The Matrix Approach for Abstract Argumentation Framework

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Abstract. Matrices and the operation of dual interchange are introduced into the study of Dung's argumentation frameworks. It is showed that every argumentation framework can be represented by a matrix, and the basic extensions (such as admissible, stable, complete) can be determined by sub-blocks of its matrix. In particular, an efficient approach for determining the basic extensions has been developed using two types of standard matrix. Furthermore, we develop the topic of matrix reduction along two different lines. The first one enables to reduce the matrix into a less order matrix playing the same role for the determination of extensions.
The second one enables to decompose an extension into several extensions of different sub-argumentation frameworks. It makes us not only solve the problem of determining grounded and preferred extensions, but also obtain results about dynamics of argumentation frameworks.

Keywords: matrix, argumentation, extension, reduction, dynamics

1 Introduction

In recent years, the area of argumentation begins to become increasingly central as a core study within Artificial Intelligence. A number of papers investigated and compared the properties of different semantics which have been proposed for abstract argumentation frameworks [1–4, 7, 13, 14, 20, 22, 23].

Directed graphs have been widely used for modeling and analyzing argumentation frameworks (AFs for short) because of the feature of visualization [3, 10, 12, 14]. Furthermore, the labeling and game approach developed by Modgil and Caminada [7, 8, 18, 19] respectively are two excellent methods for the proof theories and algorithms of AFs. In this paper, we propose another novel idea, that is, the matrix representation of AFs.

Our aim is to introduce matrices and the operation of dual interchange into the study of AFs so as to propose new efficient approaches for determining basic extensions. First, we assign a matrix of order n for each AF with n arguments.This representation enables to establish links between extensions (under various semantics) of the AF and the internal structure of the matrix, namely sub-blocks of the matrix. Moreover, the matrix of an AF can be turned into a standard form, from which the determination of admissible and complete extensions can be easily achieved through checking some sub-blocks of this standard form. Furthermore, we propose the reduced matrix *wrt* conflict-free subsets, by which the determination of various extensions becomes more efficient. This approach has not been mentioned in the literature as we know. Finally, we present the reduced matrix *wrt* extensions and give the decomposition theory for extensions. It can be used to handle the semantics based on minimality and maximality criteria, for example, to determine the preferred extensions. It can also be related to the topic of directionality and enables us to obtain results about dynamics of AFs, which improve main results by Liao and Koons [17].

The paper is organized as follows. Section 2 recalls the basic definitions on abstract AFs. Section 3 introduces the matrix representation of AFs and the operation of dual interchange of matrices. Section 4 describes the characterization theorems for stable, admissible and complete extensions. Furthermore, we integrate these theorems and obtain two kinds of standard forms for matrices by dual interchanges. Section 5 presents the matrix reductions of AFs based on contraction and division of AFs, and some applications in AFs and dynamics of AFs. The proofs can be found in [11].

2 Background on Abstract AFs

In this section, we mainly recall the basic notions of abstract AFs [13, 20].

Definition 1 *An abstract AF is a pair $AF = (A, R)$, where A is a finite set of arguments and $R \subseteq A \times A$ represents the attack relation. For any $S \subseteq A$, we say that S is conflict-free if there are no $a, b \in S$ such that $(a, b) \in R$; $a \in A$ is attacked by S if there is some $b \in S$ such that $(b, a) \in R$; $a \in A$ attacks S if there is some $b \in S$ such that $(a, b) \in R$; $a \in A$ is defended by (or acceptable wrt) S if for each $b \in A$ with $(b, a) \in R$, we have that b is attacked by S.*

We use the following notations inspired from graph theory. Let $AF = (A, R)$ be an AF and $S \subseteq A$. $R^+(S)$ denotes the set of arguments attacked by S. $R^-(S)$ denotes the set of arguments attacking S. I_{AF} denotes the set of arguments which are not attacked (also called initial arguments of AF).

An argumentation semantics is the formal definition of a method ruling the argument evaluation process. Two main styles of semantics can be identified in the literature: extension-based and labelling-based. Here, we only recall the common extension-based semantics of AF.

Definition 2 *Let $AF = (A, R)$ be an AF and $S \subseteq A$.*

- *S is a stable extension of AF if S is conflict-free and each $a \in A \setminus S$ is attacked by S.*
- *S is admissible in AF if S is conflict-free and each $a \in S$ is defended by S. Let $a(AF)$ denote the set of admissible subsets in AF.*

– *S is a preferred extension of AF if $S \in a(AF)$ and S is a maximal element (wrt set-inclusion) of $a(AF)$.*
– *S is a complete extension of AF if $S \in a(AF)$ and for each $a \in A$ defended by S, we have $a \in S$.*
– *S is a grounded extension of AF if S is the least (wrt set-inclusion) complete extension of AF.*

The common extension-based semantics can be characterized in terms of subsets of attacked/attacking arguments, due to the following results:

Proposition 1 *Let $AF = (A, R)$ be an AF and S a subset of A.*

– *S is conflict-free if and only if (iff for short) $S \cap R^+(S) = \emptyset$ (or equivalently $R^+(S) \subseteq A \setminus S$)*
– *S is stable iff $R^+(S) = A \setminus S$*
– *S is admissible iff $R^-(S) \subseteq R^+(S) \subseteq A \setminus S$*

Definition 3 ([23]) *Let $AF = (A, R)$ be an AF, S a subset of A. The restriction of AF to S, denoted by $AF\mid_S$, is the sub-argumentation framework (sub-AF for short) $(S, R \cap (S \times S))$.*

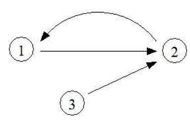
Remark 1 For any nonempty subset S of A, the set A can be divided into three disjoint parts: S, $R^+(S)$ and $A \setminus (S \cup R^+(S))$. In our discussion on division of AF, the sub-AF $AF\mid_{A \setminus (S \cup R^+(S))}$ will play an important role. We call it the remaining sub-AF *wrt* S, or remaining sub-AF for short.

3 The matrix Representation

Let $AF = (A, R)$ be an AF. It is convenient to put $A = \{1, 2, ..., n\}$ whenever the cardinality of A is large. Furthermore, we usually give the set A a permutation, for example $(i_1, i_2, ..., i_n)$, when dealing with the AF practically.

Definition 4 *Let $AF = (A, R)$ be an AF with $A = \{1, 2, ..., n\}$. The matrix of AF corresponding to the permutation $(i_1, i_2, ..., i_n)$ of A, denoted by $M(i_1, i_2, ..., i_n)$[1], is a boolean matrix of order n, its elements being determined by the following rules: (1) $a_{s,t} = 1$ iff $(i_s, i_t) \in R$ (2) $a_{s,t} = 0$ iff $(i_s, i_t) \notin R$. We usually denote the matrix $M(1, 2, ..., n)$ by $M(AF)$.*

Example 1 *Given $AF = (A, R)$ with $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 1), (3, 2)\}$, represented by the following graph:*



[1] strictly speaking, it should be denoted by $M_{AF}(i_1, i_2, ..., i_n)$

According to Definition 4, the matrices of AF corresponding to the permutations $(1,2,3)$ and $(1,3,2)$ are

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Definition 5 *Let $AF = (A, R)$ be an AF with $A = \{1, 2, ..., n\}$. A dual interchange on the matrix $M(i_1, ..., i_k, ..., i_l, ..., i_n)$ between k and l, denoted by $k \rightleftharpoons l$, consists of two interchanges: interchanging k-th row and l-th row; interchanging k-th column and l-th column.*

Lemma 1 *Let $AF = (A, R)$ be an AF with $A = \{1, 2, ..., n\}$, then $k \rightleftharpoons l$ turns the matrix $M(i_1, ..., i_k, ..., i_l, ..., i_n)$ into the matrix $M(i_1, ..., i_l, ..., i_k, ..., i_n)$.*

The dual interchange $k \rightleftharpoons l$ also turns the matrix $M(i_1, \cdots, i_l, \cdots, i_k, \cdots, i_n)$ into the matrix $M(i_1, ..., i_k, ..., i_l, ..., i_n)$. So, for any two matrices of an AF corresponding to different permutations of A we can turn one matrix into another by a sequence of dual interchanges. In this sense, we may call them to be equivalent matrix representations of the AF.

Example 1 (cont'd) By the dual interchange $1 \rightleftharpoons 2$, we can turn the matrix $M(1, 2, 3)$ into the matrix $M(2, 1, 3)$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \underline{1 \rightleftharpoons 2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

4 Characterizing the Extensions of an *AF*

In this section, we mainly focus on the characterization of various extensions in the matrix $M(AF)$. The idea is to establish the relation between the extensions (viewed as subsets) of $AF = (A, R)$ and the sub-blocks of $M(AF)$.

4.1 Characterizing the Conflict-Free Subsets

The basic requirement for extensions is conflict-freeness. So, we will discuss the matrix condition which insures that a subset of an AF is conflict-free.

Definition 6 *Let $AF = (A, R)$ be an AF with $A = \{1, 2, ..., n\}$, and $S = \{i_1, i_2, ..., i_k\} \subseteq A$. The $k \times k$ sub-block*

$$M^{i_1, i_2, ..., i_k}_{i_1, i_2, ..., i_k} = \begin{pmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_k} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i_k,i_1} & a_{i_k,i_2} & \cdots & a_{i_k,i_k} \end{pmatrix}$$

of $M(AF)$ is called the cf-sub-block of S, and denoted by $M^{cf}(S)$ for short.

Theorem 1 *Given $AF = (A, R)$ with $A = \{1, 2, ..., n\}$, $S = \{i_1, i_2, ..., i_k\} \subseteq A$ is conflict-free iff the cf-sub-block $M^{cf}(S)$ is zero.*

Example 1 (cont'd) $M^{cf}(\{1,3\}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $M^{cf}(\{1,2\}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $M^{cf}(\{2,3\}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. By Theorem 1, $\{1,3\}$ is conflict free, $\{1,2\}$ and $\{2,3\}$ are not.

4.2 Characterizing the Stable Extensions

As shown in Section 2, a subset S of A is stable iff $R^+(S) = A \setminus S$. So, except for the conflict-freeness of S, we only need to concentrate on whether the arguments in $A \setminus S$ are attacked by S. This suggests the following definition:

Definition 7 *Let $AF = (A, R)$ be an AF with $A = \{1, 2, ..., n\}$, $S = \{i_1, i_2, ..., i_k\} \subseteq A$ and $A \setminus S = \{j_1, j_2, ..., j_h\}$. The $k \times h$ sub-block*

$$M^{i_1,i_2,...,i_k}_{j_1,j_2,...,j_h} = \begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_h} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_h} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i_k,j_1} & a_{i_k,j_2} & \cdots & a_{i_k,j_h} \end{pmatrix}$$

of $M(AF)$ is called the s-sub-block of S and denoted by $M^s(S)$ for short.

In other words, we take the elements at the rows $i_1, i_2, ..., i_k$ and the columns $j_1, j_2, ..., j_h$ in the matrix $M(AF)$. For any matrix or its sub-block, the i-th row is called the i-th row vector and denoted by $M_{i,*}$, the j-th column is called j-th column vector and denoted by $M_{*,j}$.

Theorem 2 *Given $AF = (A, R)$ with $A = \{1, 2, ..., n\}$. A conflict-free subset $S = \{i_1, i_2, ..., i_k\} \subseteq A$ is a stable extension iff each column vector of the s-sub-block $M^s(S) = M^{i_1,i_2,...,i_k}_{j_1,j_2,...,j_h}$ of $M(AF)$ is non-zero, where $(j_1, j_2, ..., j_h)$ is a permutation of $A \setminus S$.*

Example 1 (cont'd) We consider the conflict-free subsets $\{1\}$ and $\{1,3\}$. Since the second column vector of $M^s(\{1\}) = \begin{pmatrix} 1 & 0 \end{pmatrix}$ is zero and the only column vector of $M^s(\{1,3\}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is non-zero, we claim that $\{1,3\}$ is a stable extension of AF but $\{1\}$ is not, according to Theorem 2.

4.3 Characterizing the Admissible Subsets

As shown in Section 2, a subset S of A is admissible if and only if $R^-(S) \subseteq R^+(S) \subseteq A \setminus S$. There may be arguments in $A \setminus S$ which are not attacked by S. Such arguments should not attack S. This suggests to explore the representation in $M(AF)$ of the relation between $R^-(S)$ and $R^+(S)$.

Definition 8 *Let $AF = (A, R)$ be an AF with $A = \{1, 2, ..., n\}$, $S = \{i_1, i_2, ..., i_k\} \subseteq A$ and $A \setminus S = \{j_1, j_2, ..., j_h\}$. The $h \times k$ sub-block*

$$M^{j_1,j_2,...,j_h}_{i_1,i_2,...,i_k} = \begin{pmatrix} a_{j_1,i_1} & a_{j_1,i_2} & \cdots & a_{j_1,i_k} \\ a_{j_2,i_1} & a_{j_2,i_2} & \cdots & a_{j_2,i_k} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j_h,i_1} & a_{j_h,i_2} & \cdots & a_{j_h,i_k} \end{pmatrix}$$

of $M(AF)$ is called the a-sub-block of S and denoted by $M^a(S)$.

In other words, we take the elements at the rows $j_1, j_2, ..., j_h$ and the columns $i_1, i_2, ..., i_k$ in the matrix $M(AF)$.

Theorem 3 *Given $AF = (A, R)$ with $A = \{1, 2, ..., n\}$. A conflict-free subset $S = \{i_1, i_2, ..., i_k\} \subseteq A$ is admissible iff any column vector of the s-sub-block $M^s(S)$ corresponding to a non-zero row vector of the a-sub-block $M^a(S)$ is non-zero, where $(j_1, j_2, ..., j_h)$ is a permutation of $A \setminus S$.*

Example 1 (cont'd) We consider the conflict-free subsets $\{1\}$ and $\{2\}$. Since $M^s(\{1\}) = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $M^a(\{1\}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the column vector $M^s_{*,1}$ of $M^s(\{1\})$ corresponding to the non-zero row vector $M^a_{1,*}$ of $M^a(\{1\})$ is non-zero, we claim that $\{1\}$ is admissible in AF by Theorem 3.
However, from $M^s(\{2\}) = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $M^a(\{2\}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we know that the column vector $M^s_{*,2}$ of $M^s(\{2\})$ corresponding to the non-zero row vector $M^a_{2,*}$ of $M^a(\{2\})$ is zero. So, $\{2\}$ is not admissible in AF according to Theorem 3.

4.4 Characterizing the Complete Extensions

From the viewpoint of set theory, every complete extension S separates A into three disjoint parts: S, $R^+(S)$ and $A \setminus (S \cup R^+(S))$. Except for the conflict-freeness of S, we need not only to consider whether S is attacked by the arguments in $A \setminus (S \cup R^+(S))$, but also to see if every argument in $A \setminus (S \cup R^+(S))$ is attacked by some others in $A \setminus (S \cup R^+(S))$. This suggests the following definition.

Definition 9 *Let $AF = (A, R)$ be an AF with $A = \{1, 2, ..., n\}$, $S = \{i_1, i_2, ..., i_k\} \subseteq A$ and $A \setminus S = \{j_1, j_2, ..., j_h\}$. The $h \times h$ sub-block*

$$M^{j_1,j_2,...,j_h}_{j_1,j_2,...,j_h} = \begin{pmatrix} a_{j_1,j_1} & a_{j_1,j_2} & \cdots & a_{j_1,j_h} \\ a_{j_2,j_1} & a_{j_2,j_2} & \cdots & a_{j_2,j_h} \\ \cdot & \cdot & \cdots & \cdot \\ a_{j_h,j_1} & a_{j_h,j_2} & \cdots & a_{j_h,j_h} \end{pmatrix}$$

of $M(AF)$ is called the c-sub-block of S and denoted by $M^c(S)$ for short.

In other words, we take the elements at the rows $j_1, j_2, ..., j_h$ and the columns $j_1, j_2, ..., j_h$ in the matrix $M(AF)$.

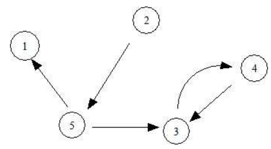
Theorem 4 *Given $AF = (A, R)$ with $A = \{1, 2, ..., n\}$. An admissible extension $S = \{i_1, i_2, ..., i_k\} \subseteq A$ is complete iff*

(1) if some column vector $M^s_{*,p}$ of the s-sub-block $M^s(S)$ is zero, then its corresponding column vector $M^c_{*,p}$ of the c-sub-block $M^c(S)$ is non-zero and

(2) for each non-zero column vector $M^c_{*,p}$ of the c-sub-block $M^c(S)$ appearing in (1), there is at least one non-zero element a_{j_q,j_p} of $M^c_{*,p}$ such that the corresponding column vector $M^s_{*,q}$ of the s-sub-block $M^s(S)$ is zero, where $\{j_1, j_2, ..., j_h\} = A \setminus S$ and $1 \leq q, p \leq h$.

Example 2 *Let $AF = (A, R)$ with $A = \{1, 2, 3, 4, 5\}$ and $R = \{(2, 5), (3, 4), (4, 3), (5, 1), (5, 3)\}$. The matrix and graph of AF are as follows:*

$$M(AF) = \begin{pmatrix} 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1 \\ 0\,0\,0\,1\,0 \\ 0\,0\,1\,0\,0 \\ 1\,0\,1\,0\,0 \end{pmatrix}$$



By Theorem 3, we have that $S = \{1, 2\}$ is admissible. Let $i_1 = 1, i_2 = 2, j_1 = 3, j_2 = 4, j_3 = 5$. Note that $M^s(\{1, 2\}) = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,1 \end{pmatrix}$ has two zero column vectors $M^s_{*,1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $M^s_{*,2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Their corresponding column vectors in $M^c(\{1, 2\}) = \begin{pmatrix} 0\,1\,0 \\ 1\,0\,0 \\ 1\,0\,0 \end{pmatrix}$ are $M^c_{*,1} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $M^c_{*,2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ respectively, which are all non-zero. For $a_{j_2 j_1} = a_{43} = 1$ in $M^c_{*,1}$, the corresponding column vector $M^s_{*,2}$ in $M^s(\{1, 2\})$ is zero. For $a_{j_1 j_2} = a_{34} = 1$ in $M^c_{*,2}$, the corresponding column vector $M^s_{*,1}$ in $M^s(\{1, 2\})$ is also zero. According to Theorem 4, we claim that $\{1, 2\}$ is a complete extension of AF.

By now, we can determine three basic extensions by checking the sub-blocks of the matrix $M(AF)$. Note that in each theorem the rules are obtained directly from the corresponding definition of extensions. So, there is no more advantage than judging by definitions. In the next subsection, we will improve the rules to achieve some standard form by which one can determine the extensions easily.

4.5 The Standard Forms of the Matrix $M(AF)$

In linear algebra, one can reduce the matrix of a system of linear equations into row echelon form by row transformations in order to find the solution easily. Similarly, we will use dual interchanges to reduce the matrix of AFs into standard forms, by which the extensions discussed above can be easily determined. In the sequel, two standard forms are introduced *wrt* different semantics.

Theorem 5 *Given $AF = (A, R)$ with $A = \{1, 2, ..., n\}$, $S = \{i_1, i_2, ..., i_k\} \subseteq A$ and $A \setminus S = \{j_1, ..., j_h\}$. By a sequence of dual interchanges $M(AF)$ can be turned into the matrix $M(i_1, i_2, ..., i_k, j_1, j_2, ..., j_h)$, which has the following form*

$$\begin{pmatrix} M^{cf}(S) & M^{s}(S) \\ M^{a}(S) & M^{c}(S) \end{pmatrix},$$

where $M^{cf}(S), M^{s}(S), M^{a}(S)$, $M^{c}(S)$ *are the* cf*- sub-block, s-sub-block, a-sub-block, c-sub-block of* S *respectively.*

Corollary 1 *Given* $AF = (A, R)$ *with* $A = \{1, 2, ..., n\}$, $S = \{i_1, i_2, ..., i_k\}$, $A \setminus S = \{j_1, ..., j_h\}$. *Let* $M(i_1, i_2, ..., i_k, j_1, ..., j_h)$ *be the matrix of* AF *corresponding to the permutation* $(i_1, i_2, ..., i_k, j_1, ..., j_h)$, *as in Theorem 5.*

1. S *is conflict-free iff the* cf*-sub-block* $M^{cf}(S) = 0$
2. S *is stable iff the* cf*-sub-block* $M^{cf}(S) = 0$ *and every column vector of the s-sub-block* $M^{s}(S)$ *is non-zero.*

Example 1 (cont'd) $S = \{1, 3\}$ is a conflict-free subset of AF. By the dual interchange $2 \rightleftarrows 3$, $M(AF)$ can be turned into the following matrix:

$$M(1,3,2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since $M^{s}(S) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\{1, 3\}$ is a stable extension of AF by Corollary 1.

We have obtained a partition matrix of order two, composed by four kinds of sub-blocks, from which we can determine the conflict-free status and stable status of S. However, there is no new information about the admissible and complete status of S. We can go further since, for any conflict-free subset S, A can be divided into three disjoint subsets: S, $R^{+}(S)$ and $A \setminus (S \cup R^{+}(S))$. So we obtain a new partition of order three.

Theorem 6 *Given* $AF = (A, R)$ *with* $A = \{1, 2, ..., n\}$ *and* $S = \{i_1, i_2, ..., i_k\} \subseteq A$ *a conflict-free subset. By a sequence of dual interchanges* $M(AF)$ *can be turned into the matrix* $M(i_1, i_2, ..., i_k, j_{t_1}, ..., j_{t_q}, j_{s_1}, ..., j_{s_l})$

$$= \begin{pmatrix} 0_{k,k} & 0_{k,q} & S_{k,l} \\ A_{q,k} & C_{q,q} & E_{q,l} \\ F_{l,k} & G_{l,q} & H_{l,l} \end{pmatrix} = \begin{pmatrix} 0_{k,k} & M^{s}(S) \\ M^{a}(S) & M^{c}(S) \end{pmatrix}$$

where $A \setminus S = \{j_{t_1}, ..., j_{t_q}, j_{s_1}, ..., j_{s_l}\}$, $k + q + l = k + h = n$, *and each column vector of* $S_{k,l}$ *is non-zero.*

Corollary 2 *Given* $AF = (A, R)$ *with* $A = \{1, 2, ..., n\}$, $S = \{i_1, i_2, ..., i_k\}$, $A \setminus S = \{j_{t_1}, ..., j_{t_q}, j_{s_1}, ..., j_{s_l}\}$. *Let* $M(i_1, i_2, ..., i_k, j_{t_1}, ..., j_{t_q}, j_{s_1}, ..., j_{s_l})$ *be the matrix of* AF *corresponding to the permutation* $(i_1, i_2, ..., i_k, j_{t_1}, ..., j_{t_q}, j_{s_1}, ..., j_{s_l})$ *as in Theorem 6.*

1. S is an admissible extension iff $A_{q,k} = 0$
2. S is complete iff $A_{q,k} = 0$ and each column vector of $C_{q,q}$ is not zero.

Example 1 (cont'd) $S = \{1\}$ is conflict-free. By the dual interchange $2 \rightleftarrows 3$, $M(AF)$ can be turned into the following matrix:

$$M(1,3,2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that here $i_1 = 1, j_{t_1} = 3$ and $j_{s_1} = 2$ with $k = 1, q = 1, l = 1$. Since $S_{k,l} = S_{1,1} = \begin{pmatrix} 1 \end{pmatrix}$, $A_{q,k} = A_{1,1} = \begin{pmatrix} 0 \end{pmatrix}$, we claim that $\{1\}$ is an admissible extension of AF according to the first item of Corollary 2.

Example 2 (cont'd) $S = \{1, 2\}$ is conflict-free. Note that $M(AF)$ has already the standard form we need for S. Here, $i_1 = 1, i_2 = 2, j_{t_1} = 3, j_{t_2} = 4$ and $j_{s_1} = 5$ with $k = 2, q = 2, l = 1$. Because $S_{k,l} = S_{2,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $A_{q,k} = A_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $C_{q,q} = C_{2,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we conclude that $\{1, 2\}$ is a complete extension of AF according to the second item of Corollary 2.

5 Matrix Reduction

For some purposes or under some conditions, we can simplify the AFs and their matrices. In this section, we will mainly discuss the matrix reduction *wrt* conflict-free subsets and *wrt* some extensions. Related results can be applied to the computation of various extensions and to the dynamics of AFs.

5.1 Matrix Reduction Based on Contraction of AFs

In Section 4, we proposed to characterize the stable (admissible, complete) extensions of an AF by dividing A into two or three parts, and then considering the interaction between these different parts. This suggests to contract one part of an AF (namely a conflict-free subset) into a single argument by drawing up some rules. And thus, the matrix can be reduced into another matrix of less order which plays the same role for our purpose.

Definition 10 *Let $M(AF)$ be the matrix of an AF. The addition of two rows of the matrix $M(AF)$ consists in adding the elements in the same position of the rows, with the rules $0+0 = 0, 0+1 = 1, 1+1 = 1$. The addition of two columns of the matrix $M(AF)$ is similar as the addition of two rows.*

For a conflict-free subset $S = \{i_1, i_2, ..., i_k\}$, we try to contract the sub-block $M^{cf}(S)$ into a single entry in the matrix and make this entry share the same status as $M^{cf}(S)$ *wrt* extension-based semantics. The matrix $M(AF)$ can be reduced into another matrix $M^r_S(AF)$ of order $n - k + 1$ by the following rules: Let $1 \leq t \leq k$. For each s such that $1 \leq s \leq k$ and $s \neq t$,

1. adding row i_s to the row i_t,
2. adding column i_s to the column i_t, then
3. deleting row i_s and column i_s.

The matrix $M^r_S(AF)$ is called the reduced matrix *wrt* the conflict-free subset S, or the reduced matrix *wrt* S for short.

Correspondingly, the original AF can be reduced into a new one with $n-k+1$ arguments by the following rules:
Let $A \setminus S = \{j_1, j_2, ..., j_h\}$ and $1 \leq t \leq k$. For each s such that $1 \leq s \leq k$ and $s \neq t$, and each q such that $1 \leq q \leq h$,

1. adding (i_t, j_q) to R if $(i_s, j_q) \in R$,
2. adding (j_q, i_t) to R if $(j_q, i_s) \in R$, then
3. deleting all (i_s, j_q) and (j_q, i_s) from R.

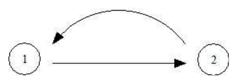
Let R^r_S denote the new relation and $A^r_S = \{i_t\} \cup (A \setminus S)$, then (A^r_S, R^r_S) is a new AF called the reduced AF *wrt* S. Obviously, the reduced matrix $M^r_S(AF)$ is exactly the matrix of (A^r_S, R^r_S).

Theorem 7 *Given AF* $= (A, R)$ *with* $A = \{1, 2, ..., n\}$. *Let* $S = \{i_1, i_2, ..., i_k\} \subseteq A$ *be conflict-free and* $1 \leq t \leq k$. *Then* S *is stable (resp. admissible, complete, preferred) in AF iff* $\{i_t\}$ *is stable (respectively admissible, complete, preferred) in the reduced AF* (A^r_S, R^r_S).

Example 1 (cont'd) Since $S = \{1, 3\}$ is conflict-free, $M(AF)$ can be turned into the following reduced matrix according to the above rules (S is contracted into $\{1\}$):

$$M^r_S(AF) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding reduced AF is (A^r_S, R^r_S) where $A^r_S = \{1, 2\}$ and $R^r_S = \{(1, 2), (2, 1)\}$. The graph of (A^r_S, R^r_S) is as follows:



Note that $\{1\}$ is stable in (A^r_S, R^r_S), and $S = \{1, 3\}$ is stable in AF.

Furthermore, we can extend the above idea to two disjoint conflict-free subsets and turn the matrix of AF into a reduced matrix of less order.

Let $S_1 = \{i_1, i_2, ..., i_k\}$ and $S_2 = \{j_1, j_2, ..., j_h\}$ be two conflict-free subsets of A such that $S_1 \cap S_2 = \emptyset$. We try to contract the sub-block $M^{cf}(S_1)$ and $M^{cf}(S_2)$ into two entries in the matrix and make them share the same status as $M^{cf}(S_1)$ and $M^{cf}(S_2)$ *wrt* extension-based semantics. The matrix $M(AF)$ can be reduced into another matrix $M^r_{S_1,S_2}(AF)$ of order $n - k - h + 2$ by the following rules:
Let $1 \leq t \leq k$ and $1 \leq s \leq h$. For each p such that $1 \leq p \leq k$ and $p \neq t$, and each q such that $1 \leq q \leq h$ and $q \neq s$,

1. for S_1, adding row i_p to the row i_t, adding column i_p to the column i_t,
2. for S_2, adding row j_q to the row j_s, adding column j_q to the column j_s, then
3. deleting row i_p and column i_p,
4. deleting row j_q and column j_q.

The matrix $M^r_{S_1,S_2}(AF)$ is called the reduced matrix *wrt* the disjoint conflict-free subsets S_1 and S_2, or the reduced matrix *wrt* (S_1, S_2) for short.
Correspondingly, the original AF can be reduced into a new one with $n-k-h+2$ arguments by the following rules:
Let $1 \le t \le k$ and $1 \le s \le h$. For each p such that $1 \le p \le k$ and $p \ne t$, each q such that $1 \le q \le h$ and $q \ne s$, each $i \in A \setminus S_1$, and each $j \in A \setminus S_2$,

1. adding (i_t, i) to R if $(i_p, i) \in R$, adding (i, i_t) to R if $(i, i_p) \in R$,
2. adding (j_s, j) to R if $(j_q, j) \in R$, adding (j, j_s) to R if $(j, j_q) \in R$,
3. deleting all (i_p, i) and (i, i_p) from R,
4. deleting all (j_q, j) and (j, j_q) from R.

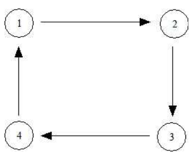
Let $R^r_{S_1,S_2}$ denote the new relation and $A^r_{S_1,S_2} = \{i_t, j_s\} \cup (A \setminus (S_1 \cup S_2))$, then $(A^r_{S_1,S_2}, R^r_{S_1,S_2})$ is a new AF called the reduced AF *wrt* (S_1, S_2). Obviously, $M^r_{S_1,S_2}(AF)$ is exactly the matrix of $(A^r_{S_1,S_2}, R^r_{S_1,S_2})$.

Theorem 8 *Given* $AF = (A, R)$ *with* $A = \{1, 2, ..., n\}$. *Let* $S_1 = \{i_1, i_2, ..., i_k\}$ *and* $S_2 = \{j_1, j_2, ..., j_h\}$ *be two conflict-free subsets of AF such that* $S_1 \cap S_2 = \emptyset$. *Let* $1 \le t \le k$ *and* $1 \le s \le h$, *then*

- S_1 *is stable (respectively admissible, complete, preferred) in AF if and only if* $\{i_t\}$ *is stable (respectively admissible, complete, preferred) in* $(A^r_{S_1,S_2}, R^r_{S_1,S_2})$,
- S_2 *is stable (respectively admissible, complete, preferred) in AF if and only if* $\{j_s\}$ *is stable (respectively admissible, complete, preferred) in* $(A^r_{S_1,S_2}, R^r_{S_1,S_2})$.

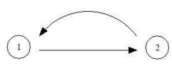
Example 3 *Let* $AF = (A, R)$ *with* $A = \{1, 2, 3, 4\}$ *and* $R = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. *The matrix and graph of AF are as follows.*

$$M(AF) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$



Since $S_1 = \{1, 3\}$ *and* $S_2 = \{2, 4\}$ *are two disjoint conflict-free subsets of* AF, $M(AF)$ *can be turned into the following reduced matrix according to the above rules(*S_1 *is contracted into* $\{1\}$ *and* S_2 *is contracted into* $\{2\}$*):*

$$M^r_{S_1,S_2}(AF) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



Obviously, $\{1\}$ *and* $\{2\}$ *are stable in* $(A^r_{S_1,S_2}, R^r_{S_1,S_2})$. *By Theorem 8,* $S_1 = \{1, 3\}$ *and* $S_2 = \{2, 4\}$ *are stable in AF.*

Theorem 7 and Theorem 8 make it more efficient for us to determine whether a conflict-free subset is one of the basic extensions.

5.2 Matrix Reduction Based on Division of AFs

The division of AFs into sub-AFs has already been considered [17] for handling dynamics of AFs. Indeed many other issues in AFs can be dealt with by the division of AFs. For example, the grounded extension can be viewed as the union of two subsets I_{AF} and E: I_{AF} consists of the initial arguments of AF and E is the grounded extension of the remaining sub-AF $AF\mid_B$ *wrt* I_{AF} (where $B = A \setminus (I_{AF} \cup R^+(I_{AF})))$.

According to the maximality criterion, a preferred extension coincides with an admissible extension E from which the associated remaining sub-AF $AF\mid_C$ (where $C = A \setminus (E \cup R^+(E)))$ has no nonempty admissible extension.

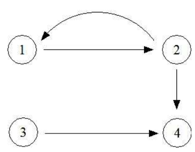
Building Grounded and Preferred Extensions Let S be an admissible extension of $AF = (A, R)$, and AF_1 be the remaining sub-AF *wrt* S. The basic extensions of AF_1 can be determined by applying the theorems obtained in Section 4. So, the matrix $M(AF_1)$ becomes the main object of our concentration. We call it the reduced matrix *wrt* the extension S.

For each extension T of AF_1, the matrix $M(AF)$ can be turned into a standard form *wrt* $S \cup T$ by a sequence of dual interchanges. Based on the results obtained in Section 4, we have the following theorem.

Theorem 9 *Let $AF = (A, R)$, $S \subseteq A$ be an admissible extension of AF, and $B = A \setminus (S \cup R^+(S))$. If $T \subseteq B$ is an admissible (resp. stable, complete, preferred) extension of the remaining sub-AF $AF\mid_B$ wrt S, then $S \cup T$ is an admissible (resp. stable, complete, preferred) extension of AF.*

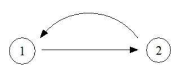
Example 4 *Let $AF = (A, R)$ with $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 1), (2, 4), (3, 4)\}$. The matrix and graph of AF are as follows.*

$$M(AF) = \begin{pmatrix} 0\,1\,0\,0 \\ 1\,0\,0\,1 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \end{pmatrix}$$



$S = \{3\}$ is an admissible extension of AF, $R^+(S) = \{4\}$ and $B = A \setminus (S \cup R^+(S)) = \{1, 2\}$. So, the matrix and graph of the remaining sub-AF wrt S are as follows:

$$M(AF\mid_B) = \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix}$$



Since $T = \{2\}$ is admissible in $AF\mid_B$, by Theorem 9, we conclude that $S \cup T = \{2, 3\}$ is admissible in AF.

These combination properties of extensions can also be used for computing related extensions.

A grounded extension can be built incrementally starting from an admissible extension. If AF has no initial argument, then the grounded extension S of AF is empty. Otherwise, let I_1 be the set of initial arguments of AF, then I_1 is an admissible extension of AF. Next, we consider the sub-AF $AF\mid_{B_1}$ where $B_1 = A \setminus (I_1 \cup R^+(I_1))$. If it has no initial argument, then the grounded extension $S = I_1$. Otherwise, let I_2 be the set of initial arguments of $AF\mid_{B_1}$ and $B_2 = B_1 \setminus (I_2 \cup R^+(I_2))$. By Theorem 9, $I_1 \cup I_2$ is an admissible extension of AF. This process can be done repeatedly, until some $AF\mid_{B_t}$ has no initial argument, where $1 \leq t \leq n$. It is easy to verify that $S = I_1 \cup ... \cup I_t$ is the grounded extension of AF.

A preferred extension is defined as a maximal (*wrt* set inclusion) admissible extension. So, it can be also built incrementally starting from some admissible extension. Let S_1 be any admissible extension of AF, and $B_1 = A \setminus (S_1 \cup R^+(S_1))$. If $B_1 = \emptyset$ or the sub-AF $AF\mid_{B_1}$ does not have nonempty admissible extension, then S_1 is a preferred extension of AF. Otherwise, let S_2 be an nonempty admissible extension. Then, $S_1 \cup S_2$ is an admissible extension of AF by Theorem 9. Let $B_2 = B_1 \setminus (S_2 \cup R^+(S_2))$, then it is a sub-AF of $AF\mid_{B_1}$. This process can be done repeatedly, until some sub-AF $AF\mid_{B_s}$ has no nonempty admissible extension where $1 \leq s \leq n$. It is easy to verify that $S = S_1 \cup ... \cup S_t$ is a preferred extension of AF.

Handling Dynamics of Argumentation Frameworks In recent years, the research on dynamics of AFs has become more and more active [5, 6, 9, 10, 15, 17, 21]. In [10] Cayrol et al. introduced change operations to describe the dynamics of AFs, and systematically studied the structural properties for change operations. Based on these notions, Liao et al.[17] concentrated their attention on the directionality of AFs and constructed a division-based method for dynamics of AFs. In the following, we introduce the reduction of a matrix *wrt* an extension in an unattacked subset of the AF and give the decomposition theorem of extensions for dynamics of AFs.

Directionality is a basic principle for extension-based semantics. According to [1, 3], the following semantics have been proved to satisfy the directionality criterion: grounded semantics, complete semantics, preferred semantics and ideal semantics. Directionality is based on the unattacked subsets. So, we recall the definition of unattacked subset.

Definition 11 *Given $AF = (A, R)$, a non-empty set $U \in A$ is unattacked if and only if there is no $a \in A \setminus U$ such that a attacks U.*

Let U be an unattacked subset of $AF = (A, R)$. Let E_1 be an admissible extension in the sub-AF $AF\mid_U$, then we have a remaining sub-AF $AF\mid_T$ with $T = A \setminus (E_1 \cup R^+(E_1))$. In order to determine the extensions of $AF\mid_T$, we can apply the theorems obtained in Section 4. So, the matrix $M(AF\mid_T)$ becomes the main object of our concentration. We call it the reduced matrix *wrt* E_1. For each conflict-free subset E_2, we can turn the matrix $M(AF)$ into one of the standard forms *wrt* $E_1 \cup E_2$ by a sequence of dual interchanges. Based on the results obtained in Section 4, we derive the following theorem.

Theorem 10 *Let $AF = (A, R)$ and U an unattacked subset of AF. $E \subseteq A$ is an admissible extension of AF iff $E_1 = E \cap U$ is admissible in the sub-AF $AF\mid_U$ and $E_2 = E \cap T$ is admissible in the remaining sub-AF $AF\mid_T$ wrt E_1 (where $T = A \setminus (E_1 \cup R^+(E_1))$).*

Example 4 (cont'd) $U = \{1, 2\}$ is an unattacked subset of AF, and $E_1 = \{1\}$ is an admissible extension in the sub-AF $AF\mid_U$. Since $T = A \setminus (E_1 \cup R^+(E_1)) = \{3, 4\}$, the matrix and graph of the remaining sub-AF *wrt* E_1 are as follows:

$$M(AF\mid_T) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Obviously, $\{3\}$ is admissible in $AF\mid_T$. According to Theorem 10, $\{1, 3\}$ is admissible in AF.

Remark 4 Theorem 10 still holds for other extensions which satisfy the directionality principle. Namely, we can replace "admissible" by "complete, preferred, grounded or ideal".

Theorem 10 provides a general result for AFs. However it happens that this result plays an important role when applied to dynamics of AFs. In order to describe this application, we need to present basic notions related to dynamics of AFs. We focus on the work described in [17].

Let U_{arg} be the universe of arguments. Different kinds of change can be considered on $AF = (A, R)$. (1) adding (or deleting) a set of interactions between the arguments in A, we denote this set by $\mathcal{I}_A$. (2) adding a set $B \subseteq U_{arg} \setminus A$ of arguments, we can also add some interactions related to it, including a set of interactions between A and B and a set of interactions between the arguments in B. The union of these two sets of interactions is denoted by $\mathcal{I}_{A:B}$. (3) deleting a set $B \subseteq A$ of arguments, we will also delete all the interactions related to it, including the set of interactions between $A \setminus B$ and B and the set of interactions between the arguments in B. The union of these two sets of interactions is denoted by $\mathcal{I}_{A\setminus B:B}$. (4) after deleting the set $B \subseteq A$ of arguments, we can continue to delete some interactions between the arguments in $A \setminus B$. This set of interactions is denoted by $\mathcal{I}_{A\setminus B}$, similar as in (1).

An addition is represented by a tuple $(B, I_{A:B} \cup I_A)$ with $B \subseteq U_{arg} \setminus A$, and a deletion is represented by a tuple $(B, I_{A\setminus B:B} \cup I_{A\setminus B})$ with $B \subseteq A$.

Definition 12 ([17]) *Given $AF = (A, R)$. Let $(B, I_{A:B} \cup I_A)$ be an addition and $(B, I_{A\setminus B:B} \cup I_{A\setminus B})$ be a deletion. The updated AF wrt $(B, I_{A:B} \cup I_A)$ and $(B, I_{A\setminus B:B} \cup I_{A\setminus B}))$ is defined as follows:*

$$AF^{\oplus} = (A, R) \oplus (B, I_{A:B} \cup I_A) = (A \cup B, R \cup I_{A:B} \cup I_A)$$
$$AF^{\ominus} = (A, R) \ominus (B, I_{A\setminus B:B} \cup I_{A\setminus B}) = (A \setminus B, R \setminus (I_{A\setminus B:B} \cup I_{A\setminus B}))$$

Now, let us apply Theorem 10 to the study of dynamics of AFs. The following two corollaries can be obtained directly.

Corollary 3 *Let $AF = (A, R)$, $AF^{\oplus}$ be the updated AF wrt an addition and U an unattacked subset in $AF^{\oplus}$. If E_1 is admissible in the sub-AF $AF^{\oplus} |_U$, and E_2 is admissible in the remaining sub-AF wrt E_1, then $E_1 \cup E_2$ is admissible in $AF^{\oplus}$. Conversely, for each admissible extension E of $AF^{\oplus}$, $E_1 = E \cap U$ is admissible in $AF^{\oplus} |_U$ and $E_2 = E \cap T$ is admissible in $AF^{\oplus} |_T$.*

Corollary 4 *Let $AF = (A, R)$, $AF^{\ominus}$ be the updated AF wrt a deletion and U an unattacked subset in $AF^{\ominus}$. If E_1 is admissible in the sub-AF $AF^{\ominus} |_U$, and E_2 is admissible in the remaining sub-AF $AF^{\ominus} |_T$ wrt E_1, then $E_1 \cup E_2$ is admissible in $AF^{\ominus}$; Conversely, for each admissible extension E of $AF^{\ominus}$, $E_1 = E \cap U$ is admissible in $AF^{\ominus} |_U$ and $E_2 = E \cap T$ is admissible in $AF^{\ominus} |_T$.*

Remark 5 The above two corollaries still hold if we replace "admissible" by "complete, preferred, grounded or ideal".

Since they are based on the division of AF and the directionality principle, the above two corollaries play a similar role as the main results in [17] when applied to dynamics of AFs. The basic idea in [17] is to divide an updated AF into three parts: an unaffected, an affected, and a conditioning part. The status of arguments in the unaffected sub-framework remains unchanged, while the status of the affected arguments is computed in a special argumentation framework (called a conditioned argumentation framework) that is composed of an affected part and a conditioning part. [17] has proved that under semantics that satisfy the directionality principle the extensions of the updated framework can be obtained by combining the extensions of an unaffected subframework and the extensions of the conditioning part.

However, in our approach, the remaining sub-AF $AF^{\oplus} |_T$ (or $AF^{\ominus} |_T$) has a simpler structure (and so is easier to compute) than the conditioning subframework of [17].

6 Concluding Remarks and Future Works

The matrix approach of AFs was constructed as a new method for computing basic extensions of AFs. For any conflict-free subset S, the matrix $M(AF)$ can be turned into one of the two standard forms by a series of dual interchanges. And thus, determining whether S is an extension can be achieved by checking some sub-blocks related to S. The underlying set A of arguments can be divided into three parts: the conflict-free set S, the attacked set $R^+(S)$ and the remaining set $A \setminus (S \cup R^+(S))$. Deciding whether S is admissible only requires to check whether the remaining set $A \setminus (S \cup R^+(S))$ attacks S. In this sense, the matrix approach is a structural (or integrated) method, which is different from checking the defended status of every argument of S.

The matrix approach of AFs can be applied to find new theories of AFs. For any conflict-free subset S of an AF, the matrix $M(AF)$ can be turned into a reduced matrix *wrt* S. The reduced matrix corresponds to a new AF with less arguments obtained by contracting the conflict-free subset S into one argument. This method has not appeared in the literature as we know. Moreover, for any admissible extension E of an AF, we can turn the matrix $M(AF)$ into a reduced matrix *wrt* E. The reduced matrix wrt extensions, when combining with the division of AFs, can be used to handle topics related to the maximality and directionality criteria. For example, we can compute the preferred extensions, and deal with the dynamics of AFs. It remains to evaluate the computational complexity of the operations. That is a first direction for further development of our work.

The matrix approach can be used for other applications. One direction for further research is to study the structural properties and status-based properties of dynamics of AFs as defined by [10]. Another topic is related to the matrix equation of AFs. We plan to find the equational representation of various extensions, by the solution of which we can obtain all the extensions *wrt* a fixed semantics. An interesting attempt has been made in this direction by [16].

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