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Louis De Mesnard

LATEC

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UNIVERSITE
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Pôle d'Economie et de Gestion

2, boulevard Gabriel - 21000 DIJON - Tél. 03 80 39 54 30 - Fax 03 80 39 54 43

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INTERN CONFLICTS AND A PRODUCTION FUNCTION WITH COMBINING SUB-FACTORS

Louis de MESNARD*

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LATEC - URA 342 CNRS, Université de Bourgogne

ABSTRACT.

Considering sub-factors, like different categories of capital or different categories of work, a production function is proposed: each sub-factor of one factor may be combined with each sub-factor of the other factor. Different types of programs exists at different levels. The conditions to obtain identical solution for these programs are studied. Considering that there is one responsible manager by program, conflicts between managers of each program are possible because these programs give non identical solutions in general. The important fact is that these intern conflicts appear naturally without extra economical considerations.

KEYWORDS.

Production function, factors of production, firm, conflicts.

1. INTRODUCTION

In the neo-classical production functions, there are some factors, capital, labour, etc. It is the case of the Cobb-Douglas and CES production functions). Both of these functions can be generalised to an undetermined number of factors. However, the factors are globally combined, and are not combined separately, one with another, that is to say by couple.

For example, in the production function of Uzawa [UZAWA 1962], there are groups of factors. Into the groups, the function is CES (there is a different function by group), between the groups, the function is Cobb-Douglas. Into a group, the combination of factors remains global, between the groups, the combination of groups remains global. It is the same thing for the production function of Sato [SATO 1967]. We have groups of factors; into each groups, the factors are globally combined by a CES function (there is a different function by group), and between the groups, the groups are combined by a CES function. It is interesting to point out that these two production functions were created to introduce an elasticity of substitution variable from one couple of factors to one another couple of factors, in the CES (were the elasticity of substitution is normally constant), and not to introduce couple combination of factors.

In these production functions, factors remain combined globally into one group. Factors are not combined two by two: if one factor is present into one group, it is not present into any other group. Moreover, the logic of division of factors into groups is not clear. Are they sub-processes of production?

For the following, we will think in terms of factors and sub-factors. Factors are classical: labour, capital, and so on. Sub-factors of a factors are some type of this factor: non-qualified labour and qualified labour are sub-factors of the factor labour, machines and immaterial capital are sub-factors of the factor capital.

In the classical approach, we cannot combine sub-factors of one type with sub-factors of one another type, because every sub-factors are on the same set. For example, if we decompose labour into non-qualified labour and qualified labour, and capital into machines and immaterial capital, we have one single set of factors:

{ non-qualified labour, qualified labour, machines, immaterial capital }.

Each of these sub-factors contributes to the production in the same way: the manner that factors are combined does not play any role. It is the principle of the Leontief approach: for example, we have many sectors of services (to firms, to households, etc.) instead of one sector of services.

It would be better to have a production functions allowing to combine sub-factors two by two. For example, we may want to combine non-qualified labour with machines, non-qualified labour with immaterial capital, qualified labour with machines and qualified labour with immaterial capital. That is to say, we may want to form a type of matrix:

	non-qualified labour	**qualified labour**
machines	X	X
immaterial capital	X	X

Table 1. The combinations

Note that couple combination of inputs may exists in the literature. For example, in the function of Gutenberg [KRELLE 1969 and 1970], a production function with complementary factors which is a generalisation of the Leontief production function, there are aggregate corresponding to stock factors. Each aggregate, like welding, provide services, like a certain number of welds. As each aggregate needs some consumer inputs, like work, raw materials, etc., we have a matrix aggregates / consumer inputs. Even if each consumer input provide services to each agregate, there is no symmetry of treatment between aggregates and consumer inputs. The aim of Gutenberg is to introduce stock mean of production: its production function is suitable to take into account only these two special categories of factors, and not to take into account every category of factors.

2. DEFINITION OF THE PRODUCTION FUNCTION WITH COMBINING SUB-FACTORS

We call this production function the *Production Function with Combining Sub-Factors*; its name is summarised by the acronym *CSF* .

2.1. The production function

In our production function, we consider a function of production of a ***single output***, in a ***single process of production***, with two types of factors, for example, machines and labour. Each machine uses any types of labour, any type of labour is used on any machine. Two tables will indicate how each sub-factor of one factor uses each sub-factor of the other factor.

Consider two types of factors. The first type of factor has I sub-factors noted i. The second type of factors has J sub-factors noted j.

Denote x_{ij} the quantity of sub-factor i , of the first type, combined with the sub-factor j , of the second type, measured in unit of sub-factor i.

Denote y_{ij} the quantity of sub-factor j , of the second type, combined with the sub-factor i , of the first type, measured in unit of sub-factor j.

- To produce, we combine each sub-factor of one factor with each sub-factor of the other factor: we form every couple $\{i, j\}$ of sub-factors. When the sub-factor i and the sub-factor j are combined, we put x_{ij} units of i and y_{ij} units of j. These two quantities of factors contribute to the production of the single output for an amount:

  $Q_{ij} = \mathrm{f}(x_{ij}, y_{ij})$.

We call it the contribution of the couple of sub-factors $\{i, j\}$.

- We do the hypothesis that the total production is the sum of the partial production, i.e.:

$$Q = \sum_{i=1}^{I} \sum_{j=1}^{J} Q_{ij} = \sum_{i=1}^{I} \sum_{j=1}^{J} \mathrm{f}(x_{ij}, y_{ij}).$$

It is obvious that there are many other ways to combine elementary productions.

- Denote $Q_i^x = \sum_{j=1}^{J} Q_{ij} = \sum_{j=1}^{J} \mathrm{f}(x_{ij}, y_{ij})$ as the contribution of the sub-factor i (of the first factor) to the production.

- Denote $Q_j^y = \sum_{i=1}^{I} Q_{ij} = \sum_{i=1}^{I} \mathrm{f}(x_{ij}, y_{ij})$ as the contribution of the sub-factor j (of the second factor) to the production.

Remarks.

- The CSF remains neo-classical.

- If $J = 1$ and $I = 1$, we have a simple neo-classical production function.

- For the production function of many outputs, or for a multistage process of production, we have many matrices (for two factors).

- The CSF may be applied to macroeconomics. Sub-factors are aggregates, like individual firms, small firms, great firms, banks, etc, or like households, rentals, etc.

2.2. Generalisation

It is possible to generalise it to the case of three or more factors, strongly increasing the number of inter conflicts. This is specially suited for macroeconomics.

For three factors, we have:

$$Q=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}Q_{ijk}=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\mathrm{f}\left(x_{ijk},y_{ijk},z_{ijk}\right)$$

$$C=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}C_{ij}=\sum_{i=1}^{I}C_i^x+\sum_{j=1}^{J}C_j^y=\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\left(x_{ijk}\,p_i^x+y_{ijk}\,p_j^y+z_{ijk}\,p_k^z\right)$$

3. MAXIMISATION OF PRODUCTION UNDER A CONSTRAINT OF COST

3.1. The isoquants

We have three great types of isoquants:

- An elementary isoquant is the set of points obtained when x_{ij} and y_{ij} vary, Q_{ij} remaining fixed.
- A partial isoquant is the set of points obtained when x_{ij} and y_{ij} vary, Q_i^x or Q_j^y remaining fixed. There are two sets of partial isoquants.
- A global isoquant is the set of points obtained when x_{ij} and y_{ij} vary, Q remaining fixed.

Note that, these three categories of isoquants are confused in a simple production function with $J=1$ and $I=1$.

3.2. The costs

We have three great types of costs:

- The elementary costs, that is to say the cost attached to the contribution of the couple of sub-factors $\{i,j\}$ is:

  $C_{ij}=x_{ij}\,p_i^x+y_{ij}\,p_j^y$,

  where p_i^x is the price of the sub-factor i of type x and p_j^y is the price of the sub-factor j of type y.

- The total cost attached to the combination of i with all j i.e. the cost attached to the total contribution of the sub-factor i (i belonging to the factor x) is:

$$C_i^x = \sum_{j=1}^{J} \left(x_{ij}\, p_i^x + y_{ij}\, p_j^y \right)$$

and the total cost attached to the combination of j with all i i.e. the cost attached to the total contribution of the sub-factor j (j belonging to the factor y) is:

$$C_j^y = \sum_{i=1}^{I} \left(x_{ij}\, p_i^x + y_{ij}\, p_j^y \right)$$

We call these the partial costs.

- The total cost of production is:

$$C = \sum_{i=1}^{I} \sum_{j=1}^{J} C_{ij} = \sum_{i=1}^{I} C_i^x = \sum_{j=1}^{J} C_j^y = \sum_{i=1}^{I} \sum_{j=1}^{J} \left(x_{ij}\, p_i^x + y_{ij}\, p_j^y \right).$$

Remark. The cost in each sub-factor may be defined: for example, the cost in sub-factor i, that is to say $\sum_{i=1}^{I} x_{ij}\, p_i^x$ or $\sum_{j=1}^{J} y_{ij}\, p_j^y$. Thus some other type of programs may be defined.

3.3. Maximisation

The different types of costs and the different type of isoquants may be combined to calculate an optimum, giving sixteen great types of possible substitutions.

Costs / *Contributions*	Elementary costs of couple $\{i, j\}$	Partial costs for sub-factor i	Partial costs for sub-factor j	Total Costs
Contribution of couple $\{i, j\}$	Max Q_{ij} under C_{ij}	Max Q_{ij} under C_i^x	Max Q_{ij} under C_j^y	Max Q_{ij} under C
Contribution of sub-factor i	Max Q_i^x under C_{ij}	Max Q_i^x under C_i^x	Max Q_i^x under C_j^y	Max Q_i^x under C
Contribution of sub-factor j	Max Q_j^y under C_{ij}	Max Q_j^x under C_i^x	Max Q_j^y under C_j^y	Max Q_j^y under C
Production	Max Q under C_{ij}	Max Q under C_i^x	Max Q under C_j^y	Max Q under C

Table 2. Typology of programs

In fact, programs in a same column are identical: there are four different types of program only, that is to say $I\,J + I + J + 1$ programs: if we have only two sub-factors by factors, we have the amount of nine programs to solve!

4. MANAGERS AND CONFLICTS

4.1. Comparison of programs

What are the conditions to obtain identical solutions for the different programs? For example, let us study three programs, an elementary program (the first in the table 2), a partial program (a program in the middle of the table 2), the global program (the last in the table 2). If elementary contribution to production are in a Cobb-Douglas form [1], the annexe shows that the solutions are:

- Max $Q_{ij} = \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$ under $C_{ij} = x_{ij}\, p_i^x + y_{ij}\, p_j^y$:

$$\begin{cases} x_{ij} = \dfrac{\beta_{ij}}{\beta_{ij}+\gamma_{ij}} \dfrac{C_{ij}}{p_i^x} \\ y_{ij} = \dfrac{\gamma_{ij}}{\beta_{ij}+\gamma_{ij}} \dfrac{C_{ij}}{p_j^y} \end{cases} \text{, for every } i \text{ and } j$$

- Max $Q_i^x = \sum_{j=1}^{J} \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$, for every i , under $C_i^x = \sum_{j=1}^{J} \left(x_{ij}\, p_i^x + y_{ij}\, p_j^y \right)$:

$$\begin{cases} x_{ij} = \dfrac{\beta_{ij}\, Q_{ij}}{\sum_{j=1}^{J} \left(\beta_{ij}+\gamma_{ij}\right) Q_{ij}} \dfrac{C_i^x}{p_i^x} \\ y_{ij} = \dfrac{\gamma_{ij}\, Q_{ij}}{\sum_{j=1}^{J} \left(\beta_{ij}+\gamma_{ij}\right) Q_{ij}} \dfrac{C_i^x}{p_j^y} \end{cases} \text{, for every } i \text{ and } j$$

- Max $Q = \sum_{i=1}^{I} \sum_{j=1}^{J} \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$ under $C = \sum_{i=1}^{I} \sum_{j=1}^{J} C_{ij} = \sum_{i=1}^{I} \sum_{j=1}^{J} \left(x_{ij}\, p_i^x + y_{ij}\, p_j^y \right)$:

$$\begin{cases} x_{ij} = \dfrac{\beta_{ij}\, Q_{ij}}{\sum_{i=1}^{I} \sum_{j=1}^{J} \left(\beta_{ij}+\gamma_{ij}\right) Q_{ij}} \dfrac{C}{p_i^x} \\ y_{ij} = \dfrac{\gamma_{ij}\, Q_{ij}}{\sum_{i=1}^{I} \sum_{j=1}^{J} \left(\beta_{ij}+\gamma_{ij}\right) Q_{ij}} \dfrac{C}{p_j^y} \end{cases} \text{, for every } i \text{ and } j$$

The first two programs provide identical solutions when

[1] However, the production is not a simple generalisation of the ordinary Cobb-Douglas function.

$$\frac{\left(\beta_{ij}+\gamma_{ij}\right)Q_{ij}}{\sum_{j=1}^{J}\left(\beta_{ij}+\gamma_{ij}\right)Q_{ij}}=\frac{C_{ij}}{C_i^x}$$

and if $\beta_{ij}+\gamma_{ij}=1$ for every i and j (first degree homogeneity), when

$$\frac{Q_{ij}}{Q_i^x}=\frac{C_{ij}}{C_i^x}\Leftrightarrow\overline{C}_{ij}=\overline{C}_i^x \text{ denoting } \overline{C}_{ij}=\frac{C_{ij}}{Q_{ij}} \text{ and } \overline{C}_i^x=\frac{C_i^x}{Q_i^x}.$$

The two last program provide identical solutions when

$$\frac{\sum_{j=1}^{J}\left(\beta_{ij}+\gamma_{ij}\right)Q_{ij}}{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\beta_{ij}+\gamma_{ij}\right)Q_{ij}}=\frac{C_i^x}{C}$$

and if $\beta_{ij}+\gamma_{ij}=1$ for every i and j, when

$$\frac{Q_i^x}{Q}=\frac{C_i^x}{C}\Leftrightarrow\overline{C}_i^x=\overline{C} \text{ denoting } \overline{C}=\frac{C}{Q}.$$

The first and the last program provide identical solutions when

$$\frac{\left(\beta_{ij}+\gamma_{ij}\right)Q_{ij}}{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\beta_{ij}+\gamma_{ij}\right)Q_{ij}}=\frac{C_{ij}}{C}$$

and if $\beta_{ij}+\gamma_{ij}=1$ for every i and j, when

$$\frac{Q_{ij}}{Q}=\frac{C_{ij}}{C}\Leftrightarrow\overline{C}_{ij}=\overline{C}.$$

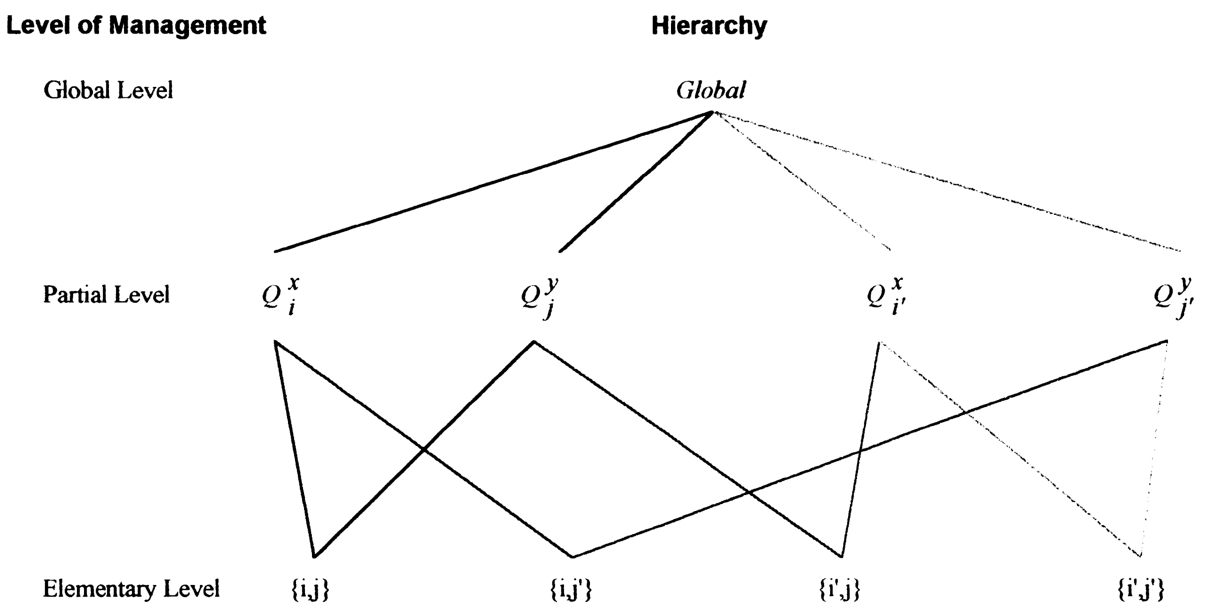
4.2. The managers

We may suppose that there is **a manager by program** inside the firm, responsible for the correct maximisation of the production under the constraint of costs. Each program looks like a "center of cost". In the organisational practice, often people consider "centers of profit".

There is a slight difference: a center of profit has under its control its receipts and its costs (even if its receipts are theoretical when the center of profit sells to another service into the firm).

There are the general manager for the general program, a sub-factor manager for each contribution of sub-factor Q_i^x and Q_j^y , a manager of each contribution of couple of sub-factors Q_{ij} .

When indices are corresponding, for example a couple $\{i, j\}$ and a sub-factor i , managers are in a **hierarchy**. Naturally, when indices are not corresponding, for example a couple $\{i, j\}$ and a sub-factor i', managers are not hierarchically placed but transversally. The following diagram illustrates this.



<u>Remark</u>. Here the hierarchy concerns only one process of production, and not vertically integrated processes into one firm. The hierarchy is not the hierarchy of processes, but it is only a hierarchy between managers combining two sub-factors, managers responsible of one sub-factor and the general manager, into one process.

4.3. Possibility of conflicts

We say that there is a conflict when the program of one manager is not compatible with the program of one another manager.

The conditions to have no **hierarchical conflicts** are clear (with Cobb-Douglas form and first degree homogeneity): all mean cost (elementary, partial and total) must be equal. These conditions seem to be very difficult to reach in practical situations. We summarise it into the following table (with Cobb-Douglas form and first degree homogeneity):

Program / ***Program***	$\{i,j\}$	**sub-factor** i	**sub-factor** j	**General**
$\{i,j\}$	no conditions	$\overline{C}_{ij} = \overline{C}_i^x$	$\overline{C}_{ij} = \overline{C}_j^y$	$\overline{C}_{ij} = \overline{C}$
sub-factor i	$\overline{C}_{ij} = \overline{C}_i^x$	no conditions	not hierarchical	$\overline{C}_i^x = \overline{C}$
sub-factor j	$\overline{C}_{ij} = \overline{C}_j^y$	not hierarchical	no conditions	$\overline{C}_j^y = \overline{C}$
General	$\overline{C}_{ij} = \overline{C}$	$\overline{C}_i^x = \overline{C}$	$\overline{C}_j^y = \overline{C}$	no conditions

Table 3. Conditions to obtain identical solutions with two programs hierarchically putted

The conditions to have no **transversal conflicts** are more complex than conditions to have no hierarchical conflicts: transversal conflicts are much more probable than hierarchical conflicts. For example, between the intermediary level of sub-factor i of factor 1 and the intermediary level of sub-factor j of factor 2, the conditions are $\frac{\overline{C}_i^x}{p_i^x} = \frac{\overline{C}_j^y}{p_j^y}$.

With this type of production function, it appears the possibility of conflicts between the managers. The firm is no more a black box, a single point: agents and conflicts naturally appear inside it. This is very important: the combination of inputs is no more a simple problem, but a complex problem, into which there are possibilities of contradictory choices.

Remarks.

1. If we think in terms of duality, that is to say if the cost are minimised under a constraint of production (i.e. on an isoquant), we have sixteen types of programs, but programs in a same row are identical. This shows that with CSF, to maximise production under a constraint of cost is not identical to minimise cost under a constraint of production: conflicts will be not the same.

2. If there is no manager for a program, there is no possibility of conflict with this program. It is not a difficulty. For example, we may think that there is no manager for the labor sub-factors.

3. In macroeconomics, conflicts between sub-aggregates may be seen as conflicts between social classes (middle class against workers, rentals against active capitalists, great firms against small firms, etc.).

5. CONCLUSION

The classical microeconomic firm looks punctual: it is a black box, we do not know what is inside it. The classical microeconomic production function put each factor in the same plan: when there are more than two factors, only the global combination of factors are taken into account, and not the combinations between factors taken two by two.

Thus, we propose a type of production function, the *Production Function with Combining Sub-Factors* (*CSF*), allowing to combine sub-factors two by two. For example, if factors are work and capital, sub-factors are different categories of work and different categories of capital. In itself, it is interesting to know the consequences of the combination of different categories of factors taken two by two. However, the principal message of this paper is not there.

Indeed, many type of programs are possible, like maximise production under total cost, maximise production under the cost of the combination of one sub-factor (of one factor) with every sub-factors (of the other factor), maximise production under the cost of the combination of one sub-factor (of one factor) with one sub-factor (of the other factor).

Conditions to obtain the same optimum for each program are studied: these conditions are often unattainable (in the simplest cases, the optimal mean cost of production must be equal in each programs).

Thus it appear possibilities of conflicts between managers, assuming that there is a manager responsible of each program. Intern conflicts are not artificially introduced, they come naturally without extra-economical or sociological considerations. This is very important because it make a bridge between the theories of intern organisation of the firm (Chandler, etc.) and the microeconomic theory.

The results may be enlarged to another production function like CES and all this may be extended to macroeconomics, with the possibility of a conflicts between aggregates.

6. ANNEXE

6.1. Maximisation of production under elementary cost

It is obviously the simple case of a simple production function.

Max $Q_{ij} = \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$

(or Max $Q_i^x = \sum_{j=1}^{J} \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$, or Max $Q = \sum_{i=1}^{I} \sum_{j=1}^{J} \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$)

under $C_{ij} = x_{ij}\, p_i^x + y_{ij}\, p_j^y$

$$L = \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}} + \lambda\left(C - x_{ij}\, p_i^x + y_{ij}\, p_j^y\right)$$

$$\begin{cases} \dfrac{\partial L}{\partial x_{ij}} = \dfrac{\partial Q_{ij}}{\partial x_{ij}} - \lambda\, p_i^x = 0 \Rightarrow x_{ij} = \dfrac{\beta_{ij}\, Q_{ij}}{\lambda\, p_i^x} \\ \dfrac{\partial L}{\partial y_{ij}} = \dfrac{\partial Q_{ij}}{\partial y_{ij}} - \lambda\, p_j^y = 0 \Rightarrow y_{ij} = \dfrac{\gamma_{ij}\, Q_{ij}}{\lambda\, p_j^y} \end{cases}, \text{ for every } i \text{ and } j$$

and $\dfrac{\partial L}{\partial \lambda} = C_{ij} - x_{ij}\, p_i^x + y_{ij}\, p_j^y = 0 \Rightarrow \lambda = \left(\beta_{ij} + \gamma_{ij}\right)\dfrac{Q_{ij}}{C_{ij}}$

$$\Rightarrow \begin{cases} x_{ij} = \dfrac{\beta_{ij}}{\beta_{ij} + \gamma_{ij}}\, \dfrac{C_{ij}}{p_i^x} \\ y_{ij} = \dfrac{\gamma_{ij}}{\beta_{ij} + \gamma_{ij}}\, \dfrac{C_{ij}}{p_j^y} \end{cases}, \text{ for every } i \text{ and } j$$

6.2. Maximisation of production under partial cost

For the partial isoquants Q_i^x, the problem is:

Max $Q_i^x = \sum_{j=1}^{J} \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$

(or Max $Q_{ij} = \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$, or Max $Q = \sum_{i=1}^{I} \sum_{j=1}^{J} \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$)

under $C_i^x = \sum_{j=1}^{J} \left(x_{ij}\, p_i^x + y_{ij}\, p_j^y\right)$

The Lagrangian is:

$$L = \sum_{j=1}^{J} \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}} + \sum_{i=1}^{I} \lambda_i \left(C_i^x - \sum_{j=1}^{J} \left(x_{ij}\, p_i^x + y_{ij}\, p_j^y\right)\right)$$

and:

$$\begin{cases} \dfrac{\partial L}{\partial x_{ij}} = \dfrac{\partial Q_{ij}}{\partial x_{ij}} - \lambda\, p_i^x = 0 \Rightarrow x_{ij} = \dfrac{\beta_{ij}\, Q_{ij}}{\lambda\, p_i^x} \\ \dfrac{\partial L}{\partial y_{ij}} = \dfrac{\partial Q_{ij}}{\partial y_{ij}} - \lambda\, p_j^y = 0 \Rightarrow y_{ij} = \dfrac{\gamma_{ij}\, Q_{ij}}{\lambda\, p_j^y} \end{cases}, \text{ for every } i \text{ and } j$$

and $\dfrac{\partial L}{\partial \lambda} = C_i^x - \sum_{j=1}^{J}\left(x_{ij}\, p_i^x + y_{ij}\, p_j^y\right) = 0 \Rightarrow \lambda = \dfrac{1}{C_i^x} \sum_{j=1}^{J}\left(\beta_{ij} + \gamma_{ij}\right) Q_{ij}$

$$\Rightarrow \begin{cases} x_{ij} = \dfrac{\beta_{ij}\, Q_{ij}}{\sum_{j=1}^{J}\left(\beta_{ij} + \gamma_{ij}\right) Q_{ij}} \dfrac{C_i^x}{p_i^x} \\ y_{ij} = \dfrac{\gamma_{ij}\, Q_{ij}}{\sum_{j=1}^{J}\left(\beta_{ij} + \gamma_{ij}\right) Q_{ij}} \dfrac{C_i^x}{p_j^y} \end{cases}, \text{ for every } i \text{ and } j$$

Obviously, the problem is symmetrical for Q_j^y and C_j^y.

6.3. Maximisation of production under total cost

The problem is:

Max $Q = \sum_{i=1}^{I} \sum_{j=1}^{J} \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$

(or Max $Q_{ij} = \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$, or Max $Q_i^x = \sum_{j=1}^{J} \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}}$)

under $C = \sum_{i=1}^{I} \sum_{j=1}^{J} \left(x_{ij}\, p_i^x + y_{ij}\, p_j^y\right)$.

The Lagrangian is:

$$L = \sum_{i=1}^{I} \sum_{j=1}^{J} \alpha_{ij}\, x^{\beta_{ij}}\, y^{\gamma_{ij}} + \lambda \left(C - \sum_{i=1}^{I} \sum_{j=1}^{J} \left(x_{ij}\, p_i^x + y_{ij}\, p_j^y\right) \right)$$

and at equilibrium:

$$\begin{cases} \dfrac{\partial L}{\partial x_{ij}} = \dfrac{\partial Q_{ij}}{\partial x_{ij}} - \lambda\, p_i^x = 0 \Rightarrow x_{ij} = \dfrac{\beta_{ij}\, Q_{ij}}{\lambda\, p_i^x} \\ \dfrac{\partial L}{\partial y_{ij}} = \dfrac{\partial Q_{ij}}{\partial y_{ij}} - \lambda\, p_j^y = 0 \Rightarrow y_{ij} = \dfrac{\gamma_{ij}\, Q_{ij}}{\lambda\, p_j^y} \end{cases}, \text{ for every } i \text{ and } j$$

and $\dfrac{\partial L}{\partial \lambda} = C - \sum_{i=1}^{I}\sum_{j=1}^{J}\left(x_{ij}\, p_i^x + y_{ij}\, p_j^y\right) = 0 \Rightarrow \lambda = \dfrac{1}{C}\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\beta_{ij} + \gamma_{ij}\right) Q_{ij}$

$$\Rightarrow \begin{cases} x_{ij} = \dfrac{\beta_{ij}\, Q_{ij}}{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\beta_{ij} + \gamma_{ij}\right) Q_{ij}} \dfrac{C}{p_i^x} \\ y_{ij} = \dfrac{\gamma_{ij}\, Q_{ij}}{\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\beta_{ij} + \gamma_{ij}\right) Q_{ij}} \dfrac{C}{p_j^y} \end{cases}, \text{ for every } i \text{ and } j$$

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