_{n+1}^+$ and $k \in \mathbb{Z}$. Then $\Delta^{-k} a$ is s_1-negative if and only if $k \ge \max\{1, \mathrm{bh}(a) - 1\}$.*

3 Orders on Garside groups

We consider a Garside structure (G, M, Δ) on a Garside group G and two parabolic substructures (H, N, Λ) and (G_1, M_1, Δ_1). We assume that $N \neq M$, $M_1 \neq M$, $N \cup M_1$ generates M, Δ is central in G, and Δ_1 is central in G_1. Note that the assumption "Δ is central in G" is not so restrictive since, by Dehornoy [5], if (G, M, Δ) is a Garside structure, then (G, M, Δ^k) is also a Garside structure for each $k \ge 1$, and there exists $k \ge 1$ such that Δ^k is central in G. We will consider alternating forms with respect to (N, M_1).

The *depth* of an element $a \in M$, denoted by $\mathrm{dpt}(a)$, is $\mathrm{dpt}(a) = \frac{\mathrm{bh}(a)-1}{2}$ if $\mathrm{bh}(a)$ is odd and is $\mathrm{dpt}(a) = \frac{\mathrm{bh}(a)}{2}$ if $\mathrm{bh}(a)$ is even. In other words, if $a = a_p \cdots a_2 a_1$ is the alternating form of a, then $\mathrm{dpt}(a)$ is the number of indices $i \in \{1, \dots, p\}$ such that $a_i \notin M_1$ (that is, the number of even indices). Note that $a \in M_1$ if and only if $\mathrm{dpt}(a) = 0$.

Definition Let $\alpha \in G$ and let $\alpha = a\Delta^{-k}$ be its Δ-form. We say that α is (H, G_1)-*negative* if $k \geq 1$ and $\mathrm{dpt}(a) < \mathrm{dpt}(\Delta^k)$. We say that α is (H, G_1)-*positive* if α^{-1} is (H, G_1)-negative. We denote by $P = P_{H,G_1}$ the set (H, G_1)-positive elements and by P^{-1} the set of (H, G_1)-negative elements.

Definition We say that (H, G_1) is a *Dehornoy structure* if P satisfies the following conditions:

(a) $PP \subset P$,

(b) $G_1 P G_1 \subset P$,

(c) we have the disjoint union $G = P \sqcup P^{-1} \sqcup G_1$.

Our goal in this section is to prove a criterion for (H, G_1) to be a Dehornoy structure. But, before, we show how the orders appear in this context.

Suppose given two sequences of parabolic subgroups $G_0 = G, G_1, \dots, G_n$ and $H_1, \dots, H_n$ such that $G_{i+1}, H_{i+1} \subset G_i$ and (H_{i+1}, G_{i+1}) is a Dehornoy structure on G_i for all $i \in \{0, 1, \dots, n-1\}$ and $G_n \simeq \mathbb{Z}$. For each $i \in \{0, 1, \dots, n-1\}$ we denote by P_i the set of (H_{i+1}, G_{i+1})-positive elements of G_i. On the other hand, we choose a generator α_n of G_n and we set $P_n = \{\alpha_n^k \mid k \geq 1\}$. For each $\epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^{n+1}$ we set $P^\epsilon = P_0^{\epsilon_0} \sqcup P_1^{\epsilon_1} \sqcup \cdots \sqcup P_n^{\epsilon_n}$.

Proposition 3.1 *Under the above assumptions P^ϵ is the positive cone for a left-order on G.*

Proof We must prove that we have a disjoint union $G = P^\epsilon \sqcup (P^\epsilon)^{-1} \sqcup \{1\}$ and that $P^\epsilon P^\epsilon \subset P^\epsilon$. The fact that we have a disjoint union $G = P^\epsilon \sqcup (P^\epsilon)^{-1} \sqcup \{1\}$ follows directly from Condition (c) of the definition. Let $\alpha, \beta \in P^\epsilon$. Let $i, j \in \{0, 1, \dots, n\}$ such that $\alpha \in P_i^{\epsilon_i}$ and $\beta \in P_j^{\epsilon_j}$. If $i < j$, then, by Condition (b) of the definition, $\alpha\beta \in P_i^{\epsilon_i} \subset P^\epsilon$. Similarly, if $i > j$, then $\alpha\beta \in P_j^{\epsilon_j} \subset P^\epsilon$. If $i = j$, then, by Condition (a) of the definition, $\alpha\beta \in P_i^{\epsilon_i} \subset P^\epsilon$. □

Definition Let $\zeta \geq 1$ be an integer. We say that the pair (H, G_1) satisfies *Condition A with constant* ζ if $\mathrm{dpt}(\Delta^k) = \zeta k + 1$ for all $k \geq 1$.

We set $\theta = \Delta\Delta_1^{-1} = \Delta_1^{-1}\Delta \in M$. We say that an element $a \in M$ is a *theta element* if it is of the form $a = \theta^k a_0$ with $k \geq 1$ and $a_0 \in M_1$. We denote by Θ the set of theta elements of M and we set $\bar{\Theta} = \Theta \cup M_1$.

Definition Let $\zeta \geq 1$ be an integer. Let $(a,b) \in (M \times M) \setminus (\bar{\Theta} \times \bar{\Theta})$ such that a,b are both unmovable. Let $ab = c\Delta^t$ be the Δ-form of ab. We say that (a,b) satisfies *Condition B with constant* ζ if there exists $\varepsilon \in \{0,1\}$ such that

(a) $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - \zeta t - \varepsilon$,

(b) $\varepsilon = 1$ if either $a \in \Theta$, or $b \in \Theta$, or $c \in M_1$.

We say that (H, G_1) satisfies *Condition B with constant* ζ if each pair $(a,b) \in (M \times M) \setminus (\bar{\Theta} \times \bar{\Theta})$ as above satisfies Condition B with constant ζ.

Theorem 3.2 *If there exists a constant $\zeta \geq 1$ such that (H, G_1) satisfies Condition A with constant ζ and Condition B with constant ζ, then (H, G_1) is a Dehornoy structure.*

Let $\zeta \geq 1$ be an integer. From here until the end of the section we assume that (H, G_1) satisfies Condition A with constant ζ and Condition B with constant ζ. Our goal is then to prove that (H, G_1) is a Dehornoy structure, that is, to prove Theorem 3.2.

Let a be an unmovable element of M and let $p = \lg(a)$. Then p is the smallest integer ≥ 0 such that $a \leq_R \Delta^p$. Let $\mathrm{com}(a) \in M$ such that $a\,\mathrm{com}(a) = \Delta^p$. Then, by El-Rifai–Morton [12], $\mathrm{com}(a)$ is unmovable, $\lg(\mathrm{com}(a)) = p$, and $a^{-1} = \mathrm{com}(a)\Delta^{-p}$ is the Δ-form of a^{-1}. Note that $a\,\mathrm{com}(a) = \mathrm{com}(a)\,a = \Delta^p$ since Δ is central. In particular, $\mathrm{com}(\mathrm{com}(a)) = a$.

Lemma 3.3 *(1) Let $a \in M_1$. Then $\theta \wedge_R a = 1$ and $\theta \vee_R a = \theta a = a\theta$.*

(2) Let $a = \theta^k a_0$ be a theta element, where $k \geq 1$ and $a_0 \in M_1$. Then $\mathrm{dpt}(a) = \zeta k + 1$.

(3) Let $a = \theta^k a_0$ be a theta element, where $k \geq 1$ and $a_0 \in M_1$. Then a is unmovable if and only if a_0 is unmovable in M_1 (that is, if and only if $\Delta_1 \not\leq_R a_0$).

(4) Let a be an unmovable element of M. We have $a \in \bar{\Theta}$ if and only if $\mathrm{com}(a) \in \bar{\Theta}$.

(5) Let $\alpha \in G_1 \setminus M_1$. Then α has a Δ-form of the form $\alpha = a\Delta^{-k}$ where $k \geq 1$ and $a = \theta^k a_0 \in \Theta$ with $a_0 \in M_1$.

(6) Let $a \in \bar{\Theta}$ and $b \in M \setminus \bar{\Theta}$. Then $ab \in M \setminus \bar{\Theta}$ and $ba \in M \setminus \bar{\Theta}$.

Proof *Part (1):* Let $a \in M_1$. Let $u = a \wedge_R \theta$. We have $u \leq_R \theta$, hence $u\Delta_1 \leq_R \theta\Delta_1 = \Delta$, and therefore $u\Delta_1 \in \mathrm{Div}(\Delta)$. On the other hand, since $u \leq_R a$, we have $u \in M_1$, hence $u\Delta_1 \in M_1$. So, $u\Delta_1 \in \mathrm{Div}(\Delta) \cap M_1 = \mathrm{Div}(\Delta_1)$, thus $u = 1$. Let $v = a \vee_R \theta$. Since Δ and Δ_1 commute with a, we have $\theta a = a\theta$. In particular,

$v \le_R a\theta$. Let $x_1 \in M$ such that $v = x_1\theta$. Then $x_1 \le_R a$ and, since M_1 is a parabolic submonoid, $x_1 \in M_1$ and there exists $x_2 \in M_1$ such that $x_2x_1 = a$. So, $a = x_2x_1 \le_R v = x_1\theta = \theta x_1$, hence $x_2 \le_R \theta$, and therefore, since $a \wedge_R \theta = 1$, we have $x_2 = 1$. Thus $x_1 = a$ and $v = a\theta = \theta a$.

Part (2): It is clear that $\mathrm{dpt}(a) = \mathrm{dpt}(aa_0)$ for all $a \in M$ and all $a_0 \in M_1$. Let $a = \theta^k a_0$ be a theta element. Then $\mathrm{dpt}(a) = \mathrm{dpt}(\theta^k) = \mathrm{dpt}(\theta^k\Delta_1^k) = \mathrm{dpt}(\Delta^k) = \zeta k + 1$.

Part (3): Let $a = \theta^k a_0$ be a theta element. Suppose that $\Delta_1 \le_R a_0$. Let $a_1 \in M_1$ such that $a_0 = a_1\Delta_1$. Then $a = \theta^k a_1\Delta_1 = \theta^{k-1}a_1\theta\Delta_1 = \theta^{k-1}a_1\Delta$, hence $\Delta \le_R a$. Now suppose that $\Delta \le_R a$. By Part (1) we have $\tau_{M_1}(a) = a_0$. Since $\Delta \le_R a$, we have $\Delta_1 \le_R a$, hence $\Delta_1 \le_R \tau_{M_1}(a) = a_0$.

Part (4): Let a be an unmovable element of M and let $p = \mathrm{lg}(a)$. Suppose that $a \in M_1$. Let $b \in M_1$ such that $ab = \Delta_1^p$. Then $a\theta^p b = \theta^p ab = \theta^p\Delta_1^p = \Delta^p$, hence $\mathrm{com}(a) = \theta^p b \in \bar\Theta$. Suppose that $a = \theta^k a_0$ where $k \ge 1$ and $a_0 \in M_1$. We have $a = \theta^k a_0 \le_R \Delta^p = \theta^p\Delta_1^p$ hence, by Part (1), $a_0 \le_R \Delta_1^p$ and $k \le p$. Let $b_0 \in M_1$ such that $a_0b_0 = \Delta_1^p$. Then $a\theta^{p-k}b_0 = \theta^k a_0\theta^{p-k}b_0 = \theta^p a_0b_0 = \theta^p\Delta_1^p = \Delta^p$, hence $\mathrm{com}(a) = \theta^{p-k}b_0 \in \bar\Theta$. So, if $a \in \bar\Theta$, then $\mathrm{com}(a) \in \bar\Theta$. Now, since $\mathrm{com}(\mathrm{com}(a)) = a$ for each unmovable element a of M, we have $a \in \bar\Theta$ if and only if $\mathrm{com}(a) \in \bar\Theta$.

Part (5): Let $\alpha \in G_1 \setminus M_1$. Since $\alpha \notin M_1$ the Δ_1-form of α is of the form $\alpha = a\Delta_1^{-k}$ with $a \in M_1$, $\Delta_1 \not\le_R a$ and $k \ge 1$. Then $\alpha = a(\theta\Delta^{-1})^k = \theta^k a\Delta^{-k}$ and $\theta^k a$ is unmovable by Part (3) of the lemma.

Part (6): Take $a, b \in M$. We assume that $a, ab \in \bar\Theta$ and we turn to prove that $b \in \bar\Theta$. We write $ab = \theta^t c$ where $t \ge 0$ and $c \in M_1$. On the other hand we know by Part (4) that $\mathrm{com}(a) \in \bar\Theta$, hence $\mathrm{com}(a)$ is of the form $\mathrm{com}(a) = \theta^k a_0$ with $k \ge 0$ and $a_0 \in M_1$, and therefore a^{-1} is of the form $a^{-1} = \theta^k a_0\Delta^{-\ell} = \theta^{k-\ell}a_0\Delta_1^{-\ell}$ where $\ell = \mathrm{lg}(a)$. So, $b\Delta_1^\ell = \theta^{t+k-\ell}a_0c$. If we had $t+k-\ell < 0$, then we would have $\theta^{\ell-t-k}b\Delta_1^\ell = a_0c \in M_1$, hence we would have $\theta^{\ell-t-k} \in M_1$, which contradicts Part (1). So, $t+k-\ell \ge 0$. By Part (1) we have $\tau_{M_1}(\theta^{t+k-\ell}a_0c) = a_0c$, hence $\Delta_1 \le_R a_0c$. Let $b_0 \in M_1$ such that $b_0\Delta_1^\ell = a_0c$. Then $b = \theta^{t+k-\ell}b_0 \in \bar\Theta$. We show in the same way that, if $a, ba \in \bar\Theta$, then $b \in \bar\Theta$. □

Lemma 3.4 *We have $P^{-1}P^{-1} \subset P^{-1}$.*

Proof Let $\alpha, \beta \in P^{-1}$. Let $\alpha = a\Delta^{-k}$ and $\beta = b\Delta^{-\ell}$ be the Δ-forms of α and β, respectively. Since $\alpha, \beta \in P^{-1}$, we have $k, \ell \ge 1$, $\mathrm{dpt}(a) \le \mathrm{dpt}(\Delta^k) - 1 = \zeta k$ and $\mathrm{dpt}(b) \le \mathrm{dpt}(\Delta^\ell) - 1 = \zeta\ell$. Let $ab = c\Delta^t$ be the Δ-form of ab. Then the Δ-form of $\alpha\beta$ is $\alpha\beta = c\Delta^{-k-\ell+t}$. We must show that $\alpha\beta \in P^{-1}$, that is, $k+\ell-t \geq 1$ and $\mathrm{dpt}(c) \leq \mathrm{dpt}(\Delta^{k+\ell-t}) - 1 = \zeta(k+\ell-t)$.

Case 1: $a, b \in M_1$. Then $t = 0$ and $c = ab$, hence $k+\ell-t = k+\ell \geq 1$ and $\mathrm{dpt}(c) = 0 \leq \zeta(k+\ell) = \zeta(k+\ell-t)$.

Case 2: $a \in M_1$ and $b \in \Theta$. We write $b = \theta^u b_0$ where $u \geq 1$ and $b_0 \in M_1$. By Lemma 3.3 (3) we have $\mathrm{dpt}(b) = \zeta u + 1 \leq \zeta\ell$, hence $u < \ell$. Let $ab_0 = c_0\Delta_1^v$ be the Δ_1-form of ab_0. If $v < u$, then $t = v$ and $c = \theta^{u-v}c_0$, hence $k+\ell-t \geq \ell-v \geq \ell-u \geq 1$ and $\mathrm{dpt}(c) = \zeta(u-v)+1 = \zeta u - \zeta t + 1 \leq \zeta\ell - \zeta t \leq \zeta(k+\ell-t)$. If $v \geq u$, then $t = u$ and $c = \Delta_1^{v-u}c_0 \in M_1$, hence $k+\ell-t = k+\ell-u \geq \ell-u \geq 1$ and $\mathrm{dpt}(c) = 0 \leq \zeta(k+\ell-t)$. The case "$a \in \Theta$ and $b \in M_1$" can be proved in a similar way.

Case 3: $a, b \in \Theta$. We set $a = \theta^u a_0$ and $b = \theta^v b_0$, where $u, v \geq 1$ and $a_0, b_0 \in M_1$. Since $\mathrm{dpt}(a) = \zeta u + 1 \leq \zeta k$, we have $u < k$. Similarly, we have $v < \ell$. Let $a_0b_0 = c_0\Delta_1^w$ be the Δ_1-form of a_0b_0. If $w < u+v$, then $t = w$ and $c = \theta^{u+v-w}c_0$, hence $k+\ell-t \geq k+\ell-(u+v) = (k-u)+(\ell-v) \geq 1$ and $\mathrm{dpt}(c) = \zeta(u+v-w)+1 = \zeta u + 1 + \zeta v - \zeta t \leq \zeta k + \zeta\ell - \zeta t = \zeta(k+\ell-t)$. If $w \geq u+v$, then $t = u+v$ and $c = c_0\Delta_1^{w-u-v} \in M_1$, hence $k+\ell-t = k+\ell-(u+v) = (k-u)+(\ell-v) \geq 1$ and $\mathrm{dpt}(c) = 0 \leq \zeta(k+\ell-t)$.

Case 4: either $a \notin \bar{\Theta}$, or $b \notin \bar{\Theta}$. Since (H, G_1) satisfies Condition B with constant ζ, there exists $\varepsilon \in \{0,1\}$ such that $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - \zeta t - \varepsilon$. If $c \in M_1$, then $\varepsilon = 1$ and

$$0 = \mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - \zeta t - 1 \leq \zeta k + \zeta\ell - \zeta t - 1 < \zeta(k+\ell-t)\,.$$

This (strict) inequality also implies that $k+\ell-t \geq 1$. If $c \notin M_1$, then

$$1 \leq \mathrm{dpt}(c) \leq \mathrm{dpt}(a) + \mathrm{dpt}(b) - \zeta t \leq \zeta k + \zeta\ell - \zeta t = \zeta(k+\ell-t)\,.$$

Again, this inequality also implies that $k+\ell-t \geq 1$. □

Lemma 3.5 *We have $G_1 P^{-1} G_1 \subset P^{-1}$.*

Proof We take $\alpha \in G_1$ and $\beta \in P^{-1}$ and we turn to prove that $\alpha\beta \in P^{-1}$. The proof of the inclusion $\beta\alpha \in P^{-1}$ is made in a similar way. Let $\alpha = a\Delta^{-k}$ and $\beta = b\Delta^{-\ell}$ be the Δ-forms of α and β, respectively. Since $\beta \in P^{-1}$ we have $\ell \geq 1$ and $\mathrm{dpt}(b) \leq \mathrm{dpt}(\Delta^\ell) - 1 = \zeta\ell$. Let $ab = c\Delta^t$ be the Δ-form of ab. Then the Δ-form of $\alpha\beta$ is $\alpha\beta = c\Delta^{-k-\ell+t}$. We must show that $k+\ell-t \geq 1$ and $\mathrm{dpt}(c) \leq \zeta(k+\ell-t)$.

Case 1: $\alpha \in M_1$ and $b \in M_1$. We have $k = 0$, $\alpha = a$, $t = 0$ and $c = ab \in M_1$. Thus $k+\ell-t = \ell \geq 1$ and $0 = \mathrm{dpt}(c) \leq \zeta(k+\ell-t)$.

Case 2: $\alpha \in M_1$ and $b \in \Theta$. We have $k = 0$, $\alpha = a$ and $b = \theta^v b_0$ where $v \geq 1$ and $b_0 \in M_1$. We also have $\mathrm{dpt}(b) = \zeta v + 1 \leq \zeta\ell$, hence $v < \ell$. Let $ab_0 = c_0\Delta_1^u$ be the Δ_1-form of ab_0. If $u < v$, then $t = u$ and $c = \theta^{v-u}c_0$, hence $k+\ell-t = \ell-u \geq \ell-v \geq 1$ and $\mathrm{dpt}(c) = \zeta(v-u)+1 = \zeta v+1-\zeta t \leq \zeta\ell-\zeta t = \zeta(k+\ell-t)$. If $u \geq v$, then $t = v$ and $c = \Delta_1^{u-v}c_0 \in M_1$, hence $k+\ell-t = \ell-v \geq 1$ and $0 = \mathrm{dpt}(c) \leq \zeta(k+\ell-t)$.

Case 3: $\alpha \in M_1$ and $b \in M \setminus \bar{\Theta}$. We have $k = 0$ and $\alpha = a$. On the other hand, by Lemma 3.3 (6), we have $ab \in M \setminus \bar{\Theta}$, hence $c \notin M_1$, and therefore $\mathrm{dpt}(c) \geq 1$. Since (H, G_1) satisfies Condition B with constant ζ, there exists $\varepsilon \in \{0, 1\}$ such that $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - \zeta t - \varepsilon$. So,

$$1 \leq \mathrm{dpt}(c) \leq 0 + \zeta\ell - \zeta t = \zeta(k+\ell-t)\,.$$

This inequality also implies that $k+\ell-t \geq 1$.

Case 4: $\alpha \notin M_1$ and $b \in M_1$. By Lemma 3.3 (5) we have $k \geq 1$ and $a = \theta^k a_0$ with $a_0 \in M_1$. Let $a_0 b = c_0\Delta_1^u$ be the Δ_1-form of $a_0 b$. If $u < k$, then $t = u$ and $c = \theta^{k-u}c_0$, hence $k+\ell-t \geq \ell \geq 1$ and $\mathrm{dpt}(c) = \zeta(k-u)+1 \leq \zeta k - \zeta t + \zeta\ell \leq \zeta(k+\ell-t)$. If $u \geq k$, then $t = k$ and $c = c_0\Delta_1^{u-k} \in M_1$, hence $k+\ell-t = \ell \geq 1$ and $0 = \mathrm{dpt}(c) \leq \zeta(k+\ell-t)$.

Case 5: $\alpha \notin M_1$ and $b \in \Theta$. By Lemma 3.3 (5) we have $k \geq 1$ and $a = \theta^k a_0$ with $a_0 \in M_1$. On the other hand, b is written $b = \theta^v b_0$ with $v \geq 1$ and $b_0 \in M_1$. Since $\mathrm{dpt}(b) = \zeta v + 1 \leq \zeta\ell$, we have $v < \ell$. Let $a_0 b_0 = c_0\Delta_1^w$ be the Δ_1-form of $a_0 b_0$. If $w < k+v$, then $t = w$ and $c = \theta^{k+v-w}c_0$, hence $k+\ell-t \geq k+v-w \geq 1$ and $\mathrm{dpt}(c) = \zeta(k+v-w)+1 = \zeta k + \zeta v + 1 - \zeta t \leq \zeta k + \zeta\ell - \zeta t = \zeta(k+\ell-t)$. If $w \geq k+v$, then $t = k+v$ and $c = c_0\Delta_1^{w-k-v} \in M_1$, hence $k+\ell-t = \ell-v \geq 1$ and $0 = \mathrm{dpt}(c) \leq \zeta(k+\ell-t)$.

Case 6: $\alpha \notin M_1$ and $b \in M \setminus \bar{\Theta}$. By Lemma 3.3 (5) we have $k \geq 1$ and $a = \theta^k a_0$ with $a_0 \in M_1$. On the other hand, by Lemma 3.3 (6), we have $ab \in M \setminus \bar{\Theta}$, hence $c \notin M_1$, and therefore $\mathrm{dpt}(c) \geq 1$. Since (H, G_1) satisfies Condition B with constant ζ and $a \in \Theta$, $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - \zeta t - 1$. So,

$$1 \leq \mathrm{dpt}(c) \leq \zeta k + 1 + \zeta\ell - \zeta t - 1 = \zeta(k+\ell-t)\,.$$

This inequality also implies that $k+\ell-t \geq 1$. $\square$

Lemma 3.6 *We have $G_1 \cap (P \cup P^{-1}) = \emptyset$.*

Proof Let $\alpha \in G_1$ and let $\alpha = a\Delta^{-k}$ be the Δ-form of α. If $\alpha \in M_1$, then $k = 0$ and $\alpha = a$, thus $\alpha \notin P^{-1}$. If $\alpha \notin M_1$, then, by Lemma 3.3 (5), we have $k \geq 1$ and $a = \theta^k a_0$ where $a_0 \in M_1$, hence $\mathrm{dpt}(a) = \zeta k + 1 = \mathrm{dpt}(\Delta^k)$, and therefore $\alpha \notin P^{-1}$. Since $\alpha^{-1} \in G_1$, we also have $\alpha^{-1} \notin P^{-1}$, hence $\alpha \notin P$. $\square$

Lemma 3.7 *We have $P \cap P^{-1} = \emptyset$.*

Proof Let $\alpha \in P^{-1}$ and let $\alpha = a\Delta^{-k}$ be its Δ-form. By definition we have $k \geq 1$ and $\mathrm{dpt}(a) < \mathrm{dpt}(\Delta^k) = \zeta k + 1$. Let $\ell = \mathrm{lg}(a)$. Then the Δ-form of α^{-1} is $\alpha^{-1} = \mathrm{com}(a)\Delta^{k-\ell}$. We are going to show that $\alpha^{-1} \notin P^{-1}$, that is, either $k - \ell \geq 0$ or $\mathrm{dpt}(\mathrm{com}(a)) \geq \zeta(\ell - k) + 1$.

Case 1: $a \in M_1$. Let $b \in M_1$ such that $ab = \Delta_1^\ell$. We have $a^{-1} = b\Delta_1^{-\ell} = b\theta^\ell\Delta^{-\ell} = \theta^\ell b\Delta^{-\ell}$, hence $\mathrm{com}(a) = \theta^\ell b$, and therefore, $\mathrm{dpt}(\mathrm{com}(a)) = \zeta\ell + 1 > \zeta(\ell - k) + 1$, since $k \geq 1$. So, $\alpha^{-1} \notin P^{-1}$.

Case 2: $a \in \Theta$. We write $a = \theta^u a_0$ where $a_0 \in M_1$ and $u \geq 1$. We have $\mathrm{dpt}(a) = \zeta u + 1 \leq \zeta k$, hence $u < k$. Let $t \geq 0$ be the length of a_0 and let $b_0 \in M_1$ such that $a_0 b_0 = \Delta_1^t$. We have $a_0^{-1} = b_0\Delta_1^{-t} = \theta^t b_0\Delta^{-t}$, hence $a^{-1} = \theta^{t-u}b_0\Delta^{-t}$, and therefore $\alpha^{-1} = \theta^{t-u}b_0\Delta^{k-t}$. If $u < t$, then $\mathrm{com}(a) = \theta^{t-u}b_0$ and $\mathrm{dpt}(\mathrm{com}(a)) = \zeta(t-u) + 1 > \zeta(t-k) + 1$, hence $\alpha^{-1} \notin P^{-1}$. If $u \geq t$, then $\alpha^{-1} = \theta^{-u+t}b_0\Delta^{k-t} = b_0\Delta_1^{u-t}\Delta^{k-t-u+t} = b_0\Delta_1^{u-t}\Delta^{k-u}$ and $k - u \geq 1$, hence $\alpha^{-1} \notin P^{-1}$.

Case 3: $a \in M \setminus \bar{\Theta}$. Recall that $a\,\mathrm{com}(a) = \Delta^\ell$. Since (H, G_1) satisfies Condition B with constant ζ and $1 \in M_1$, we have $0 = \mathrm{dpt}(1) = \mathrm{dpt}(a) + \mathrm{dpt}(\mathrm{com}(a)) - \zeta\ell - 1$, hence

$$\mathrm{dpt}(\mathrm{com}(a)) = \zeta\ell + 1 - \mathrm{dpt}(a) \geq \zeta\ell + 1 - \zeta k = \zeta(\ell - k) + 1\,,$$

and therefore $\alpha^{-1} \notin P^{-1}$. □

Lemma 3.8 *We have $G = P \cup P^{-1} \cup G_1$.*

Proof We take $\alpha \in G$ and we assume that $\alpha \notin (P^{-1} \cup G_1)$. We are going to show that $\alpha \in P$, that is, $\alpha^{-1} \in P^{-1}$. Let $\alpha = a\Delta^k$ be the Δ-form of α and let ℓ be the length of a. Then the Δ-form of α^{-1} is $\mathrm{com}(a)\Delta^{-k-\ell}$.

Case 1: $a \in M_1$. Then $k \geq 1$ because $\alpha \notin (P^{-1} \cup G_1)$. If $a = 1$, then $\alpha^{-1} = \Delta^{-k} \in P^{-1}$. So, we can assume that $a \neq 1$, and therefore $\ell \geq 1$. Let $b \in M_1$ such that $ab = \Delta_1^\ell$. We have $a^{-1} = \theta^\ell b\Delta^{-\ell}$, hence $\alpha^{-1} = \theta^\ell b\Delta^{-k-\ell}$ and $\mathrm{com}(a) = \theta^\ell b$. Then $k + \ell \geq 1$ and $\mathrm{dpt}(\mathrm{com}(a)) = \zeta\ell + 1 \leq \zeta\ell + \zeta k = \zeta(\ell + k)$, hence $\alpha^{-1} \in P^{-1}$.

Case 2: $a \in \Theta$. We write $a = \theta^u a_0$ where $u \geq 1$ and $a_0 \in M_1$. Since $\alpha \notin P^{-1}$ we have $\mathrm{dpt}(a) = \zeta u + 1 \geq \zeta(-k) + 1$, hence $u \geq -k$. We also have $u \neq -k$, otherwise we would have $\alpha = a_0\Delta_1^{-u} \in G_1$. So, $u > -k$. Let t be the length of a_0 and let $b_0 \in M_1$ such that $a_0 b_0 = \Delta_1^t$. We have $a_0^{-1} = b_0\Delta_1^{-t}$, hence $a^{-1} = \theta^{t-u}b_0\Delta^{-t}$, and therefore $\alpha^{-1} = \theta^{t-u} b_0 \Delta^{-k-t}$. If $u < t$, then $\mathrm{com}(a) = \theta^{t-u} b_0$, $k+t > k+u \ge 1$ and $\mathrm{dpt}(\mathrm{com}(a)) = \zeta(t-u)+1 < \zeta(t+k)+1 = \mathrm{dpt}(\Delta^{t+k})$, hence $\alpha^{-1} \in P^{-1}$. If $u \ge t$, then $\alpha^{-1} = b_0 \Delta_1^{u-t} \Delta^{-k-u}$, $\mathrm{com}(a) = b_0 \Delta_1^{u-t} \in M_1$, $k+u \ge 1$, and $\mathrm{dpt}(\mathrm{com}(a)) = 0 \le \zeta(k+u)$, hence $\alpha^{-1} \in P^{-1}$.

Case 3: $a \in M \setminus \bar{\Theta}$. Since (H, G_1) satisfies Condition B with constant ζ, we have $0 = \mathrm{dpt}(1) = \mathrm{dpt}(a) + \mathrm{dpt}(\mathrm{com}(a)) - \zeta\ell - 1$. On the other hand, since $\Delta^\ell \in \bar{\Theta}$, by Lemma 3.3 (6), $\mathrm{com}(a) \notin \bar{\Theta}$, hence $\mathrm{com}(a) \notin M_1$, and therefore $\mathrm{dpt}(\mathrm{com}(a)) \ge 1$. Moreover, since $\alpha \notin P^{-1}$, we have $\mathrm{dpt}(a) \ge \zeta(-k)+1$. So,

$$1 \le \mathrm{dpt}(\mathrm{com}(a)) = \zeta\ell + 1 - \mathrm{dpt}(a) \le \zeta\ell + 1 + \zeta k - 1 = \zeta(\ell + k)\,.$$

This inequality also implies that $\ell + k \ge 1$. Thus, $\alpha^{-1} \in P^{-1}$. □

Proof of Theorem 3.2 We have $PP \subset P$ by Lemma 3.4, we have $G_1 P G_1 \subset P$ by Lemma 3.5, and we have the disjoint union $G = P \sqcup P^{-1} \sqcup G_1$ by Lemma 3.6, Lemma 3.7 and Lemma 3.8. □

4 Artin groups of type A

In this section we assume that G and M are the Artin group and the Artin monoid of type A_n, respectively, where $n \ge 2$. Recall that G is defined by the presentation

$$G = \langle s_1, \dots, s_n \mid s_i s_j s_i = s_j s_i s_j \text{ for } |i-j| = 1,\ s_i s_j = s_j s_i \text{ for } |i-j| \ge 2 \rangle\,,$$

and that M is the submonoid of G generated by $s_1, s_2, \dots, s_n$. Recall also that G is the braid group $\mathcal{B}_{n+1}$ on $n+1$ strands and M is the positive braid monoid $\mathcal{B}_{n+1}^+$. By Brieskorn–Saito [1] and Deligne [10], (G, M, Ω) is a Garside structure, where $\Omega = (s_1 \cdots s_n) \cdots (s_1 s_2 s_3)(s_1 s_2) s_1$. The element Ω is not central in G but $\Delta = \Omega^2 = (s_1 \cdots s_n)^{n+1}$ is central and, by Dehornoy [5], (G, M, Δ) is also a Garside structure on G. The latter is the Garside structure that we consider in this section.

We denote by G_1 (resp. M_1) the subgroup of G (resp. the submonoid of M) generated by $s_2, \dots, s_n$ and we set $\Delta_1 = (s_2 \cdots s_n)^n$. Then (G_1, M_1, Δ_1) is a parabolic substructure of (G, M, Δ) and Δ_1 is central in G_1. On the other hand, we denote by H (resp. N) the subgroup of G (resp. the submonoid of M) generated by $s_1, \dots, s_{n-1}$ and we set $\Lambda = (s_1 \cdots s_{n-1})^n$. Again, (H, N, Λ) is a parabolic substructure of (G, M, Δ). Observe that $M_1 \cup N$ generates M.

The purpose of this section is to prove the following.

Theorem 4.1 *The pair (H, G_1) satisfies Condition A with constant $\zeta = 1$ and Condition B with constant $\zeta = 1$.*

By applying Theorem 3.2 we deduce the following.

Corollary 4.2 *The pair (H, G_1) is a Dehornoy structure.*

For $1 \le i \le n-1$ we set $G_i = \langle s_{i+1}, \dots, s_n \rangle$, $M_i = \langle s_{i+1}, \dots, s_n \rangle^+$, $\Delta_i = (s_{i+1} \cdots s_n)^{n+1-i}$ and $H_i = \langle s_i, \dots, s_{n-1} \rangle$. By iterating Corollary 4.2 and applying Proposition 3.1 we get the following.

Corollary 4.3 *(1) For each $1 \le i \le n-1$ the pair (H_i, G_i) is a Dehornoy structure on $(G_{i-1}, M_{i-1}, \Delta_{i-1})$, where $(G_0, M_0, \Delta_0) = (G, M, \Delta)$.*

(2) For each $1 \le i \le n-1$ we denote by P_i the set of (H_i, G_i)-positive elements of G_{i-1}. Furthermore we set $P_n = \{s_n^k \mid k \ge 1\}$. For each $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$ the set $P^\epsilon = P_1^{\epsilon_1} \sqcup \cdots \sqcup P_n^{\epsilon_n}$ is the positive cone for a left-order on G.

Before proving Theorem 4.1 we show that the orders on G given in Corollary 4.3 (2) coincide with those obtained using Theorem 1.1. More precisely we prove the following.

Proposition 4.4 *The set $P = P_{H,G_1}$ of (H, G_1)-positive elements is equal to the set of s_1-positive elements of $G = \mathcal{B}_{n+1}$.*

Proof Let P' denote the set of s_1-positive elements of G. We know by Dehornoy [4] that we have the disjoint union $G = P' \sqcup P'^{-1} \sqcup G_1$. We also know by Corollary 4.2 that $PP \subset P$, $G_1 P G_1 \subset P$ and $G = P \sqcup P^{-1} \sqcup G_1$. Let $\alpha \in P'$. By definition α is written $\alpha = \alpha_0 s_1 \alpha_1 \cdots s_1 \alpha_p$ where $p \ge 1$ and $\alpha_0, \alpha_1, \dots, \alpha_p \in G_1$. The Δ-form of s_1 is $s_1 = s_1 \Delta^0$, hence s_1 does not lie in P^{-1}. The element s_1 does not lie in G_1 either, hence s_1 lies in P. Since $PP \subset P$ and $G_1 P G_1 \subset P$ we deduce that α lies in P. So, $P' \subset P$ and therefore $P'^{-1} \subset P^{-1}$. Since we have disjoint unions $G = P \sqcup P^{-1} \sqcup G_1$ and $G = P' \sqcup P'^{-1} \sqcup G_1$ we conclude that $P = P'$ and $P^{-1} = P'^{-1}$. □

The rest of the section is dedicated to the proof of Theorem 4.1. We recall once for all the expressions of Δ and θ over the standard generators.

$$\Delta = (s_1 s_2 \cdots s_n)^{n+1} = (s_1 \cdots s_{n-1} s_n^2 s_{n-1} \cdots s_1) \cdots (s_{n-1} s_n^2 s_{n-1}) s_n^2 \,,$$
$$\theta = s_1 \cdots s_{n-1} s_n^2 s_{n-1} \cdots s_1 \,.$$

Proposition 4.5 *The pair (H, G_1) satisfies Condition A with constant $\zeta = 1$.*

Proof Let $k \geq 1$. Then, by Dehornoy [6], $\mathrm{bh}(\Delta^k) = \mathrm{bh}(\Omega^{2k}) = 2k + 2$, hence $\mathrm{dpt}(\Delta^k) = k + 1$. $\square$

It remains to show that (H, G_1) satisfies Condition B with constant $\zeta = 1$ (see Proposition 4.12). This is the goal of the rest of the section.

An (N, M_1)*-expression of length* p of an element $a \in M$ is defined to be an expression of a of the form $a = a_p \cdots a_2 a_1$ with $a_i \in N$ if i is even and $a_i \in M_1$ if i is odd.

Lemma 4.6 (Dehornoy [6], Burckel [2]) *Let $a \in M$ and let $a = a_p \cdots a_2 a_1$ be an (N, M_1)-expression of a. Then $p \geq \mathrm{bh}(a)$.*

Let $a \in M$. Choose an expression $a = s_{i_\ell} \cdots s_{i_2} s_{i_1}$ of a over S and set $\mathrm{rev}(a) = s_{i_1} s_{i_2} \cdots s_{i_\ell}$. Since the relations that define M are symmetric, the definition of $\mathrm{rev}(a)$ does not depend on the choice of the expression of a. It is easily checked that $\mathrm{rev}(\Omega) = \Omega$, $\mathrm{rev}(\Delta) = \Delta$ and $\mathrm{rev}(\theta) = \theta$. Moreover, $\mathrm{rev}(a) \in M_1$ for all $a \in M_1$ and $\mathrm{rev}(a) \in N$ for all $a \in N$.

Lemma 4.7 *Let $a \in M$. Then* $\mathrm{dpt}(\mathrm{rev}(a)) = \mathrm{dpt}(a)$.

Proof Let $a = a_p \cdots a_2 a_1$ be the alternating form of a. If p is even, then $\mathrm{rev}(a) = \mathrm{rev}(a_1)\,\mathrm{rev}(a_2) \cdots \mathrm{rev}(a_p)\,1$ is a (N, M_1)-expression of $\mathrm{rev}(a)$ hence, by Lemma 4.6, $p+1 \geq \mathrm{bh}(\mathrm{rev}(a))$, and therefore $\mathrm{dpt}(a) = \frac{p}{2} \geq \mathrm{dpt}(\mathrm{rev}(a))$. If p is odd, then $\mathrm{rev}(a) = \mathrm{rev}(a_1)\,\mathrm{rev}(a_2) \cdots \mathrm{rev}(a_p)$ is a (N, M_1)-expression of $\mathrm{rev}(a)$ hence, by Lemma 4.6, $p \geq \mathrm{bh}(\mathrm{rev}(a))$, and therefore $\mathrm{dpt}(a) = \frac{p-1}{2} \geq \mathrm{dpt}(\mathrm{rev}(a))$. So, $\mathrm{dpt}(a) \geq \mathrm{dpt}(\mathrm{rev}(a))$ in both cases. Since $\mathrm{rev}(\mathrm{rev}(a)) = a$, we also have $\mathrm{dpt}(\mathrm{rev}(a)) \geq \mathrm{dpt}(a)$, hence $\mathrm{dpt}(\mathrm{rev}(a)) = \mathrm{dpt}(a)$. $\square$

Lemma 4.8 *Let $a \in M \setminus M_1$ and $k \geq 1$. Then* $\mathrm{dpt}(a\theta^k) = \mathrm{dpt}(a) + k$.

Proof Let $a \in M \setminus M_1$. It suffices to show that $\mathrm{bh}(a\theta) = \mathrm{bh}(a) + 2$. Let $a = a_p \cdots a_2 a_1$ be the alternating form of a. Note that, since $a \notin M_1$, we have $p \geq 2$. Note also that, by Lemma 3.3 (1), we have $a_1 \theta = \theta a_1$. Then $a\theta = a_p \cdots a_3 a_2 \theta a_1 = a_p \cdots a_3 b_4 b_3 b_2 a_1$, where $b_4 = a_2 s_1 \in N$, $b_3 = s_2 \cdots s_{n-1} s_n^2 \in M_1$ and $b_2 = s_{n-1} \cdots s_2 s_1 \in N$. We turn to show that $a\theta = a_p \cdots a_2 b_4 b_3 b_2 a_1$ is the alternating form of $a\theta$. This will prove the lemma.

Let $x = \tau_{M_1}(a_p \cdots a_3 b_4 b_3 b_2) = \tau_{M_1}(a_p \cdots a_3 a_2 \theta)$. We know by Lemma 3.3 (1) that $x \vee_R \theta = \theta x = x\theta$, hence $x \le_R a_p \cdots a_2$, and therefore $x = 1$, since $\tau_{M_1}(a_p \cdots a_3 a_2) = 1$. We have $a_p \cdots a_3 b_4 b_3 = a_p \cdots a_3 a_2 s_1 s_2 \cdots s_{n-1} s_n^2$. It is easily checked that $(s_1 \cdots s_{n-1} s_n^2) \vee_R s_i = s_{i+1}(s_1 \cdots s_{n-1} s_n^2)$ for all $i \in \{1, \dots, n-1\}$. Thus, if there exists $i \in \{1, \dots, n-1\}$ such that $s_i \le_R a_p \cdots a_3 b_4 b_3$, then there exists $j \in \{2, \dots, n\}$ such that $s_j \le_R a_p \cdots a_3 a_2$. But, since $\tau_{M_1}(a_p \cdots a_3 a_2) = 1$, such a j does not exist, hence such an i does not exist either, hence $\tau_N(a_p \cdots a_3 b_4 b_3) = 1$. We have $a_p \cdots a_3 b_4 = a_p \cdots a_3 a_2 s_1$. We have $s_1 \vee_R s_i = s_i s_1$ for all $i \in \{3, \dots, n\}$, and $s_1 \vee_R s_2 = s_1 s_2 s_1$. Thus, for $i \in \{2, \dots, n\}$, if $s_i \le_R a_p \cdots a_3 b_4$, then $s_i \le_R a_p \cdots a_3 a_2$. Since such an i does not exist, we have $\tau_{M_1}(a_p \cdots a_3 b_4) = 1$. This finishes the proof that $a_p \cdots a_3 b_4 b_3 b_2 a_1$ is the alternating form of $a\theta$ since $a_p \cdots a_3$ is an alternating form and $\tau_N(a_p \cdots a_3) = 1$. □

Lemma 4.9 *(1) Let $a \in M_1$ and $b \in M \setminus M_1$. Then $\mathrm{dpt}(ab) = \mathrm{dpt}(ba) = \mathrm{dpt}(b)$.*

(2) Let $a \in \Theta$ and $b \in M \setminus M_1$. Then $\mathrm{dpt}(ab) = \mathrm{dpt}(ba) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - 1$.

Proof Let $a \in M_1$ and $b \in M \setminus M_1$. We obviously have $\mathrm{bh}(ba) = \mathrm{bh}(b)$, hence $\mathrm{dpt}(ba) = \mathrm{dpt}(b)$. On the other hand, since $\mathrm{rev}(a) \in M_1$, By Lemma 4.7 we have $\mathrm{dpt}(ab) = \mathrm{dpt}(\mathrm{rev}(ab)) = \mathrm{dpt}(\mathrm{rev}(b)\,\mathrm{rev}(a)) = \mathrm{dpt}(\mathrm{rev}(b)) = \mathrm{dpt}(b)$.

Let $a \in \Theta$ and $b \in M \setminus M_1$. Write $a = \theta^k a_0$ with $a_0 \in M_1$ and $k \ge 1$. By the above and Proposition 4.5 we have $\mathrm{dpt}(a) = \mathrm{dpt}(\theta^k) = \mathrm{dpt}(\Delta^k) = k+1$. Then, by the above and Lemma 4.8, $\mathrm{dpt}(ba) = \mathrm{dpt}(b\theta^k) = \mathrm{dpt}(b) + k = \mathrm{dpt}(a) + \mathrm{dpt}(b) - 1$. On the other hand, since $\mathrm{rev}(a) \in \Theta$, we have $\mathrm{dpt}(ab) = \mathrm{dpt}(\mathrm{rev}(ab)) = \mathrm{dpt}(\mathrm{rev}(b)\,\mathrm{rev}(a)) = \mathrm{dpt}(\mathrm{rev}(a)) + \mathrm{dpt}(\mathrm{rev}(b)) - 1 = \mathrm{dpt}(a) + \mathrm{dpt}(b) - 1$. □

Lemma 4.10 *Let $a \in M$ and $k \ge 0$. If $a\Delta^{-k} \in G_1$ then $a \in \bar\Theta$.*

Proof Let $a\Delta^{-k} = a_0 \Delta_1^{-t}$ be the Δ_1-form of $a\Delta^{-k}$. We have $a = a_0 \Delta_1^{-t} \Delta^k = \theta^k a_0 \Delta_1^{k-t}$. If $k \ge t$ then we clearly have $a \in \bar\Theta$. Suppose that $k < t$. Then $a\Delta_1^{t-k} = \theta^k a_0$, hence $\Delta_1^{t-k} \le_R \tau_{M_1}(\theta^k a_0)$. By Lemma 3.3 (1) we have $\tau_{M_1}(\theta^k a_0) = a_0$, hence $\Delta_1^{t-k} \le_R a_0$. Let $b_0 \in M_1$ such that $a_0 = b_0 \Delta_1^{t-k}$. Then $a = \theta^k b_0 \in \bar\Theta$. □

Lemma 4.11 *Let $a, b \in M \setminus M_1$, $c \in M_1$ and $k \ge 0$ such that $ab = c\Delta^k$ and $\mathrm{dpt}(a) + \mathrm{dpt}(b) = k+2$. Then $(a,b) \in (\Theta \times \Theta)$.*

Proof Let $p = \mathrm{dpt}(a)$ and $q = \mathrm{dpt}(b)$. Note that, since $a, b \notin M_1$, we have $p, q \ge 1$. We have $\mathrm{bh}(a) \ge 2p$, hence $\mathrm{bh}(a) - 1 > 2p - 2$, and therefore, by Theorem 2.5,

$\Omega^{-2p+2}a = a\Delta^{-p+1}$ either lies in G_1 or is s_1-positive. Similarly, $b\Delta^{-q+1}$ either lies in G_1 or is s_1-positive. If either $a\Delta^{-p+1}$ was s_1-positive or $b\Delta^{-q+1}$ was s_1-positive, then $c = ab\Delta^{-k} = (a\Delta^{-p+1})(b\Delta^{-q+1})$ would be s_1-positive. Since $c \in M_1$, c cannot be s_1-positive, hence both $a\Delta^{-p+1}$ and $b\Delta^{-q+1}$ lie in G_1. We conclude by Lemma 4.10 that $a, b \in \bar{\Theta}$, hence $a, b \in \Theta$ since we assumed that $a, b \notin M_1$. □

Now we are ready to prove the second part of Theorem 4.1.

Proposition 4.12 *The pair (H, G_1) satisfies Condition B with constant $\zeta = 1$.*

Proof We take $(a, b) \in (M \times M) \setminus (\bar{\Theta} \times \bar{\Theta})$ such that a and b are unmovable. We must show that (a, b) satisfies Condition B with constant $\zeta = 1$. Let $ab = c\Delta^t$ be the Δ-form of ab. So, we must show that there exists $\varepsilon \in \{0, 1\}$ such that $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - t - \varepsilon$, and $\varepsilon = 1$ if either $a \in \Theta$, or $b \in \Theta$, or $c \in M_1$.

Case 1: $a \in M_1$ and $b \in M \setminus \bar{\Theta}$. By Lemma 3.3 (6) we have $ab \notin \bar{\Theta}$, hence $c \notin M_1$. Then, by Lemma 4.9, $\mathrm{dpt}(a) + \mathrm{dpt}(b) = \mathrm{dpt}(b) = \mathrm{dpt}(ab) = \mathrm{dpt}(c) + t$, hence $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - t - 0$. The case $a \in M \setminus \bar{\Theta}$ and $b \in M_1$ is proved in a similar way.

Case 2: $a \in \Theta$ and $b \in M \setminus \bar{\Theta}$. We write $a = \theta^k a_0$ where $k \geq 1$ and $a_0 \in M_1$. Again, by Lemma 3.3 (6) we have $ab \notin \bar{\Theta}$, hence $c \notin M_1$. Then, by Lemma 4.9, $\mathrm{dpt}(a) + \mathrm{dpt}(b) - 1 = \mathrm{dpt}(ab) = \mathrm{dpt}(c) + t$, hence $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - t - 1$. The case $a \in M \setminus \bar{\Theta}$ and $b \in \Theta$ is proved in a similar way.

Case 3: $a, b \in M \setminus \bar{\Theta}$. Set $p = \mathrm{dpt}(a)$ and $q = \mathrm{dpt}(b)$. We have $\mathrm{bh}(a) \in \{2p, 2p+1\}$ hence, by Theorem 2.5, $\Omega^{-2p}a$ is s_1-negative and $\Omega^{-2p+2}a$ either lies in G_1 or is s_1-positive. Similarly, $\Omega^{-2q}b$ is s_1-negative and $\Omega^{-2q+2}b$ either lies in G_1 or is s_1-positive. So, $\Omega^{-2p-2q}ab$ is s_1-negative and $\Omega^{-2p-2q+4}ab$ either lies in G_1 or is s_1-positive. By Theorem 2.5 it follows that $\mathrm{bh}(ab) - 1 \leq 2p + 2q$ and $2p + 2q - 4 < \mathrm{bh}(ab) - 1$, hence $2p + 2q - 2 \leq \mathrm{bh}(ab) \leq 2p + 2q + 1$, and therefore $p + q - 1 \leq \mathrm{dpt}(ab) \leq p + q$. So, there exists $\varepsilon \in \{0, 1\}$ such that $\mathrm{dpt}(ab) = p + q - \varepsilon = \mathrm{dpt}(a) + \mathrm{dpt}(b) - \varepsilon$.

Suppose that $c \notin M_1$. By Lemma 4.9 (2), $\mathrm{dpt}(c) + t = \mathrm{dpt}(c) + \mathrm{dpt}(\Delta^t) - 1 = \mathrm{dpt}(c\Delta^t) = \mathrm{dpt}(ab) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - \varepsilon$, hence $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - t - \varepsilon$. Suppose that $c \in M_1$. By Lemma 4.9 (1), $\mathrm{dpt}(a) + \mathrm{dpt}(b) - \varepsilon = \mathrm{dpt}(ab) = \mathrm{dpt}(c\Delta^t) = \mathrm{dpt}(\Delta^t) = t + 1$, hence $\mathrm{dpt}(a) + \mathrm{dpt}(b) = t + 1 + \varepsilon$. Since $a, b \notin \bar{\Theta}$ Lemma 4.11 implies that $\varepsilon = 0$. So, $\mathrm{dpt}(c) = 0 = \mathrm{dpt}(a) + \mathrm{dpt}(b) - t - 1$. □

5 Artin groups of dihedral type, the even case

Let $m \geq 4$ be an integer. Recall that the *Artin group of type* $I_2(m)$ is the group $G = A_{I_2(m)}$ defined by the presentation $G = \langle s, t \mid \Pi(s,t,m) = \Pi(t,s,m) \rangle$. Let M be the submonoid of G generated by $\{s, t\}$ and let $\Omega = \Pi(s,t,m)$. Then, by Brieskorn–Saito [1] and Deligne [10], the triple (G, M, Ω) is a Garside structure on G. If m is even then $\Delta = \Omega$ is central. However, if m is odd then Ω is not central but $\Delta = \Omega^2$ is central. In both cases, by Dehornoy [5], the triple (G, M, Δ) is a Garside structure on G. In this section we study the case where m is even and in the next one we will study the case where m is odd. So, from now until the end of the section we assume that $m = 2k$ is even and $\Delta = \Pi(s,t,m) = (st)^k = (ts)^k$.

Remark By setting $\Delta = \Omega^2$ in the even case as in the odd case we could state global results valid for all $m \geq 4$, but it would be still necessary to differentiate the even case from the odd case in the proofs, and this would lengthen the proofs for the even case.

We denote by G_1 (resp. M_1) the subgroup of G (resp. submonoid of M) generated by t, and by H (resp. N) the subgroup of G (resp. submonoid of M) generated by s. We set $\Delta_1 = t$ and $\Lambda = s$. By Brieskorn–Saito [1] the triples (G_1, M_1, Δ_1) and (H, N, Λ) are parabolic substructures of (G, M, Δ). On the other hand it is obvious that $M_1 \cup N$ generates M. The main result of the present section is the following.

Theorem 5.1 *The pair (H, G_1) satisfies Condition A with constant $\zeta = k - 1$ and Condition B with constant $\zeta = k - 1$.*

By Theorem 3.2 this implies the following.

Corollary 5.2 *The pair (H, G_1) is a Dehornoy structure on G.*

We denote by P_1 the set of (H, G_1)-positive elements of G and we set $P_2 = \{t^n \mid n \geq 1\}$. For each $\epsilon = (\epsilon_1, \epsilon_2) \in \{\pm 1\}^2$ we set $P^\epsilon = P_1^{\epsilon_1} \cup P_2^{\epsilon_2}$. Then, by Proposition 3.1, we have the following.

Corollary 5.3 *The set P^ϵ is the positive cone for a left-order on G.*

In this section we denote by $r_1, \dots, r_{2k-1}$ the standard generators of the braid group $\mathcal{B}_{2k}$ on $2k = m$ strands. By Crisp [3] we have an embedding $\iota : G \to \mathcal{B}_{2k}$ which sends s to $\prod_{i=0}^{k-1} r_{2i+1}$ and sends t to $\prod_{i=1}^{k-1} r_{2i}$. In the second part of the section we will show that the orders obtained from Corollary 5.3 can be deduced from ι together with the Dehornoy order. More precisely, we show the following.

Proposition 5.4 *Let $\alpha \in G$. Then α is (H, G_1)-negative if and only if $\iota(\alpha)$ is r_1-negative.*

The proof of Theorem 5.1 is based on the following observation whose proof is left to the reader.

Lemma 5.5 *Let a be an unmovable element of M. Then a is uniquely written in the form $a = t^{u_p} s^{v_p} \cdots t^{u_1} s^{v_1} t^{u_0}$ with $u_0, u_p \geq 0$, $u_1, \dots, u_{p-1} \geq 1$ and $v_1, \dots, v_p \geq 1$. In this case $\mathrm{dpt}(a) = p$.*

The first part of Theorem 5.1 is a straightforward consequence of this lemma.

Proposition 5.6 *The pair (H, G_1) satisfies Condition A with constant $\zeta = k - 1$.*

Proof Let $p \geq 1$ be an integer. We have $\theta = s(ts)^{k-1}$, hence $\theta^p = (s(ts)^{k-1})^p$. By Lemma 5.5 it follows that $\mathrm{dpt}(\theta^p) = p(k-1) + 1$, hence $\mathrm{dpt}(\Delta^p) = \mathrm{dpt}(\theta^p t^p) = \mathrm{dpt}(\theta^p) = p(k-1) + 1$. □

If $a \in M \setminus \{1\}$ is written as in Lemma 5.5 we set $\sigma(a) = t$ if $u_p \neq 0$ and $\sigma(a) = s$ if $u_p = 0$. Similarly we set $\tau(a) = t$ if $u_0 \neq 0$ and $\tau(a) = s$ if $u_0 = 0$. In other words $\sigma(a)$ is the first letter of a and $\tau(a)$ is the last one. The following is a straightforward consequence of Lemma 5.5.

Lemma 5.7 *(1) Let a, b be two unmovable elements of M such that ab is unmovable. Then*

$$\mathrm{dpt}(ab) = \begin{cases} \mathrm{dpt}(a) + \mathrm{dpt}(b) - 1 & \text{if } a \neq 1,\ b \neq 1 \text{ and } \tau(a) = \sigma(b) = s\,, \\ \mathrm{dpt}(a) + \mathrm{dpt}(b) & \text{otherwise}\,. \end{cases}$$

(2) Let $a, b \in M$ such that $ab = \Delta$. Then $\mathrm{dpt}(a) + \mathrm{dpt}(b) = \mathrm{dpt}(\Delta) = k$.

Now we can prove the second part of Theorem 5.1.

Proposition 5.8 *The pair (H, G_1) satisfies Condition B with constant $\zeta = k - 1$.*

Proof We take two unmovable elements $a, b \in M$ such that $(a, b) \notin \bar{\Theta} \times \bar{\Theta}$ and we denote by $ab = c\Delta^p$ the Δ-form of ab. We must show that there exists $\varepsilon \in \{0, 1\}$ such that $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - p(k-1) - \varepsilon$ and that $\varepsilon = 1$ if either $a \in \Theta$, or $b \in \Theta$, or $c \in M_1$. We write $a = a_{p+1} a_p \cdots a_1$ and $b = b_1 \cdots b_p b_{p+1}$ so that:

- $a_i \neq 1$, $b_i \neq 1$ and $a_i b_i = \Delta$ for all $i \in \{1, \dots, p\}$;

- $a_{p+1}b_{p+1} = c$;
- We set $x_i = \tau(a_i)$, $x'_i = \sigma(a_i)$, $y_i = \sigma(b_i)$, $y'_i = \tau(b_i)$ for all $i \in \{1, \dots, p+1\}$. Then $x'_i = x_{i+1}$ for all $i \in \{1, \dots, p-1\}$.

We denote by $\varphi : M \to M$ the isomorphism that sends s to t and t to s. Since $a_i b_i = \Delta$, we have $y_i = \varphi(x_i)$ and $y'_i = \varphi(x'_i)$ for all $i \in \{1, \dots, p\}$. In particular, $y'_i = \varphi(x'_i) = \varphi(x_{i+1}) = y_{i+1}$ for all $i \in \{1, \dots, p-1\}$.

Let $u = |\{i \in \{1, \dots, p\} \mid x'_i = s\}|$. By Lemma 5.7, $\mathrm{dpt}(a) = \mathrm{dpt}(a_{p+1}) + \sum_{i=1}^{p} \mathrm{dpt}(a_i) - u + \varepsilon_a$, where ε_a is as follows. If $p \geq 1$ and $a_{p+1} \neq 1$, then: $\varepsilon_a = 0$ if $(x'_p, x_{p+1}) \in \{(s,s),(t,s),(t,t)\}$ and $\varepsilon_a = 1$ if $(x'_p, x_{p+1}) = (s,t)$. If $p \geq 1$ and $a_{p+1} = 1$, then: $\varepsilon_a = 0$ if $x'_p = t$ and $\varepsilon_a = 1$ if $x'_p = s$. If $p = 0$, then $\varepsilon_a = 0$.

Let $v = |\{i \in \{1, \dots, p\} \mid y'_i = s\}|$. As for a, by applying Lemma 5.7 we obtain $\mathrm{dpt}(b) = \mathrm{dpt}(b_{p+1}) + \sum_{i=1}^{p} \mathrm{dpt}(b_i) - v + \varepsilon_b$ where ε_b is as follows. If $p \geq 1$ and $b_{p+1} \neq 1$, then: $\varepsilon_b = 0$ if $(y'_p, y_{p+1}) \in \{(s,s),(t,s),(t,t)\}$ and $\varepsilon_b = 1$ if $(y'_p, y_{p+1}) = (s,t)$. If $p \geq 1$ and $b_{p+1} = 1$, then: $\varepsilon_b = 0$ if $y'_p = t$ and $\varepsilon_b = 1$ if $y'_p = s$. If $p = 0$, then $\varepsilon_b = 0$.

By applying again Lemma 5.7 we obtain $\mathrm{dpt}(c) = \mathrm{dpt}(a_{p+1}) + \mathrm{dpt}(b_{p+1}) + \varepsilon_c$ where ε_c is as follows. If $a_{p+1} \neq 1$ and $b_{p+1} \neq 1$, then: $\varepsilon_c = -1$ if $(x_{p+1}, y_{p+1}) = (s,s)$ and $\varepsilon_c = 0$ if $(x_{p+1}, y_{p+1}) \in \{(s,t),(t,s),(t,t)\}$. If either $a_{p+1} = 1$ or $b_{p+1} = 1$, then $\varepsilon_c = 0$.

Finally, by Lemma 5.7 (2), we have $\sum_{i=1}^{p}(\mathrm{dpt}(a_i) + \mathrm{dpt}(b_i)) = pk$. On the other hand, since $y'_i = \varphi(x'_i)$ for all $i \in \{1, \dots, p\}$, we have $u + v = p$.

Set $\varepsilon = \varepsilon_a + \varepsilon_b - \varepsilon_c$. By the above we have $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - p(k-1) - \varepsilon$ and ε is as follows. If $p \geq 1$, $a_{p+1} \neq 1$ and $b_{p+1} \neq 1$, then: $\varepsilon = 0$ if $(x'_p, x_{p+1}, y_{p+1}) \in \{(s,s,t),(t,t,s)\}$ and $\varepsilon = 1$ otherwise. If $p \geq 1$, $a_{p+1} \neq 1$ and $b_{p+1} = 1$, then: $\varepsilon = 0$ if $(x'_p, x_{p+1}) = (s,s)$ and $\varepsilon = 1$ otherwise. If $p \geq 1$, $a_{p+1} = 1$ and $b_{p+1} \neq 1$, then: $\varepsilon = 0$ if $(x'_p, y_{p+1}) = (t,s)$ and $\varepsilon = 1$ otherwise. If $p \geq 1$, $a_{p+1} = 1$ and $b_{p+1} = 1$, then $\varepsilon = 1$. If $p = 0$, $a \neq 1$ and $b \neq 1$, then: $\varepsilon = 0$ if $(x_{p+1}, y_{p+1}) \in \{(s,t),(t,s),(t,t)\}$ and $\varepsilon = 1$ otherwise. If $p = 0$ and either $a = 1$ or $b = 1$, then $\varepsilon = 0$.

Suppose that $a \in \Theta$. Then a is written $a = \theta^q$ with $q \geq 1$. Set $b = t^r b'$ where $b' \neq 1$ (since $b \notin \bar{\Theta}$) and $\sigma(b') = s$. If $r = 0$, then $p = 0$, $a = \theta^q \neq 1$, $b = b' \neq 1$ and $(x_{p+1}, y_{p+1}) = (s,s)$, hence $\varepsilon = 1$. If $0 < r < q$, then $r = p > 0$, $a_{p+1} = \theta^{q-p} \neq 1$, $b_{p+1} = b' \neq 1$ and $(x'_p, x_{p+1}, y_{p+1}) = (s,s,s)$, hence $\varepsilon = 1$. If $r = q$, then $r = p = q$, $a_{p+1} = 1$, $b_{p+1} = b' \neq 1$ and $(x'_p, y_{p+1}) = (s,s)$, hence $\varepsilon = 1$. If $r > q$, then $p = q$,

$a_{p+1} = 1$, $b_{p+1} = t^{r-q}b'$ and $(x'_p, y_{p+1}) = (s,t)$, hence $\varepsilon = 1$. The case $b \in \Theta$ is proved in the same way.

Suppose that $c \in M_1$. Then $p \geq 1$, since $(a,b) \notin (\bar\Theta \times \bar\Theta)$. If $a_{p+1} \neq 1$ and $b_{p+1} \neq 1$, then $(x_{p+1}, y_{p+1}) = (t,t)$, hence $\varepsilon = 1$. If $a_{p+1} \neq 1$ and $b_{p+1} = 1$, then $x_{p+1} = t$, hence $\varepsilon = 1$. If $a_{p+1} = 1$ and $b_{p+1} \neq 1$, then $y_{p+1} = t$, hence $\varepsilon = 1$. If $a_{p+1} = 1$ and $b_{p+1} = 1$, then $\varepsilon = 1$. □

We turn now to the proof of Proposition 5.4. We denote by G'_1 (resp M'_1) the subgroup of $\mathcal{B}_{2k}$ (resp. the submonoid of $\mathcal{B}_{2k}^+$) generated by $r_2, \dots, r_{2k-1}$ and we denote by H' (resp. N') the subgroup of $\mathcal{B}_{2k}$ (resp. the submonoid of $\mathcal{B}_{2k}^+$) genetared by $r_1, \dots, r_{2k-2}$. Note that $\iota(t) \in G'_1$, hence $\iota(G_1) \subset G'_1$. We denote by $\Omega_{\mathcal{B}} = (r_1 r_2 \cdots r_{2k-1}) \cdots (r_1 r_2) r_1$ the standard Garside element of $\mathcal{B}_{2k}$ and by $\Phi : \mathcal{B}_{2k} \to \mathcal{B}_{2k}$, $\alpha \mapsto \Omega_{\mathcal{B}} \alpha \Omega_{\mathcal{B}}^{-1}$, the conjugation by $\Omega_{\mathcal{B}}$. Recall that $\Phi(r_i) = r_{2k-i}$ for all $i \in \{1, \dots, 2k-1\}$. So, $\Phi(G'_1) = H'$ and $\Phi(H') = G'_1$.

Lemma 5.9 *Let a be an unmovable element of M such that $\mathrm{dpt}(a) \leq k-1$. Then there exist $b_1 \in M'_1$ and $b_2 \in N'$ such that $\iota(a) = b_1 b_2$.*

Proof Let $p = \mathrm{dpt}(a)$. By Lemma 5.5, a can be written $a = t^{u_0} s^{v_1} t^{u_1} \cdots s^{v_p} t^{u_p}$ where $u_0, u_p \geq 0$, $u_1, \dots, u_{p-1} \geq 1$ and $v_1, \dots, v_p \geq 1$. We show by induction on p that there exist $b_1 \in M'_1$ and $b_2 \in \langle r_1, \dots, r_{2p} \rangle^+$ such that $\iota(a) = b_1 b_2$. Since $p \leq k-1$ this proves the lemma. The case $p = 0$ is obvious because $\iota(t) \in M'_1$. We assume that $1 \leq p \leq k-1$ and that the inductive hypothesis holds. Set $a' = t^{u_0} s^{v_1} t^{u_1} \cdots s^{v_{p-1}} t^{u_{p-1}}$. By induction there exist $b'_1 \in M'_1$ and $b'_2 \in \langle r_1, \dots, r_{2p-2} \rangle^+$ such that $\iota(a') = b'_1 b'_2$. Note that b'_2 commutes with r_i for all $i \geq 2p$. So,

$$\iota(a) = b'_1 b'_2 \left(\prod_{i=0}^{k-1} r_{2i+1}^{v_p} \right) \left(\prod_{i=1}^{k-1} r_{2i}^{u_p} \right) = b'_1 \left(\prod_{i=p}^{k-1} r_{2i+1}^{v_p} \right) b'_2 \left(\prod_{i=0}^{p-1} r_{2i+1}^{v_p} \right) \left(\prod_{i=1}^{k-1} r_{2i}^{u_p} \right) =$$

$$b'_1 \left(\prod_{i=p}^{k-1} r_{2i+1}^{v_p} \right) \left(\prod_{i=p+1}^{k-1} r_{2i}^{u_p} \right) b'_2 \left(\prod_{i=0}^{p-1} r_{2i+1}^{v_p} \right) \left(\prod_{i=1}^{p} r_{2i}^{u_p} \right) = b_1 b_2 \,,$$

where

$$b_1 = b'_1 \left(\prod_{i=p}^{k-1} r_{2i+1}^{v_p} \right) \left(\prod_{i=p+1}^{k-1} r_{2i}^{u_p} \right) \in M'_1 \,,$$

$$b_2 = b'_2 \left(\prod_{i=0}^{p-1} r_{2i+1}^{v_p} \right) \left(\prod_{i=1}^{p} r_{2i}^{u_p} \right) \in \langle r_1, \dots, r_{2p} \rangle^+ \,.$$

□

Proof of Proposition 5.4 We denote by P the set of (H, G_1)-positive elements of G and by P' the set of r_1-positive elements of $\mathcal{B}_{2k}$. By Corollary 5.2 we have the disjoint union $G = P \sqcup P^{-1} \sqcup G_1$ and by Dehornoy [4] we have the disjoint union $\mathcal{B}_{2k} = P' \sqcup P'^{-1} \sqcup G'_1$. It suffices to show that $\iota(P^{-1}) \subset P'^{-1}$. Indeed, suppose that $\iota(P^{-1}) \subset P'^{-1}$. Since ι is a homomorphism we also have $\iota(P) \subset P'$. Since we also know that $\iota(G_1) \subset G'_1$, from the disjoint unions given above follows that $\alpha \in P^{-1}$ if and only if $\iota(\alpha) \in P'^{-1}$.

Let α be an element of P^{-1}. Let $\alpha = a\Delta^{-p}$ be the Δ-form of α. By definition we have $p \geq 1$ and $\mathrm{dpt}(a) \leq p(k-1)$. Suppose first that $p = 1$ and $\mathrm{dpt}(a) \leq k-1$. By Lemma 5.9 there exist $b_1 \in M'_1$ and $b_2 \in N'$ such that $\iota(a) = b_1 b_2$. Moreover, by Crisp [3], $\iota(\Delta) = \Omega_{\mathcal{B}}$. Thus $\iota(\alpha) = b_1 b_2 \Omega_{\mathcal{B}}^{-1} = b_1 \Omega_{\mathcal{B}}^{-1} \Phi(b_2)$. Since $b_1, \Phi(b_2) \in M'_1$ and $\Omega_{\mathcal{B}}^{-1} \in P'^{-1}$, it follows that $\iota(\alpha) \in P'^{-1}$.

Now we consider the general case where $p \geq 1$ and $\mathrm{dpt}(a) \leq p(k-1)$. It is easily deduced from Lemma 5.5 that a can be written $a = a_1 a_2 \cdots a_p$ where a_i is an unmovable element of M such that $\mathrm{dpt}(a_i) \leq k-1$ for all $i \in \{1, \dots, p\}$. Note that a_i may be equal to 1 in the above expression. We have $\alpha = (a_1\Delta^{-1})(a_2\Delta^{-1}) \cdots (a_p\Delta^{-1})$ and, by the above, $\iota(a_i\Delta^{-1}) \in P'^{-1}$ for all $i \in \{1, \dots, p\}$, hence $\iota(\alpha) \in P'^{-1}$. □

6 Artin groups of dihedral type, the odd case

Let $m = 2k+1 \geq 5$ be an odd integer and let $G = A_{I_2(m)} = \langle s, t \mid \Pi(s,t,m) = \Pi(t,s,m) \rangle$ be the Artin group of type $I_2(m)$. Let M be the submonoid of G generated by $\{s, t\}$ and let $\Omega = \Pi(s,t,m) = (st)^k s = (ts)^k t$. Recall that, by Brieskorn–Saito [1] and Deligne [10], the triple (G, M, Ω) is a Garside structure on G. As pointed out in Section 5, Ω is not central but $\Delta = \Omega^2$ is, and, by Dehornoy [5], (G, M, Δ) is also a Garside structure on G. This is the Garside structure on G that will be considered in the present section.

We denote by G_1 (resp. M_1) the subgroup of G (resp. submonoid of M) generated by t, and by H (resp. N) the subgroup of G (resp. submonoid of M) generated by s. Set $\Delta_1 = t^2$ and $\Lambda = s^2$. Then, by Brieskorn–Saito [1], the triples (G_1, M_1, Δ_1) and (H, N, Λ) are parabolic substructures of (G, M, Δ). Moreover, $M_1 \cup N$ obviously generates M. The main result of this section is the following.

Theorem 6.1 *The pair (H, G_1) satisfies Condition A with constant $\zeta = 2k-1$ and Condition B with constant $\zeta = 2k-1$.*

By Theorem 3.2 this implies the following.

Corollary 6.2 *The pair (H, G_1) is a Dehornoy structure on G.*

We denote by P_1 the set of (H, G_1)-positive elements of G and we set $P_2 = \{t^n \mid n \geq 1\}$. For each $\epsilon = (\epsilon_1, \epsilon_2) \in \{\pm 1\}^2$ we set $P^\epsilon = P_1^{\epsilon_1} \cup P_2^{\epsilon_2}$. Then by Proposition 3.1 we have the following.

Corollary 6.3 *The set P^ϵ is the positive cone for a left-order on G.*

Let $r_1, \dots, r_{2k}$ be the standard generators of the braid group $\mathcal{B}_{2k+1}$ on $m = 2k+1$ strands. Again, by Crisp [3], we have an embedding $\iota : G \to \mathcal{B}_{2k+1}$ which sends s to $\prod_{i=0}^{k-1} r_{2i+1}$ and t to $\prod_{i=1}^{k} r_{2i}$. The proof of the following is substantially the same as the proof of Proposition 5.4, hence it is left to the reader.

Proposition 6.4 *Let $\alpha \in G$. Then α is (H, G_1)-negative if and only if $\iota(\alpha)$ is r_1-negative.*

We start now the proof of Theorem 6.1. We say that an element $a \in M$ is Ω*-unmovable* if $\Omega \not\leq_L a$ or, equivalently, if $\Omega \not\leq_R a$. The following is an observation.

Lemma 6.5 *(1) Let a be an Ω-unmovable element of M. Then a is uniquely written in the form $a = t^{u_p} s^{v_p} \cdots t^{u_1} s^{v_1} t^{u_0}$, where $u_0, u_p \geq 0$, $u_1, \dots, u_{p-1} \geq 1$ and $v_1, \dots, v_p \geq 1$. In this case we have $\mathrm{dpt}(a) = p$.*

(2) Let a be an unmovable element of M. Then a is uniquely written in the form $a = a'\Omega^\varepsilon$ where a' is Ω-unmovable and $\varepsilon \in \{0, 1\}$.

The first part of Theorem 6.1 is a direct consequence of this lemma.

Proposition 6.6 *The pair (H, G_1) satisfies Condition A with constant $\zeta = 2k - 1$.*

Proof Let $p \geq 1$ be an integer. We have $\theta = (st)^k (ts)^k$, hence $\theta^p = ((st)^k (ts)^k)^p$. By Lemma 6.5 (1) it follows that $\mathrm{dpt}(\theta^p) = p(2k-1) + 1$, hence $\mathrm{dpt}(\Delta^p) = \mathrm{dpt}(\theta^p t^{2p}) = \mathrm{dpt}(\theta^p) = p(2k-1) + 1$. □

The second part of Theorem 6.1 will be much more difficult to prove. Let $a \in M \setminus \{1\}$ be an Ω-unmovable element that we write as in Lemma 6.5 (1). Then we set $\sigma(a) = t$ if $u_p \neq 0$ and $\sigma(a) = s$ if $u_p = 0$. Similarly, we set $\tau(a) = t$ if $u_0 \neq 1$ and $\tau(a) = s$ if $u_0 = 0$. In other words, $\sigma(a)$ is the first letter of a and $\tau(a)$ is its last one. On the other hand, we denote by $\varphi : G \to G$ the automorphism which sends s to t and t to s. Note that φ is the conjugation by Ω, that is, $\varphi(\alpha) = \Omega \alpha \Omega^{-1}$ for all $\alpha \in G$. The following is again a direct consequence of Lemma 6.5.

Lemma 6.7 *(1) Let $a, b \in M$ such that ab is Ω-unmovable. Then*

$$\operatorname{dpt}(ab) = \begin{cases} \operatorname{dpt}(a) + \operatorname{dpt}(b) - 1 & \text{if } a \neq 1,\ b \neq 1, \text{ and } \tau(a) = \sigma(b) = s\,, \\ \operatorname{dpt}(a) + \operatorname{dpt}(b) & \text{otherwise}\,. \end{cases}$$

(2) Let $a, b \in M \setminus \{1\}$ such that $ab = \Omega$. Then

$$\operatorname{dpt}(a) + \operatorname{dpt}(b) = \begin{cases} k+1 & \text{if } \sigma(a) = s\,, \\ k & \text{if } \sigma(a) = t\,. \end{cases}$$

(3) Let c be an Ω-unmovable element of M. Then

$$\operatorname{dpt}(\varphi(c)) = \begin{cases} \operatorname{dpt}(c) + 1 & \text{if } c \neq 1 \text{ and } \sigma(c) = \tau(c) = t\,, \\ \operatorname{dpt}(c) - 1 & \text{if } c \neq 1 \text{ and } \sigma(c) = \tau(c) = s\,, \\ \operatorname{dpt}(c) & \text{otherwise}\,. \end{cases}$$

(4) Let c be an Ω-unmovable element of M. Then

$$\operatorname{dpt}(c\Omega) = \begin{cases} \operatorname{dpt}(c) + k - 1 & \text{if } c \neq 1 \text{ and } \tau(c) = s\,, \\ \operatorname{dpt}(c) + k & \text{otherwise}\,. \end{cases}$$

(5) Let $a, b \in M \setminus \{1\}$ such that $a\,\varphi(b) = \Omega$. Then

$$\operatorname{dpt}(a) + \operatorname{dpt}(b) = \begin{cases} k+1 & \text{if } \tau(a) = s\,, \\ k & \text{if } \tau(a) = t\,. \end{cases}$$

Lemma 6.8 *Let a_1, a_2, b_1, b_2 be four non-trivial Ω-unmovable elements of M such that $\sigma(a_1) = \tau(a_2)$, $\tau(b_1) = \sigma(b_2)$, $a_1 b_1 = \Omega$ and $a_2\,\varphi(b_2) = \Omega$. Set $u = |\{i \in \{1,2\} \mid \sigma(a_i) = s\}|$ and $v = |\{i \in \{1,2\} \mid \tau(b_i) = s\}|$. Then $\operatorname{dpt}(a_1) + \operatorname{dpt}(a_2) + \operatorname{dpt}(b_1) + \operatorname{dpt}(b_2) = 2k - 1 + u + v$.*

Proof If $\sigma(a_1) = s$ and $\sigma(a_2) = s$, then $\tau(a_2) = s$, $\tau(b_1) = s$ and $\tau(b_2) = t$, hence $u = 2$, $v = 1$ and, by Lemma 6.7, $\operatorname{dpt}(a_1) + \operatorname{dpt}(a_2) + \operatorname{dpt}(b_1) + \operatorname{dpt}(b_2) = 2k + 2 = 2k - 1 + u + v$. If $\sigma(a_1) = s$ and $\sigma(a_2) = t$, then $\tau(a_2) = s$, $\tau(b_1) = s$ and $\tau(b_2) = s$, hence $u = 1$, $v = 2$ and, by Lemma 6.7, $\operatorname{dpt}(a_1) + \operatorname{dpt}(a_2) + \operatorname{dpt}(b_1) + \operatorname{dpt}(b_2) = 2k + 2 = 2k - 1 + u + v$. If $\sigma(a_1) = t$ and $\sigma(a_2) = s$, then $\tau(a_2) = t$, $\tau(b_1) = t$ and $\tau(b_2) = t$, hence $u = 1$, $v = 0$ and, by Lemma 6.7, $\operatorname{dpt}(a_1) + \operatorname{dpt}(a_2) + \operatorname{dpt}(b_1) + \operatorname{dpt}(b_2) = 2k = 2k - 1 + u + v$. If $\sigma(a_1) = t$ and $\sigma(a_2) = t$, then $\tau(a_2) = t$, $\tau(b_1) = t$ and $\tau(b_2) = s$, hence $u = 0$, $v = 1$ and, by Lemma 6.7, $\operatorname{dpt}(a_1) + \operatorname{dpt}(a_2) + \operatorname{dpt}(b_1) + \operatorname{dpt}(b_2) = 2k = 2k - 1 + u + v$. □

Lemma 6.9 *Let a, b be two Ω-unmovable elements in M. We assume that the Δ-form of ab is in the form $ab = c\Delta^p$ where c is Ω-unmovable.*

(1) *Suppose that* $(a,b) \notin (\bar\Theta \times \bar\Theta)$. *There exists* $\varepsilon \in \{0,1\}$ *such that* $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - p(2k-1) - \varepsilon$. *Moreover,* $\varepsilon = 1$ *if either* $a \in \Theta$ *or* $b \in \Theta$ *or* $c \in M_1$.

(2) *Suppose that* $(a\Omega, \varphi(b)) \notin (\bar\Theta \times \bar\Theta)$. *The exists* $\varepsilon \in \{0,1\}$ *such that* $\mathrm{dpt}(c\Omega) = \mathrm{dpt}(a\Omega) + \mathrm{dpt}(\varphi(b)) - p(2k-1) - \varepsilon$. *Moreover,* $\varepsilon = 1$ *if either* $a\Omega \in \Theta$ *or* $\varphi(b) \in \Theta$.

(3) *Suppose that* $(a, b\Omega) \notin (\bar\Theta \times \bar\Theta)$. *There exists* $\varepsilon \in \{0,1\}$ *such that* $\mathrm{dpt}(c\Omega) = \mathrm{dpt}(a) + \mathrm{dpt}(b\Omega) - p(2k-1) - \varepsilon$. *Moreover,* $\varepsilon = 1$ *if either* $a \in \Theta$ *or* $b\Omega \in \Theta$.

(4) *Suppose that* $(a\Omega, \varphi(b)\,\Omega) \notin (\bar\Theta \times \bar\Theta)$. *There exists* $\varepsilon \in \{0,1\}$ *such that* $\mathrm{dpt}(c) = \mathrm{dpt}(a\Omega) + \mathrm{dpt}(\varphi(b)\,\Omega) - (p+1)(2k-1) - \varepsilon$. *Moreover,* $\varepsilon = 1$ *if either* $a\Omega \in \Theta$ *or* $\varphi(b)\,\Omega \in \Theta$ *or* $c \in M_1$.

Proof We write a and b in the form $a = a_{2p+1}a_{2p} \cdots a_2 a_1$ and $b = b_1 b_2 \cdots b_{2p} b_{2p+1}$ so that:

- $a_i \neq 1$, $b_i \neq 1$, $a_i b_i = \Omega$ if i is odd, and $a_i\,\varphi(b_i) = \Omega$ if i is even, for all $i \in \{1, \dots, 2p\}$;
- $c = a_{2p+1} b_{2p+1}$;
- Set $x_i = \tau(a_i)$, $x'_i = \sigma(a_i)$, $y_i = \sigma(b_i)$, $y'_i = \tau(b_i)$, for all $i \in \{1, \dots, 2p+1\}$. Then $x_{i+1} = x'_i$ for all $i \in \{1, \dots, 2p-1\}$.

We have $y_i = \varphi(x_i)$ and $y'_i = x'_i$ if i is odd, and $y_i = x_i$ and $y'_i = \varphi(x'_i)$ if i is even, for all $i \in \{1, \dots, 2p\}$. Thus, if i is odd, then $y_{i+1} = x_{i+1} = x'_i = y'_i$, and if i is even, then $y_{i+1} = \varphi(x_{i+1}) = \varphi(x'_i) = y'_i$, for $i \in \{1, \dots, 2p-1\}$.

Let $u = |\{i \in \{1, \dots, 2p\} \mid x'_i = s\}|$. By using Lemma 6.7 we show successively the following equalities.

$$\mathrm{dpt}(a) = \mathrm{dpt}(a_{2p+1}) + \textstyle\sum_{i=1}^{2p} \mathrm{dpt}(a_i) - u + \varepsilon_{1,a}\,,$$
$$\mathrm{dpt}(a\Omega) = \mathrm{dpt}(a_{2p+1}) + \textstyle\sum_{i=1}^{2p} \mathrm{dpt}(a_i) - u + k + \varepsilon_{2,a}\,,$$

where $\varepsilon_{1,a}$ and $\varepsilon_{2,a}$ are as follows. If $p \geq 1$ and $a_{2p+1} \neq 1$, then:

$$\varepsilon_{1,a} = \begin{cases} 0 & \text{if } (x'_{2p}, x_{2p+1}) \in \{(s,s),(t,s),(t,t)\}\,,\\ 1 & \text{if } (x'_{2p}, x_{2p+1}) = (s,t)\,,\end{cases}$$
$$\varepsilon_{2,a} = \begin{cases} -1 & \text{if } (x_1, x'_{2p}, x_{2p+1}) \in \{(s,s,s),(s,t,s),(s,t,t)\}\,,\\ 0 & \text{if } (x_1, x'_{2p}, x_{2p+1}) \in \{(s,s,t),(t,s,s),(t,t,s),(t,t,t)\}\,,\\ 1 & \text{if } (x_1, x'_{2p}, x_{2p+1}) = (t,s,t)\,.\end{cases}$$

If $p \geq 1$ and $a_{2p+1} = 1$, then:

$$\varepsilon_{1,a} = \begin{cases} 0 & \text{if } x'_{2p} = t\,, \\ 1 & \text{if } x'_{2p} = s\,, \end{cases} \quad \varepsilon_{2,a} = \begin{cases} -1 & \text{if } (x_1, x'_{2p}) = (s,t)\,, \\ 0 & \text{if } (x_1, x'_{2p}) \in \{(s,s),(t,t)\}\,, \\ 1 & \text{if } (x_1, x'_{2p}) = (t,s)\,. \end{cases}$$

If $p = 0$ and $a \neq 1$, then:

$$\varepsilon_{1,a} = 0\,,\ \varepsilon_{2,a} = \begin{cases} -1 & \text{if } x_{2p+1} = s\,, \\ 0 & \text{if } x_{2p+1} = t\,. \end{cases}$$

If $p = 0$ and $a = 1$, then $\varepsilon_{1,a} = \varepsilon_{2,a} = 0$.

Let $v = |\{i \in \{1, \dots, 2p\} \mid y'_i = s\}|$. Similarly, by using Lemma 6.7 we prove successively the following equalities.

$$\begin{gathered} \mathrm{dpt}(b) = \mathrm{dpt}(b_{2p+1}) + \textstyle\sum_{i=1}^{2p} \mathrm{dpt}(b_i) - v + \varepsilon_{1,b}\,, \\ \mathrm{dpt}(\varphi(b)) = \mathrm{dpt}(b_{2p+1}) + \textstyle\sum_{i=1}^{2p} \mathrm{dpt}(b_i) - v + \varepsilon_{2,b}\,, \\ \mathrm{dpt}(b\Omega) = \mathrm{dpt}(b_{2p+1}) + \textstyle\sum_{i=1}^{2p} \mathrm{dpt}(b_i) - v + k + \varepsilon_{3,b}\,, \\ \mathrm{dpt}(\varphi(b)\,\Omega) = \mathrm{dpt}(b_{2p+1}) + \textstyle\sum_{i=1}^{2p} \mathrm{dpt}(b_i) - v + k + \varepsilon_{4,b}\,, \end{gathered}$$

where $\varepsilon_{1,b}$, $\varepsilon_{2,b}$, $\varepsilon_{3,b}$ and $\varepsilon_{4,b}$ are as follows. If $p \geq 1$ and $b_{2p+1} \neq 1$, then:

$$\begin{aligned} \varepsilon_{1,b} &= \begin{cases} 0 & \text{if } (y'_{2p}, y_{2p+1}) \in \{(s,s),(t,s),(t,t)\}\,, \\ 1 & \text{if } (y'_{2p}, y_{2p+1}) = (s,t)\,, \end{cases} \\ \varepsilon_{2,b} &= \begin{cases} -1 & \text{if } (y_1, y'_{2p}, y_{2p+1}, y'_{2p+1}) \in \{(s,s,s,s),(s,t,s,s),(s,t,t,s)\}\,, \\ 0 & \text{if } (y_1, y'_{2p}, y_{2p+1}, y'_{2p+1}) \in \{(s,s,s,t),(s,s,t,s),(s,t,s,t), \\ & (s,t,t,t),(t,s,s,s),(t,t,s,s),(t,t,t,s)\}\,, \\ 1 & \text{if } (y_1, y'_{2p}, y_{2p+1}, y'_{2p+1}) \in \{(s,s,t,t),(t,s,s,t),(t,s,t,s), \\ & (t,t,s,t),(t,t,t,t)\}\,, \\ 2 & \text{if } (y_1, y'_{2p}, y_{2p+1}, y'_{2p+1}) = (t,s,t,t)\,, \end{cases} \\ \varepsilon_{3,b} &= \begin{cases} -1 & \text{if } (y'_{2p}, y_{2p+1}, y'_{2p+1}) \in \{(s,s,s),(t,s,s),(t,t,s)\}\,, \\ 0 & \text{if } (y'_{2p}, y_{2p+1}, y'_{2p+1}) \in \{(s,s,t),(s,t,s),(t,s,t),(t,t,t)\}\,, \\ 1 & \text{if } (y'_{2p}, y_{2p+1}, y'_{2p+1}) = (s,t,t)\,, \end{cases} \\ \varepsilon_{4,b} &= \begin{cases} -1 & \text{if } (y_1, y'_{2p}, y_{2p+1}) \in \{(s,s,s),(s,t,s),(s,t,t)\}\,, \\ 0 & \text{if } (y_1, y'_{2p}, y_{2p+1}) \in \{(s,s,t),(t,s,s),(t,t,s),(t,t,t)\}\,, \\ 1 & \text{if } (y_1, y'_{2p}, y_{2p+1}) = (t,s,t)\,. \end{cases} \end{aligned}$$

If $p \geq 1$ and $b_{2p+1} = 1$, then:

$$\varepsilon_{1,b} = \begin{cases} 0 & \text{if } y'_{2p} = t\,, \\ 1 & \text{if } y'_{2p} = s\,, \end{cases} \quad \varepsilon_{2,b} = \begin{cases} 0 & \text{if } y_1 = s\,, \\ 1 & \text{if } y_1 = t\,, \end{cases}$$

$$\varepsilon_{3,b} = 0\,,\ \varepsilon_{4,b} = \begin{cases} -1 & \text{if } (y_1, y'_{2p}) = (s,t)\,, \\ 0 & \text{if } (y_1, y'_{2p}) \in \{(s,s),(t,t)\}\,, \\ 1 & \text{if } (y_1, y'_{2p}) = (t,s)\,. \end{cases}$$

If $p = 0$ and $b \neq 1$, then:

$$\varepsilon_{1,b} = 0\,,\ \varepsilon_{2,b} = \begin{cases} -1 & \text{if } (y_{2p+1}, y'_{2p+1}) = (s,s)\,, \\ 0 & \text{if } (y_{2p+1}, y'_{2p+1}) \in \{(s,t),(t,s)\}\,, \\ 1 & \text{if } (y_{2p+1}, y'_{2p+1}) = (t,t)\,, \end{cases}$$

$$\varepsilon_{3,b} = \begin{cases} -1 & \text{if } y'_{2p+1} = s\,, \\ 0 & \text{if } y'_{2p+1} = t\,, \end{cases} \quad \varepsilon_{4,b} = \begin{cases} -1 & \text{if } y_{2p+1} = s\,, \\ 0 & \text{if } y_{2p+1} = t\,. \end{cases}$$

If $p = 0$ and $b = 1$, then $\varepsilon_{1,b} = \varepsilon_{2,b} = \varepsilon_{3,b} = \varepsilon_{4,b} = 0$.

Again, by applying Lemma 6.7 we prove successively the following equalities.

$$\begin{aligned} \mathrm{dpt}(c) &= \mathrm{dpt}(a_{2p+1}) + \mathrm{dpt}(b_{2p+1}) + \varepsilon_{1,c}\,, \\ \mathrm{dpt}(c\Omega) &= \mathrm{dpt}(a_{2p+1}) + \mathrm{dpt}(b_{2p+1}) + k + \varepsilon_{2,c}\,, \end{aligned}$$

where $\varepsilon_{1,c}$ and $\varepsilon_{2,c}$ are as follows. If $a_{2p+1} \neq 1$ and $b_{2p+1} \neq 1$, then:

$$\varepsilon_{1,c} = \begin{cases} -1 & \text{if } (x_{2p+1}, y_{2p+1}) = (s,s)\,, \\ 0 & \text{if } (x_{2p+1}, y_{2p+1}) \in \{(s,t),(t,s),(t,t)\}\,, \end{cases}$$

$$\varepsilon_{2,c} = \begin{cases} -2 & \text{if } (x_{2p+1}, y_{2p+1}, y'_{2p+1}) = (s,s,s)\,, \\ -1 & \text{if } (x_{2p+1}, y_{2p+1}, y'_{2p+1}) \in \{(s,s,t),(s,t,s),(t,s,s),(t,t,s)\}\,, \\ 0 & \text{if } (x_{2p+1}, y_{2p+1}, y'_{2p+1}) \in \{(s,t,t),(t,s,t),(t,t,t)\}\,. \end{cases}$$

If $a_{2p+1} \neq 1$ and $b_{2p+1} = 1$, then:

$$\varepsilon_{1,c} = 0\,,\ \varepsilon_{2,c} = \begin{cases} -1 & \text{if } x_{2p+1} = s\,, \\ 0 & \text{if } x_{2p+1} = t\,. \end{cases}$$

If $a_{2p+1} = 1$ and $b_{2p+1} \neq 1$, then:

$$\varepsilon_{1,c} = 0\,,\ \varepsilon_{2,c} = \begin{cases} -1 & \text{if } y'_{2p+1} = s\,, \\ 0 & \text{if } y'_{2p+1} = t\,. \end{cases}$$

If $a_{2p+1} = 1$ and $b_{2p+1} = 1$, then $\varepsilon_{1,c} = \varepsilon_{2,c} = 0$.

From Lemma 6.8 we also get $\sum_{i=1}^{2p}(\mathrm{dpt}(a_i) + \mathrm{dpt}(b_i)) = p(2k-1) + u + v$.

Part (1): Let $\varepsilon = \varepsilon_{1,a} + \varepsilon_{1,b} - \varepsilon_{1,c}$. By the above we have $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - p(2k-1) - \varepsilon$, and ε is as follows. If $p \ge 1$, $a_{2p+1} \neq 1$ and $b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $(x'_{2p}, x_{2p+1}, y_{2p+1}) \in \{(s,s,t),(t,t,s)\}$, and $\varepsilon = 1$ otherwise. If $p \ge 1$, $a_{2p+1} \neq 1$ and $b_{2p+1} = 1$, then: $\varepsilon = 0$ if $(x'_{2p}, x_{2p+1}) = (s,s)$, and $\varepsilon = 1$ otherwise. If $p \ge 1$, $a_{2p+1} = 1$ and $b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $(x'_{2p}, y_{2p+1}) = (t,s)$, and $\varepsilon = 1$ otherwise. If $p \ge 1$, $a_{2p+1} = 1$ and $b_{2p+1} = 1$, then $\varepsilon = 1$. If $p = 0$, $a = a_{2p+1} \neq 1$ and $b = b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $(x_{2p+1}, y_{2p+1}) \in \{(s,t),(t,s),(t,t)\}$, and $\varepsilon = 1$ otherwise. If $p = 0$ and $a = a_{2p+1} = 1$, then $\varepsilon = 0$. If $p = 0$ and $b = b_{2p+1} = 1$, then $\varepsilon = 0$.

Suppose that $a \in \Theta$. Then a is written $a = \theta^q$ with $q \ge 1$. On the other hand we write $b = t^r b'$ where $b' \neq 1$ (since $b \notin \bar{\Theta}$) and $\sigma(b') = s$. If $r = 0$, then $p = 0$, $x_{2p+1} = s$ and $y_{2p+1} = s$, hence $\varepsilon = 1$. If $1 \le r < 2q$, then $r = 2p$, $a_{2p+1} = \theta^{q-p}$, $b_{2p+1} = b'$ and $(x'_{2p}, x_{2p+1}, y_{2p+1}) = (s,s,s)$, hence $\varepsilon = 1$. If $r \ge 2q$, then $q = p$, $a_{2p+1} = 1$, $b_{2p+1} \neq 1$ and $x'_{2p} = s$, hence $\varepsilon = 1$. The case $b \in \Theta$ is proved in the same way.

Suppose that $c \in M_1$. Then $p \ge 1$, since $(a,b) \notin \bar{\Theta} \times \bar{\Theta}$. If $a_{2p+1} \neq 1$ and $b_{2p+1} \neq 1$, then $(x_{2p+1}, y_{2p+1}) = (t,t)$, hence $\varepsilon = 1$. If $a_{2p+1} \neq 1$ and $b_{2p+1} = 1$, then $x_{2p+1} = t$, hence $\varepsilon = 1$. If $a_{2p+1} = 1$ and $b_{2p+1} \neq 1$, then $y_{2p+1} = t$, hence $\varepsilon = 1$. If $a_{2p+1} = 1$ and $b_{2p+1} = 1$, then $\varepsilon = 1$.

Part (2): Let $\varepsilon = \varepsilon_{2,a} + \varepsilon_{2,b} - \varepsilon_{2,c}$. By the above we have $\mathrm{dpt}(c\Omega) = \mathrm{dpt}(a\Omega) + \mathrm{dpt}(\varphi(b)) - p(2k-1) - \varepsilon$, and ε is as follows. If $p \ge 1$, $a_{2p+1} \neq 1$ and $b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $(x'_{2p}, x_{2p+1}, y_{2p+1}) \in \{(s,s,t),(t,t,s)\}$, and $\varepsilon = 1$ otherwise. If $p \ge 1$, $a_{2p+1} \neq 1$ and $b_{2p+1} = 1$, then: $\varepsilon = 0$ if $(x_1, x'_{2p}, x_{2p+1}) \in \{(t,s,s),(t,t,s),(t,t,t)\}$, and $\varepsilon = 1$ otherwise. If $p \ge 1$, $a_{2p+1} = 1$ and $b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $(x'_{2p}, y_{2p+1}) = (t,s)$, and $\varepsilon = 1$ otherwise. If $p \ge 1$, $a_{2p+1} = 1$ and $b_{2p+1} = 1$, then: $\varepsilon = 0$ if $x'_{2p} = t$, and $\varepsilon = 1$ otherwise. If $p = 0$, $a = a_{2p+1} \neq 1$ and $b = b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $(x_{2p+1}, y_{2p+1}) \in \{(s,s),(s,t),(t,s)\}$, and $\varepsilon = 1$ otherwise. If $p = 0$, $a = a_{2p+1} = 1$ and $b = b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $y_{2p+1} = s$, and $\varepsilon = 1$ otherwise. If $p = 0$ and $b = b_{2p+1} = 1$, then $\varepsilon = 0$.

Suppose that $a\Omega \in \Theta$. Then $a\Omega$ is written $a\Omega = \theta^q t$ with $q \ge 1$, hence $a = \theta^{q-1}(st)^k$. On the other hand we write $b = s^r b'$, where $b' \neq 1$ (since $\varphi(b') \notin \bar{\Theta}$) and $\sigma(b') = t$. We necessarily have $r = 2p \le 2(q-1)$, hence $a_{2p+1} = \theta^{q-p-1}(st)^k$ and $b_{2p+1} = b'$. If $p \ge 1$, then $(x'_{2p}, x_{2p+1}, y_{2p+1}) = (t,t,t)$, hence $\varepsilon = 1$. If $p = 0$, then $(x_{2p+1}, y_{2p+1}) = (t,t)$, hence $\varepsilon = 1$.

Suppose that $\varphi(b) \in \Theta$. Then $\varphi(b)$ is written $\varphi(b) = \theta^q$ with $q \ge 1$, hence $b = ((ts)^k(st)^k)^q$. On the other hand we write $a = a's^r$ where either $a' = 1$ or

$\tau(a') = t$. If $r = 0$ and $a' = 1$, then $p = 0$, $b_{2p+1} = b \neq 1$ and $y_{2p+1} = t$, hence $\varepsilon = 1$. If $r = 0$ and $a' \neq 1$, then $p = 0$, $a_{2p+1} = a' \neq 1$, $b_{2p+1} = b \neq 1$ and $(x_{2p+1}, y_{2p+1}) = (t,t)$, hence $\varepsilon = 1$. If $0 < r < 2q$ and $a' = 1$, then $r = 2p$, $a_{2p+1} = 1$, $b_{2p+1} = ((ts)^k(st)^k)^{q-p} \neq 1$ and $(x'_{2p}, y_{2p+1}) = (s,t)$, hence $\varepsilon = 1$. If $0 < r < 2q$ and $a' \neq 1$, then $r = 2p$, $a_{2p+1} = a' \neq 1$, $b_{2p+1} = ((ts)^k(st)^k)^{q-p} \neq 1$ and $(x'_{2p}, x_{2p+1}, y_{2p+1}) = (s,t,t)$, hence $\varepsilon = 1$. If $r = 2q$ and $a' = 1$, then $r = 2p$, $a_{2p+1} = 1$, $b_{2p+1} = 1$ and $x'_{2p} = s$, hence $\varepsilon = 1$. If $r = 2q$ and $a' \neq 1$, then $r = 2p$, $a_{2p+1} = a' \neq 1$, $b_{2p+1} = 1$ and $(x_1, x'_{2p}, x_{2p+1}) = (s,s,t)$, hence $\varepsilon = 1$. If $r > 2q$, then $p = q$, $a_{2p+1} = a's^{r-2q} \neq 1$, $b_{2p+1} = 1$, and $(x_1, x'_{2p}, x_{2p+1}) = (s,s,s)$, hence $\varepsilon = 1$.

Part (3): Let $\varepsilon = \varepsilon_{1,a} + \varepsilon_{3,b} - \varepsilon_{2,c}$. By the above we have $\mathrm{dpt}(c\Omega) = \mathrm{dpt}(a) + \mathrm{dpt}(b\Omega) - p(2k-1) - \varepsilon$, and ε is as follows. If $p \geq 1$, $a_{2p+1} \neq 1$ and $b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $(x'_{2p}, x_{2p+1}, y_{2p+1}) \in \{(s,s,t),(t,t,s)\}$, and $\varepsilon = 1$ otherwise. If $p \geq 1$, $a_{2p+1} \neq 1$ and $b_{2p+1} = 1$, then: $\varepsilon = 0$ if $(x'_{2p}, x_{2p+1}) = (t,t)$, and $\varepsilon = 1$ otherwise. If $p \geq 1$, $a_{2p+1} = 1$ and $b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $(x'_{2p}, y_{2p+1}) = (t,s)$, and $\varepsilon = 1$ otherwise. If $p \geq 1$, $a_{2p+1} = 1$ and $b_{2p+1} = 1$, then: $\varepsilon = 0$ if $x'_{2p} = t$, and $\varepsilon = 1$ otherwise. If $p = 0$, $a \neq 1$ and $b \neq 1$, then: $\varepsilon = 0$ if $(x_{2p+1}, y_{2p+1}) \in \{(s,t),(t,s),(t,t)\}$, and $\varepsilon = 1$ otherwise. If $p = 0$, $a \neq 1$ and $b = 1$, then: $\varepsilon = 0$ if $x_{2p+1} = t$, and $\varepsilon = 1$ otherwise. If $p = 0$ and $a = 1$, then $\varepsilon = 0$.

Suppose that $a \in \Theta$. Then a is written $a = \theta^q$ with $q \geq 1$. On the other hand we write $b = t^r b'$ where either $b' = 1$ or $\sigma(b') = s$. If $r = 0$ and $b' = 1$, then $p = 0$, $a_{2p+1} = \theta^q$, $b_{2p+1} = 1$ and $x_{2p+1} = s$, hence $\varepsilon = 1$. If $r = 0$ and $b' \neq 1$, then $p = 0$, $a_{2p+1} = \theta^q$, $b_{2p+1} = b' \neq 1$ and $(x_{2p+1}, y_{2p+1}) = (s,s)$, hence $\varepsilon = 1$. If $0 < r < 2q$ and $b' = 1$, then $r = 2p$, $a_{2p+1} = \theta^{q-p} \neq 1$, $b_{2p+1} = 1$ and $(x'_{2p}, x_{2p+1}) = (s,s)$, hence $\varepsilon = 1$. If $0 < r < 2q$ and $b' \neq 1$, then $r = 2p$, $a_{2p+1} = \theta^{q-p} \neq 1$, $b_{2p+1} = b' \neq 1$ and $(x'_{2p}, x_{2p+1}, y_{2p+1}) = (s,s,s)$, hence $\varepsilon = 1$. If $r = 2q$ and $b' = 1$, then $p = q$, $a_{2p+1} = 1$, $b_{2p+1} = 1$ and $x'_{2p} = s$, hence $\varepsilon = 1$. If $r = 2q$ and $b' \neq 1$, then $p = q$, $a_{2p+1} = 1$, $b_{2p+1} = b' \neq 1$ and $(x'_{2p}, y_{2p+1}) = (s,s)$, hence $\varepsilon = 1$. If $r > 2q$, then $a_{2p+1} = 1$, $b_{2p+1} = t^{r-2q}b' \neq 1$ and $(x'_{2p}, y_{2p+1}) = (s,t)$, hence $\varepsilon = 1$.

Suppose that $b\Omega \in \Theta$. Then $b\Omega$ is written $b\Omega = \theta^q t$ with $q \geq 1$, hence $b = \theta^{q-1}(st)^k$. On the other hand we write $a = a't^r$ where $a' \neq 1$ (since $a \notin \bar{\Theta}$) and $\tau(a') = s$. If $r = 0$, then $p = 0$, $a_{2p+1} = a' \neq 1$, $b_{2p+1} = \theta^{q-1}(st)^k$ and $(x_{2p+1}, y_{2p+1}) = (s,s)$, hence $\varepsilon = 1$. If $r > 0$, then $r = 2p \leq 2(q-1)$, $a_{2p+1} = a' \neq 1$, $b_{2p+1} = \theta^{q-p-1}(st)^k$ and $(x'_{2p}, x_{2p+1}, y_{2p+1}) = (t,s,s)$, hence $\varepsilon = 1$.

Part (4): Let $\varepsilon = 1 + \varepsilon_{2,a} + \varepsilon_{4,b} - \varepsilon_{1,c}$. By the above we have $\mathrm{dpt}(c) = \mathrm{dpt}(a\Omega) + \mathrm{dpt}(\varphi(b)\Omega) - (p+1)(2k-1) - \varepsilon$, and ε is as follows. If $p \geq 1$, $a_{2p+1} \neq 1$ and

$b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $(x'_{2p}, x_{2p+1}, y_{2p+1}) \in \{(s,s,t),(t,t,s)\}$, and $\varepsilon = 1$ otherwise. If $p \geq 1$, $a_{2p+1} \neq 1$ and $b_{2p+1} = 1$, then: $\varepsilon = 0$ if $(x'_{2p}, x_{2p+1}) = (s,s)$, and $\varepsilon = 1$ otherwise. If $p \geq 1$, $a_{2p+1} = 1$ and $b_{2p+1} \neq 1$, then: $\varepsilon = 0$ if $(x'_{2p}, y_{2p+1}) = (t,s)$, and $\varepsilon = 1$ otherwise. If $p \geq 1$, $a_{2p+1} = 1$ and $b_{2p+1} = 1$, then $\varepsilon = 1$. If $p = 0$, $a \neq 1$ and $b \neq 1$, then: $\varepsilon = 0$ if $(x_{2p+1}, y_{2p+1}) \in \{(s,s),(s,t),(t,s)\}$, and $\varepsilon = 1$ otherwise. If $p = 0$, $a \neq 1$ and $b = 1$, then: $\varepsilon = 0$ if $x_{2p+1} = s$, and $\varepsilon = 1$ otherwise. If $p = 0$, $a = 1$ and $b \neq 1$, then: $\varepsilon = 0$ if $y_{2p+1} = s$, and $\varepsilon = 1$ otherwise. If $p = 0$, $a = 1$ and $b = 1$, then $\varepsilon = 1$.

Suppose that $a\Omega \in \Theta$. Then $a\Omega$ is written $a\Omega = \theta^q t$ with $q \geq 1$, hence $a = \theta^{q-1}(st)^k$. On the other hand we write $b = s^r b'$, where either $b' = 1$ or $\sigma(b') = t$. If $r = 0$ and $b' = 1$, then $p = 0$, $a = \theta^{q-1}(st)^k \neq 1$, $b = 1$ and $x_{2p+1} = t$, hence $\varepsilon = 1$. If $r = 0$ and $b' \neq 1$, then $p = 0$, $a = \theta^{q-1}(st)^k \neq 1$, $b = b' \neq 1$ and $(x_{2p+1}, y_{2p+1}) = (t,t)$, hence $\varepsilon = 1$. If $r > 0$ and $b' = 1$, then $r = 2p \leq 2(q-1)$, $a_{2p+1} = \theta^{q-p-1}(st)^k \neq 1$, $b_{2p+1} = b' = 1$ and $(x'_{2p}, x_{2p+1}) = (t,t)$, hence $\varepsilon = 1$. If $r > 0$ and $b' \neq 1$, then $r = 2p \leq 2(q-1)$, $a_{2p+1} = \theta^{q-p-1}(st)^k \neq 1$, $b_{2p+1} = b' \neq 1$ and $(x'_{2p}, x_{2p+1}, y_{2p+1}) = (t,t,t)$, hence $\varepsilon = 1$.

Suppose that $\varphi(b)\Omega \in \Theta$. Then $\varphi(b)\Omega$ is written $\varphi(b)\Omega = \theta^q t$ with $q \geq 1$, hence $b = (ts)^k \theta^{q-1}$. On the other hand we write $a = a's^r$ where either $a' = 1$ or $\tau(a') = t$. If $r = 0$ and $a' = 1$, then $p = 0$, $a = 1$, $b = (ts)^k\theta^{q-1} \neq 1$ and $y_{2p+1} = t$, hence $\varepsilon = 1$. If $r = 0$ and $a' \neq 1$, then $p = 0$, $a = a' \neq 1$, $b = (ts)^k\theta^{q-1} \neq 1$ and $(x_{2p+1}, y_{2p+1}) = (t,t)$, hence $\varepsilon = 1$. If $r > 0$ and $a' = 1$, then $r = 2p \leq 2(q-1)$, $a_{2p+1} = 1$, $b_{2p+1} = (ts)^k\theta^{q-p-1} \neq 1$ and $(x'_{2p}, y_{2p+1}) = (s,t)$, hence $\varepsilon = 1$. If $r > 0$ and $a' \neq 1$, then $r = 2p \leq 2(q-1)$, $a_{2p+1} = a' \neq 1$, $b_{2p+1} = (ts)^k\theta^{q-p-1} \neq 1$ and $(x'_{2p}, x_{2p+1}, y_{2p+1}) = (s,t,t)$, hence $\varepsilon = 1$.

Suppose that $c \in M_1$. If $p \geq 1$, $a_{2p+1} \neq 1$ and $b_{2p+1} \neq 1$, then $(x_{2p+1}, y_{2p+1}) = (t,t)$, hence $\varepsilon = 1$. If $p \geq 1$, $a_{2p+1} \neq 1$ and $b_{2p+1} = 1$, then $x_{2p+1} = t$, hence $\varepsilon = 1$. If $p \geq 1$, $a_{2p+1} = 1$ and $b_{2p+1} \neq 1$, then $y_{2p+1} = t$, hence $\varepsilon = 1$. If $p \geq 1$, $a_{2p+1} = 1$ and $b_{2p+1} = 1$, then $\varepsilon = 1$. If $p = 0$, $a \neq 1$ and $b \neq 1$, then $(x_{2p+1}, y_{2p+1}) = (t,t)$, hence $\varepsilon = 1$. If $p = 0$, $a \neq 1$ and $b = 1$, then $x_{2p+1} = t$, hence $\varepsilon = 1$. If $p = 0$, $a = 1$ and $b \neq 1$, then $y_{2p+1} = t$, hence $\varepsilon = 1$. If $p = 0$, $a = 1$ and $b = 1$, then $\varepsilon = 1$. $\square$

Lemma 6.10 *Let a, b be two Ω-unmovable elements of M. We assume that the Δ-form of ab is in the form $ab = (c\Omega)\Delta^p$ where c is an Ω-unmovable element of M and $p \geq 0$.*

(1) Suppose that $(a,b) \notin (\bar{\Theta} \times \bar{\Theta})$. There exists $\varepsilon \in \{0,1\}$ such that $\mathrm{dpt}(c\Omega) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - p(2k-1) - \varepsilon$. Moreover, $\varepsilon = 1$ if either $a \in \Theta$ or $b \in \Theta$.

(2) *Suppose that* $(a\Omega, \varphi(b)) \notin (\bar\Theta \times \bar\Theta)$. *There exists* $\varepsilon \in \{0,1\}$ *such that* $\mathrm{dpt}(c) = \mathrm{dpt}(a\Omega) + \mathrm{dpt}(\varphi(b)) - (p+1)(2k-1) - \varepsilon$. *Moreover,* $\varepsilon = 1$ *if either* $a\Omega \in \Theta$ *or* $\varphi(b) \in \Theta$ *or* $c \in M_1$.

(3) *Suppose that* $(a, b\Omega) \notin (\bar\Theta \times \bar\Theta)$. *There exists* $\varepsilon \in \{0,1\}$ *such that* $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b\Omega) - (p+1)(2k-1) - \varepsilon$. *Moreover,* $\varepsilon = 1$ *if either* $a \in \Theta$ *or* $b\Omega \in \Theta$ *or* $c \in M_1$.

(4) *Suppose that* $(a\Omega, \varphi(b)\Omega) \notin (\bar\Theta \times \bar\Theta)$. *There exists* $\varepsilon \in \{0,1\}$ *such that* $\mathrm{dpt}(c\Omega) = \mathrm{dpt}(a\Omega) + \mathrm{dpt}(\varphi(b)\,\Omega) - (p+1)(2k-1) - \varepsilon$. *Moreover,* $\varepsilon = 1$ *if either* $a\Omega \in \Theta$ *or* $\varphi(b)\,\Omega \in \Theta$.

Proof We write a and b in the form $a = a_{2p+2}a_{2p+1}\cdots a_2a_1$ and $b = b_1b_2\cdots b_{2p+1}b_{2p+2}$ so that:

- $a_i \neq 1$, $b_i \neq 1$, $a_ib_i = \Omega$ if i is odd, and $a_i\,\varphi(b_i) = \Omega$ if i is even, for all $i \in \{1, \dots, 2p+1\}$;
- $c = a_{2p+2}\,\varphi(b_{2p+2})$.
- Set $x_i = \tau(a_i)$, $x'_i = \sigma(a_i)$, $y_i = \sigma(b_i)$ and $y'_i = \tau(b_i)$ for all $i \in \{1, \dots, 2p+2\}$. Then $x_{i+1} = x'_i$ for all $i \in \{1, \dots, 2p\}$.

For $i \in \{1, \dots, 2p+1\}$ we have $y_i = \varphi(x_i)$ and $y'_i = x'_i$ if i is odd and $y_i = x_i$ and $y'_i = \varphi(x'_i)$ if i is even. So, if $i \in \{1, \dots, 2p\}$, then $y_{i+1} = x_{i+1} = x'_i = y'_i$ if i is odd, and $y_{i+1} = \varphi(x_{i+1}) = \varphi(x'_i) = y'_i$ if i is even.

Let $u = |\{i \in \{1, \dots, 2p+1\} \mid x'_i = s\}|$. By using Lemma 6.7 we obtain successively the following equalities.

$$\mathrm{dpt}(a) = \mathrm{dpt}(a_{2p+2}) + \textstyle\sum_{i=1}^{2p+1} \mathrm{dpt}(a_i) - u + \varepsilon_{1,a}\,,$$
$$\mathrm{dpt}(a\Omega) = \mathrm{dpt}(a_{2p+2}) + \textstyle\sum_{i=1}^{2p+1} \mathrm{dpt}(a_i) - u + k + \varepsilon_{2,a}\,,$$

where $\varepsilon_{1,a}$ and $\varepsilon_{2,a}$ are as follows. If $a_{2p+2} \neq 1$, then:

$$\varepsilon_{1,a} = \begin{cases} 0 & \text{if } (x'_{2p+1}, x_{2p+2}) \in \{(s,s),(t,s),(t,t)\}\,, \\ 1 & \text{if } (x'_{2p+1}, x_{2p+2}) = (s,t)\,, \end{cases}$$
$$\varepsilon_{2,a} = \begin{cases} -1 & \text{if } (x_1, x'_{2p+1}, x_{2p+2}) \in \{(s,s,s),(s,t,s),(s,t,t)\}\,, \\ 0 & \text{if } (x_1, x'_{2p+1}, x_{2p+2}) \in \{(s,s,t),(t,s,s),(t,t,s),(t,t,t)\}\,, \\ 1 & \text{if } (x_1, x'_{2p+1}, x_{2p+2}) = (t,s,t)\,. \end{cases}$$

If $a_{2p+2} = 1$, then:

$$\varepsilon_{1,a} = \begin{cases} 0 & \text{if } x'_{2p+1} = t\,, \\ 1 & \text{if } x'_{2p+1} = s\,, \end{cases} \qquad \varepsilon_{2,a} = \begin{cases} -1 & \text{if } (x_1, x'_{2p+1}) = (s,t)\,, \\ 0 & \text{if } (x_1, x'_{2p+1}) \in \{(s,s),(t,t)\}\,, \\ 1 & \text{if } (x_1, x'_{2p+1}) = (t,s)\,. \end{cases}$$

Let $v = |\{i \in \{1, \dots, 2p+1\} \mid y'_i = s\}|$. Similarly, by using Lemma 6.7 we obtain successively the following equalities.

$$\begin{gathered}
\mathrm{dpt}(b) = \mathrm{dpt}(b_{2p+2}) + \textstyle\sum_{i=1}^{2p+1} \mathrm{dpt}(b_i) - v + \varepsilon_{1,b}\,, \\
\mathrm{dpt}(\varphi(b)) = \mathrm{dpt}(b_{2p+2}) + \textstyle\sum_{i=1}^{2p+1} \mathrm{dpt}(b_i) - v + \varepsilon_{2,b}\,, \\
\mathrm{dpt}(b\Omega) = \mathrm{dpt}(b_{2p+2}) + \textstyle\sum_{i=1}^{2p+1} \mathrm{dpt}(b_i) - v + k + \varepsilon_{3,b}\,, \\
\mathrm{dpt}(\varphi(b)\,\Omega) = \mathrm{dpt}(b_{2p+2}) + \textstyle\sum_{i=1}^{2p+1} \mathrm{dpt}(b_i) - v + k + \varepsilon_{4,b}\,,
\end{gathered}$$

where $\varepsilon_{1,b}$, $\varepsilon_{2,b}$, $\varepsilon_{3,b}$ and $\varepsilon_{4,b}$ are as follows. If $b_{2p+2} \neq 1$, then:

$$\begin{aligned}
\varepsilon_{1,b} &= \begin{cases} 0 & \text{if } (y'_{2p+1}, y_{2p+2}) \in \{(s,s),(t,s),(t,t)\}\,, \\ 1 & \text{if } (y'_{2p+1}, y_{2p+2}) = (s,t)\,, \end{cases} \\
\varepsilon_{2,b} &= \begin{cases} -1 & \text{if } (y_1, y'_{2p+1}, y_{2p+2}, y'_{2p+2}) \in \{(s,s,s,s),(s,t,s,s),(s,t,t,s)\}\,, \\ 0 & \text{if } (y_1, y'_{2p+1}, y_{2p+2}, y'_{2p+2}) \in \{(s,s,s,t),(s,s,t,s),(s,t,s,t), \\ & (s,t,t,t),(t,s,s,s),(t,t,s,s),(t,t,t,s)\}\,, \\ 1 & \text{if } (y_1, y'_{2p+1}, y_{2p+2}, y'_{2p+2}) \in \{(s,s,t,t),(t,s,s,t),(t,s,t,s), \\ & (t,t,s,t),(t,t,t,t)\}\,, \\ 2 & \text{if } (y_1, y'_{2p+1}, y_{2p+2}, y'_{2p+2}) = (t,s,t,t)\,, \end{cases} \\
\varepsilon_{3,b} &= \begin{cases} -1 & \text{if } (y'_{2p+1}, y_{2p+2}, y'_{2p+2}) \in \{(s,s,s),(t,s,s),(t,t,s)\}\,, \\ 0 & \text{if } (y'_{2p+1}, y_{2p+2}, y'_{2p+2}) \in \{(s,s,t),(s,t,s),(t,s,t),(t,t,t)\}\,, \\ 1 & \text{if } (y'_{2p+1}, y_{2p+2}, y'_{2p+2}) = (s,t,t)\,, \end{cases} \\
\varepsilon_{4,b} &= \begin{cases} -1 & \text{if } (y_1, y'_{2p+1}, y_{2p+2}) \in \{(s,s,s),(s,t,s),(s,t,t)\}\,, \\ 0 & \text{if } (y_1, y'_{2p+1}, y_{2p+2}) \in \{(s,s,t),(t,s,s),(t,t,s),(t,t,t)\}\,, \\ 1 & \text{if } (y_1, y'_{2p+1}, y_{2p+2}) = (t,s,t)\,. \end{cases}
\end{aligned}$$

If $b_{2p+2} = 1$, then:

$$\begin{gathered}
\varepsilon_{1,b} = \begin{cases} 0 & \text{if } y'_{2p+1} = t\,, \\ 1 & \text{if } y'_{2p+1} = s\,, \end{cases} \quad \varepsilon_{2,b} = \begin{cases} 0 & \text{if } y_1 = s\,, \\ 1 & \text{if } y_1 = t\,, \end{cases} \\
\varepsilon_{3,b} = 0\,,\ \varepsilon_{4,b} = \begin{cases} -1 & \text{if } (y_1, y'_{2p+1}) = (s,t)\,, \\ 0 & \text{if } (y_1, y'_{2p+1}) \in \{(s,s),(t,t)\}\,, \\ 1 & \text{if } (y_1, y'_{2p+1}) = (t,s)\,. \end{cases}
\end{gathered}$$

Again, by using Lemma 6.7 we obtain the following equalities.

$$\begin{gathered}
\mathrm{dpt}(c) = \mathrm{dpt}(a_{2p+2}) + \mathrm{dpt}(b_{2p+2}) + \varepsilon_{1,c}\,, \\
\mathrm{dpt}(c\Omega) = \mathrm{dpt}(a_{2p+2}) + \mathrm{dpt}(b_{2p+2}) + k + \varepsilon_{2,c}\,,
\end{gathered}$$

where $\varepsilon_{1,c}$ and $\varepsilon_{2,c}$ are as follows. If $a_{2p+2} \neq 1$ and $b_{2p+2} \neq 1$, then:

$$\varepsilon_{1,c} = \begin{cases} -1 & \text{if } (x_{2p+2}, y_{2p+2}, y'_{2p+2}) \in \{(s,s,s),(s,t,s),(t,s,s)\}\,, \\ 0 & \text{if } (x_{2p+2}, y_{2p+2}, y'_{2p+2}) \in \{(s,s,t),(s,t,t),(t,s,t),(t,t,s)\}\,, \\ 1 & \text{if } (x_{2p+2}, y_{2p+2}, y'_{2p+2}) = (t,t,t)\,, \end{cases}$$
$$\varepsilon_{2,c} = \begin{cases} -1 & \text{if } (x_{2p+2}, y_{2p+2}) \in \{(s,s),(s,t),(t,s)\}\,, \\ 0 & \text{if } (x_{2p+2}, y_{2p+2}) = (t,t)\,. \end{cases}$$

If $a_{2p+2} \neq 1$ and $b_{2p+2} = 1$, then:

$$\varepsilon_{1,c} = 0\,,\ \varepsilon_{2,c} = \begin{cases} -1 & \text{if } x_{2p+2} = s\,, \\ 0 & \text{if } x_{2p+2} = t\,. \end{cases}$$

If $a_{2p+2} = 1$ and $b_{2p+2} \neq 1$, then:

$$\varepsilon_{1,c} = \begin{cases} -1 & \text{if } (y_{2p+2}, y'_{2p+2}) = (s,s)\,, \\ 0 & \text{if } (y_{2p+2}, y'_{2p+2}) \in \{(s,t),(t,s)\}\,, \\ 1 & \text{if } (y_{2p+2}, y'_{2p+2}) = (t,t)\,, \end{cases} \quad \varepsilon_{2,c} = \begin{cases} -1 & \text{if } y_{2p+2} = s\,, \\ 0 & \text{if } y_{2p+2} = t\,. \end{cases}$$

If $a_{2p+2} = 1$ and $b_{2p+2} = 1$, then $\varepsilon_{1,c} = \varepsilon_{2,c} = 0$.

Finally, from Lemma 6.7 and Lemma 6.8 follows that

$$\sum_{i=1}^{2p+1} (\operatorname{dpt}(a_i) + \operatorname{dpt}(b_i)) = p(2k-1) + k + u + v + \varepsilon_d\,,$$

where $\varepsilon_d = -1$ if $x'_{2p+1} = s$, and $\varepsilon_d = 0$ if $x'_{2p+1} = t$.

Part (1): Let $\varepsilon = \varepsilon_{1,a} + \varepsilon_{1,b} - \varepsilon_{2,c} + \varepsilon_d$. By the above we have $\operatorname{dpt}(c\Omega) = \operatorname{dpt}(a) + \operatorname{dpt}(b) - p(2k-1) - \varepsilon$, where ε is as follows. If $a_{2p+2} \neq 1$ and $b_{2p+2} \neq 1$, then $\varepsilon = 0$ if $(x'_{2p+1}, x_{2p+2}, y_{2p+2}) \in \{(s,s,s),(t,t,t)\}$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} \neq 1$ and $b_{2p+2} = 1$, then $\varepsilon = 0$ if $(x'_{2p+1}, x_{2p+2}) = (t,t)$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} = 1$ and $b_{2p+2} \neq 1$, then $\varepsilon = 0$ if $(x'_{2p+1}, y_{2p+2}) = (t,t)$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} = 1$ and $b_{2p+2} = 1$, then $\varepsilon = 0$ if $x'_{2p+1} = t$, and $\varepsilon = 1$ otherwise.

Suppose that $a \in \Theta$. Then a is written $a = \theta^q$ with $q \geq 1$. On the other hand we write $b = t^r b'$ where $b' \neq 1$ (since $b \notin \bar{\Theta}$) and $\sigma(b') = s$. We necessarily have $r = 2p+1 < 2q$, $a_{2p+2} = \theta^{q-p-1}(st)^k$, $b_{2p+2} = b'$, and $(x'_{2p+1}, x_{2p+2}, y_{2p+2}) = (t,t,s)$, hence $\varepsilon = 1$. The case $b \in \Theta$ is proved in a similar way.

Part (2): Let $\varepsilon = 1 + \varepsilon_{2,a} + \varepsilon_{2,b} - \varepsilon_{1,c} + \varepsilon_d$. By the above we have $\operatorname{dpt}(c) = \operatorname{dpt}(a\Omega) + \operatorname{dpt}(\varphi(b)) - (p+1)(2k-1) - \varepsilon$, and ε is as follows. If $a_{2p+2} \neq 1$ and $b_{2p+2} \neq 1$, then $\varepsilon = 0$ if $(x'_{2p+1}, x_{2p+2}, y_{2p+2}) \in \{(s,s,s),(t,t,t)\}$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} \neq 1$ and $b_{2p+2} = 1$, then $\varepsilon = 0$ if $(x'_{2p+1}, x_{2p+2}) = (s,s)$, and

$\varepsilon = 1$ otherwise. If $a_{2p+2} = 1$ and $b_{2p+2} \neq 1$, then $\varepsilon = 0$ if $(x'_{2p+1}, y_{2p+2}) = (t, t)$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} = 1$ and $b_{2p+2} = 1$, then $\varepsilon = 1$.

Suppose that $a\Omega \in \Theta$. Then $a\Omega$ is written $a\Omega = \theta^q t$ with $q \geq 1$, hence $a = \theta^{q-1}(st)^k$. On the other hand we write $b = s^r b'$ where $b' \neq 1$ (since $\varphi(b) \notin \bar{\Theta}$) and $\sigma(b') = t$. If $r < 2(q-1)+1$, then $r = 2p+1$, $a_{2p+2} = \theta^{q-1-p} \neq 1$, $b_{2p+2} = b' \neq 1$, and $(x'_{2p+1}, x_{2p+2}, y_{2p+2}) = (s, s, t)$, hence $\varepsilon = 1$. If $r \geq 2(q-1)+1$, then $a_{2p+2} = 1$, $b_{2p+2} \neq 1$ and $x'_{2p+1} = s$, hence $\varepsilon = 1$.

Suppose that $\varphi(b) \in \Theta$. Then $\varphi(b)$ is written $\varphi(b) = \theta^q$ with $q \geq 1$, hence $b = ((ts)^k(st)^k)^q$. On the other hand we write $a = a's^r$ where either $a' = 1$ or $\tau(a') = t$. We necessarily have $r = 2p+1 < 2q$, $a_{2p+2} = a'$ and $b_{2p+2} = ((ts)^k(st)^k)^{q-p-1}(ts)^k$, hence $x'_{2p+1} = s$ and $x_{2p+2} = t$ if $a' \neq 1$, and therefore $\varepsilon = 1$.

Suppose that $c \in M_1$. If $a_{2p+2} \neq 1$ and $b_{2p+2} \neq 1$, then $x_{2p+2} = t$ and $y_{2p+2} = s$, hence $\varepsilon = 1$. If $a_{2p+2} \neq 1$ and $b_{2p+2} = 1$, then $x_{2p+2} = t$, hence $\varepsilon = 1$. If $a_{2p+2} = 1$ and $b_{2p+2} \neq 1$, then $y_{2p+2} = s$, hence $\varepsilon = 1$. If $a_{2p+2} = 1$ and $b_{2p+2} = 1$, then $\varepsilon = 1$.

Part (3): Let $\varepsilon = 1 + \varepsilon_{1,a} + \varepsilon_{3,b} - \varepsilon_{1,c} + \varepsilon_d$. By the above we have $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b\Omega) - (p+1)(2k-1) - \varepsilon$, and ε is as follows. If $a_{2p+2} \neq 1$ and $b_{2p+2} \neq 1$, then $\varepsilon = 0$ if $(x'_{2p+1}, x_{2p+2}, y_{2p+2}) \in \{(s,s,s), (t,t,t)\}$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} \neq 1$ and $b_{2p+2} = 1$, then $\varepsilon = 0$ if $(x'_{2p+1}, x_{2p+2}) = (s, s)$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} = 1$ and $b_{2p+2} \neq 1$, then $\varepsilon = 0$ if $(x'_{2p+1}, y_{2p+2}) = (t, t)$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} = 1$ and $b_{2p+2} = 1$, then $\varepsilon = 1$.

Suppose that $a \in \Theta$. Then a is written $a = \theta^q$ with $q \geq 1$. On the other hand we write $b = t^r b'$ where either $b' = 1$ or $\sigma(b') = s$. We necessarily have $r = 2p+1 < 2q$, hence $a_{2p+2} = \theta^{q-p-1}(st)^k$ and $b_{2p+2} = b'$. If $b' \neq 1$, then $(x'_{2p+1}, x_{2p+2}, y_{2p+2}) = (t, t, s)$, hence $\varepsilon = 1$. If $b' = 1$, then $(x'_{2p+1}, x_{2p+2}) = (t, t)$, hence $\varepsilon = 1$.

Suppose that $b\Omega \in \Theta$. Then $b\Omega$ is written $b\Omega = \theta^q t$ with $q \geq 1$, hence $b = \theta^{q-1}(st)^k$. On the other hand we write $a = a't^r$ where $a' \neq 1$ (since $a \notin \bar{\Theta}$) and $\tau(a') = s$. If $r \geq 2q-1$, then $p = q-1$, $a_{2p+2} = a't^{r-2p-1}$ and $b_{2p+2} = 1$, hence $x'_{2p+1} = t$, and therefore $\varepsilon = 1$. If $r < 2q-1$, then $r = 2p+1$, $a_{2p+2} = a'$ and $b_{2p+2} = \theta^{q-p-1} \neq 1$, hence $(x'_{2p+1}, x_{2p+2}, y_{2p+2}) = (t, s, s)$, and therefore $\varepsilon = 1$.

Suppose that $c \in M_1$. If $b_{2p+2} \neq 1$ and $a_{2p+2} \neq 1$, then $x_{2p+2} = t$ and $y_{2p+2} = s$, hence $\varepsilon = 1$. If $a_{2p+2} \neq 1$ and $b_{2p+2} = 1$, then $x_{2p+2} = t$, hence $\varepsilon = 1$. If $a_{2p+2} = 1$ and $b_{2p+2} \neq 1$, then $y_{2p+2} = s$, hence $\varepsilon = 1$. If $a_{2p+2} = 1$ and $b_{2p+2} = 1$, then $\varepsilon = 1$.

Part (4): Let $\varepsilon = 1 + \varepsilon_{2,a} + \varepsilon_{4,b} - \varepsilon_{2,c} + \varepsilon_d$. By the above we have $\mathrm{dpt}(c\Omega) = \mathrm{dpt}(a\Omega) + \mathrm{dpt}(\varphi(b)\Omega) - (p+1)(2k-1) - \varepsilon$, and ε is as follows. If $a_{2p+2} \neq 1$ and $b_{2p+2} \neq 1$, then: $\varepsilon = 0$ if $(x'_{2p+1}, x_{2p+2}, y_{2p+2}) \in \{(s,s,s),(t,t,t)\}$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} \neq 1$ and $b_{2p+2} = 1$, then: $\varepsilon = 0$ if $(x'_{2p+1}, x_{2p+2}) = (t,t)$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} = 1$ and $b_{2p+2} \neq 1$, then: $\varepsilon = 0$ if $(x'_{2p+1}, y_{2p+2}) = (t,t)$, and $\varepsilon = 1$ otherwise. If $a_{2p+2} = 1$ and $b_{2p+2} = 1$, then $\varepsilon = 0$ if $x'_{2p+1} = t$, and $\varepsilon = 1$ otherwise.

Suppose that $a\Omega \in \Theta$. Then $a\Omega$ is written $a\Omega = \theta^q t$ with $q \geq 1$, hence $a = \theta^{q-1}(st)^k$. On the other hand we write $b = s^r b'$ where either $b' = 1$ or $\sigma(b') = t$. If $r \geq 2q-1$, then $a_{2p+2} = 1$ and $x'_{2p+1} = s$, hence $\varepsilon = 1$. If $r < 2q-1$, then $r = 2p+1$, $a_{2p+2} = \theta^{q-p-1}$ and $b_{2p+2} = b'$, hence $x'_{2p+1} = s$, $x_{2p+2} = s$ and either $b_{2p+2} = 1$ or $y_{2p+2} = t$, and therefore $\varepsilon = 1$.

Suppose that $\varphi(b)\Omega \in \Theta$. Then $\varphi(b)\Omega$ is written $\varphi(b)\Omega = \theta^q t$ with $q \geq 1$, hence $b = ((ts)^k(st)^k)^{q-1}(ts)^k$. On the other hand we write $a = a's^r$ where either $a' = 1$ or $\tau(a') = t$. If $r \geq 2q-1$, then $b_{2p+2} = 1$ and $x'_{2p+1} = s$, hence $\varepsilon = 1$. If $r < 2q-1$, then $r = 2p+1$, $a_{2p+2} = a'$ and $b_{2p+2} = ((ts)^k(st)^k)^{q-p-1}$, hence $x'_{2p+1} = s$ and $y_{2p+2} = t$, and therefore $\varepsilon = 1$. □

Now, the second part of Theorem 6.1 is a direct consequence of the previous two lemmas.

Proposition 6.11 *The pair (H, G_1) satisfies Condition B with constant $\zeta = 2k-1$.*

Proof We take two unmovable elements $a, b \in M$, and we consider the Δ-form $ab = c\Delta^p$ of ab. We should prove that there exists $\varepsilon \in \{0,1\}$ such that $\mathrm{dpt}(c) = \mathrm{dpt}(a) + \mathrm{dpt}(b) - p(2k-1) - \varepsilon$, and $\varepsilon = 1$ if either $a \in \Theta$ or $b \in \Theta$ or $c \in M_1$. Clearly, there exist two Ω-unmovable elements $a', b' \in M$ such that $(a,b) \in \{(a',b'),(a'\Omega,\varphi(b')),(a',b'\Omega),(a'\Omega,\varphi(b')\Omega)\}$. Let $a'b' = d\Delta^q$ be the Δ-form of $a'b'$. Then, again, there exists an Ω-unmovable element $c' \in M$ such that $d \in \{c', c'\Omega\}$. Suppose that $d = c'$. Then: $c = c'$ and $p = q$ if $(a,b) = (a',b')$, $c = c'\Omega$ and $p = q$ if either $(a,b) = (a'\Omega,\varphi(b'))$ or $(a,b) = (a',b'\Omega)$, and $c = c'$ and $p = q+1$ if $(a,b) = (a'\Omega,\varphi(b')\Omega)$. These four cases are covered by Lemma 6.9. Suppose that $d = c'\Omega$. Then: $c = c'\Omega$ and $p = q$ if $(a,b) = (a',b')$, $c = c'$ and $p = q+1$ if either $(a,b) = (a'\Omega,\varphi(b'))$ or $(a,b) = (a',b'\Omega)$, and $c = c'\Omega$ and $p = q+1$ if $(a,b) = (a'\Omega,\varphi(b')\Omega)$. These four cases are covered by Lemma 6.10. □

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