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Joint distribution of a Lévy process and its running supremum

Laure Coutin*Waly Ngom†Monique Pontier‡

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Abstract

Let X be a jump-diffusion process and X^* its running supremum. In this paper, we first show that for any $t > 0$, the law of the pair (X_t^*, X_t) has a density with respect to Lebesgue measure and compute this one. This allows us to show that for any $t > 0$, the pair formed by the random variable X_t and the running supremum X_t^* of X at time t can be characterized as a solution of a weakly valued-measure partial differential equation. Then we compute the marginal density of X_t^* for all $t > 0$.

Keywords: Lévy process, partial differential equation, running supremum process, first hitting time.

A.M.S. Classification: 60G51, 60H20, 60H99.

1 Introduction

Consider a Lévy process $(X_t, t \geq 0)$, starting from zero, which is right continuous left limited. If moreover X is the sum of a drifted Brownian motion and a compound Poisson process, it is called a mixed diffusive-jump process. As any Lévy process, X has stationary and independent increments and is characterized by its Laplace transform. The mixed diffusive-jump processes and the notion of first passage time (behavior of certain processes at first passage time) are very useful and widely studied.

Introducing the running supremum at time t , X_t^* and the first passage τ_b of X at level b , the probability $\mathbb{P}(X_t \geq a, X_t^* \geq b) = \mathbb{P}(X_t \geq a, \tau_b \leq t)$ for some fixed real numbers (a, b) , $a \leq b$ and $b > 0$, is of great importance, for example, in pricing barrier options while the logarithm of the underlying asset price is modeled by a jump-diffusion process. In this idea, Kou and Wang [5] give the explicit expression of the Laplace transform of the joint distribution of the double exponential mixed diffusive-jump process and its running supremum.

In [4], Jeanblanc et al. consider the first passage time by a diffusion at a deterministic function h that depends on time and they define a function of τ_h and X which satisfies the Fokker-Planck Equation.

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In [1], it is well noted (Theorem 2.2.9 and Exercise 2.2.10) that the $\frac{1}{2}$ -stable subordinator is the first passage time of a standard Brownian motion and the inverse Gaussian subordinator is the first passage time of standard Brownian motion with a drift.

Mark Veillette and Murad S. Taqqu study in [9] the first passage time of a subordinator D . Since D is in general non-Markovian with non-stationary and non-independent increments, they derive a partial differential equation for the Laplace transform of the n -time tail distribution $\mathbb{P}(\tau_{t_1} > s_1, \dots, \tau_{t_n} > s_n)$ where $\tau_{t_k} = \inf\{s : D_s > t_k\}$ for a subordinator $(D_s, s \geq 0)$. With this result, they give a recursive formula for multiple-time moments of the local time of a Markov process in terms of its transition density.

The authors of [2] use a partial differential equation (PDE) approach to show that the calibration of an implied volatility surface and the pricing of contingent claims can be as simple in mixed diffusive-jump framework as it is in a diffusion framework.

This work characterizes the law of the pair U_t formed by the random variable X_t and the running supremum X_t^* of X at time t , with a valued-measure partial differential equation and gives an explicit expression for the density function of this pair. Then the marginal density of X_t^* is given. The paper is organized as follows: Section 2 provides the main result. Section 3 gives the density function of the pair formed by the random variable X_t and its running supremum X_t^* and Section 4 is devoted to the proof of the main result of Section 2. To finish, one concludes and gives some auxiliary results in Appendix.

2 Valued measure differential equation for the joint law

We introduce some preliminary concepts for the diffusion part: for a standard Brownian motion W and a real number m , let be

$$\tilde{X}_t = mt + W_t, \quad \tilde{X}_t^* = \sup_{s \leq t} \tilde{X}_s. \quad (1)$$

In [4] page 147, Jeanblanc et al. show that the pair $(\tilde{X}_t^*, \tilde{X}_t)$ has a density with respect to Lebesgue measure on $\mathbb{R}^2$ noted $\tilde{p}(\cdot, \cdot; t)$ where

$$\tilde{p}(b, a; t) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp \left[-\frac{(2b-a)^2}{2t} + ma - m^2 \frac{t}{2} \right] \mathbf{1}_{\{\max(0, a) < b\}}. \quad (2)$$

In all the following, Φ_G means the standard normal Gaussian distribution and one often uses the following:

$$1 - \Phi_G(x) = \Phi_G(-x) \leq \frac{1}{x\sqrt{2\pi}} \exp -\frac{x^2}{2}, \quad \forall x > 0. \quad (3)$$

In order to have a Lévy process with non zero jump part, let us introduce

$$X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i, \quad X_t^* = \sup_{s \leq t} X_s,$$

where N is a Poisson process with constant positive intensity λ , $(Y_i, i \in \mathbb{N}^*)$ is a sequence of independent and identically distributed random variables with the distribution function F_Y and

the sequence of jump times of N is denoted by $(T_i), i \geq 1$. Let θ be the shift operator and $(U_t; t \geq 0)$ be the $\mathbb{R}^2$ -value process defined by

$$U_t = (X_t^*, X_t), \quad t \geq 0. \quad (4)$$

The aim is to prove the theorem:

Theorem 2.1. (i) For all $t > 0$, the law of the pair (X_t^*, X_t) has a density with respect to the Lebesgue measure, denoted by $p(b, a; t)$.

(ii) For all $t > 0, a \in \mathbb{R}$ the map $h \mapsto p(a + h, a; t)$ has a limit when h goes to 0 denoted by $p(a+, a; t)$.

(iii) Let be $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ a C_b^3 -bounded function with a support in $\{(b, a), b > 0, b \geq a\}$ such that there exists $\delta > 1$ satisfying $\int_{\mathbb{R}} |\partial_1 \varphi(a, a)|^\delta da < \infty$.

For any $t > 0$,

$$\begin{aligned} \mathbb{E}(\varphi(U_t)) &= \varphi(0, 0) + \int_0^t \mathbb{E} \left[m \partial_2 \varphi(U_s) + \frac{1}{2} \partial_{22}^2 \varphi(U_s) \right] ds \\ &+ \int_0^t \frac{1}{2} \mathbb{E} \left[\partial_1 \varphi(X_s, X_s) \frac{p(X_s+, X_s; s)}{g(X_s; s)} \right] ds + \lambda \int_0^t \mathbb{E} \left(\int_{\mathbb{R}} [\varphi(U_s(y)) - \varphi(U_s)] F_Y(dy) \right) ds. \end{aligned} \quad (5)$$

where $g(\cdot; s)$ is the density of the random variable X_s and

$$U_s = (X_s^*, X_s), \quad U_s(y) = (\max(X_s^*, X_s + y), X_s + y), \quad s \geq 0. \quad (6)$$

In the next sections, details of the proof of Theorem 2.1 are given.

3 Existence of the density of the law of (X_t^*, X_t) and its properties

We note that

$$X_t^* = \max\left\{ \left(\sup_{u \in [T_i, \inf(T_{i+1}, t)]} X_u, i = 0, \dots, N_t \right), X_t \right\}$$

and use the joint density of $(\tilde{X}_t^*, \tilde{X}_t)$ given by (2) to show that the pair (X_t^*, X_t) law has a density which is right continuous on the diagonal, see Proposition 3.1 below which actually is the proof of Theorem 2.1 (i) and (ii).

Proposition 3.1. (i) For all $t > 0$, the law of the random vector (X_t^*, X_t) admits a density with respect to the Lebesgue measure given by

$$p(b, a, t) = \mathbb{E} \left(\sum_{k=0}^{N_t} \tilde{p}(b - X_{T_k}, a - X_{T_k} - Y_{k+1} \mathbf{1}_{\{T_{k+1} \leq t\}} - (X_t - X_{T_{k+1} \wedge t}), t \wedge T_{k+1} - T_k) \mathbf{1}_{\Delta_{k,t}}(b, a) \right)$$

where $\tilde{p}$ is given by (2) and

$$\Delta_{k,t} = \left\{ (b, a) \mid b > \max \left(X_{T_k}^*, a + [X_{t \wedge T_{k+1}} - \sup_{u \in [T_{k+1}, t]} X_u] \mathbf{1}_{\{T_{k+1} < t\}} \right) \right\}. \quad (7)$$

(ii) Moreover, for all $a \in \mathbb{R}$, $t > 0$ the map $h \mapsto p(a+h, a; t)$ has a limit when h goes to 0 denoted by $p(a+, a; t)$ and

$$p(a+, a; t) = \mathbb{E} \left(\sum_{k=0}^{N_t} \tilde{p}((a - X_{T_k})^+, a - X_{T_k} - Y_{k+1} \mathbf{1}_{\{T_{k+1} \leq t\}} - (X_t - X_{T_{k+1} \wedge t}), t \wedge T_{k+1} - T_k) \mathbf{1}_{a \in D_{k,t}} \right)$$

where $D_{k,t} := \{a : a \geq \max(X_{T_k}^*, a + [X_{t \wedge T_{k+1}} - \sup_{u \in [T_{k+1}, t]} X_u] \mathbf{1}_{\{T_{k+1} < t\}})\}$

The proof of (i) relies on the following lemma:

Lemma 3.2. *Almost surely, for all t ,*

$$X_t^* = \max \left(X_{T_k} + \sup_{u \in [T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}), \quad k = 0, \dots, N_t \right). \quad (8)$$

Moreover, almost surely, for all t , there exists a unique k denoted as N_t^* such that

$$X_t^* = X_{T_k} + \sup_{u \in [T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}). \quad (9)$$

Proof. Let t be fixed.

(a) Note that

$$X_t^* = \max \left\{ \max \left(X_{T_k} + \sup_{u \in [T_k, T_{k+1} \wedge t]} (X_u - X_{T_k}), \quad k = 0, \dots, N_t \right), X_t \right\}. \quad (10)$$

For $k \in \mathbb{N}$, for all $u \in [T_k, T_{k+1}[$, $X_u - X_{T_k} = \tilde{X}_u - \tilde{X}_{T_k}$ where $\tilde{X}$ is the continuous process defined in (1), thus for $k \leq N_t$,

$$\sup_{u \in [T_k, T_{k+1} \wedge t]} (X_u - X_{T_k}) = \sup_{u \in [T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \quad (11)$$

and

$$\max \left(X_{T_{N_t}} + \sup_{u \in [T_{N_t}, T_{N_t+1} \wedge t]} (X_u - X_{T_{N_t}}), X_t \right) = X_{T_{N_t}} + \sup_{u \in [T_{N_t}, T_{N_t+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_{N_t}}). \quad (12)$$

Plugging identities (11) and (12) in equality (10) yields (8).

(b) Let two integers $i < j$ then,

$$X_{T_j} + \sup_{u \in [T_j, T_{j+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) = X_{T_i} + (\tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}) + Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u \in [T_j, T_{j+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_j})$$

and

$$\begin{aligned} X_{T_j} + \sup_{u \in [T_j, T_{j+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) - X_{T_i} - \sup_{u \in [T_i, T_{i+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_i}) = \\ \left\{ Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u \in [T_j, T_{j+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) \right\} + \left\{ (\tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}) - \sup_{u \in [T_i, T_{i+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_i}) \right\}. \end{aligned}$$

The two following random vectors are independent:

$$\left(\sup_{u \in [T_i, T_{i+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_i}); \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i} \right), Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u \in [T_j, T_{j+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_j})$$

and the law of the vector $\left(\sup_{u \in [T_i, T_{i+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_i}); \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i} \right)$ admits a density with respect to the Lebesgue measure, hence the law of the random variable

$$\sup_{u \in [T_i, T_{i+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_i}) + \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}$$

has a density with respect to the Lebesgue measure and is independent of

$$Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u \in [T_j, T_{j+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}).$$

Therefore, $X_{T_j} + \sup_{u \in [T_j, T_{j+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) - X_{T_i} - \sup_{u \in [T_i, T_{i+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_i})$ is the sum of two independent random variables, one having a density, then also has a density. So for all t , almost surely, whenever $i \neq j$

$$X_{T_j} + \sup_{u \in [T_j, T_{j+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) \neq X_{T_i} + \sup_{u \in [T_i, T_{i+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_i})$$

(c) Above, we can exchange $\forall t > 0$ and almost surely, since the processes $(N_t, t \geq 0)$ and $\left(\left(\max \left(X_{T_k} + \sup_{u \in [T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \right), k \leq N_t \right), t \geq 0 \right)$ are right continuous. $\square$

Proof. of Proposition 3.1 (i): According to Lemma 3.2, let N_t^* denoting the index k where the maximum below is reached,

$$X_t^* = \max \left(X_{T_k} + \sup_{u \in [T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}), k = 0, \dots, N_t \right).$$

The fact $N_t^* = k$ is equivalent to: the supremum is reached on the interval $[T_k, T_{k+1} \wedge t]$, actually meaning $X_t^* = \sup_{[T_k, T_{k+1} \wedge t]} X_u$ and remark that $\sup_{[T_k, T_{k+1} \wedge t]} X_u \geq X_{T_k}^* \vee \sup_{[T_{k+1} \wedge t, t]} X_u$.

On the interval $[T_k, T_{k+1} \wedge t]$, $X_u = X_{T_k} + \tilde{X}_u - \tilde{X}_{T_k}$. Thus the following inequalities are equivalent to $N_t^* = k$:

$$(a) \quad X_{T_k} + \sup_{[T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq X_{T_k}^*,$$

$$(b) \quad X_{T_k} + \sup_{[T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq \sup_{[T_{k+1} \wedge t, t]} X_u = [X_{T_{k+1} \wedge t} + \sup_{[T_{k+1}, t]} (X_u - X_{T_{k+1} \wedge t})] \mathbf{1}_{\{T_{k+1} < t\}} + X_t \mathbf{1}_{\{T_{k+1} \geq t\}}.$$

Using $X_{T_{k+1}} = X_{T_k} + \tilde{X}_{T_{k+1}} - \tilde{X}_{T_k} + Y_{k+1}$, (b) is equivalent to

$$\sup_{[T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq [\tilde{X}_{T_{k+1}} - \tilde{X}_{T_k} + Y_{k+1} + \sup_{[T_{k+1}, t]} (X_u - X_{T_{k+1}})] \mathbf{1}_{\{T_{k+1} < t\}} + (\tilde{X}_t - \tilde{X}_{T_k}) \mathbf{1}_{\{T_{k+1} \geq t\}}.$$

As a conclusion we get $\{N_t^* = k\} =$

$$\left\{ \sup_{[T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq X_{T_k}^* - X_{T_k} \right\} \cap \left\{ \sup_{[T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq \tilde{X}_{t \wedge T_{k+1}} - \tilde{X}_{T_k} + [Y_{k+1} + \sup_{[T_{k+1}, t]} (X_u - X_{T_{k+1}})] \mathbf{1}_{\{T_{k+1} \leq t\}} \right\}.$$

Thus

$$\{N_t^* = k\} = \left\{ \left(\sup_{u \in [T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}), \tilde{X}_{t \wedge T_{k+1}} - \tilde{X}_{T_k} \right) \in \bar{\Delta}_{k,t} \right\}$$

where

$$\bar{\Delta}_{k,t} = \left\{ (b, a) : |b| > \max \left(X_{T_k}^* - X_{T_k}, a + [Y_{k+1} + \sup_{u \in [T_{k+1}, t]} (X_u - X_{T_{k+1}})] \mathbf{1}_{\{T_{k+1} \leq t\}} \right) \right\}. \quad (13)$$

Moreover on $\{k \leq N_t\}$ so on $\{N_t^* = k\} \subset \{k \leq N_t\}$

$$X_t = X_{T_k} + (\tilde{X}_{t \wedge T_{k+1}} - \tilde{X}_{T_k}) + Y_{k+1} \mathbf{1}_{\{t \geq T_{k+1}\}} + (X_t - X_{t \wedge T_{k+1}}). \quad (14)$$

Let Φ be a bounded Borel function, hence

$$\begin{aligned} \mathbb{E}[\Phi(X_t^*, X_t)] &= \mathbb{E} \left[\sum_{k=0}^{N_t} \Phi(X_t^*, X_t) \mathbf{1}_{\{N_t^* = k\}} \right] = \\ &= \mathbb{E} \left[\sum_{k=0}^{N_t} \mathbf{1}_{\{N_t^* = k\}} \Phi \left(X_{T_k} + \sup_{u \in [T_k, T_{k+1} \wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}), X_{T_k} + (\tilde{X}_{t \wedge T_{k+1}} - \tilde{X}_{T_k}) + Y_{k+1} \mathbf{1}_{\{t \geq T_{k+1}\}} + (X_t - X_{t \wedge T_{k+1}}) \right) \right]. \end{aligned}$$

The four following random vectors are independent:

$$(X_{T_k}, X_{T_k}^*), Y_{k+1}, \left(X_t - X_{t \wedge T_{k+1}}, \sup_{u \in [T_{k+1} \wedge t, t]} (X_u - X_{T_{k+1} \wedge t}) \right), \left(\sup_{u \in [T_k, T_{k+1} \wedge t]} \tilde{X}_u - \tilde{X}_{T_k}, \tilde{X}_{t \wedge T_{k+1}} - \tilde{X}_{T_k} \right)$$

and conditionally to $\sigma(\mathcal{F}_{T_k}, Y_{k+1}, (X_u - X_{T_{k+1}}, u \geq T_{k+1} \wedge t), T_k, T_{k+1})$, the law of the random vector

$$\left(\sup_{u \in [T_k, T_{k+1} \wedge t]} \tilde{X}_u - \tilde{X}_{T_k}, \tilde{X}_{t \wedge T_{k+1}} - \tilde{X}_{T_k} \right)$$

has a density with respect to the Lebesgue measure given by $\tilde{p}(b, a, T_{k+1} \wedge t - T_k)$ where $\tilde{p}$ is defined by (2). We obtain that $\mathbb{E}(\Phi(X_t^*, X_t)) =$

$$\int \mathbb{E} \left[\sum_{k=0}^{N_t} \Phi(X_{T_k} + b, X_{T_k} + a + Y_{k+1} \mathbf{1}_{\{t \geq T_{k+1}\}} + (X_t - X_{T_{k+1} \wedge t})) \tilde{p}(b, a, T_{k+1} \wedge t - T_k) \mathbf{1}_{\bar{\Delta}_{k,t}}(b, a) \right] da db.$$

The change of variable formula $v = b + X_{T_k}$ and $u = X_{T_k} + a + Y_{k+1} \mathbf{1}_{\{t \geq T_{k+1}\}} + (X_t - X_{T_{k+1} \wedge t})$ concludes the proof.

Proof of Proposition 3.1 (ii): Let $a \in \mathbb{R}$, $t > 0$, the map

$$h \mapsto \sum_{k=0}^{N_t} \tilde{p}(a + h - X_{T_k}, a - X_{T_k} - Y_{k+1} \mathbf{1}_{\{T_{k+1} \leq t\}} - (X_t - X_{T_{k+1} \wedge t}), t \wedge T_{k+1} - T_k) \mathbf{1}_{\Delta_{k,t}}(b, a)$$

has a right limit when h goes to 0 since both functions $h \mapsto \tilde{p}(a + h, a; t)$ and $h \mapsto \mathbf{1}_{\Delta_{k,t}}(a + h, a)$ admit a limit when h decreases to 0. According to Proposition 6.2 in Appendix the family

$$\left(\sum_{k=0}^{N_t} \tilde{p}(a + h - X_{T_k}, a - X_{T_k} - Y_{k+1} \mathbf{1}_{\{T_{k+1} \leq t\}} - (X_t - X_{T_{k+1} \wedge t}), t \wedge T_{k+1} - T_k) \mathbf{1}_{\Delta_{k,t}}(b, a) \right)_{h \in [0,1]}$$

is uniformly integrable.

Then, we can exchange the limit and the expectation and $h \mapsto p(a+h, a; t)$ has a limit when h decreases to 0 and

$$p(a+, a; t) = \mathbb{E} \left(\sum_{k=0}^{N_t} \tilde{p}((a - X_{T_k})_+, a - X_{T_k} - Y_{k+1} \mathbf{1}_{\{T_{k+1} \leq t\}} - (X_t - X_{T_{k+1} \wedge t}), t \wedge T_{k+1} - T_k) \mathbf{1}_{a \in D_{k,t}} \right)$$

where $D_{k,t} := \{a : a \geq \max(X_{T_k}^*, a + [X_{t \wedge T_{k+1}} - \sup_{u \in [T_{k+1}, t]} X_u] \mathbf{1}_{\{T_{k+1} < t\}})\}$

□

As a corollary the law of X_t^* is deduced:

Corollary 3.3. *For any $t > 0$, the law of the random variable X_t^* has a density $p^*(\cdot, t)$ given by*

$$p^*(b, t) = 2\mathbb{E} \left(\sum_{k=0}^{N_t} 2e^{2(b-X_{T_k})m} H_m[(C_{t,k})^+ \mathbf{1}_{T_{k+1} < t} + (b - X_{T_k}) + m(t \wedge T_{k+1} - T_k), t \wedge T_{k+1} - T_k] \mathbf{1}_{\{b > X_{T_k}^*\}} \right) \quad (15)$$

where $H_m : (x, t) \rightarrow \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{x^2}{2t}\right] - m\Phi_G\left(-\frac{x}{\sqrt{t}}\right)$ and $C_{t,k} = (Y_{k+1} + \sup_{u \in [T_{k+1}, t]} (X_u - X_{T_{k+1}})) \mathbf{1}_{\{T_{k+1} \leq t\}}$.

Proof. Let $\tilde{q}$ be the function such that $\tilde{p}(b, a, \cdot) = \tilde{q}(b, a, \cdot) \mathbf{1}_{b > a \vee 0}$ where

$$\tilde{p}(b, a; t) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left[-\frac{(2b-a)^2}{2t} + ma - m^2 \frac{t}{2}\right] \mathbf{1}_{\{\max(0, a) < b\}}.$$

Remark that

$$(2b-a)^2 - 2mta + m^2 t^2 = [a - (2b+mt)]^2 - 4bmt, \quad (16)$$

thus we obtain

$$\tilde{q}(b, a, t) = \frac{2e^{2bm}}{\sqrt{2\pi t}} \left(\frac{2b+mt-a}{t} \exp\left[-\frac{[a - (2b+mt)]^2}{2t}\right] - m \exp\left[-\frac{[a - (2b+mt)]^2}{2t}\right] \right).$$

Hence, for any A ,

$$0 \leq \int_{-\infty}^A \tilde{p}(b, a, t) da = 2e^{2bm} \mathbf{1}_{b > 0} H_m(x, t), \quad x = b \wedge A - 2b - mt. \quad (17)$$

Let k be fixed and $P_k^*(b, t)$ be given by

$$P_k^*(b, t) := \int_{\mathbb{R}} \tilde{p}(b - X_{T_k}, a - X_{T_k} - Y_{k+1} \mathbf{1}_{T_{k+1} \leq t} - (X_t - X_{T_{k+1} \wedge t}), t \wedge T_{k+1} - T_k) \mathbf{1}_{\Delta_{k,t}}(b, a) da$$

then the density of X_t^* is given by

$$p^*(b, t) = \mathbb{E} \left(\sum_{k=0}^{N_t} P_k^*(b, t) \right).$$

With the change of variables $u = a - X_{T_k} - Y_{k+1} \mathbf{1}_{T_{k+1} \leq t} - (X_t - X_{t \wedge T_{k+1}})$, it follows

$$P_k^*(b, t) := \int_{\mathbb{R}} \tilde{p}(b - X_{T_k}, u, t \wedge T_{k+1} - T_k) \mathbf{1}_{\Delta_{k,t}}(b, u + X_{T_k} + Y_{k+1} \mathbf{1}_{T_{k+1} \leq t} + (X_t - X_{t \wedge T_{k+1}})) du.$$

According to the definition of $\Delta_{k,t}$ (7)

$$\begin{aligned} \mathbf{1}_{\Delta_{k,t}}(b, u + X_{T_k} + Y_{k+1} \mathbf{1}_{\{T_{k+1} \leq t\}} + (X_t - X_{t \wedge T_{k+1}})) = \\ \mathbf{1}_{\{b > X_{T_k}^*\}} \mathbf{1}_{\{b > u + X_{T_k} + [Y_{k+1} + \sup_{u \in [T_{k+1}, t]} (X_u - X_t)] \mathbf{1}_{\{T_{k+1} \leq t\}} \} \end{aligned}$$

On the event $T_{k+1} \leq t$ (id est $k < N_t$)

$$P_k^*(b, t) = \int_{\mathbb{R}} \tilde{q}(b - X_{T_k}, u, T_{k+1} - T_k) \mathbf{1}_{b > X_{T_k}^*} \mathbf{1}_{]-\infty, \min(b - X_{T_k}, b - X_{T_k} - C_{t,k}[}(u) du$$

since $C_{t,k} = (Y_{k+1} + \sup_{u \in [T_{k+1}, t]} (X_u - X_t)) \mathbf{1}_{\{T_{k+1} \leq t\}}$.

And on the event $T_{k+1} > t$

$$P_k^*(b, t) = \int_{\mathbb{R}} \tilde{q}(b - X_{T_k}, u, t - T_k) \mathbf{1}_{b > X_{T_k}^*} \mathbf{1}_{]-\infty, b - X_{T_k}[}(u) du.$$

(a) On the event $T_{k+1} \leq t$ applying (17) to $A = b - X_{T_k} - C_{t,k}$, with $T_{k+1} - T_k$ and $b - X_{T_k}$ instead of t and b , $(b - X_{T_k}) \wedge A = b - X_{T_k} - (C_{t,k})^+$ (since $0 \wedge (-x) = -x^+$) so on this event

$$P_k^*(b, t) = 2e^{2m(b - X_{T_k})} H_m [(C_{t,k})^+ + (b - X_{T_k}) + m(T_{k+1} - T_k), T_{k+1} - T_k].$$

(b) On the event $T_{k+1} > t$, applying (17) to $A = b - X_{T_k}$ and taking $t - T_k$ and $b - X_{T_k}$ instead of t and b , so on this event

$$P_k^*(b, t) = 2e^{2(b - X_{T_k})m} H_m [-(b - X_{T_k}) - m(t - T_k), t - T_k].$$

To summarize both cases

$$P_k^*(b, t) = 2e^{2(b - X_{T_k})m} H_m [(C_{t,k})^+ \mathbf{1}_{\{T_{k+1} \leq t\}} + (b - X_{T_k}) + m(t \wedge T_{k+1} - T_k), t \wedge T_{k+1} - T_k]$$

and the proof is achieved. $\square$

4 Proof of Theorem 2.1 (iii)

To prove the end of this theorem, we proceed as follows: we will compute $\lim_{h \rightarrow 0} h^{-1} A(t, h) = a(t)$ where

$$A(t, h) := \mathbb{E} [\varphi(U_{t+h}) - \varphi(U_t)]. \quad (18)$$

After that, we will use [8] 11.82 p. 368: If f is a function such that f' is finite everywhere and integrable, then for all $a \leq b$, $f(b) - f(a) = \int_a^b f'(s) ds$. The study of $a(t) := \lim_{h \rightarrow 0} h^{-1} A(t, h)$ could prove that for all $t > t_0 > 0$,

$$\mathbb{E} (\varphi(U_t)) = \int_{t_0}^t a(s) ds + \mathbb{E} (\varphi(X_{t_0}^*, X_{t_0})), \quad \forall t > t_0.$$

A last step will be to prove the convenient properties of the function a , for all $T > 0$, there exist $v \in]0, 1[$, $\xi \geq 0$ and a constant C such that

$$|a(t)| \leq C \left[\frac{1}{t^v} + 1 + t^\xi \right], \quad \forall t \in]0, T]. \quad (19)$$

Then letting t_0 going to 0 and using the fact that φ is continuous bounded, X and X^* are right continuous, estimation (19) and Lebesgue dominated theorem, for all $t > 0$

$$\mathbb{E}(\varphi(U_t)) = \int_0^t a(s)ds + \varphi(0, 0).$$

Proof of the function $a : t \mapsto \mathbb{E}(\varphi(U_t))$ admits a derivative on $]0, +\infty[$ denoted as a and satisfying (19).

The idea is to split $A(t, h)$ in three parts according to the values of $N_{t+h} - N_t$:

$$A(t, h) = \sum_{i=0}^2 A_i(t, h) \quad (20)$$

where

$$\begin{aligned} A_i(t, h) &:= \mathbb{E}([\varphi(U_{t+h}) - \varphi(U_t)] \mathbf{1}_{\{N_{t+h} - N_t = i\}}), \quad i = 0, 1 \\ A_2(t, h) &:= \mathbb{E}([\varphi(U_{t+h}) - \varphi(U_t)] \mathbf{1}_{\{N_{t+h} - N_t \geq 2\}}). \end{aligned} \quad (21)$$

Lemma 4.1. *Under the hypothesis of Theorem 2.1*

$$\lim_{h \rightarrow 0} h^{-1} A_2(t, h) = 0. \quad (22)$$

Proof. By hypothesis φ is bounded and we get

$$|A_2(t, h)| \leq 2\|\varphi\|_\infty \mathbb{P}(N_{t+h} - N_t \geq 2) \leq 2\|\varphi\|_\infty (1 - e^{-\lambda h} - \lambda h e^{-\lambda h}).$$

Thus, $\lim_{h \rightarrow 0} h^{-1} A_2(t, h) = 0$. □

This lemma added to the three next propositions proves Theorem 2.1 (iii): this lemma treats the term $A_2(t, h)$ while Proposition 4.2 treats the term $A_1(t, h)$. Propositions 4.3 and 4.4 treat the term $A_0(t, h)$.

Proposition 4.2. *Let be $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ a C_b^3 - bounded function. Then*

$$\lim_{h \rightarrow 0} h^{-1} A_1(t, h) = \lambda \mathbb{E} \int_{\mathbb{R}} [\varphi(U_t(y)) - \varphi(U_t)] F_Y(dy).$$

where U_t is defined by (4) and $U_t(y)$ by (6).

This proposition gets the last term on the right hand in (5).

Proof. Introducing the term $\varphi(U_t(Y_{N_{t+h}}))$, let be $A_1(t, h) := A_{1,1}(t, h) + A_{1,2}(t, h)$ where

$$\begin{aligned} A_{1,1}(t, h) &= \mathbb{E}(\{\varphi(U_t(Y_{N_{t+h}})) - \varphi(U_t)\} \mathbf{1}_{\{N_{t+h} = N_t + 1\}}), \\ A_{1,2}(t, h) &= \mathbb{E}(\{\varphi(U_{t+h}) - \varphi(U_t(Y_{N_{t+h}}))\} \mathbf{1}_{\{N_{t+h} = N_t + 1\}}). \end{aligned}$$

- Since φ is $\mathcal{C}^1$ class with bounded derivative, Lemma 6.1 (Appendix) implies that on the event $\{N_{t+h} - N_t = 1\}$ we have

$$|\varphi(U_{t+h}) - \varphi(U_t(Y_{N_{t+h}}))| \leq \|\nabla\varphi\|_\infty (2 \sup_{0 \leq u \leq h} |\tilde{X}_{t+u} - \tilde{X}_t| + h). \text{ Thus}$$

$$h^{-1} A_{1,2}(t, h) \leq \lambda e^{-\lambda h} \|\nabla\varphi\|_\infty (2 \sup_{0 \leq u \leq h} |\tilde{X}_{t+u} - \tilde{X}_t| + h) \rightarrow 0$$

when $h \rightarrow 0$: Indeed, the process $\tilde{X}$ satisfies $0 \leq \tilde{X}_h \leq |m|h + W_h^*$ and Burkholder Davis Gundy's inequality for all $i \geq 1$ implies there exists a constant $C_i > 0$ such that $E[(W_h^*)^i] \leq C_i h^{i/2}$, hence for $h \leq 1$,

$$\mathbb{E} \left((\tilde{X}_h^*)^i \right) \leq C_i h^{i/2}, \quad (23)$$

thus $E(\tilde{X}_h^*) \leq C\sqrt{h}$.

- Let us deal with $A_{1,1}(t, h)$ to show

$$\lim_{h \rightarrow 0} h^{-1} A_{1,1}(t, h) = \lambda \mathbb{E} \int_{\mathbb{R}} [\varphi(U_t(y)) - \varphi(U_t)] F_Y(dy). \quad (24)$$

On the event $\{N_t = n, N_{t+h} = n+1\}$, the equality $U_t(Y_{N_{t+h}}) = U_t(Y_{n+1})$ holds. The independence properties arising from the structure of the process X , the use of the laws of Y_i , T_n , the decomposition of T_{n+1} , $T_{n+1} = T_n + S_{n+1}$, and the conditioning to $\mathcal{F}_{T_n}$, yield:

$$\begin{aligned} A_{1,1}(t, h) &= \sum_{n=0}^{+\infty} \mathbb{E} \left[\mathbf{1}_{\{T_n \leq t < T_{n+1} \leq t+h < T_{n+2}\}} (\varphi(U_t(Y_{n+1})) - \varphi(U_t)) \right] = \\ &= \sum_{n=0}^{+\infty} \mathbb{E} \left[\mathbf{1}_{\{T_n \leq t\}} \int_{t-T_n}^{t+h-T_n} ds \int_{\mathbb{R}} (\varphi(U_t(y)) - \varphi(U_t)) F_Y(dy) \lambda e^{-\lambda(t+h-T_n)} \right]. \end{aligned}$$

By hypothesis, the function φ is bounded and when h goes to 0, Lebesgue's dominated convergence theorem yields

$$\lim_{h \rightarrow 0} h^{-1} A_{1,1}(t, h) = \sum_{n=0}^{+\infty} \mathbb{E} \left[\lambda e^{-\lambda(t-T_n)} \mathbf{1}_{\{T_n \leq t\}} \int_{\mathbb{R}} (\varphi(U_t(y)) - \varphi(U_t)) F_Y(dy) \right]. \quad (25)$$

Since $e^{-\lambda(t-T_n)} \mathbf{1}_{\{T_n \leq t\}} = \mathbb{E}[\mathbf{1}_{\{T_n \leq t < T_{n+1}\}} / \mathcal{F}_t]$, it follows

$$\begin{aligned} \lim_{h \rightarrow 0} h^{-1} A_{1,1}(t, h) &= \sum_{n=0}^{+\infty} \mathbb{E} \left[\lambda \mathbf{1}_{\{T_n \leq t < T_{n+1}\}} \int_{\mathbb{R}} (\varphi(U_t(y)) - \varphi(U_t)) F_Y(dy) \right] = \\ &= \mathbb{E} \left[\lambda \int_{\mathbb{R}} (\varphi(U_t(y)) - \varphi(U_t)) F_Y(dy) \right]. \end{aligned} \quad (26)$$

□

We now turn to the study of $h^{-1} A_0(t, h)$ when h goes to 0. On the event $\{N_{t+h} - N_t = 0\}$, $T_{N_t} = T_{N_{t+h}}$, hence $X_{T_{N_{t+h}}}^* = X_{T_{N_t}}^*$ and $X_{T_{N_{t+h}}} = X_{T_{N_t}}$,

$$X_{t+h} = X_t + \tilde{X}_h \circ \theta_t, X_{t+h}^* = \max(X_t^*, X_t + \tilde{X}_h^* \circ \theta_t).$$

Using Markov property at t and the fact that the processes N and $\tilde{X}$ are independent

$$A_0(t, h) = e^{-\lambda h} \mathbb{E} \left(\mathbb{E} \left(\varphi(\max(x^*, x + \tilde{X}_h^*), x + \tilde{X}_h) - \varphi(x^*, x) \right)_{|x^*=X_t^*, x=X_t} \right).$$

Let us introduce

$$a_0(h, x^*, x) := \mathbb{E} \left(\varphi(\max(x^*, x + \tilde{X}_h^*), x + \tilde{X}_h) - \varphi(x^*, x) \right).$$

To study the term $a_0(h, x^*, x)$, we make a Taylor expansion at a neighborhood of (x^*, x) :

$$\begin{aligned} a_0(h, x^*, x) &:= \partial_2 \varphi(x^*, x) m h + \frac{1}{2} \partial_{22}^2 \varphi(x^*, x) [m^2 h^2 + h] + \\ &\quad + \partial_{1,2}^2 \varphi(x^*, x) \mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right] \tilde{X}_h \right) \\ &\quad + \frac{1}{2} \partial_{1,1}^2 \varphi(x^*, x) \mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right]^2 \right) + R_0(h, x^*, x), \\ &\quad + \partial_1 \varphi(x^*, x) \mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right] \right) \end{aligned}$$

where, using ∇^i the tensor of order i ,

$$|R_0(h, x^*, x)| \leq 4 \|\nabla^3 \varphi\|_\infty \left[\mathbb{E} \left(\left| \max(x^*, x + \tilde{X}_h^*) - x^* \right|^3 \right) + \mathbb{E} \left(\left| \tilde{X}_h \right|^3 \right) \right].$$

This allows us to write:

$$A_0(t, h) = \sum_{i=1}^3 A_{0,i}(t, h), \quad A_{0,i}(t, h) := \mathbb{E} \left(a_{0,i}(h, x^*, x)_{|x^*=X_t^*, x=X_t} \right) \quad (27)$$

where

$$\begin{aligned} a_{0,1}(h, x^*, x) &:= \partial_2 \varphi(x^*, x) m h + \frac{1}{2} \partial_{22}^2 \varphi(x^*, x) [m^2 h^2 + h] \\ a_{0,2}(h, x^*, x) &:= \partial_{1,2}^2 \varphi(x^*, x) \mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right] \tilde{X}_h \right) \\ &\quad + \frac{1}{2} \partial_{1,1}^2 \varphi(x^*, x) \mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right]^2 \right) + R_0(h, x^*, x), \\ a_{0,3}(h, x^*, x) &:= \partial_1 \varphi(x^*, x) \mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right] \right). \end{aligned}$$

Proposition 4.3. *Let be $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ a C_b^3 - bounded function. Then for any $t > 0$*

$$\lim_{h \rightarrow 0} h^{-1} (A_{0,1} + A_{0,2})(t, h) = \mathbb{E} \left(\partial_2 \varphi(U_t) m + \frac{1}{2} \partial_{22}^2 \varphi(U_t) \right) \quad (28)$$

Proof. (a) Since $\tilde{X}$ and $\tilde{X}^*$ are continuous processes and φ a three times differentiable function with bounded differential, it follows

$$\lim_{h \rightarrow 0} h^{-1} A_{0,1}(t, h) = \mathbb{E} \left(\partial_2 \varphi(U_t) m + \frac{1}{2} \partial_{22}^2 \varphi(U_t) \right). \quad (29)$$

(b) The second term satisfies: Under hypothesis of Theorem 2.1,

$$\lim_{h \rightarrow 0} h^{-1} A_{0,2}(t, h) = 0.$$

Indeed, we first note that $\max(x^*, x + \tilde{X}_h^*) - x^* = (\tilde{X}_h^* - (x^* - x))^+ \leq \tilde{X}_h^* \mathbf{1}_{\{\tilde{X}_h^* > x^* - x\}}$. Using Cauchy-Schwarz inequality and (23) there exists a constant D_i such that

$$\mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right]^i \right) \leq D_i h^{i/2} \sqrt{\mathbb{P}(\tilde{X}_h^* > x^* - x)}.$$

The function φ is three times differentiable with bounded differential, we deduce from the expression of $a_{0,2}$ that there exists a constant $C > 0$ such that

$$A_{0,2}(t, h) \leq \left[\sum_{i=2}^3 \|\nabla^i \varphi\| \right] C \sum_{i=2}^3 h^{i/2} \mathbb{E} \left(\sqrt{\mathbb{P}(\tilde{X}_h^* > x^* - x)}_{|x^*=X_t^*, x=X_t} \right).$$

The law of the pair (X_t^*, X_t) has a density with respect to Lebesgue measure on $\mathbb{R}^2$, (cf. Proposition 3.1) almost surely $X_t^* > X_t$, it follows with Lebesgue dominated convergence Theorem

$$\lim_{h \rightarrow 0} h^{-1} A_{0,2}(t, h) = 0.$$

□

We now deal with the term $h^{-1} A_{0,3}(t, h)$.

Proposition 4.4. *Let be $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ a C_b^3 - bounded function such that there exists $\delta > 1$ satisfying $\int_{\mathbb{R}} |\partial_1 \varphi(a, a)|^\delta da < \infty$. Then for any $t > 0$*

$$\lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} \left(\partial_1 \varphi(U_t) \mathbb{E} \left(\max(x^*, x + \tilde{X}_h^*) - x^* \right)_{x^*=X_t^*, x=X_t} \right) = \frac{1}{2} \mathbb{E} \left[\partial_1 \varphi(X_t, X_t) \frac{p(X_t, X_t, t)}{g(X_t, t)} \right].$$

Proof. We first need the following lemmas:

Lemma 4.5. *For any $t > 0$, the law of $\tilde{X}_t^*$ has the density with respect to Lebesgue measure on $\mathbb{R}$,*

$$\tilde{p}^*(b, t) := 2 \left[\frac{1}{\sqrt{2\pi t}} \exp - \frac{(b - mt)^2}{2t} - m e^{2bm} \Phi_G \left(\frac{-b - mt}{\sqrt{t}} \right) \right] \mathbf{1}_{]0, +\infty[}(b). \quad (30)$$

Remark 4.6. *This result is consistent with the fact that when $m = 0$, $\tilde{X}_t^*$ and $|\tilde{X}_t|$ have the same law (cf. Proposition 3.7, Revuz-Yor [7]).*

Proof. This is obviously the derivative with respect to b of the law provided in [4] page 147. □

Lemma 4.7. *Let be $h > 0$ and $H(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} - x \Phi_G(-x)$:*

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right] \right) = \\ & - mh \int_0^\infty e^{2bm\sqrt{h}} \left(b - \frac{(x^* - x)}{\sqrt{h}} \right)_+ \Phi_G(-b - m\sqrt{h}) db + \sqrt{h} H \left(\frac{(x^* - x - mh)}{\sqrt{h}} \right). \end{aligned}$$

Proof. Recall that

$$\mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right] \right) = \mathbb{E} \left(\left[\tilde{X}_h^* - (x^* - x) \right] \mathbf{1}_{\{\tilde{X}_h^* > x^* - x\}} \right).$$

Lemma 4.5 gives the density of $\tilde{X}_h^*$ and the change of variable $b \rightarrow \sqrt{h}b$ yields

$$\begin{aligned} \frac{1}{2} \mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right] \right) &= \\ \int_{\frac{x^* - x}{\sqrt{h}}}^{\infty} \sqrt{h} \left[b - \frac{(x^* - x)}{\sqrt{h}} \right]_+ \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{(b - m\sqrt{h})^2}{2}} - m\sqrt{h} e^{2bm\sqrt{h}} \Phi_G(-b - m\sqrt{h}) \right] db. \end{aligned}$$

This can be written again as

$$\begin{aligned} \frac{1}{2} \mathbb{E} \left(\left[\max(x^*, x + \tilde{X}_h^*) - x^* \right] \right) &= \int_{\frac{x^* - x}{\sqrt{h}}}^{\infty} \sqrt{h} \left[b - m\sqrt{h} - \frac{(x^* - x) - mh}{\sqrt{h}} \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{(b - m\sqrt{h})^2}{2}} db \\ &\quad - mh \int_{\frac{x^* - x}{\sqrt{h}}}^{\infty} e^{2bm\sqrt{h}} \left(b - \frac{x^* - x}{\sqrt{h}} \right) \Phi_G(-b - m\sqrt{h}) db. \end{aligned}$$

The lemma is proved using the integration by parts formula and the definition of H . $\square$

Lemma 4.7 allows to compute $h^{-1}A_{0,3}(t, h)$ including $\mathcal{F}_t$ -conditional expectation under the expectation:

$$\begin{aligned} \frac{1}{h} \mathbb{E} \left(\partial_1 \varphi(U_t) \mathbb{E} \left(\max(x^*, x + \tilde{X}_h^*) - x^* \right)_{x^* = X_t^*, x = X_t} \right) &= \frac{2}{\sqrt{h}} \mathbb{E} \left(\partial_1 \varphi(U_t) H \left(\frac{(X_t^* - X_t - mh)}{\sqrt{h}} \right) \right) \\ &\quad - 2m \mathbb{E} \left(\partial_1 \varphi(U_t) \int_0^{\infty} e^{2bm\sqrt{h}} \Phi_G(-b - m\sqrt{h}) \left(b - \frac{X_t^* - X_t}{\sqrt{h}} \right)_+ db \right). \end{aligned}$$

(a) Firstly, we show that

$$\lim_{h \rightarrow 0} -m \mathbb{E} \left[\partial \varphi_1(U_t) \int_0^{+\infty} e^{2mb\sqrt{h}} \Phi_G(-b - m\sqrt{h}) \left(b - \frac{X_t^* - X_t}{\sqrt{h}} \right)_+ db \right] = 0.$$

The term $\partial \varphi_1(U_t) \int_0^{+\infty} e^{2mb\sqrt{h}} \Phi_G(-b - m\sqrt{h}) \left(b - \frac{X_t^* - X_t}{\sqrt{h}} \right)_+$ is uniformly bounded with respect to h : Indeed, $b > 0$ and let $0 < h \leq 1$,

$$0 \leq e^{2mb\sqrt{h}} \Phi_G(-b - m\sqrt{h}) \left(b - \frac{X_t^* - X_t}{\sqrt{h}} \right)_+ \leq e^{2mb\sqrt{h}} \Phi_G(-b - m\sqrt{h}) b.$$

The function $(h, b) \mapsto e^{2mb\sqrt{h}} \Phi_G(-b - m\sqrt{h}) b$ is continuous on the compact interval $[0, 1] \times [0, 2|m|]$, then it is bounded on this interval.

Now, consider $b > 2|m|$. Therefore $b + m\sqrt{h} > |m| > 0$, $b - m\sqrt{h} > \frac{b}{2}$ and $\frac{b}{(b + m\sqrt{h})} \leq 2$. We use the inequality (3): $1 - \Phi_G(x) = \Phi_G(-x) \leq \frac{1}{x\sqrt{2\pi}} \exp -\frac{x^2}{2}$, $\forall x > 0$, to obtain for $b > 2|m|$, $h \in [0, 1]$

$$e^{2mb\sqrt{h}} \Phi_G(-b - m\sqrt{h}) b \leq \frac{b}{(b + m\sqrt{h})\sqrt{2\pi}} e^{2mb\sqrt{h}} e^{-\frac{(b + m\sqrt{h})^2}{2}} \leq \frac{2}{\sqrt{2\pi}} e^{-\frac{b^2}{8}}.$$

This implies that the term $\partial\varphi_1(U_t) \int_0^{+\infty} e^{2mb\sqrt{h}} \Phi_G(-b - m\sqrt{h})(b - \frac{X_t^* - X_t}{\sqrt{h}})_+ db$ is uniformly bounded by a constant. The result follows by Lebesgue dominated convergence Theorem: Indeed, almost surely $X_t^* - X_t > 0$ and on this set, the integrand goes almost surely to 0. .

(b) Secondly our goal is to compute the limit when h goes to 0 of the term

$$B_1^*(t, h) = E[\partial_1\varphi(U_t) \frac{1}{\sqrt{h}} H(\frac{X_t^* - X_t - mh}{\sqrt{h}})].$$

The proof is divided into four steps.

1. Firstly, we prove that

$$\lim_{h \rightarrow 0} |B_1^*(t, h) - E[\partial_1\varphi(U_t) \frac{1}{\sqrt{h}} H(\frac{X_t^* - X_t}{\sqrt{h}})]| = 0. \quad (31)$$

2. Secondly, we prove that

$$\lim_{h \rightarrow 0} \left| B_1^*(t, h) - E[\partial_1\varphi(X_t, X_t) \frac{1}{\sqrt{h}} H(\frac{X_t^* - X_t}{\sqrt{h}})] \right| = 0. \quad (32)$$

3. Thirdly, we prove that

$$\lim_{h \rightarrow 0} B_1^*(t, h) = \frac{1}{2} \int_{\mathbb{R}} \partial_1\varphi(a, a; t) p(a+, a; t) da.$$

4. Finally we observe that

$$\lim_{h \rightarrow 0} B_1^*(t, h) = \frac{1}{2} E[\partial_1\varphi(X_t, X_t) \frac{p(X_t+, X_t; t)}{g(X_t, t)}].$$

Step 1: The function H defined by $H(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} - x\Phi_G(-x)$ is differentiable with differential given by $x \mapsto \Phi_G(-x)$, which is positive and bounded by 1. Hence for all $(x, y) \in \mathbb{R}$, there exists $\lambda \in [0, 1]$ such that $H(y) - H(x) = (y - x)H'(\lambda x + (1 - \lambda)y)$. Thus there exists $\lambda \in [0, 1]$ such that

$$\frac{1}{\sqrt{h}} \left[H(\frac{X_t^* - X_t}{\sqrt{h}}) - H(\frac{X_t^* - X_t - mh}{\sqrt{h}}) \right] = mH'(\frac{X_t^* - X_t}{\sqrt{h}} + \lambda m\sqrt{h})$$

Then for $h > 0$,

$$\left| B_1^*(t, h) - E[\partial_1\varphi(U_t) \frac{1}{\sqrt{h}} H(\frac{X_t^* - X_t}{\sqrt{h}})] \right| \leq |m| \|\partial_1\varphi\|_{\infty} E\left[\sup_{x \geq \frac{X_t^* - X_t}{\sqrt{h}} - |m|\sqrt{h}} H'(x) \right].$$

The fact that almost surely $X_t^* - X_t > 0$ proves that the almost sure limit of $\frac{X_t^* - X_t}{\sqrt{h}} - |m|\sqrt{h}$ is $+\infty$. Moreover, $H' = \Phi_G$ is positive bounded and satisfies $\lim_{x \rightarrow \infty} H'(x) = 0$, so Lebesgue theorem achieves the proof of (31).

Step 2: Using regularity assumption on φ

$$\begin{aligned} & \left| B_1^*(t, h) - E[\partial_1\varphi(X_t, X_t) \frac{1}{\sqrt{h}} H(\frac{X_t^* - X_t}{\sqrt{h}})] \right| \leq \\ & |m|\sqrt{h} \|\partial_1\varphi\|_{\infty} + \|\partial_{11}^2\varphi\|_{\infty} E \left[\frac{X_t^* - X_t}{\sqrt{h}} H(\frac{X_t^* - X_t}{\sqrt{h}}) \right] \end{aligned}$$

The function $x \mapsto xH(x)$ is bounded on $\mathbb{R}^+$, and $\lim_{x \rightarrow \infty} xH(x) = 0$. Then, since $X_t^* - X_t$ law has a density and

$$\text{almost sure } \lim_{h \rightarrow 0} \frac{X_t^* - X_t}{\sqrt{h}} H\left(\frac{X_t^* - X_t}{\sqrt{h}}\right) = 0,$$

the dominated Lebesgue Theorem yields:

$$\lim_{h \rightarrow 0} E \left(\frac{X_t^* - X_t}{\sqrt{h}} H\left(\frac{X_t^* - X_t}{\sqrt{h}}\right) \right) = 0.$$

The proof of (32) is achieved.

Step 3: Introducing the density of the law of (X_t^*, X_t) according to Proposition 3.1 (i):

$$B_1^*(t, h) = \int_{\mathbb{R}^2} \partial \varphi_1(a, a) \frac{1}{\sqrt{h}} H\left(\frac{b-a}{\sqrt{h}}\right) p(b, a; t) da db + o(1).$$

We perform the change of variable $b = a + u\sqrt{h}$ and

$$B_1^*(t, h) = \int_{\mathbb{R} \times \mathbb{R}^+} \partial_1 \varphi(a, a) H(u) p(a + u\sqrt{h}, a, t) da du + o(1)$$

Note that, for all $a > 0, u > 0$, according to the Proposition 3.1 (ii),

$$\lim_{h \rightarrow 0} \partial_1 \varphi(a, a) H(u) p(a + u\sqrt{h}, a, u) = \partial_1 \varphi(a, a) H(u) p(a, a, t).$$

Jensen inequality induces

$$(B_1^*(t, h))^\delta \leq \int_{\mathbb{R} \times \mathbb{R}^+} (\partial_1 \varphi)^\delta(a, a) H^\delta(u) p^\delta(a + u\sqrt{h}, a, t) da du + o(1).$$

According to Proposition 6.2 and (39)

$$p^\delta(a + u\sqrt{h}, a, t) \leq C(\delta, T, m) \left[\frac{1}{t^\beta} \frac{1}{|a + 2u\sqrt{h} + mt|^\gamma} + \frac{1}{t^\beta} + 1 + t^\xi \right] e^{6(m_+)^2 \delta t}$$

Integrability and boundedness assumptions on $\partial_1 \varphi$ induces that for all h there exists D_i such that

$$\int_{\mathbb{R} \times \mathbb{R}^+} (\partial_1 \varphi)^\delta(a, a) H^\delta(u) p^\delta(a + u\sqrt{h}, a, t) da du \leq D_1 + D_2 \int_{\mathbb{R} \times \mathbb{R}^+} |\partial_1 \varphi|^\delta(a, a) \frac{1}{|a + 2u\sqrt{h} + mt|^\gamma} H^\delta(u) da du.$$

The integral with respect to da is shared in two parts factor, of $\int_{\mathbb{R}^+} H^\delta(u) du$:

$$\int_{|a + 2u\sqrt{h} + mt| \leq 1} |\partial_1 \varphi|^\delta(a, a) \frac{1}{|a + 2u\sqrt{h} + mt|^\gamma} da + \int_{|a + 2u\sqrt{h} + mt| > 1} |\partial_1 \varphi|^\delta(a, a) \frac{1}{|a + 2u\sqrt{h} + mt|^\gamma} da.$$

The second term is bounded by $\int_{\mathbb{R}} |\partial_1 \varphi|^\delta(a, a) da < \infty$, the first one is bounded by

$$\|\partial_1 \varphi\|_\infty \int_{|a + 2u\sqrt{h} + mt| \leq 1} \frac{1}{|a + 2u\sqrt{h} + mt|^\gamma} da = \|\partial_1 \varphi\|_\infty \int_{|x| \leq 1} |x|^{-\gamma} dx = \frac{2}{1 - \gamma} \|\partial_1 \varphi\|_\infty.$$

Thus for all $T > 0$ there exist $v \in]0, 1[$, $\xi \geq 0$ and a constant C such that

$$\sup_{h \in]0, 1]} \int_{\mathbb{R} \times \mathbb{R}^+} |\partial_1 \varphi(a, a)|^\delta H^\delta(u) p^\delta(a + u\sqrt{h}, a, t) dadu \leq C \left[\frac{1}{t^v} + 1 + t^\xi \right], \quad \forall t \in]0, T]. \quad (33)$$

The family $\{(a, u) \rightarrow \partial_1 \varphi(a, a) H(u) p(a + 2u\sqrt{h}, a, t), h \in [0, 1]\}$ is uniformly integrable with respect to Lebesgue measure $dadu$, so we can exchange the limit and the integral:

$$\lim_{h \rightarrow 0} B_1^*(t, h) = \int_{\mathbb{R} \times \mathbb{R}^+} \partial \varphi_1(a, a) H(u) p(a+, a; t) dadu.$$

Note that $\int_{\mathbb{R}^+} H(u) du = \frac{1}{2}$ ends the proof of this step.

Step 4: Propositions 4.2 4.3 and 4.4, it is proved that

$$a(t) = \lambda \mathbb{E} \int_{\mathbb{R}} [\varphi(U_t(y)) - \varphi(U_t)] F_Y(dy) + \mathbb{E} \left(\partial_2 \varphi(U_t) m + \frac{1}{2} \partial_{22}^2 \varphi(U_t) \right) + \frac{1}{2} \mathbb{E} \left[\partial_1 \varphi(X_t, X_t) \frac{p(X_t+, X_t, t)}{g(X_t, t)} \right].$$

The two first terms are bounded, so we have only to check (19) on the third term.

Similarly to Step 3, we get

$$\lim_{h \rightarrow 0} E[|\partial_1 \varphi(U_t)| \frac{1}{\sqrt{h}} H(\frac{X_t^* - X_t - mh}{\sqrt{h}})] = \int_{\mathbb{R} \times \mathbb{R}^+} |\partial \varphi_1(a, a)| H(u) p(a+, a; t) dadu.$$

so $a \mapsto \partial \varphi_1(a, a) p(a+, a; t)$ belongs to $L^1(\mathbb{R}, da)$. Then, since $g(., t)$ denotes the density of the law of X_t $a \mapsto \partial \varphi_1(a, a) \frac{p(a+, a; t)}{g(a; t)}$ belongs to $L^1(\mathbb{R}, P_{X_t})$.

Note that from estimation (33) for all $T > 0$ there exist $v \in]0, 1[$, $\xi \geq 0$ and a constant C such that

$$\left| \frac{1}{2} \mathbb{E} \left[\partial_1 \varphi(X_t, X_t) \frac{p(X_t+, X_t, t)}{g(X_t, t)} \right] \right| \leq C \left[\frac{1}{t^v} + 1 + t^\xi \right], \quad \forall t \in]0, T],$$

and the function a satisfies (19). □

5 Conclusion

The aim of this paper is to have a complete study of the law of one Lévy process X and its running supremum. Recall that X^* is not Markovian, but the pair $U := (X^*, X)$ is. In the second section, we give the main result (Theorem 2.1): the density of the law of U_t , its right continuity on the diagonal and a weakly valued-measure differential equation which characterizes the law of U_t .

To complete the study of the survival probability initiated by Coutin and Dorobantu [3], as a consequence, one gives the marginal density of the law of X_t^* (Corollary 3.3). A perspective could be the proof of regularity of the survival probability on $\mathbb{R}^+ \times \mathbb{R}$, meaning

$$(b, t) \rightarrow \mathbb{P}(\tau_b > t) = \mathbb{P}(X_t^* \leq b) = \int_0^b p^*(x, t) dx.$$

Another perspective is to study filtering, for instance to generalize W. Ngom [6] to incomplete observation.

6 Appendix

Lemma 6.1. *On the event $\{N_{t+h} = N_t + 1\}$, $|U_t(Y_{N_{t+h}}) - U_{t+h}| \leq 2 \sup_{0 \leq u \leq h} |\tilde{X}_{t+u} - \tilde{X}_t|$.*

Proof. : (a) On the event $\{N_{t+h} - N_t = 1\} = \cup_n \{N_t = n, N_{t+h} - N_t = 1\}$, we compute $U_t(Y_{N_{t+h}}) = (\max(X_t^*, X_t + Y_{N_{t+h}}); X_t + Y_{N_{t+h}})$ and $U_{t+h} = (X_{t+h}^*, X_{t+h})$.

(b) On the event $\{N_t = n, N_{t+h} = n + 1\}$,

$$U_t(Y_{N_{t+h}}) = (\max(X_t^*, X_t + Y_{n+1}); X_t + Y_{n+1}), \quad U_{t+h} = (X_{t+h}^*, X_{t+h}).$$

(c) We bound up $|U_t(Y_{N_{t+h}}) - U_{t+h}|$ component by component:

- Concerning the second component, on the event $\{N_t = n, N_{t+h} = n + 1\}$, we have

$$X_{t+h} = X_t + Y_{n+1} + (X_{t+h} - X_t - Y_{n+1}).$$

Since there is one only jump at time T_{n+1} for the process X between t and $t + h$, hence $X_{t+h} - X_t - Y_{n+1} = \tilde{X}_{t+h} - \tilde{X}_t$ and

$$|X_{t+h} - X_t - Y_{n+1}| \mathbf{1}_{\{N_t=n, N_{t+h}=n+1\}} \leq \sup_{0 \leq u \leq h} |\tilde{X}_{t+u} - \tilde{X}_t| \mathbf{1}_{\{N_t=n, N_{t+h}=n+1\}}. \quad (34)$$

- The first component is

$$X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1})$$

with

$$\begin{aligned} X_{t+h}^* &= \max(X_t^*, X_t + \sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t), X_t + Y_{n+1} + (\tilde{X}_{T_{n+1}} - \tilde{X}_t) + \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}})) \\ &= \max(X_t^*, X_t + \sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t), X_t + Y_{n+1} + \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t)). \end{aligned}$$

(a) On the event $\{X_t^* \geq X_t + Y_{n+1}\}$,

$$\begin{aligned} X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1}) &= \\ 0 \vee (X_t + \sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - X_t^*) \vee (X_t + Y_{n+1} + \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t) - X_t^*). \end{aligned}$$

Since $X_t \leq X_t^*$:

$$X_t + \sup_{t \leq u \leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - X_t^* \leq \sup_{t \leq u \leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t) \leq \sup_{t \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t)$$

and on the event $\{X_t^* \geq X_t + Y_{n+1}\}$

$$X_t + Y_{n+1} + \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t) - X_t^* \leq \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t) \leq \sup_{t \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t).$$

On this event, globally

$$0 \leq X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1}) \leq \sup_{0 \leq u \leq h} |\tilde{X}_{t+u} - \tilde{X}_t|. \quad (35)$$

(b) On the event $\{X_t^* < X_t + Y_{n+1}\}$, the first component is equal to

$$(X_t^* - X_t - Y_{n+1}) \vee \left(\sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - Y_{n+1} \right) \vee \left(\sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}}) \right). \quad (36)$$

On this event, the first element in (36) $(X_t^* - X_t - Y_{n+1}) \leq 0$ and the third one being non negative, thus the first component is $(\sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - Y_{n+1}) \vee (\sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}}))$.

As a conclusion, globally:

$$|X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1})| \mathbf{1}_{\{N_t=n, N_{t+h}=n+1\}} \leq \sup_{t \leq u \leq t+h} |\tilde{X}_u - \tilde{X}_t|. \quad (37)$$

Inequalities (34), (35) and (37) lead to the result. $\square$

6.1 Integrability properties

In the sequel, P is the random field defined by

$$P(b, a; t) := \sum_{k=0}^{N_t} \tilde{p}(b - X_{T_k}, a - X_{T_k} - Y_{k+1} \mathbf{1}_{\{t > T_{k+1}\}} - (X_t - X_{\min(t, T_{k+1})}); \min(t, T_{k+1}) - T_k) \quad (38)$$

which satisfies

$$p(b, a, t) \leq \mathbb{E}[P(b, a; t)]. \quad (39)$$

Proposition 6.2. *Let $\delta \in [1, \frac{3}{2}]$, there exists $(v, \gamma, \beta) \in [0, 1]^3$ and $\xi \geq 0$, such that for all $T > 0$ there exists a constant $C(\delta, T, m)$ satisfying for all $t \in]0, T]$, $b > \max(a, 0)$*

$$\mathbb{E} \left(P(b, a; t)^\delta \right) \leq C(\delta, T, m) \left[\frac{1}{t^v} \frac{1}{(2b - a + mt)^\gamma} + \frac{1}{t^\beta} + 1 + t^\xi \right] e^{6(m_-)^2 \delta t} \quad (40)$$

where $x_+ = \max(x, 0)$ and $x_- = \max(-x, 0)$ for any $x \in \mathbb{R}$.

The proof is based on the following three lemmas.

Recall that for all $t > 0$ the law of the pair $(\tilde{X}_t^*, \tilde{X}_t)$ has a density with respect to the Lebesgue measure given by

$$\tilde{p}(b, a; t) = \frac{2(2b - a)}{\sqrt{2\pi t^3}} e^{-\frac{(2b-a)^2}{2t} + ma - \frac{m^2 t}{2}} \mathbf{1}_{\{b > \max(0, a)\}}.$$

We have the following estimations on $\tilde{p}$.

Lemma 6.3. *For all $\alpha \in]0, 1[$, there exists a constant $C(\alpha, m)$ such that for all $t > 0$*

$$\tilde{p}(b, a; t) \leq C(\alpha, m) \left[\frac{1}{t^{1-\frac{\alpha}{2}} (2b - a + mt)^\alpha} + (m_-) \frac{1}{\sqrt{t}} \right] e^{-\frac{(2b-a+mt)^2}{8t} + 6(m_+)^2 t} \mathbf{1}_{\{b > \max(0, a)\}}. \quad (41)$$

Proof. We factorize

$$(2b - a)^2 - 2mat + m^2 t^2 = [a - (2b + mt)]^2 - 4bmt$$

and write

$$\tilde{p}(b, a; t) = \frac{2}{\sqrt{2\pi}} \left[\frac{(2b - a + mt)}{t\sqrt{t}} - \frac{m}{\sqrt{t}} \right] e^{-\frac{(2b-a+mt)^2}{2t} + 2bm} \mathbf{1}_{\{b > \max(0, a)\}}. \quad (42)$$

Let be $C_\beta := \sup_{x \in [0, +\infty[} x^\beta e^{-\frac{x^2}{4}} < +\infty$.

- For $m \leq 0$ and since $b > 0$ we obtain

$$\tilde{p}(b, a; t) \leq \frac{2}{\sqrt{2\pi}} \left[\frac{C_{1+\alpha}}{t^{1-\frac{\alpha}{2}}(2b-a+mt)^\alpha} + \frac{m_-}{\sqrt{t}} \right] e^{-\frac{(2b-a+mt)^2}{4t}} \mathbf{1}_{\{b > \max(0, a)\}},$$

and estimation (41) for $m \leq 0$.

- For $m > 0$, using (42)

$$\tilde{p}(b, a; t) \leq \frac{2}{\sqrt{2\pi}} \left[\frac{C_{1+\alpha}}{t^{1-\frac{\alpha}{2}}(2b-a+mt)^\alpha} \right] e^{-\frac{(2b-a+mt)^2}{4t} + 2bm} \mathbf{1}_{\{b > \max(0, a)\}}.$$

Since

$$\frac{(2b-a+mt)^2}{8t} - 2bm = \frac{1}{8t} ((2b-a)^2 + 2(2b-a)mt + m^2t^2 - 16bmt),$$

using $2b-a \geq b$ and $m > 0$,

$$\frac{(2b-a+mt)^2}{8t} + 2bm \geq \frac{1}{8t} (b^2 + 2bmt + m^2t^2 - 16bmt) = \frac{1}{8t} ((b-7mt)^2 - 48m^2t^2) \geq -6m^2t^2.$$

Thus

$$\tilde{p}(b, a; t) \leq \frac{2}{\sqrt{2\pi}} \left[\frac{C_{1+\alpha}}{t^{1-\frac{\alpha}{2}}(2b-a+mt)^\alpha} \right] e^{-\frac{(2b-a+mt)^2}{8t} + 6m^2t} \mathbf{1}_{\{b > \max(0, a)\}}.$$

We obtain the estimation (41) for $m > 0$.

□

Lemma 6.4. *Let $0 < \alpha < 1$ there exists a constant $C(\alpha)$ such that for all $T > 0, \sigma > 0$*

$$\sup_{c \in \mathbb{R}} \mathbb{E} \left(|c + \sigma G|^{-\alpha} e^{-\frac{(c+\sigma G)^2}{2T}} \right) \leq C(\alpha) \frac{T^{\frac{1-\alpha}{2}}}{\sigma}, \quad (43)$$

where G is a standard Gaussian variable.

Proof. First we prove inequality (43) for $\sigma = 1$. Let

$$I(c, T) := \mathbb{E} \left(|c + G|^{-\alpha} e^{-\frac{(c+G)^2}{2T}} \right) = I(c, T, +) + I(c, T, -) \quad (44)$$

where

$$I(c, T, \pm) := \mathbb{E} \left((c + G)_\pm^{-\alpha} e^{-\frac{(c+G)^2}{2T}} \right). \quad (45)$$

Using the density of G

$$I(c, T, +) = \int_{-c}^{\infty} (c + g)^{-\alpha} e^{-\frac{(c+g)^2}{2T}} \frac{e^{-\frac{g^2}{2}}}{\sqrt{2\pi}} dg$$

and an integration by part

$$I(c, T, +) = \int_{-c}^{\infty} \frac{1}{1-\alpha} \left[\frac{(c+g)^{2-\alpha}}{T} + g(c+g)^{1-\alpha} \right] e^{-\frac{(c+g)^2}{2T}} \frac{e^{-\frac{g^2}{2}}}{\sqrt{2\pi}} dg.$$

Note that $x \mapsto x^\beta e^{-\frac{x^2}{4}}$ is bounded on $\mathbb{R}^+$ by C_β then

$$I(c, T, +) \leq \int_{-c}^{\infty} \frac{1}{1-\alpha} \left[C_{2-\alpha} T^{-\frac{\alpha}{2}} \frac{e^{-\frac{(c+g)^2}{4T}}}{\sqrt{2\pi}} + C_1 C_{1-\alpha} T^{\frac{1-\alpha}{2}} \frac{e^{-\frac{g^2}{4}}}{\sqrt{2\pi}} \right] dg.$$

Integrating with respect to g yields

$$I(c, T, +) \leq 2 \left[C_{2-\alpha} T^{\frac{1-\alpha}{2}} + C_1 C_{1-\alpha} T^{\frac{1-\alpha}{2}} \right]. \quad (46)$$

Using the same lines we obtain

$$I(c, T, -) \leq 2 \left[C_{2-\alpha} T^{\frac{1-\alpha}{2}} + C_1 C_{1-\alpha} T^{\frac{1-\alpha}{2}} \right]. \quad (47)$$

Plugging inequalities (46) and (47) into (44) yields (43) for $\sigma = 1$.

Replace c by $\frac{c}{\sigma}$ and T by $\frac{T}{\sigma^2}$ yields (43). $\square$

Lemma 6.5. *For all $\alpha < 1, \beta < 1, \delta > 0, T > 0$ there exists a constant $C(\alpha, \beta, \delta, T)$ such that for all $t \in]0, T]$,*

$$\mathbb{E} \left(\mathbf{1}_{\{N_t > 0\}} (N_t + 1)^\delta \left[T_{N_t}^{-\beta} (t - T_{N_t})^{-\alpha} + \sum_{k=0}^{N_t-1} (t - T_{k+1})^{-\alpha} (T_{k+1} - T_k)^{-\beta} \right] \right) \leq C(\alpha, \beta, \delta, T) t^{1-\alpha-\beta}. \quad (48)$$

Proof. Let $S(1, \alpha, \beta, \delta, t)$ be the random variable defined by

$$S(1, \alpha, \beta, \delta, t) := \mathbb{E} \left(\mathbf{1}_{\{N_t > 0\}} (N_t + 1)^\delta \left[T_{N_t}^{-\beta} (t - T_{N_t})^{-\alpha} \right] \right). \quad (49)$$

Note that $(T_{k+1} - T_k)_{k \geq 0}$ are independent variables, identically distributed and T_1 follows an exponential distribution with parameter λ .

Then, $S(1, \alpha, \beta, \delta, t) = \sum_{n=1}^{\infty} S(1, \alpha, \beta, \delta, t; n)$ where for $n \geq 1$:

$$S(1, \alpha, \beta, \delta, t; n) = (n+1)^\delta \int_{[0, \infty[^{n+1}} (u_1 + \dots + u_n)^{-\beta} (t - u_1 - \dots - u_n)^{-\alpha} \lambda^{n+1} e^{-\lambda(u_1 + \dots + u_{n+1})} \mathbf{1}_{\{u_1 + \dots + u_n \leq t \leq u_1 + \dots + u_{n+1}\}} du_1 \dots du_n. \quad (50)$$

We integrate with respect to u_{n+1} between $t - u_1 - \dots - u_n$ and infinity

$$S(1, \alpha, \beta, \delta, t; n) = \lambda^n (n+1)^\delta e^{-\lambda t} \int_{[0, \infty[^n} (u_1 + \dots + u_n)^{-\beta} (t - u_1 - \dots - u_n)^{-\alpha} \mathbf{1}_{\{u_1 + \dots + u_n \leq t\}} du_1 \dots du_n.$$

We perform the change of variable $tv_i = u_1 + \dots + u_i, i = 1, \dots, n$

$$S(1, \alpha, \beta, \delta, t; n) = \lambda^n t^{n-\alpha-\beta} e^{-\lambda t} \int_{[0, 1]^n} (v_n)^{-\beta} (1 - v_n)^{-\alpha} \mathbf{1}_{\{v_1 < \dots < v_n \leq 1\}} dv_1 \dots dv_n.$$

We integrate with respect to $v_1, \dots, v_{n-1}$

$$\begin{aligned} S(1, \alpha, \beta, \delta, t; n) &= \frac{(n+1)^\delta \lambda^n t^{n-\alpha-\beta}}{(n-1)!} e^{-\lambda t} \int_0^1 (v_n)^{-\beta+n-1} (1-v_n)^{-\alpha} dv_n \\ &= \frac{(n+1)^\delta \lambda^n t^{n-\alpha-\beta}}{(n-1)!} e^{-\lambda t} B(n-\beta, 1-\alpha) \end{aligned} \quad (51)$$

The radius of convergence of entire series $(\sum_{n \geq 1} \frac{(n+1)^\delta \lambda^n t^{n-1}}{(n-1)!} B(n-\beta, 1-\alpha))$ is infinite, so the series is continuous on $\mathbb{R}$ and then bounded on $[0, T]$. There exists a constant $C_1(\alpha, \beta, \delta, T)$ such that for all $t \in]0, T]$ adding inequalities (51) yields

$$S(1, \alpha, \beta, \delta, t) \leq C_1(\alpha, \beta, \delta, T) t^{1-\alpha-\beta}. \quad (52)$$

Let $S(2, \alpha, \beta, \delta, t)$ be the random variable defined by

$$S(2, \alpha, \beta, \delta, t) := \mathbb{E} \left(\mathbf{1}_{\{N_t > 0\}} (N_t + 1)^\delta \sum_{k=0}^{N_t-1} \left[(t - T_{k+1})^{-\beta} (T_{k+1} - T_k)^{-\alpha} \right] \right). \quad (53)$$

Note that $(T_{k+1} - T_k)_{k \geq 0}$ are independent variables, identically distributed and T_1 follows an exponential distribution with parameter λ .

Then,

$$S(2, \alpha, \beta, \delta, t) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} S(2, \alpha, \beta, \delta, t; n, k)$$

where for $n \geq 1, 0 \leq k \leq n-1$

$$\begin{aligned} &S(2, \alpha, \beta, \delta, t; n, k) \\ &= (n+1)^\delta \int_{[0, \infty]^{n+1}} (t - u_1 - \dots - u_{k+1})^{-\beta} (u_{k+1})^{-\alpha} \lambda^{n+1} e^{-\lambda(u_1 + \dots + u_{n+1})} \mathbf{1}_{\{u_1 + \dots + u_n \leq t \leq u_1 + \dots + u_{n+1}\}} du_1 \dots du_{n+1}. \end{aligned} \quad (54)$$

We integrate with respect to u_{n+1} between $t - u_1 - \dots - u_n$ and infinity

$$S(2, \alpha, \beta, \delta, t; n, k) = (n+1)^\delta e^{-\lambda t} \lambda^n \int_{[0, \infty]^n} (t - u_1 - \dots - u_{k+1})^{-\beta} (u_{k+1})^{-\alpha} \mathbf{1}_{\{u_1 + \dots + u_n \leq t\}} du_1 \dots du_n.$$

We perform the change of variable $tv_i = u_1 + \dots + u_i, i = 1, \dots, n$

$$S(2, \alpha, \beta, \delta, t; n, k) = (n+1)^\delta t^{n-\alpha-\beta} e^{-\lambda t} \lambda^n \int_{[0, 1]^n} (1 - v_{k+1})^{-\beta} (v_{k+1} - v_k)^{-\alpha} \mathbf{1}_{\{v_1 \leq \dots \leq v_n \leq 1\}} dv_1 \dots dv_n.$$

We integrate with respect to $v_1, \dots, v_k$ and $v_{k+2}, \dots, v_n$

$$\begin{aligned} S(2, \alpha, \beta, \delta, t; n, k) &= \frac{(n+1)^\delta t^{n-\alpha-\beta} e^{-\lambda t} \lambda^n}{k!(n-k)!} \int_{[0, 1]^2} (1 - v_{k+1})^{-\beta+n-k} (v_{k+1} - v_k)^{-\alpha+k} dv_k dv_{k+1} \\ &\leq \frac{(n+1)^\delta t^{n-\alpha-\beta} e^{-\lambda t} \lambda^n}{n!} \frac{k!(n-k)!}{n!} \max(1, B(1-\beta, 1-\alpha)) \end{aligned} \quad (55)$$

since

$$0 \leq \int_{[0, 1]^2} (1 - v_{k+1})^{-\beta+n-k} (v_{k+1} - v_k)^{-\alpha+k} dv_k dv_{k+1} \leq \max(1, \int_{[0, 1]^2} (1 - v_{k+1})^{-\beta} (v_{k+1} - v_k)^{-\alpha} dv_k dv_{k+1}).$$

Then, adding inequalities (55) from $k = 0$ to $n - 1$

$$\sum_{k=0}^n S(2, \alpha, \beta, \delta, t; n, k) \leq \max \left(1, \frac{\Gamma(-\alpha + 1)\Gamma(-\beta + 1)}{\Gamma(-\alpha - \beta + 1)} \right) \frac{(n+1)^\delta 2^n t^{n-\alpha-\beta} e^{-\lambda t} \lambda^n}{n!}. \quad (56)$$

The radius of convergence of entire series $(\sum_{n \geq 1} \frac{(n+1)^\delta \lambda^n 2^n t^{n-1}}{(n)!})$ is infinite, so the series is continuous on $\mathbb{R}$ and then bounded on $[0, T]$: there exists a constant $C_2(\alpha, \beta, \delta, T)$ such that for all $t \in]0, T]$ and adding inequalities (56) yields

$$S(2, \alpha, \beta, \delta, t) \leq C_2(\alpha, \beta, \delta, T) t^{1-\alpha-\beta}. \quad (57)$$

Inequality (48) is a consequence of inequalities (52) and (57). $\square$

Proof. of the proposition: recall

$$P(b, a; t) := \sum_{k=0}^{N_t} \tilde{p} \left(b - X_{T_k}, a - X_{T_k} - Y_{k+1} \mathbf{1}_{\{t > T_{k+1}\}} - (X_t - X_{\min(t, T_{k+1})}); \min(t, T_{k+1}) - T_k \right).$$

- Note that from Lemma 6.3, there exists a constant $C(\alpha, m)$ such that

$$P(b, a; t) \mathbf{1}_{\{N_t=0\}} \leq C(\alpha, m) \left[\frac{1}{t^{1-\frac{\alpha}{2}} (2b - a + mt)^\alpha} + (m)_+ \frac{1}{\sqrt{t}} \right] e^{6(m_+)^2 t} \mathbf{1}_{\{b > \max(0, a)\}}$$

Taking $\xi = 0$, $\beta = \frac{\delta}{2}$. Since $\delta \in [1, \frac{3}{2}]$ then $2(1 - \frac{1}{\delta}) < \frac{1}{\delta}$. If $\alpha = 1 - \frac{1}{2\delta}$ then $\gamma := \alpha\delta = \delta - \frac{1}{2} < 1$ and $v := \delta(1 - \frac{\alpha}{2}) < 1$. Then, using $(x + y)^\delta \leq 2^{\delta-1} [x^\delta + y^\delta]$ there exists a constant $C_0(\delta)$ such that for all $t > 0$, $b > \max(a, 0)$

$$\mathbb{E} \left(P(b, a; t)^\delta \mathbf{1}_{\{N_t=0\}} \right) \leq C_0(\delta) \left[\frac{1}{t^v} \frac{1}{(2b - a + mt)^\gamma} + \frac{1}{t^\beta} \right] e^{6(m_+)^2 t \delta}. \quad (58)$$

- Using estimation (41) and development (38)

$$P(b, a; t) := \sum_{k=0}^{N_t} \tilde{p} \left(b - X_{T_k}, a - X_{T_k} - Y_{k+1} \mathbf{1}_{\{t > T_{k+1}\}} - (X_t - X_{\min(t, T_{k+1})}); \min(t, T_{k+1}) - T_k \right),$$

$$P(b, a; t) \mathbf{1}_{\{N_t > 0\}} \leq C(\alpha, m) \mathbf{1}_{\{N_t > 0\}} \sum_{k=0}^{N_t} P_k(b, a, t) \text{ where}$$

$$P_k(b, a; t) \leq \left[\frac{1}{(t \wedge T_{k+1} - T_k)^{1-\alpha/2} Z_k(t)} + \frac{m_-}{\sqrt{t \wedge T_{k+1} - T_k}} \right] e^{-\frac{Z_k(t)^2}{8t}} e^{6(m_-)^2 (\min(t, T_{k+1}) - T_k)}$$

$$\text{and } Z_k(t) := (2b - a - X_{T_k} + Y_{k+1} \mathbf{1}_{\{t > T_{k+1}\}} + X_t - X_{t \wedge T_{k+1}} + m(T_{k+1} \wedge t - T_k).$$

We define a family $(C_k(t), \sigma_k(t), S_k(t), G_k(t))_{k \in \mathbb{N} \cup \{0\}}$ by

- $(C_k(t), \sigma_k(t), S_k(t), G_k(t)) = (0, 0, 0, 0)$ for $k > N_t$,
- for $k < N_t$:

$$C_k(t) = 2b - a - X_{T_k} + \sum_{i=k+1}^{N_t} Y_i + m(t - T_{k+1}) + m(T_{k+1} - T_k);$$

$$\sigma_k(t) = \sqrt{t - T_{k+1}}; \quad S_k(t) = T_{k+1} - T_k; \quad G_k(t) = \frac{B_t - B_{T_{k+1}}}{\sqrt{t - T_{k+1}}}$$

• for $k = N_t$

$$C_k(t) = 2b - a - \sum_{i=0}^{N_t} Y_i - mT_{N_t} + m(t - T_{N_t}); \quad \sigma_k(t) = \sqrt{T_{N_t}}; \quad S_k(t) = t - T_{N_t}; \quad G_k(t) = \frac{B_{T_{N_t}}}{\sqrt{T_{N_t}}}.$$

Then, from estimation (41) and the definition of $(C_k(t), \sigma_k(t), S_k(t), G_k(t))$

$$P(b, a; t) \mathbf{1}_{\{N_t \neq 0\}} \leq \mathbf{1}_{\{N_t \neq 0\}} \sum_{k=0}^{N_t} \left[\frac{1}{S_k^{1-\frac{\alpha}{2}} (C_k + \sigma_k G_k)^\alpha} + (m)_- \frac{1}{\sqrt{S_k}} \right] e^{6(m_+)^2 t} \mathbf{1}_{\{b > \max(0, a)\}}$$

Note that $S_k \leq t$ and using Jensen inequality

$$P(b, a; t)^\delta \mathbf{1}_{\{N_t \neq 0\}} \leq \mathbf{1}_{\{N_t \neq 0\}} (N_t + 1)^\delta \sum_{k=0}^{N_t} \left[\frac{1}{S_k^{\delta(1-\frac{\alpha}{2})} (C_k + \sigma_k G_k)^{\alpha\delta}} + (m)_-^\delta \frac{1}{\sqrt{S_k}^\delta} \right] e^{6\delta(m_+)^2 t} \mathbf{1}_{\{b > \max(0, a)\}}$$

Conditionally to $\{N_t \neq 0\}$, the random variable G_k are independent of C_k, σ_k and S_k and their law is the standard Gaussian distribution. Using Lemma 6.5 for $\sigma = \sigma_k$ and $T = S_k$ yields there exists a constant $C(\alpha, \delta)$ such that

$$\begin{aligned} \mathbb{E} \left(P(b, a; t)^\delta \mathbf{1}_{\{N_t \neq 0\}} \right) &\leq \\ C(\alpha, \delta) \mathbb{E} \left(\mathbf{1}_{\{N_t \neq 0\}} (N_t + 1)^\delta \sum_{k=0}^{N_t-1} \left[\frac{1}{(T_{k+1} - T_k)^{\delta-\frac{1}{2}} \sqrt{t - T_{k+1}}} + (m)_-^\delta \frac{1}{\sqrt{t - T_{k+1}}} \right] \right) &e^{6\delta(m_+)^2 t} \mathbf{1}_{\{b > \max(0, a)\}} \\ + C(\alpha, \delta) \mathbb{E} \left(\mathbf{1}_{\{N_t \neq 0\}} (N_t + 1)^\delta \left[\frac{1}{T_{N_t}^{\delta-\frac{1}{2}} \sqrt{t - T_{N_t}}} + (m)_-^\delta \frac{1}{\sqrt{T_{N_t}}} \right] \right) &e^{6\delta(m_+)^2 t} \mathbf{1}_{\{b > \max(0, a)\}}. \end{aligned}$$

According to Lemma 6.5, there exists a constant $C(\alpha, \delta, T)$ such that for all $t \in]0, T]$

$$\mathbb{E} \left(P(b, a; t)^\delta \mathbf{1}_{\{N_t \neq 0\}} \right) \leq C(\alpha, \delta, T) \left[t^{1-\delta} + t^{\frac{1-\delta}{2}} \right] e^{6\delta(m_+)^2 t} \mathbf{1}_{\{b > \max(0, a)\}}. \quad (59)$$

Adding inequalities (58) and (59) yields inequality (40). □

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