Explicit formulas for $C^{1,1}$ Glaeser-Whitney extensions of 1-fields in Hilbert spaces

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Abstract. We give a simple alternative proof for the $C^{1,1}$–convex extension problem which has been introduced and studied by D. Azagra and C. Mudarra [2]. As an application, we obtain an easy constructive proof for the Glaeser-Whitney problem of $C^{1,1}$ extensions on a Hilbert space. In both cases we provide explicit formulae for the extensions. For the Glaeser-Whitney problem the obtained extension is almost minimal, that is, minimal up to a multiplicative factor in the sense of Le Gruyer [15].

Key words. Whitney extension problem, convex extension, sup-inf convolution, semiconvex function.

AMS Subject Classification *Primary* 54C20 ; *Secondary* 52A41, 26B05, 26B25, 58C25.

1 Introduction

Determining a function (or a class of functions) of a certain regularity fitting to a prescribed set of data is one of the most challenging problems in modern mathematics. The origin of this problem is very old, since this general framework encompasses classical problems of applied analysis. Depending on the requested regularity, it goes from the Tietze extension theorem in normal topological spaces, where the required regularity is minimal (continuity), to results where the requested regularity is progressively increasing: McShane results on uniformly continuous, Hölder or Lipschitz extensions [19], Lipschitz extensions for vector-valued functions (Valentine [20]), differentiable and C^k-extensions (Whitney [22], Glaeser [12], and more recently Brudnyi-Shvartsman [7], Zobin [23], Fefferman [9]), monotone multivalued extensions (Bauschke-Wang [5]), definable (in some o-minimal structure) Lipschitz extensions (Aschenbrenner-Fischer [1]), etc. In this work we are interested in the Glaeser-Whitney $C^{1,1}$-extension problem, which we describe below.

Let S be a nonempty subset of a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, |\cdot|)$ and assume $\alpha : S \to \mathbb{R}$ and $v : S \to \mathcal{H}$ satisfy the so-called Glaeser-Whitney conditions:

$$\begin{cases} \sup_{s_1,s_2\in S, s_1\neq s_2} \dfrac{|\alpha(s_2)-\alpha(s_1)-\langle v(s_1), s_2-s_1\rangle|}{|s_1-s_2|^2} := K_1 < +\infty, \\ \\ \sup_{s_1,s_2\in S, s_1\neq s_2} \dfrac{|v(s_1)-v(s_2)|}{|s_1-s_2|} := K_2 < +\infty. \end{cases} \tag{1.1}$$

In [12, 22] it has been shown that under the above conditions, in case $\mathcal{H} = \mathbb{R}^n$, there exists a $C^{1,1}$-smooth function $F : \mathbb{R}^n \to \mathbb{R}$ such that the prescribed 1-Taylor field $(\alpha(s), v(s))$ coincides, at every $s \in S$, with the 1-Taylor field $(F(s), \nabla F(s))$ of F. The above result has been extended to Hilbert spaces in Wells [21] and Le Gruyer [15]. In particular, in [15] the following constant has been introduced:

$$\Gamma^1(S,(\alpha,v)) := \sup_{s_1,s_2\in S, s_1\neq s_2} \left(\sqrt{A_{s_1s_2}^2 + B_{s_1s_2}^2} + |A_{s_1s_2}|\right), \tag{1.2}$$

where

$$A_{s_1s_2} = \frac{2(\alpha(s_1)-\alpha(s_2)) + \langle v(s_1)+v(s_2), s_2-s_1\rangle}{|s_1-s_2|^2}, \quad B_{s_1s_2} = \frac{v(s_1)-v(s_2)}{|s_1-s_2|}.$$

It has been shown in [15] that $\Gamma^1(S,(\alpha,v)) < +\infty$ if and only if conditions (1.1) hold. Moreover, in this case, the existence of a $C^{1,1}$ function $F : \mathcal{H} \to \mathbb{R}$ such that $F|_S = \alpha$, $\nabla F|_S = v$ and

$$\Gamma^1(\mathcal{H},(F,\nabla F)) = \Gamma^1(S,(\alpha,v)), \tag{1.3}$$

has been established. Henceforth, every $C^{1,1}$-extension of (α, v) satisfying (1.3) will be called a minimal Glaeser-Whitney extension. The terminology is justified by the fact that, for every $C^{1,1}$ function $G : \mathcal{H} \to \mathbb{R}$, we have $\Gamma^1(\mathcal{H},(G,\nabla G)) = \mathrm{Lip}(\nabla G)$ (see [15, Proposition 2.4]). Thus $\mathrm{Lip}(\nabla F) \leq \mathrm{Lip}(\nabla G)$ for any $C^{1,1}$-extension G of the prescribed 1-Taylor field $(\alpha(s), v(s))$. If for some universal constant $K \geq 1$ (not depending on the data) we have $\Gamma^1(\mathcal{H},(G,\nabla G)) \leq K\,\Gamma^1(S,(\alpha,v))$ then the extension G will be called almost minimal.

Recently, several authors have been interested in extensions that are subject to additional constraints: extensions which preserve positivity [10, 11] or convexity [2, 3]. In [2], D. Azagra and C. Mudarra considered the problem of finding a *convex* $C^{1,1}$-smooth extention over a prescribed Taylor polynomial $(\alpha(s), v(s))_{s\in S}$ in a Hilbert space $\mathcal{H}$ and established that the condition

$$\alpha(s_2) \geq \alpha(s_1) + \langle v(s_1), s_2 - s_1\rangle + \frac{1}{2M}|v(s_1) - v(s_2)|^2, \quad \forall s_1, s_2 \in S, \tag{1.4}$$

is necessary and sufficient for the existence of such extension.

Inspired by the recent work [2] concerning $C^{1,1}$-convex extensions, we revisit the classical Glaeser-Whitney problem. We first provide an alternative shorter proof of the result of [2] concerning $C^{1,1}$-convex extensions in Hilbert spaces by giving a simple explicit formula. This formula is heavily based on the regularization via sup-inf convolution in the spirit of Lasry-Lions [14] and can be efficiently computed, see Remark 2.2. As an easy consequence, we obtain a direct proof for the classical $C^{1,1}$-Glaeser-Whitney problem in Hilbert spaces, which goes together with an explicit formula of the same type as for the convex extension problem. Let us mention that the previous proofs are quite involved both in finite dimension [12, 22] and Hilbert spaces [15, 21]. In the finite dimensional case, a construction of the extension is proposed in [21] and some explicit formulae can be found in [16] but both are not tractable (see however the work [13] for concrete computations). Our approach also compares favorably to the result of [15], in which the existence of minimal extensions is established. On the other hand, the extension given by our explicit formula may fail to be minimal —though it is almost minimal up to a universal multiplicative factor.

Before we proceed, we recall that a function $f : \mathcal{H} \to \mathbb{R}$ is called C_*-semiconvex (resp., C^*-semiconcave) when, for all $x, y \in H$,

$$f(y) - f(x) - \langle\nabla f(x), y - x\rangle \geq -\frac{C_*}{2}|x - y|^2 \quad (\text{resp.}, \leq \frac{C^*}{2}|x - y|^2).$$

This is equivalent to assert that $f + \frac{C_*}{2}|x|^2$ is convex (respectively $f - \frac{C^*}{2}|x|^2$ is concave). When f is both C-semiconvex and C-semiconcave, then f is $C^{1,1}$ in $\mathcal{H}$ with $\mathrm{Lip}(\nabla f) \leq C$ (for a proof of this latter result in finite dimension, see [8] and use the arguments of [14] to extend the result to Hilbert spaces).

2 Convex $C^{1,1}$ extension of 1-fields

For any $f : \mathcal{H} \to \mathbb{R}$ and $\varepsilon > 0$, we define respectively the sup and the inf-convolution of f by

$$f^\varepsilon(x) = \sup_{y\in\mathcal{H}} \left\{ f(y) - \frac{|y - x|^2}{2\varepsilon} \right\}, \quad f_\varepsilon(x) = \inf_{y\in\mathcal{H}} \left\{ f(y) + \frac{|y - x|^2}{2\varepsilon} \right\}.$$

Theorem 2.1 ($\mathrm{C}^{1,1}$-convex extension)**.** *Let S be any nonempty subset of the Hilbert space $\mathcal{H}$ and $(\alpha(s), v(s))_{s\in S}$ be a 1-Taylor field on S satisfying* (1.4) *for some constant $M > 0$. Then*

$$f(x) = \sup_{s\in S}\{\alpha(s) + \langle v(s), x - s\rangle\} \tag{2.1}$$

is the smallest continuous convex extension of (α, v) in $\mathcal{H}$ and

$$F(x) = \lim_{\varepsilon\nearrow\frac{1}{M}}(f^{\varepsilon})_{\varepsilon}(x) = \lim_{\varepsilon\nearrow\frac{1}{M}}\inf_{z\in\mathcal{H}}\sup_{y\in\mathcal{H}}\{f(y) - \frac{|y-z|^2}{2\varepsilon} + \frac{|z-x|^2}{2\varepsilon}\} \tag{2.2}$$

is a $C^{1,1}$ convex extension of (α, v) in $\mathcal{H}$. Moreover, $\mathrm{Lip}(\nabla F) \leq M$*.*

Remark 2.2. (i) The function f given by (2.1) is the smallest convex continuous extension of (α, v) in the following sense: if g is a continuous convex function in $\mathcal{H}$, differentiable on S, satisfying $g(s) = \alpha(s)$ and $\nabla g(s) = v(s)$, for all $s \in S$, then $f \leq g$.
(ii) As we shall see in the forthcoming proof, $\varepsilon \mapsto (f^{\varepsilon})_{\varepsilon}$ is nondecreasing. Therefore, "$\lim_{\varepsilon\nearrow\frac{1}{M}}$" can be replaced by "$\sup_{\varepsilon\in(0,\frac{1}{M})}$" in formula (2.2).
(iii) The inf-convolution corresponds to the well-known Moreau-Yosida regularization in convex analysis. It is also related to the Legendre-Fenchel transform (convex conjugate). A discussion on theoretical and practical properties of this regularization can be found in [17] and references therein. In practice, f_{ε}, f^{ε} and therefore the formula (2.2) can be very efficiently computed using different techniques and algorithms such as [6] or [18].

Proof of Theorem 2.1. For all $x \in \mathcal{H}$ and $s_1, s_2 \in S$, by (1.4), we have

$$\begin{aligned}
&\alpha(s_1) + \langle v(s_1), x - s_1\rangle \\
&\leq \alpha(s_2) + \langle v(s_2), x - s_2\rangle + \langle v(s_1) - v(s_2), x - s_2\rangle - \frac{1}{2M}|v(s_1) - v(s_2)|^2 \\
&\leq \alpha(s_2) + \langle v(s_2), x - s_2\rangle + \sup_{\xi\in\mathcal{H}}\{\langle \xi, x - s_2\rangle - \frac{1}{2M}|\xi|^2\} \\
&= \alpha(s_2) + \langle v(s_2), x - s_2\rangle + \frac{M}{2}|x - s_2|^2.
\end{aligned}$$

It follows that for all $x \in \mathcal{H}$ and $s \in S$

$$\alpha(s) + \langle v(s), x - s\rangle \leq f(x) \leq \alpha(s) + \langle v(s), x - s\rangle + \frac{M}{2}|x - s|^2. \tag{2.3}$$

In particular, the function f defined by (2.1) is convex, finite in $\mathcal{H}$ and trapped between affine hyperplanes and quadratics with equality on S.

Therefore, it is differentiable on S with $f(s) = \alpha(s)$, $\nabla f(s) = v(s)$ and it is clearly the smallest continuous convex extension of the field.

Setting $q(x) = \alpha(s) + \langle v(s), x-s\rangle + \frac{M}{2}|x-s|^2$, for $\varepsilon \in (0, M^{-1})$, straightforward computations lead to the formulae:

$$q^\varepsilon(x) = \alpha(s) + \frac{1}{1-\varepsilon M}\left(\frac{M}{2}|x-s|^2 + \langle v(s), x-s\rangle + \frac{\varepsilon}{2}|v(s)|^2\right), \tag{2.4}$$
$$q_\varepsilon(x) = \alpha(s) + \frac{1}{1+\varepsilon M}\left(\frac{M}{2}|x-s|^2 + \langle v(s), x-s\rangle - \frac{\varepsilon}{2}|v(s)|^2\right).$$

In particular, after a new short computation, we deduce that

$$(q^\varepsilon)_\varepsilon = q, \tag{2.5}$$

and from (2.3), since the sup and inf-convolution are order-preserving operators, we obtain that for every $\varepsilon \in (0, M^{-1})$, $x \in \mathcal{H}$ and $s \in S$,

$$\alpha(s) + \langle v(s), x-s\rangle \le (f^\varepsilon)_\varepsilon(x) \le \alpha(s) + \langle v(s), x-s\rangle + \frac{M}{2}|x-s|^2. \tag{2.6}$$

It follows that $(f^\varepsilon)_\varepsilon$ is well-defined on $\mathcal{H}$. Notice also that

$$f \le (f^\varepsilon)_\varepsilon \qquad \text{in } \mathcal{H} \tag{2.7}$$

and that $(f^\varepsilon)_\varepsilon$ is differentiable on S with $(f^\varepsilon)_\varepsilon(s) = \alpha(s)$ and $\nabla(f^\varepsilon)_\varepsilon(s) = v(s)$, for every $s \in S$.

Notice that since f is defined as the supremum of the affine functions $\ell_s(x) = \alpha(s) + \langle v(s), x-s\rangle$ and $\ell_s^\varepsilon(x) = \ell_s(x) + \frac{\varepsilon}{2}|v(s)|^2$ by (2.4), we have

$$f^\varepsilon(x) = \sup_{s\in S}\{\ell_s^\varepsilon(x)\},$$

which proves that f^ε is convex. Therefore, $(f^\varepsilon)_\varepsilon$ is still convex, being the infimum with respect to y of the jointly convex functions

$$f^\varepsilon(y) + \frac{1}{2\varepsilon}|y-x|^2, \quad (x,y) \in \mathcal{H}\times\mathcal{H}.$$

It is well-known [14] that the sup and inf-convolution satisfy some semigroup properties,

$$f^{\varepsilon+\varepsilon'} = (f^\varepsilon)^{\varepsilon'} \text{ and } f_{\varepsilon+\varepsilon'} = (f_\varepsilon)_{\varepsilon'} \text{ for all } \varepsilon, \varepsilon' > 0.$$

Therefore, for $0 < \varepsilon < \varepsilon'$, $f^{\varepsilon'} = (f^\varepsilon)^{\varepsilon'-\varepsilon}$. By (2.7), we infer $((f^\varepsilon)^{\varepsilon'-\varepsilon})_{\varepsilon'-\varepsilon} \geq f^\varepsilon$. It follows

$$((f^{\varepsilon'})_{\varepsilon'-\varepsilon})_\varepsilon = (f^{\varepsilon'})_{\varepsilon'} \geq (f^\varepsilon)_\varepsilon \quad \text{for all } 0 < \varepsilon < \varepsilon'.$$

We conclude that $\varepsilon \mapsto (f^\varepsilon)_\varepsilon$ is nondecreasing on $(0, M^{-1})$ so F is well defined, convex and still satisfies (2.6). Therefore F is an extension of $(\alpha(s), v(s))_{s\in S}$ in $\mathcal{H}$ and is differentiable on S.

It remains to prove that F is $C^{1,1}$ in $\mathcal{H}$ and to estimate $\mathrm{Lip}(\nabla F)$. From [14], we know that the inf-convolution $(f^\varepsilon)_\varepsilon$ of f^ε is ε^{-1}-semiconcave. Since $(f^\varepsilon)_\varepsilon$ is also convex, it means that $(f^\varepsilon)_\varepsilon$ is both ε^{-1}-semiconcave and ε^{-1}-semiconvex. Therefore $(f^\varepsilon)_\varepsilon$ is $C^{1,1}$ in $\mathcal{H}$ with $\mathrm{Lip}(\nabla(f^\varepsilon)_\varepsilon) \leq \varepsilon^{-1}$. Since $(f^\varepsilon)_\varepsilon - \frac{1}{2\varepsilon}|x|^2$ is concave for every $0 < \varepsilon < M^{-1}$, sending $\varepsilon \nearrow M^{-1}$, we conclude that F is M-semiconcave. Since F is also convex, the previous arguments allow to conclude that F is $C^{1,1}$ in $\mathcal{H}$ with $\mathrm{Lip}(\nabla F) \leq M$. $\square$

Remark 2.3. In [14], the $C^{1,1}$ regularization result is stated for $(f^\varepsilon)_\delta$ with $0 < \delta < \varepsilon$. To obtain an extension in our framework, we need to take $\delta = \varepsilon$. The fact that we have been able to increase the value of δ and take it equal to ε without losing the $C^{1,1}$ regularity relies strongly on the convexity of f. Since convexity is preserved under the sup and inf-convolution operations, the inf-convolution does not affect the semiconvexity property of f^ε even for $\delta = \varepsilon$. For the same reason, one cannot reverse the above operations: more precisely, the function $(f_\varepsilon)^\varepsilon = f$ would not be semiconcave.

3 $C^{1,1}$ extension of 1-fields: explicit formulae

Let us now apply the previous result to obtain a general $C^{1,1}$-extension in the Glaeser-Whitney problem.

Theorem 3.1 ($C^{1,1}$-Glaeser-Whitney almost minimal extension)**.** *Let S be a nonempty subset of a Hilbert space $\mathcal{H}$ and $(\alpha(s), v(s))_{s\in S}$ be a 1-Taylor field on S satisfying* (1.1)*. Then, the function*

$$G(x) = F(x) - \frac{1}{2}\overline{\mu}|x|^2$$

is an explicit $C^{1,1}$ extension of the 1-Taylor field (α, v), provided that F is the convex extension of the 1-Taylor field $(\tilde{\alpha}, \tilde{v})$ where, for all $s \in S$,

$$\tilde{\alpha}(s) := \alpha(s) + \frac{1}{2}\overline{\mu}|s|^2, \quad \tilde{v}(s) := v(s) + \overline{\mu}s$$

and

$$\overline{\mu} := 2K_1 + K_2 + \sqrt{(2K_1+K_2)^2 + K_2^2}, \quad K_1, K_2 \textit{ given by } (1.1).$$

Moreover, the extension G is almost minimal, i.e.,

$$\Gamma^1(S,(\alpha,v)) \le \Gamma^1(\mathcal{H},(G,\nabla G)) = \mathrm{Lip}(\nabla G) \le \left(\frac{5+\sqrt{29}}{2}\right)\Gamma^1(S,(\alpha,v)).$$

Proof of Theorem 3.1. We check that for every $\mu > 2K_1$ the 1-Taylor field

$$(\tilde{\alpha}(s),\tilde{v}(s)) := (\alpha(s) + \frac{\mu}{2}|s|^2, v(s) + \mu s)$$

satisfies (1.4) with $M = (\mu+K_2)^2(\mu-2K_1)^{-1}$. Indeed, for any $s_1, s_2 \in S$ we obtain, using (1.1),

$$\begin{aligned}
& \tilde{\alpha}(s_2) - \tilde{\alpha}(s_1) - \langle \tilde{v}(s_1), s_2 - s_1\rangle \\
= \;& \alpha(s_2) - \alpha(s_1) - \langle v(s_1), s_2 - s_1\rangle + \frac{\mu}{2}\,\big(|s_2|^2 - |s_1|^2 - 2\langle s_1, s_2 - s_1\rangle\big) \\
\ge \;& \left(\frac{\mu - 2K_1}{2}\right)\,|s_1 - s_2|^2 \ge \frac{1}{2}\Big(\frac{\mu-2K_1}{(\mu+K_2)^2}\Big)\,|\tilde{v}(s_1) - \tilde{v}(s_2)|^2,
\end{aligned}$$

since $\mathrm{Lip}(\tilde{v}) \le \mathrm{Lip}(v) + \mu = K_2 + \mu$. Thus, the function F given by Theorem 2.1 is a $C^{1,1}$-convex extension of $(\tilde{\alpha}(s),\tilde{v}(s))$ satisfying $F|_S = \tilde{\alpha}$, $\nabla F|_S = \tilde{v}$ and $\mathrm{Lip}(\nabla F) \le (\mu+K_2)^2(\mu-2K_1)^{-1}$. Therefore $G(x) = F(x) - \frac{\mu}{2}|x|^2$ satisfies $G|_S = \alpha$, $\nabla G|_S = v$.

Moreover, G is $\left(\frac{(\mu+K_2)^2}{\mu-2K_1} - \mu\right)$-semiconcave and μ-semiconvex (since F is convex). We deduce

$$\mathrm{Lip}(\nabla G) \le \max\left\{\mu, \frac{(\mu+K_2)^2}{\mu-2K_1} - \mu\right\}.$$

Minimizing the above quantity on $\mu \in (2K_1, +\infty)$ yields

$$\begin{aligned}
\mathrm{Lip}(\nabla G) \;&\le\; \min_{\mu\in(2K_1,\infty)} \max\left\{\mu, \frac{(\mu+K_2)^2}{\mu-2K_1} - \mu\right\} \\
&=\; \overline{\mu} := 2K_1 + K_2 + \sqrt{(2K_1+K_2)^2 + K_2^2}.
\end{aligned}$$

By Lemma 5.3 (Appendix), we have $\max\{K_2, 4K_1 - 2K_2\} \le \Gamma^1(S,(\alpha,v))$. It follows that $\mathrm{Lip}(\nabla G) \le \left(\frac{5+\sqrt{29}}{2}\right)\Gamma^1(S,(\alpha,v))$. By [15, Proposition 2.4], we have $\mathrm{Lip}(\nabla G) = \Gamma^1(\mathcal{H},(G,\nabla G))$. The result follows. $\square$

4 Limitations of the sup-inf approach

The main result (Theorem 3.1) is heavily based on the explicit construction of a $C^{1,1}$-convex extension of a 1-Taylor field (α, v) satisfying (1.4), which in turn, is based on the sup-inf convolution approach. The reader might wonder whether our approach can be adapted to include cases where less regularity is required, as for instance $C^{1,\theta}$-extensions, that is, extensions to a C^1-function whose derivative has a Hölder modulus of continuity with exponent $\theta \in (0,1)$. The existence of such *convex* extensions (and even $C^{1,\omega}$ convex extensions with a general modulus of continuity ω) was established in finite dimensions in Azagra-Mudarra [3] by means of involved arguments. Indeed, it would be natural to endeavor an adaptation of formula (2.2) to treat the problem of $C^{1,\theta}$-convex extensions, for $0 < \theta < 1$. According to [3], the adequate condition, analogous to (1.4), is that the 1-Taylor field has to satisfy, for some $M > 0$,

$$\alpha(s_2) \geq \alpha(s_1) + \langle v(s_1), s_2 - s_1 \rangle + \frac{\theta}{(1+\theta)M^{1/\theta}} |v(s_1) - v(s_2)|^{1+\frac{1}{\theta}}. \quad (4.1)$$

Unfortunately, the technique developed in Section 2 is specific to the $C^{1,1}$-regularity and cannot be easily adapted to this more general case. Let us briefly explain the reason.

Considering the suitable sup and inf-convolutions

$$f^{\varepsilon}(x) = \sup_{y \in \mathcal{H}} \left\{ f(y) - \frac{|y-x|^{1+\theta}}{(1+\theta)\varepsilon^{\theta}} \right\}, \quad f_{\varepsilon}(x) = \inf_{y \in \mathcal{H}} \left\{ f(y) + \frac{|y-x|^{1+\theta}}{(1+\theta)\varepsilon^{\theta}} \right\},$$

all of the arguments of the proof of Theorem 2.1 go through except (2.5), which fails to hold in this general case. More precisely, the convex extension f defined by (2.1) satisfies

$$l(x) \leq f(x) \leq q(x) \qquad \text{for all } x \in \mathcal{H} \text{ and } s \in S, \quad (4.2)$$

with equalities for $x = s$, where

$$l(x) := \alpha(s) + \langle v(s), x - s \rangle \quad (4.3)$$

$$q(x) := \alpha(s) + \langle v(s), x - s \rangle + \frac{M}{1+\theta} |x - s|^{1+\theta}. \quad (4.4)$$

Therefore for every $\varepsilon > 0$ such that $M\varepsilon^{\theta} < 1$, we have

$$l(x) \leq (f^{\varepsilon})_{\varepsilon}(x) \leq (q^{\varepsilon})_{\varepsilon}(x)\,.$$

Nonetheless, we may now possibly have

$$(q^\varepsilon)_\varepsilon(s) \neq q(s), \tag{4.5}$$

yielding that $(f^\varepsilon)_\varepsilon$ is a $C^{1,\theta}$-convex function but may differ from f on S, hence it is not an extension of the latter. Let us underline that the problem arises even in dimension 1 and even for small ε. In particular, the sup-convolution q^ε may develop singularities for arbitrary small ε so that q^ε is not anymore in the same class as q, contrary to the quadratic case (see (2.4)).

Remark 4.1. Recalling [14] that $u(x,t) := q^t(x)$ is a viscosity solution to the Hamilton-Jacobi equation $\partial_t u - \frac{\theta}{1+\theta}|\nabla u|^{1+\frac{1}{\theta}} = 0$ in $\mathcal{H} \times (0,\varepsilon)$, we obtain an explicit example where the solutions develop singularities instantaneously, even when starting with a $C^{1,\theta}$ initial condition $u(x,0) = q(x)$. See [4] for related comments.

Sketch of proof of the Claim (4.5). Without loss of generality we may assume that $\alpha(s) = 0$ and $s = 0$ in (4.3)–(4.4). Fix $v \neq 0$. Assume by contradiction that $(q^\varepsilon)_\varepsilon(0) = q(0) = l(0) = 0$. Then, since q is a $C^{1,\theta}$ function, necessarily, $\nabla (q^\varepsilon)_\varepsilon(0) = \nabla l(0) = v$. Using that $y \mapsto q^\varepsilon(y) + \frac{|y|^{1+\theta}}{(1+\theta)\varepsilon^\theta}$ is a strictly convex function achieving a unique minimum $\overline{y}$ in $\mathcal{H}$, we obtain that

$$\begin{aligned}
(q^\varepsilon)_\varepsilon(0) &= q^\varepsilon(\overline{y}) + \frac{|\overline{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} = \sup_{y\in\mathcal{H}} \left\{ q(y) - \frac{|y-\overline{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} \right\} + \frac{|\overline{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} = 0, \\
\nabla (q^\varepsilon)_\varepsilon(0) &= -\frac{\overline{y}|\overline{y}|^{\theta-1}}{\varepsilon^\theta} = v,
\end{aligned}$$

yielding $\overline{y} = -\varepsilon v |v|^{\frac{1}{\theta}-1} \neq 0$. To prove the claim, it is enough to find some $y \in \mathcal{H}$ such that

$$\varphi(y) := q(y) - \frac{|y-\overline{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} + \frac{|\overline{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} > 0.$$

In particular, let us seek for $y = \lambda \overline{y}$ where $\lambda \in \mathbb{R}$ is small. (Notice that this guarantees that the computation would also hold when $\mathcal{H}$ is one dimensional.) We have

$$\begin{aligned}
(q^\varepsilon)_\varepsilon(0) \quad &\geq \quad \varphi(y) = \frac{|\overline{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} \left(M\varepsilon^\theta |\lambda|^{1+\theta} - (1+\theta)\lambda - |\lambda - 1|^{1+\theta} + 1 \right) \\
&= \quad \frac{|\overline{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} \left(M\varepsilon^\theta |\lambda|^{1+\theta} - \frac{1}{2}(1+\theta)\theta\lambda^2 + o(\lambda^2) \right) > 0 = q(0),
\end{aligned}$$

at least for small $\lambda > 0$. □

5 Appendix

5.1 Independence of the Glaeser-Whitney constants in (1.1)

It is worth-noticing that there is no link between K_1 and K_2 in (1.1). Each of these constants can be 0 while the other one can be very large, as it is shown in the following examples.

Example 5.1. Let $\mathcal{H} = \mathbb{R}$, $S = \{0, 1\}$ and $\alpha(0) = A > 0$, $v(0) = 0$, $\alpha(1) = 0$, $v(1) = 0$. Then it follows that $K_1 = A$ and $K_2 = 0$.

Example 5.2. Let $\mathcal{H} = \mathbb{R}^2$, $S = \{s_1, s_2\}$ with $s_1 = (-1, 0)$ and $s_2 = (1, 0)$. Set $\alpha(s_1) = \alpha(s_2) = 0$, $v(s_1) = (0, -A)$ and $v(s_2) = (0, A)$, for $A > 0$. Since $\alpha(s_1) - \alpha(s_2) = 0$ and $s_1 - s_2 \perp v(s_i)$, $i = 1, 2$, we have $K_1 = 0$. Obviously $K_2 = A$.

5.2 Inequality estimations between $\Gamma^1(S, (\alpha, v))$, K_1 and K_2.

The following result has been used in the last part of the proof of Theorem 3.1

Lemma 5.3. *Let* $\{(\alpha(s), v(s))\}_{s \in S}$ *be a* 1*-Taylor field satisfying the Glaeser-Whitney conditions* (1.1)*. Then we have:*
(i) $K_2 \le \Gamma^1(S, (\alpha, v))$*;*
(ii) $4K_1 - 2K_2 \le \Gamma^1(S, (\alpha, v))$.

Proof of Lemma 5.3. Recalling (1.2), we have

$$\Gamma^1(S, (\alpha, v)) \ge \sup_{s_1 \neq s_2} |B_{s_1 s_2}| = K_2,$$

which proves (i).

To establish (ii), we set

$$K_1 = \sup_{s_1 \neq s_2} \frac{|k_1^{s_1 s_2}|}{|s_1 - s_2|^2} \quad \text{with} \quad k_1^{s_1 s_2} = \alpha(s_2) - \alpha(s_1) - \langle v(s_1), s_2 - s_1 \rangle,$$

and we deduce that

$$\begin{aligned} |-A_{s_1 s_2}| &= \left| \frac{2k_1^{s_1 s_2}}{|s_1 - s_2|^2} + \frac{\langle v(s_1) - v(s_2), s_2 - s_1 \rangle}{|s_1 - s_2|^2} \right| \\ &\ge \frac{2|k_1^{s_1 s_2}|}{|s_1 - s_2|^2} - \frac{|v(s_1) - v(s_2)|}{|s_1 - s_2|} \\ &\ge \frac{2|k_1^{s_1 s_2}|}{|s_1 - s_2|^2} - K_2. \end{aligned}$$

It follows

$$\Gamma^1(S,(\alpha,v)) \geq \sup_{s_1\neq s_2} 2|A_{s_1s_2}| \geq \sup_{s_1\neq s_2} \frac{4|k_1^{s_1s_2}|}{|s_1-s_2|^2} - 2K_2 = 4K_1 - 2K_2.$$

This completes the proof. □

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