Bicausative matrices to measure structural change: are they a good tool?

Louis de Mesnard

LATEC

LABORATOIRE D'ANALYSE ET DE TECHNIQUES ÉCONOMIQUES

UMR 5601 CNRS

DOCUMENT DE TRAVAIL

CENTRE NATIONAL
DE LA RECHERCHE
SCIENTIFIQUE

UNIVERSITE
DE BOURGOGNE

Pôle d'Economie et de Gestion
2, boulevard Gabriel - 21000 DIJON - Tél. 03 80 39 54 30 - Fax 03 80 39 54 43

ISSN : 1260-8556

n° 9904

Bicausative matrices to measure structural change : are they a good tool ?

Louis de Mesnard

mars 1999

Bicausative matrices to measure structural change: are they a good tool?

JEL classification. C63, C67, D57.

KEYWORDS. Input-Output, Change, Causative, RAS, Biproportion.

ABSTRACT. The causative-matrix method to analyze temporal change assumes that a matrix transforms one Markovian transition matrix into another by a left multiplication of the first matrix; the method is demand-driven when applied to input-output economics. An extension is presented without assuming the demand-driven or supply-driven hypothesis. Starting from two flow matrices **X** and **Y**, two diagonal matrices are searched, one premultiplying and the second postmultiplying **X**, to obtain a result the closer as possible to **Y** by least squares. The paper proves that the method is deceptive because the diagonal matrices are unidentified and the interpretation of results is unclear.

AUTHOR.
Prof. Louis de MESNARD
LATEC (UMR CNRS 5601), Faculty of Economics, University of Burgundy
2 Bd Gabriel, 21000 Dijon, FRANCE
Tel: (33) 3 80 58 25 58
Fax: (33) 3 80 39 35 22
E-mail : *louis.de-mesnard@u-bourgogne.fr*

I. Introduction

Jackson, Rogerson, Plane and O hUallachain (1990) present an extension to input-output of the *causative matrix* approach to evaluate the change between two matrices. Denoting **A** the technical coefficient matrix and **B** the inverse Leontief matrix, $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$, there are two possibilities, working on the direct matrix **A** or the inverse matrix **B**. They choose the second possibility and they compute Markovian matrices:

$\mathbf{B}_M^t = \mathbf{B}^t \left(\mathbf{M}^t\right)^{-1}$ and $\mathbf{B}_M^{t+1} = \mathbf{B}^{t+1} \left(\mathbf{M}^{t+1}\right)^{-1}$

where **M** is the diagonal matrix whose diagonal element m_{ii} is the sum of column i of matrix **B**.

Then the transition matrix $\mathbf{B}_M^{t+1}$ is assumed to be linked to the transition matrix $\mathbf{B}_M^t$ by the formula:

(1) $$\mathbf{B}_M^{t+1} = \mathbf{C}\,\mathbf{B}_M^t$$

The matrix **C** is the causative matrix, that explain change between the Markovian matrices $\mathbf{B}_t^M$ and $\mathbf{B}_M^{t+1}$. One have $b_{M_{ij}}^{t+1} = \sum_k c_{ik}\, b_{M_{kj}}^t$.

Matrix C is found by inverting $\mathbf{B}_M^t$:

(2) $$\mathbf{C} = \mathbf{B}_M^{t+1} \left(\mathbf{B}_M^t\right)^{-1}$$

Matrix **C** should be compared to the identity matrix: all diagonal elements should be compared to 1, and all off-diagonal elements will be compared to 0. This matrix is called the *left causative matrix* by Jackson et *alii* (1990, p. 261-262). It is linked to the idea of backward linkages.

Remark. A reverse comparison is possible, $t+1$ on t instead of t on $t+1$: $\mathbf{B}_M^t = \tilde{\mathbf{C}}\,\mathbf{B}_M^{t+1}$, where $\tilde{\mathbf{C}}$ is the causative matrix for the reverse analysis. So:

(3) $$\tilde{\mathbf{C}} = \mathbf{B}_M^t \left(\mathbf{B}_M^{t+1}\right)^{-1} = \mathbf{C}^{-1}$$

This should not be confused with the idea of right causative matrix exposed by Jackson et alii (1990). ■

This approach that can be qualified as "multiplicative", is one way to compare two matrices of column coefficients. An alternative approach to do the same thing consists into calculating the difference matrix $\mathbf{D} = \mathbf{B}_M^{t+1} - \mathbf{B}_M^t$ (or $\mathbf{B}^{t+1} - \mathbf{B}^t$; and similarly with the direct matrix); this matrix should be compared to the null matrix: all elements should be compared to zero. The model can be qualified as "additive" because the model is $\mathbf{B}_M^{t+1} = \mathbf{D} + \mathbf{B}_M^t$. Matrix D is also a type of causative matrix, but additive. Note that the multiplicative model could have be chosen instead of the additive model for the biproportional filter: $\mathbf{Z}^{t+1} = \mathbf{C}\,K\!\left(\mathbf{Z}^t,\, \mathbf{Z}^{t+1}\right)$ instead of $\mathbf{Z}^{t+1} = \mathbf{D} + K\!\left(\mathbf{Z}^t,\, \mathbf{Z}^{t+1}\right)$, where K denotes the biproportional projector (Mesnard, 1990a and b, 1997). However, to follow the term chosen by Rogerson and Plane (1984), Plane and Rogerson (1986), Jackson, Rogerson, Plane and O hUallachain (1990), the term *causative* will be reserved to the multiplicative model applied to Markovian matrices found from technical coefficient matrices (direct or inverse).

Interpretation of matrix C is not easy because it is completely filled, nor sparse nor diagonal: it contains n^2 terms. So, to understand the change from one matrix with n^2 elements to another matrix with n^2 elements, this requires to analyze n^2 elements, the same number of elements than for a simple difference-based method, like the computation of $\mathbf{B}_M^t - \mathbf{B}_M^{t+1}$ or like the biproportional filter on two flow matrices $\mathbf{Z}^t$ and $\mathbf{Z}^{t+1}$ (Mesnard, 1990a and b, 1997). It is a good idea to try to reduce the complexity of the analysis. In the biproportional filtering method, the number of parameters is decreased by computing systematically only the variability of complete rows or columns, i.e., the variability of supplying or demanding sectors (technically, the Frobenius norm of row or column vectors of the difference matrix $\mathbf{Z}^{t+1} - K\left(\mathbf{Z}^t,\ \mathbf{Z}^{t+1}\right)$), that is 2 n elements. In the causative method, to reduce complexity, Jackson *et alii* (1990) focus on only two set of elements: the set of the diagonal elements of $\mathbf{C}$ and the set of the off-diagonal elements of $\mathbf{C}$, that is 2 n elements also.

Remark. Matrix $\mathbf{C}$ may contain negative terms: a positive (negative) c_{ik} implies that the probability of transition between state i and state j is increased (reduced). ■Remark. Note that a right causative matrix can also be defined for sales coefficients and forward linkages: $\mathbf{B}_M^{t+1} = \mathbf{B}_M^t \, \mathbf{R} \Rightarrow \mathbf{R} = \left(\mathbf{B}_M^t\right)^{-1} \mathbf{B}_M^{t+1}$.

Remark. In the causative method, each Markovian matrix should be square, what implies additional hypotheses.

Jackson *et alii* (1990, p. 268) have proposed a promising idea, that they called the *double causative* model:

(4) $$\mathbf{B}_M^{t+1} = \mathbf{C}_L \, \mathbf{B}_M^t \, \mathbf{C}_R$$

However, as they said themselves,

> *The use of a double causative matrix would require both additional data and additional consideration of appropriate estimation techniques*

This is obviously true but it can be the source of some complementary ideas. First, estimation problems are effectively great, because one cannot find directly $\mathbf{C}$ matrices by computing the inverse of $\mathbf{B}_M^t$ as in equation (2), and the number of parameters to be found is equal to the number of known coefficients: $2\ n^2$, so it is too large.

Second, the interpretation of the two (completely filled) causative matrices $\mathbf{C}_L$ and $\mathbf{C}_R$ is more complicated than with the simple form of equation (1) because:

$$b_{M_{ij}}^{t+1} = \sum_{k=1}^{n} \sum_{l=1}^{n} c_{L_{il}}\, b_{M_{lk}}^{t}\, c_{R_{kj}} \tag{5}$$

Third, this approach is restricted by the fact that is limits itself to Markovian chains, with matrices calculated in column, what involves a clear choice for the demand-driven model, excluding the supply-driven model, i.e. what predetermines the sense of causality. Remember that two incompatible hypotheses are possible: Leontief (technical coefficients, calculated in columns, are constant) or Ghosh (allocation coefficients, calculated in rows, are constant) and the analysis on technical coefficient cannot be compared to the analysis on allocation coefficients; this problem disappear with the biproportional approach (Mesnard, 1990a, 1990b, 1997). There is a large discussion about it (Bon, 1986), (Oosterhaven, 1988, 1989, 1996), (Miller, 1989), (Gruver, 1989), (Rose and Allison, 1989), etc., and there is a certain contradiction between the fact to introduce two causative matrices - the sense of causality seems to be undetermined - and to stay with matrices proportional by rows - the sense of causality seems to be determined - . So, it could be interesting to keep the idea of causative matrices, without predetermining the sense of causality, i.e., without predetermining the role of rows and the role of columns, by an extension of the causative-matrices method outside of the Markovian chains view. This is why one can think to propose a modification of the double causative model, in which two diagonal matrices **L** and **R** will replace matrices $\mathbf{C}_L$ and $\mathbf{C}_R$: their interpretation will be more easy because the number of parameters becomes equal to 2 n instead of n^2 or 2 n^2. This model will be called the *bicausative* model. The aim of this paper is to examine the pros and the cons of this idea.

II. Computation of bicausative matrices

The flow matrices are used or computed if necessary:

(6) $$\mathbf{Z}^t = \mathbf{A}^t \langle \mathbf{x}^t \rangle$$

(7) $$\mathbf{Z}^{t+1} = \mathbf{A}^{t+1} \langle \mathbf{x}^{t+1} \rangle$$

Remark. One can work on direct or on inverse matrices, the choice depending on what one wants to study. Direct technical coefficient matrices are close to flow matrices, i.e., $\mathbf{A} = \mathbf{Z} \langle \mathbf{x} \rangle^{-1}$, when inverse technical coefficient matrices indicate the link between final demand $\mathbf{y}$ and output $\mathbf{x}$ at equilibrium, i.e., $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{y}$. ■

Then, the change between these flow matrices is assumed to be of the form: $\mathbf{L}^t \mathbf{Z}^t \mathbf{R}^t$. This form justifies the denomination *bicausative* by analogy with the concept of biproportion, even if the form is *not* biproportional, despite its likeness: even if $\mathbf{L}^t$ and $\mathbf{R}^t$ are diagonal matrices, this is not a biproportion as the expression of the terms $\mathbf{L}^t$ and $\mathbf{R}^t$ will shown it. These terms can be obtained by minimizing the sum of squares of the differences between z_{ij}^{t+1} and $l_i^t \, z_{ij}^t \, r_j^t$:

(8) $$\min SS \, ; SS = \sum_{i=1}^{n} \sum_{j=1}^{m} \left[z_{ij}^{t+1} - l_i^t \, z_{ij}^t \, r_j^t \right]^2$$

Remark. Matrices $\mathbf{Z}^t$ and $\mathbf{Z}^{t+1}$ do not need to be square: they could be as well as rectangular because the above formulas do not imply that $n = m$. This is a real advantage on the causative method because transforming a naturally rectangular matrix into a square matrix requires additional hypothesis; this is required for the traditional Leontief model, but not for the Stone model (Make / Use matrices); the biproportional filter does not require square matrices also (Mesnard, 1997). Note that there is not matter of a "left model" and a "right model" as for causative matrix model. ■

The result is the following:

$$(9) \qquad \left\{ \begin{array}{l} l_i^t = \dfrac{\sum_{j=1}^{m} z_{ij}^{t+1} \, z_{ij}^t \, r_j^t}{\sum_{j=1}^{m} \left(z_{ij}^t\right)^2 \left(r_j^t\right)^2}, \text{ for all } i \\ r_j^t = \dfrac{\sum_{i=1}^{n} z_{ij}^{t+1} \, z_{ij}^t \, l_i^t}{\sum_{i=1}^{n} \left(z_{ij}^t\right)^2 (l_i^t)^2}, \text{ for all } j \end{array} \right\}$$

This is denoted:

$$(10) \qquad LS(\mathbf{Z}^t, \mathbf{Z}^{t+1}) = \mathbf{L}^t \, \mathbf{Z}^t \, \mathbf{R}^t$$

If $\mathbf{Z}^t$ and $\mathbf{Z}^{t+1}$ are of dimensions (n, m), $\mathbf{L}^t$ is of dimensions (n, n) and $\mathbf{R}^t$ is of dimensions (m, m).

Change can be also evaluated in reverse form from $\mathbf{Z}^{t+1}$ to $\mathbf{Z}^t$:

$$(11) \qquad LS(\mathbf{Z}^{t+1}, \mathbf{Z}^t) = \tilde{\mathbf{L}}^{t+1} \, \mathbf{Z}^{t+1} \, \tilde{\mathbf{R}}^{t+1}$$

i.e.,

$$(12) \qquad \left\{ \begin{array}{l} \tilde{l}_i^{t+1} = \dfrac{\sum_{j=1}^{m} z_{ij}^{t+1} \, z_{ij}^t \, \tilde{r}_j^{t+1}}{\sum_{j=1}^{m} \left(z_{ij}^{t+1}\right)^2 \left(\tilde{r}_j^{t+1}\right)^2}, \text{ for all } i \\ \tilde{r}_j^{t+1} = \dfrac{\sum_{i=1}^{n} z_{ij}^{t+1} \, z_{ij}^t \, \tilde{l}_i^{t+1}}{\sum_{i=1}^{n} \left(z_{ij}^{t+1}\right)^2 \left(\tilde{l}_i^{t+1}\right)^2}, \text{ for all } j \end{array} \right\}$$

Unlike with causative matrices, results will be not the same than for the direct form and the reverse form with bicausative matrices, but as equation (3) has suggested it, to compare direct and reverse results with causative matrices, it is sufficient to inverse matrices $\mathbf{L}^{t+1}$ and $\mathbf{R}^{t+1}$. Matrix $LS\left(\mathbf{Z}^t,\ \mathbf{Z}^{t+1}\right)=\mathbf{L}^t\,\mathbf{Z}^t\,\mathbf{R}^t$ is on the space of year $t+1$, when matrix $LS\left(\mathbf{Z}^{t+1},\ \mathbf{Z}^t\right)=\tilde{\mathbf{L}}^{t+1}\,\mathbf{Z}^{t+1}\,\tilde{\mathbf{R}}^{t+1}$ is on the space of year t. So, to compare direct and reverse method, that is to say to go from year t to year $t+1$, one should compute $LS\left(\mathbf{Z}^t,\ \mathbf{Z}^{t+1}\right)=\mathbf{L}^t\,\mathbf{Z}^t\,\mathbf{R}^t$ and $\left(\tilde{\mathbf{L}}^{t+1}\right)^{-1}LS\left(\mathbf{Z}^{t+1},\ \mathbf{Z}^t\right)\left(\tilde{\mathbf{R}}^{t+t}\right)^{-1}=\mathbf{Z}^{t+1}$ that is on the space of year $t+1$: matrix $\mathbf{R}^t$ should be compared to matrix $\left(\tilde{\mathbf{R}}^{t+1}\right)^{-1}$ and matrix $\left(\tilde{\mathbf{L}}^{t+1}\right)^{-1}$ should be compared to matrix $\mathbf{L}^t$.

III. Properties of bicausative matrices

A. *Iterative resolution of the algorithm*

For more generality, the problem is presented in a form that is independent to the choice made on the nature of matrices, Markovian or not, and direct or inverse. One consider two matrices $\mathbf{X}(nxm)$ and $\mathbf{Y}(nxm)$, that can be either $\mathbf{Z}^t$ or $\mathbf{Z}^{t+1}$. The problem is to find two diagonal matrices $\mathbf{U}(nxn)$ and $\mathbf{V}(mxm)$, knowing the two matrices $\mathbf{X}$ and $\mathbf{Y}$, with a symmetrical role to rows and columns and without specifying if n is greater than m or m is greater than n. The form of the model is $\mathbf{Y}=\mathbf{U}\,\mathbf{X}\,\mathbf{V}$ and one computes matrices $\mathbf{U}$ and $\mathbf{V}$ by the least squares:

(13) $$\min SS\,;SS=\sum_{i=1}^{n}\sum_{j=1}^{m}\left(y_{ij}-u_i\,x_{ij}\,v_j\right)^2$$

By deriving *SS* with respect of the u_i and v_j, one obtain:

(14)
$$\left\{ \begin{aligned} u_i &= \frac{\sum_{j=1}^{m} y_{ij}\, x_{ij}\, v_j}{\sum_{j=1}^{m} x_{ij}^2\, v_j^2}, \text{ for all } i \\ v_j &= \frac{\sum_{i=1}^{n} y_{ij}\, x_{ij}\, u_i}{\sum_{i=1}^{n} x_{ij}^2\, u_i^2}, \text{ for all } j \end{aligned} \right\}$$

We denote it: $LS(\mathbf{X},\ \mathbf{Y}) = \mathbf{U}\ \mathbf{X}\ \mathbf{V}$.

This system cannot be solved analytically: it must be solved recursively. As for the algorithms of biproportion, one calculates it recursively, beginning by a set of coefficients $v_j^{(0)} = 1$, for all j, for example, then repeating the following operations for the step k:

(15)
$$\left\{ \begin{aligned} u_i(k+1) &= \frac{\sum_{j=1}^{m} y_{ij}\, x_{ij}\, v_j(k)}{\sum_{j=1}^{m} x_{ij}^2 \left(v_j(k)\right)^2}, \text{ for all } i \\ v_j(k+1) &= \frac{\sum_{i=1}^{n} y_{ij}\, x_{ij}\, u_i(k+1)}{\sum_{i=1}^{n} x_{ij}^2 \left(u_i(k+1)\right)^2}, \text{ for all } j \end{aligned} \right\}$$

This leads to an equilibrium:

(16)
$$\left\{ \begin{aligned} u_i^* &= \frac{\sum_{j=1}^{m} y_{ij}\, x_{ij}\, v_j^*}{\sum_{j=1}^{m} x_{ij}^2 \left(v_j^*\right)^2}, \text{ for all } i \\ v_j^* &= \frac{\sum_{i=1}^{n} y_{ij}\, x_{ij}\, u_i^*}{\sum_{i=1}^{n} x_{ij}^2 \left(u_i^*\right)^2}, \text{ for all } j \end{aligned} \right\}$$

The solution exists and is unique because it is found from the minimization of a quadratic function: $SS=\sum_{i=1}^{n}\sum_{j=1}^{m}(y_{ij}-\hat{y}_{ij})^2$, by denoting $\hat{y}_{ij}=u_i x_{ij} v_j$. This function, $\Re^n x \Re^m \xrightarrow{SS} \Re$, of the $n\times m$ terms y_{ij} is continuous, derivable, on its interval of variation, what is itself compact. The function –*SS* is convex, so it has a unique maximum on its interval of variation. Finally, *SS* has a unique minimum.

However, the difficulty lies in the fact that the solution of this minimization program cannot be found analytically, but only iteratively. So some problems may occur for the initialization of the iterative process and for its convergence.

B. Non identification of bicausative matrices

1. Independence by respect to initialization

Property 1. All terms of matrices **U** and **V** are non negative if all terms of matrices **X** and **Y** and non negative.

Proof. Consider the expression of *u* and *v* terms:

$$(17)\qquad \left\{\begin{array}{l} u_i(k+1)=\dfrac{\sum_{j=1}^{m} y_{ij}\,x_{ij}\,v_j(k)}{\sum_{j=1}^{m} x_{ij}^2\left(v_j(k)\right)^2},\ \text{for all } i \\ v_j(k+1)=\dfrac{\sum_{i=1}^{n} y_{ij}\,x_{ij}\,u_i(k+1)}{\sum_{i=1}^{n} x_{ij}^2\left(u_i(k+1)\right)^2},\ \text{for all } j \end{array}\right\}$$

As the equilibrium is independent to the initialization, initial coefficients can be chosen arbitrarily. One can choose them all non negative. If all $v_j(0)$ are non negative, all $u_i(1)$ are non negative, and then all $v_j(1)$ are non negative. By recursion, all $u_i(k+1)$ are non negative and all $v_j(k+1)$ are non negative, so all u_i^* and all v_j^* are non negative. ■

The solutions of the algorithm, u_i^*, for all i and v_j^*, for all j, are independent to the initialization.

Property 2. The algorithm may be initialized by any set of identical values, $v_j(0) = \lambda$, for all j, instead of $v_j(0) = 1$, for all j, without change anything.

Proof. The initialization by $v_j(0) = \lambda$, for all j, gives:

(18)
$$\begin{cases} u_i(1) = \dfrac{\sum_{j=1}^{m} y_{ij}\, x_{ij}\, \lambda}{\sum_{j=1}^{m} x_{ij}^2\, \lambda^2} = \dfrac{1}{\lambda}\, \bar{u}_i(1), \text{ for all } i \\ v_j(1) = \dfrac{\sum_{i=1}^{n} y_{ij}\, x_{ij}\, u_i^{(1)}}{\sum_{i=1}^{n} x_{ij}^2 \left(u_i(1)\right)^2} = \lambda\, \bar{v}_j(1), \text{ for all } j \end{cases}$$

and the product $u_i^*\, x_{ij}\, v_j^*$ remains unchanged:

(19)
$$u_i^*\, x_{ij}\, v_j^* = \bar{u}_i^*\, x_{ij}\, \bar{v}_j^*, \text{ for all } i \text{ and } j,$$

$\bar{u}_i(1)$, $\bar{v}_j(1)$, $\bar{u}_i^*$ and $\bar{v}_j^*$ denoting the values obtained after an initialization by $v_j(0) = 1$, for all j.

By induction, the dual result is proved: things are unchanged if one initializes by $u_i(0) = \lambda$ instead of $u_i(0) = 1$ for all i. ■

Remark. When the computation is initialized by a set of not identical values, i.e. by $\exists j_1, j_2 \,/\, v_{j_1}(0) \neq v_{j_2}(0)$, for all j, then result is not predictable.

The above developments show that the estimators **U** and **V** are specified at a coefficient of proportionality. For example, if one multiplies the coefficients of **U** by λ, the symmetrical coefficients of **V** are divided by λ. Therefore, the true values are specified at a coefficient of proportionality. It looks like a problem of identification, to the sense of econometrics, but as the products $u_i^* v_j^*$, for all i, j, remain fixed, these products are identified. *This is the major inconvenient of the method: two vectors are searched, but they are not identified*!

Remark. When projecting a matrix **X** on the margins of a matrix **Y**, to compare **X** to **Y**, one computes $K(\mathbf{X}, \mathbf{Y}) = \mathbf{D\,X\,E}$, where K is the operator of biproportion (what is a generalization of RAS), with:

(20)
$$\left\{ \begin{array}{l} d_i = \dfrac{y_{i\bullet}}{\sum_{j=1}^{m} x_{ij}\, e_j}, \text{ for all } i \\ e_j = \dfrac{y_{\bullet j}}{\sum_{i=1}^{n} x_{ij}\, d_i}, \text{ for all } j \end{array} \right\}$$

So, even if with biproportion the same problem of identification (Mesnard, 1994) occurs, it is not a difficulty for the biproportional filtering method (Mesnard, 1990a, 1990b, 1997) because matrices **D** and **E** are not searched for themselves; what is searched is the product matrix **D X E** to be compared to **Y**: $\mathbf{Y} - \mathbf{D\,X\,E}$; but as **D X E** is identified and fixed, $\mathbf{Y} - \mathbf{D\,X\,E}$ is itself identified and fixed. ■

A very simple example will help to understand.

Example 1.

$$\mathbf{X} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 2 \\ 4 & 5 & 2 \end{bmatrix} \text{ and } \mathbf{Y} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 9 & 1 \\ 4 & 5 & 7 \end{bmatrix}$$

When initializing by $v_j(0) = 1$ for all j, one obtain:

$$LS(\mathbf{X},\ \mathbf{Y}) = \begin{bmatrix} 0.485 & & 0 \\ & 0.993 & \\ 0 & & 1.497 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 2 \\ 4 & 5 & 2 \end{bmatrix} \begin{bmatrix} 0.763 & & 0 \\ & 0.818 & \\ 0 & & 1.725 \end{bmatrix} = \begin{bmatrix} 0.741 & 0.397 & 3.350 \\ 2.271 & 0.812 & 3.425 \\ 4.567 & 6.125 & 5.165 \end{bmatrix}$$

with $SS = \sum_{i=1}^{n} \sum_{j=1}^{m} \left[y_{ij} - u_i\, x_{ij}\, v_j \right]^2 = 80.476$.

The same solution,

$$LS(\mathbf{X},\ \mathbf{Y}) = \begin{bmatrix} 1.018 & & 0 \\ & 2.081 & \\ 0 & & 3.138 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 2 \\ 4 & 5 & 2 \end{bmatrix} \begin{bmatrix} 0.364 & & 0 \\ & 0.390 & \\ 0 & & 0.823 \end{bmatrix} = \begin{bmatrix} 0.741 & 0.397 & 3.350 \\ 2.271 & 0.812 & 3.425 \\ 4.567 & 6.125 & 5.165 \end{bmatrix}$$

is found when one initialize by $v_1(0) = 0,\ v_2(0) = 0,\ v_3(0) = 1$ (remember that **U** and **V** are not identified, only the product **U X V** is identified).

This is denoted: $\begin{bmatrix} 0.741 & 0.397 & 3.350 \\ 2.271 & 0.812 & 3.425 \\ 4.567 & 6.125 & 5.165 \end{bmatrix} = LS(\mathbf{X},\ \mathbf{Y})_{80.476}$

2. Ineffectiveness of a separable modification

We call separable modification of a matrix, a modification of **X** that can be reduced to the left product by a diagonal matrix $\mathbf{\Psi}$ of size n, and to the right product by another diagonal matrix $\mathbf{\Omega}$ of size m: $\check{\mathbf{X}} = \mathbf{\Psi}\ \mathbf{X}\ \mathbf{\Omega}$.

<u>Property 3</u>. A separable modification of **X**, $\check{\mathbf{X}} = \mathbf{\Psi}\ \mathbf{X}\ \mathbf{\Omega}$, is ineffective: $LS\left(\check{\mathbf{X}},\ \mathbf{Y}\right) = LS(\mathbf{X},\ \mathbf{Y})$.

<u>Proof</u>. One denote $LS\left(\check{\mathbf{X}},\ \mathbf{Y}\right) = \check{\mathbf{U}}\ \check{\mathbf{X}}\ \check{\mathbf{V}}$ and $LS(\mathbf{X},\ \mathbf{Y}) = \mathbf{U}\ \mathbf{X}\ \mathbf{V}$; then:

(21) $$\check{u}_i(k+1) = \frac{\sum_{j=1}^{m} y_{ij}\, \check{x}_{ij}\, \check{v}_j(k)}{\sum_{j=1}^{m} \check{x}_{ij}^2\, (\check{v}_j(k))^2} = \frac{\sum_{j=1}^{m} y_{ij}\, \psi_i\, x_{ij}\, \omega_j\, \check{v}_j(k)}{\sum_{j=1}^{m} \psi_i^2\, x_{ij}^2\, \omega_j^2 \left(\check{v}_j(k)\right)^2} = \frac{\sum_{j=1}^{m} y_{ij}\, x_{ij} \left(\omega_j\ \check{v}_j(k)\right)}{\psi_i \sum_{j=1}^{m} x_{ij}^2 \left(\omega_j\ \check{v}_j(k)\right)^2}, \text{ for all } i$$

so $\breve{u}_i(k+1) = \dfrac{\sum_{j=1}^{m} y_{ij}\, x_{ij}\, \breve{v}_j(k)}{\sum_{j=1}^{m} x_{ij}^2 \left(\breve{v}_j(k)\right)^2}$,

by denoting $\breve{u}_i(k+1) = \psi_i\, \check{u}_i(k+1)$ and $\breve{v}_j(k) = \omega_j\, \check{v}_j(k)$.

Similarly,

(22) $$\breve{v}_j(k+1) = \frac{\sum_{i=1}^{n} y_{ij}\, x_{ij}\, \breve{u}_i(k+1)}{\sum_{i=1}^{n} x_{ij}^2\, (\breve{u}_i(k+1))^2}, \text{ for all } j$$

So, if one initialize by $\breve{v}_j(0) = v_j(0)$, for all j, then $\breve{u}_i(1) = u_i(1)$, for all i, and $\breve{v}_j^{(1)} = v_j^{(1)}$, for all j, and by induction, $\breve{u}_i(k) = u_i(k)$, for all i and $\breve{v}_j(k) = v_j(k)$, for all j.

Therefore, $\breve{u}_i^* = u_i^*$, for all i and $\breve{v}_j^* = v_j^*$, for all j,

and $\check{u}_i^*\, \tilde{x}_{ij}\, \check{v}_j^* = \check{u}_i^*\, \psi_i\, x_{ij}\, \omega_j\, \check{v}_j^* = \breve{u}_i^*\, x_{ij}\, \breve{v}_j^* = u_i^*\, x_{ij}\, v_j^*$ and so $\check{\mathbf{U}}\, \check{\mathbf{X}}\, \check{\mathbf{V}} = \mathbf{U\, X\, V}$. ■

However, this property does not hold with a separable modification of $\mathbf{Y}$: $\check{\mathbf{Y}} = \mathbf{\Psi\, Y\, \Omega}$ and $LS\left(\mathbf{X},\ \check{\mathbf{Y}}\right) = \check{\mathbf{U}}\, \mathbf{X}\, \check{\mathbf{V}} \neq LS(\mathbf{X},\ \mathbf{Y}) = \mathbf{U\, X\, V}$ generally because:

(23) $$\check{u}_i(k+1) = \frac{\sum_{j=1}^{m} \check{y}_{ij}\, x_{ij}\, \check{v}_j(k)}{\sum_{j=1}^{m} x_{ij}^2 \left(\check{v}_j(k)\right)^2} = \frac{\sum_{j=1}^{m} \psi_i\, y_{ij}\, \omega_j\, x_{ij}\, \check{v}_j(k)}{\sum_{j=1}^{m} x_{ij}^2 \left(\check{v}_j(k)\right)^2} = \frac{\sum_{j=1}^{m} y_{ij}\, x_{ij}\, \omega_j\, \check{v}_j(k)}{\psi_i \sum_{j=1}^{m} x_{ij}^2 \left(\check{v}_j(k)\right)^2}, \text{ for all } i$$

C. Convergence of the solution of the algorithm

Some problems of convergence may occur: there can be some problems of local equilibrium.

Example 2. It is again the following of example 1. When initializing by $v_1(0) = 1$, $v_2(0) = 0$, $v_3(0) = 0$, one obtain another solution but with a lower value of *SS*:

$$LS(\mathbf{X},\ \mathbf{Y}) = \begin{bmatrix} 0.938 & & 0 \\ & 3.172 & \\ 0 & & 0.469 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 2 \\ 4 & 5 & 2 \end{bmatrix} \begin{bmatrix} 0.408 & & 0 \\ & 2.506 & \\ 0 & & 0.438 \end{bmatrix} = \begin{bmatrix} 0.765 & 2.351 & 1.643 \\ 3.882 & 7.948 & 2.778 \\ 0.766 & 5.882 & 0.411 \end{bmatrix}$$

with $SS = 64.884$.

This is denoted: $\begin{bmatrix} 0.765 & 2.351 & 1.643 \\ 3.882 & 7.948 & 2.778 \\ 0.766 & 5.882 & 0.411 \end{bmatrix} = LS(\mathbf{X},\ \mathbf{Y})_{64.884}$

and $LS(\mathbf{X},\ \mathbf{Y})_{64.884} \neq LS(\mathbf{X},\ \mathbf{Y})_{80.476}$.

As *SS* is lower, $LS(\mathbf{X},\ \mathbf{Y})_{64.884}$ is better than for $LS(\mathbf{X},\ \mathbf{Y})_{80.476}$: this last solution can be considered as a local optimum. To see what happens, let's start from this solution and increase progressively $v_3(0)$ from 0 to 1. Between $v_3(0) = 0.436357$ and $v_3(0) = 0.436357$, one pass from the better solution $LS(\mathbf{X},\ \mathbf{Y})_{64.884}$ to the other solution $LS(\mathbf{X},\ \mathbf{Y})_{80.476}$. Figures 1 to 4 show what happens:

Figure 1 here

Figure 2 here

Figure 3 here

Figure 4 here

To avoid this, one must be careful, always looking at the value of *SS*, trying a large (an even infinite) set of initializing values and doing repeatedly the computation.

Remark. Similar problems of convergence may occur in very simple models known as chaotic. For example, in the "May equation", $x_{t+1} = a\,x_t\,(1 - x_t)$, the solution is given by solving $x^* = a\,x^*\,(1 - x^*)$, that is $x^* = \frac{a-1}{a}$; but when a exceeds 3, the solution becomes chaotic. See figures 5 and 6. ■

IV. Problems of interpretation

A. Interpretation of bicausative matrices

The two diagonal matrices $\mathbf{R}^t$ and $\mathbf{L}^t$ are summarizing the information about the change from matrices $\mathbf{Z}_t$ to $\mathbf{Z}_{t+1}$.

For example, consider a 3x3 case:

$$\mathbf{L}^t\,\mathbf{Z}^t\,\mathbf{R}^t = \begin{bmatrix} l_1^t & & 0 \\ & l_2^t & \\ 0 & & l_3^t \end{bmatrix} \begin{bmatrix} z_{11}^t & z_{12}^t & z_{13}^t \\ z_{21}^t & z_{22}^t & z_{23}^t \\ z_{31}^t & z_{32}^t & z_{33}^t \end{bmatrix} \begin{bmatrix} r_1^t & & 0 \\ & r_2^t & \\ 0 & & r_3^t \end{bmatrix} = \begin{bmatrix} l_1^t\,z_{11}^t\,r_1^t & l_1^t\,z_{12}^t\,r_2^t & l_1^t\,z_{13}^t\,r_3^t \\ l_2^t\,z_{21}^t\,r_1^t & l_2^t\,z_{22}^t\,r_2^t & l_2^t\,z_{23}^t\,r_3^t \\ l_3^t\,z_{31}^t\,r_1^t & l_3^t\,z_{32}^t\,r_2^t & l_3^t\,z_{33}^t\,r_3^t \end{bmatrix}$$

- The causal diagonal matrix $\mathbf{R}^t$ is interpreted as a *column bicausative matrix*: it affects equally all terms of a column, i.e. it affects equally the intermediary purchases of a sector: it is formally similar to an input (i.e., technical) effect.

- The causal diagonal matrix $\mathbf{L}^t$ is interpreted as a *row bicausative matrix*: it affects equally all terms of a row, i.e. it affects equally all the intermediary sales of a sector. It corresponds to an allocation or output effect and it is formally similar to a price effect. It is not exactly a price because it is possible to remove the effect of price change from $\mathbf{Z}^t$ to $\mathbf{Z}^{t+1}$ by using the same price base, for example, the price base of the date *t*.

However, the above interpretation is false because the terms l_i and r_j are not identified.

Remark. A consequence of this problem of identification is that the method -- application of least squares on two matrices $\mathbf{Z}^t$ and $\mathbf{Z}^{t+1}$ to obtain two vectors $\mathbf{l}^t$ and $\mathbf{r}^t$ -- cannot be applied to the prevision of another matrix $\mathbf{Z}^{t+2}$ at a later date *t*+2 by doing $\hat{\mathbf{Z}}^{t+1+\Delta} = \mathbf{l}^t\ \mathbf{Z}^{t+2}\ \mathbf{r}^t$.

Remark. The same interpretation could be used for the biproportional projection of a matrix to the margins of another matrix. When calculating the matrix $K\left(\mathbf{Z}^t,\ \mathbf{Z}^{t+1}\right) = \mathbf{C}^t\ \mathbf{Z}^t\ \mathbf{D}^t$, that is the biproportional projection of $\mathbf{Z}^t$ on the *margins* of $\mathbf{Z}^{t+1}$, where $\mathbf{C}^t$ and $\mathbf{D}^t$ are diagonal matrices, one can interpret the terms d_j as a technical effect and the terms c_i as a "price" effect. However, $\mathbf{C}$ and $\mathbf{D}$ are not identified and the similarity stops at this point because $\mathbf{Z}^t$ is projected on the margins of $\mathbf{Z}^{t+1}$ and not on $\mathbf{Z}^{t+1}$ itself as it is done with the bicausative method.

Note that Paelinck and Waelbroeck (1963) give a rather different interpretation, because they work on technical coefficient matrices instead of flow matrices. Considering two technical coefficient matrices, $\mathbf{A}^t$ and $\mathbf{A}^{t+1}$, the price effect from the date t to the date $t+1$ is given by expressing the matrix $\mathbf{A}^t$ to the prices of the date $t+1$: $\mathbf{A}^{t/t+1} = \hat{\mathbf{p}}\ \mathbf{A}^t\ \hat{\mathbf{p}}^{-1}$, where $\mathbf{p}$ is the price vector of the date $t+1$ with the date t as price base. Then the transformation of $\mathbf{A}^{t/t+1}$ such a this matrix has the same margins than $\mathbf{A}^{t+1}$ is given by: $K(\mathbf{A}^t,\ \mathbf{A}^{t+1}) = \mathbf{C}^t\ \mathbf{A}^{t/t+1}\ \mathbf{D}^t$. In this case the d_j's are interpreted as a substitution effect and the c_i's as a fabrication or transformation effect. However, it is partially incorrect to do a biproportional projection on a technical coefficient matrix, because the biproportional projection is yet a generalization for both column and rows of the computation of column coefficients (Mesnard, 1997). Moreover, as the terms **C** and **D** are not identified, they cannot be interpreted for themselves as a substitution effect and a fabrication effect. ■

B. Interpretation of the bias

The causative method is generalized by the bicausative method to eliminate the effect of the demand-driven-model hypothesis. The bicausative method allows to detect change in the structure by the exhibition of a left diagonal matrix and of a right diagonal matrix, but some problems may arise in comparison with other methods as the biproportional filtering method (Mesnard, 1990a and b, 1997).

Remark. Remember that in the biproportional filter method, one calculates $\mathbf{Z}^{t+1} - K\left(\mathbf{Z}^t,\ \mathbf{Z}^{t+1}\right)$ or $\mathbf{Z}^t - K\left(\mathbf{Z}^{t+1},\ \mathbf{Z}^t\right)$, that is in general terms $\mathbf{Y} - K(\mathbf{X},\ \mathbf{Y})$: as $K(\mathbf{X},\ \mathbf{Y})$ have the same row and column margins than **Y**, one is able to compare both matrices without the differential growth effect of sectors, i.e., with only the structural effect. This generalizes the simple comparison of two technical coefficient matrices, $\mathbf{A}^{t+1} - \mathbf{A}^t$ (same column margins) or two allocation coefficient matrices $\mathbf{B}^{t+1} - \mathbf{B}^t$ (same row margins), without implying a choice between demand-driven and supply-driven models, i.e. this generalizes the "shift-and-share method".

There can be a bias between **Y** and $LS(\mathbf{X}, \mathbf{Y})$: the expression $\mathbf{Y} - LS(\mathbf{X}, \mathbf{Y})$ can be more or less different to **0**. The quality of the analysis is depending on the size of this bias.

> Remark. Such a problem does not arise with the biproportional filtering method, because it is the difference $\mathbf{Y} - K(\mathbf{X}, \mathbf{Y})$ that is itself analyzed: this difference is not a bias, it is the subject of the analysis. ■

However, this manner to calculate the bias in the bicausative method could appear as mixing up two phenomenons, the bias caused by the differences in the sector size and the bias caused by the real structural effect because both matrices **X** and **Y** have not the same margins. So it could seem correct to give to both matrices the same margins by a biproportion to eliminate the size effect of the differential growth of margins and to found what can be called "the structural bias of the bicausative method"; it is logical to choose the last matrix margins of each couple of matrices, then:

(25) $$K\left[LS(\mathbf{X}, \mathbf{Y}), \mathbf{Y}\right]$$

However, as **U** and **V** are diagonal in $LS(\mathbf{X}, \mathbf{Y}) = \mathbf{U}\,\mathbf{X}\,\mathbf{V}$, because of the properties of biproportion (Mesnard, 1994),

(26) $$K\left[LS(\mathbf{X}, \mathbf{Y}), \mathbf{Y}\right] = K(\mathbf{U}\,\mathbf{X}\,\mathbf{V}, \mathbf{Y}) = K(\mathbf{X}, \mathbf{Y})$$

The bias becomes:

(27) $$\mathbf{Y} - K\left[LS(\mathbf{X}, \mathbf{Y}), \mathbf{Y}\right] = \mathbf{Y} - K(\mathbf{X}, \mathbf{Y})$$

what becomes similar to the result of the biproportional filtering method (Mesnard, 1990a and b, 1997). Then, if one tries to evaluate the structural bias of the bicausative method, one is conducted to evaluate the structural change as found by the biproportional filtering method.

Remark. An additional problem is that this hold for any method, denoted M, used instead of least squares LS, so long as M has a form like $M(\mathbf{X},\ \mathbf{Y}) = \mathbf{\Psi}\ \mathbf{X}\ \mathbf{\Phi}$, where $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are diagonal:

(28) $$K\left[M(\mathbf{X},\ \mathbf{Y}),\ \mathbf{Y}\right] = K(\mathbf{X},\ \mathbf{Y})$$

■

However, the bicausative method does not measure change in margins of matrix **X** when one tries to go to another matrix **Y** because if a matrix **X** is transformed so that it has the margins of a matrix **Y**, to obtain the matrix $K(\mathbf{X},\ \mathbf{Y})$, the bicausative method applied from **X** to **Y** gives the same results than the bicausative method applied to $\left[K(\mathbf{X},\ \mathbf{Y}),\ \mathbf{Y}\right]$:

(29) $$LS\left[K(\mathbf{X},\ \mathbf{Y}),\ \mathbf{Y}\right] = LS\left[\mathbf{P}\ \mathbf{X}\ \mathbf{Q},\ \mathbf{Y}\right] = LS(\mathbf{X},\ \mathbf{Y})$$

Here, $K(\mathbf{X}, \mathbf{Y}) = \mathbf{P}\ \mathbf{X}\ \mathbf{Q}$ have the same margins than **Y**, but the associated bicausative matrices are equivalent to those simply obtained with **X**.

Finally, what is the interpretation of the bicausative method?

Remark. One have even:

(30) $$LS\left[M(\mathbf{X}, \mathbf{Y}),\ \mathbf{Y}\right] = LS(\mathbf{\Psi}\ \mathbf{X}\ \mathbf{\Phi},\ \mathbf{Y}) = LS(\mathbf{X},\ \mathbf{Y})$$

where M is the same as above. ■

All these "curious" properties are caused by the fact that both methods are not identified in the econometric sense.

Example 3. It is the following of preceding examples.

$$K(\mathbf{X},\ \mathbf{Y}) = \begin{bmatrix} 0.888 & & 0 \\ & 2.410 & \\ 0 & & 1.326 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 2 \\ 4 & 5 & 2 \end{bmatrix} \begin{bmatrix} 0.629 & & 0 \\ & 1.511 & \\ 0 & & 0.998 \end{bmatrix}$$

$$= \begin{bmatrix} 1.116 & 1.341 & 3.543 \\ 4.548 & 3.642 & 4.810 \\ 3.336 & 10.017 & 2.647 \end{bmatrix}$$

$$LS\left[K(\mathbf{X},\ \mathbf{Y}),\ \mathbf{Y}\right] = \begin{bmatrix} 1.154 & & 0 \\ & 1.437 & \\ 0 & & 0.387 \end{bmatrix} \begin{bmatrix} 1.116 & 1.341 & 3.543 \\ 4.548 & 3.642 & 4.810 \\ 3.336 & 10.017 & 2.647 \end{bmatrix} \begin{bmatrix} 0.594 & & 0 \\ & 1.519 & \\ 0 & & 0.402 \end{bmatrix}$$

$$= \begin{bmatrix} 1.116 & 1.341 & 3.543 \\ 4.548 & 3.642 & 4.810 \\ 3.336 & 10.017 & 2.647 \end{bmatrix}$$

V. "Application" for France 1980-1996

The bicausative-matrices method will be applied to the period 1980-1996. This is not a true application, because the method is deceptive. Price effects are removed as possible by using the price base of 1980. To reduce computer time, sectors are regrouped in "10 sectors" tables (in fact, 9x10 sectors) from the French "40 sectors" tables (in fact, 34x36 sectors), following the rule indicated by table 3.

Remember that sectors T25 (*Trade*) and T38 (*Non Marketable Services*) do not appear in row, but only in column, in the French national accounting system. So, sector *Trade* in the "10 sectors" table contains sector T25 in column but not in row; sector *Non Marketable Services* do not appear in row in the "10 sectors" table.

Table 1 here

Table 2 here

Table 3 here

After 300 iterations (but convergence was obtained before), the matrices $LS(\mathbf{Z}^{1980}, \mathbf{Z}^{1996})$ and $LS(\mathbf{Z}^{1996}, \mathbf{Z}^{1980})$ are indicated by tables 4 and 5. **l** and **r** vectors are not indicated because they are not identified.

Table 4 here

Table 5 here

Moreover, the analyze of the bias is deceptive, as shown by table 6.

Table 6 here

VI. Conclusion

The bicausative-matrices method could appear as a method suitable to analyze the change from one flow matrix **X** to another flow matrix **Y**, without predetermining the orientation of the data, that is to say considering columns as variables and rows as observations, but giving to rows and columns the same role. In this sense, it allows to perform the same type of analysis than the biproportional filtering method.

However, the bicausative-matrices method is deceptive. It is based on the calculation of two diagonal matrices **U** and **V** such that the product **U X V** is a the nearest as possible of **Y** in the sense of least squares. The searched diagonal matrices **U** and **V**, that are calculated iteratively, are not identified; the product **U X V** is identified but this is not useful for the interpretation of the method, because the comparison of **U X V** and Y indicates only the bias of estimation.

When analyzing this bias, $\mathbf{Y} - \mathbf{U\,X\,V}$, if one try to give to **U X V** the same rows and column than **Y** by a biproportion, $K(\mathbf{U\,X\,V},\ \mathbf{Y})$, in order to remove the effect of the differences in sector sizes, the resulting bias $\mathbf{Y} - K(\mathbf{U\,X\,V},\ \mathbf{Y})$ becomes identical to the result of the biproportional filtering method applied from **X** to **Y**, $\mathbf{Y} - K(\mathbf{X},\ \mathbf{Y})$.

Moreover, the method does not analyze the differential growth of sectors between **X** and **Y** because if the margins of **Y** are given to **X**, i.e., if **X** is projected on **Y** by a biproportion, $K(\mathbf{X},\ \mathbf{Y})$, and if the bicausative-matrices method is applied from the resulting matrix to **Y**, the result, $LS(K(\mathbf{X},\ \mathbf{Y}),\ \mathbf{Y})$, is equal to the result of the bicausative method applied from **X** to **Y**, $LS(\mathbf{X},\ \mathbf{Y})$.

VII. Bibliographical references

BON, Ranko 1986. "Comparative Stability Analysis of Demand-Side and Supply-side Input-Output Models," *International Journal of Forecasting* 2, 231-235.

GRUVER, Gene W. 1989. "A Comment on the Plausibility of Supply-Driven Input-Output Models," *Journal of Regional Science* 29, 441-450.

JACKSON Randall W., P. ROGERSON, D. PLANE and B. O hUALLACHAIN, 1990, "A Causative Matrix Approach to Interpreting Structural Change", *Economic Systems Research,* Vol. 2, No 3, pp. 259-269.

de MESNARD, Louis 1990*a*. *Dynamique de la structure industrielle française*. Economica, Paris.

______ 1990*b*. "Biproportional Method for Analyzing Interindustry Dynamics: the Case of France," *Economic Systems Research* 2, 271-293.

______ 1994. "Unicity of Biproportion", *SIAM Journal on Matrix Analysis and Applications* 2, 15, 490-495.

______ 1997. "A biproportional filter to compare technical and allocation coefficient variations", *Journal of Regional Science*, 37, 4, pp. 541-564.

MILLER, Ronald E. 1989. "Stability of Supply Coefficients and Consistency of Supply-Driven and Demand-Driven Input-Output Models: a Comment," *Environment and Planning A* 21, 1113-1120.

OOSTERHAVEN, Jan 1988. "On the Plausibility of the Supply-Driven Input-Output Model," *Journal of Regional Science* 28, 203-217.

______ 1989. "The Supply-Driven Input-Output Model: A New Interpretation but Still Implausible," *Journal of Regional Science* 29, 459-465.

______ 1996. "Leontief Versus Ghoshian Price And Quantity Models", *Southern Economic Journal* 62-3, pp. 750-759.

PAELINCK, Jean and J. WAELBROECK, 1963. "Etude empirique sur l'évolution de coefficients «input-output». Essai d'application de la procédure «RAS» de Cambridge au tableau interindustriel belge", *Economie Appliquée*, No 6, pp. 82-111.

PLANE David, and P. ROGERSON 1986. "Dynamic Flow Modeling with Interregional Dependency Effects", *Demography*, 23, pp. 91-104.

ROGERSON, Peter and D. PLANE, 1984. "Modeling temporal change in flow matrices", *Papers of the Regional Science Association*, 54, pp. 147-164.

ROSE, Adam and T. ALLISON 1989. "On the Plausibility of the Supply-Driven Input-Output Model: Empirical Evidence on Joint Stability," *Journal of Regional Science* 29, 451-458.

Figures and tables

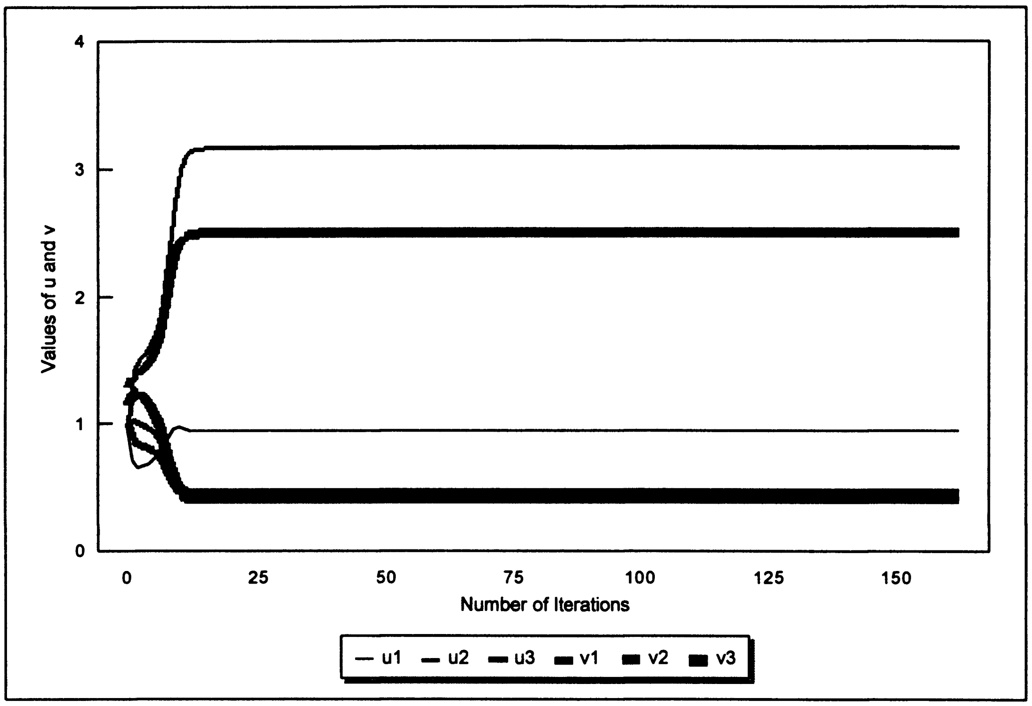

Figure 1. u and v for $v_1(0) = 1,\ v_2(0) = 0,\ v_3(0) = 0,\ LS(\mathbf{X},\ \mathbf{Y})_{64.884}$

Figures and tables

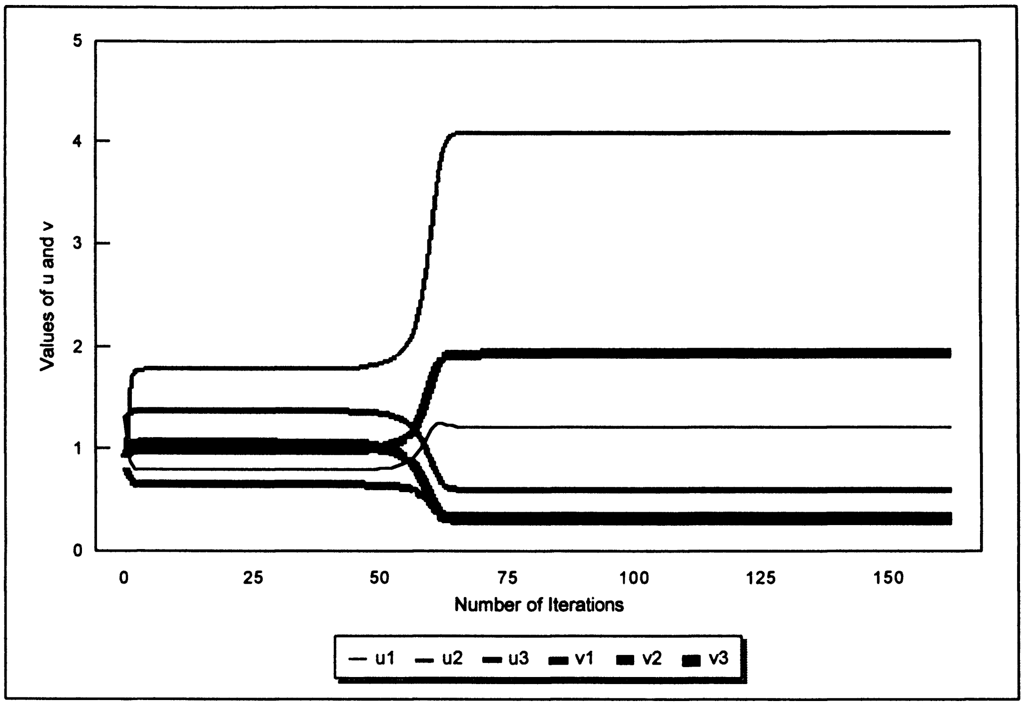

Figure 2. u and v for $v_1(0) = 1,\ v_2(0) = 0,\ v_3(0) = 0.436357,\ LS(\mathbf{X},\ \mathbf{Y})_{64.884}$

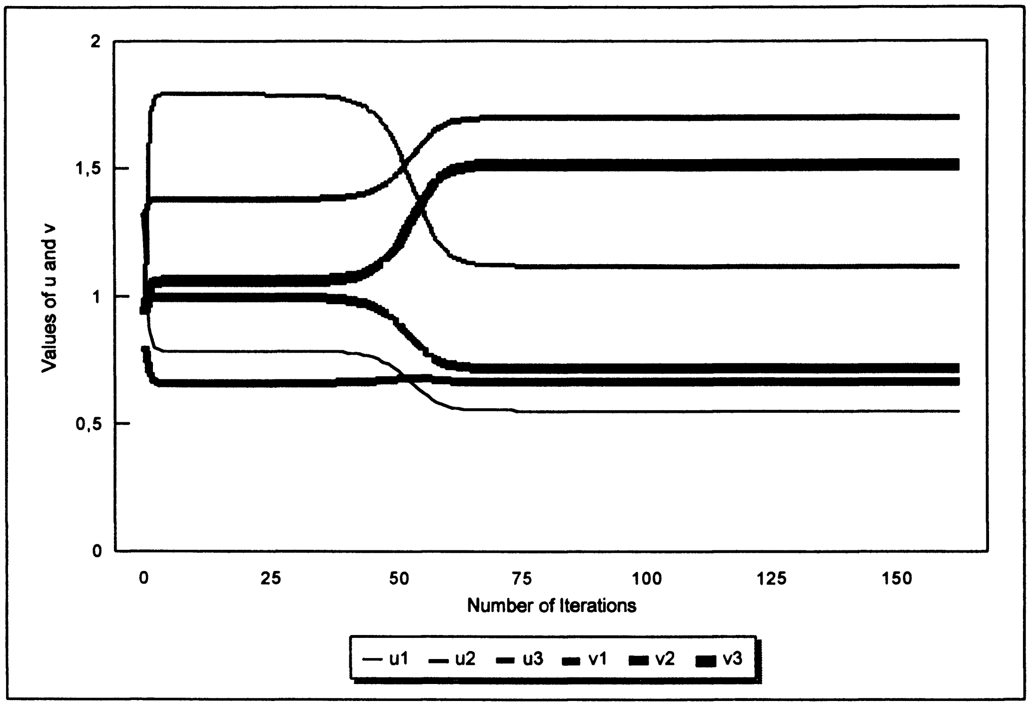

Figure 3. u and v for $v_1(0) = 1,\ v_2(0) = 0,\ v_3(0) = 0.436358,\ LS(\mathbf{X},\ \mathbf{Y})_{80.476}$

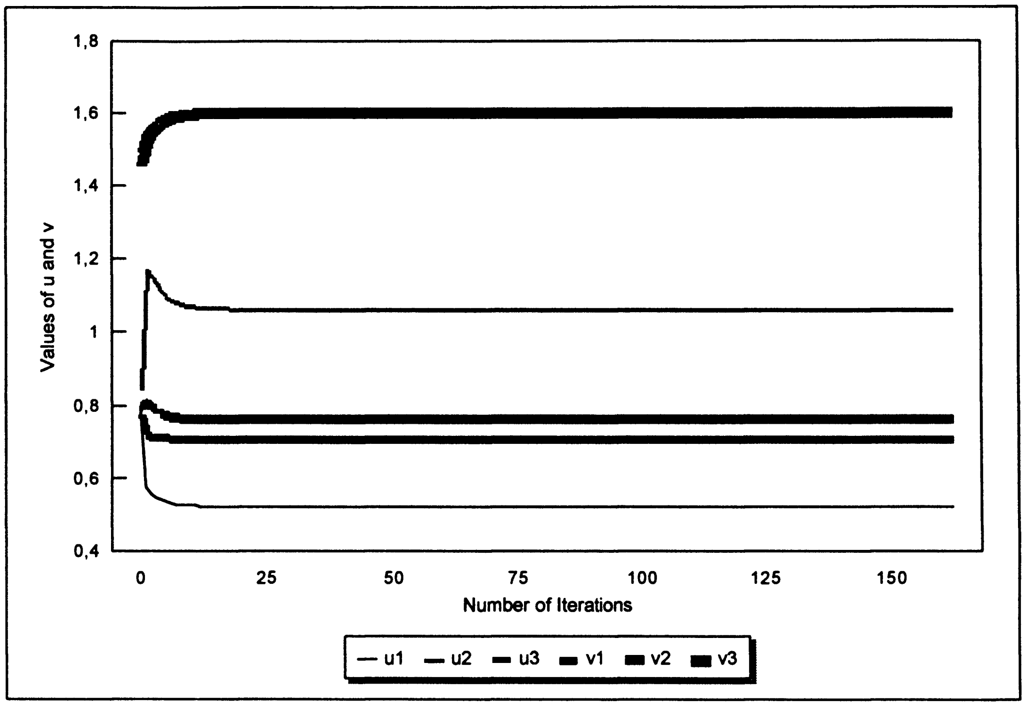

Figure 4. u and v for $v_1(0) = 1,\ v_2(0) = 0,\ v_3(0) = 1,\ LS(\mathbf{X},\ \mathbf{Y})_{80.476}$

Sectors of the "40 sectors table"	Regrouped sectors
T01 Farming, Forestry, Fishing T02 Meat and Dairy Products T03 Other Agricultural and Food Products	Agriculture, Forestry, Fishing
T04 Solid Fuels T05 Oil Products, Natural Gas T06 Electricity, Gas and Water	Energy
T07 Ores and Ferrous Metals T08 Ores and non Ferrous Metals T09 Building Materials, Miscellaneous Minerals	Minerals
T10 Glass T11 Basic Chemicals, Synthetic Fibers T12 Miscellaneous Chemicals, Pharmaceuticals T13 Smelting Works, Metal Works T14 Mechanical Engineering T15A Electric Industrial Equipment T15B Household Appliances T16 Motor Vehicles T17 Shipping, Aircrafts and Arms T18 Textile Industry, Clothing Industry T19 Leather and Shoe Industries T20 Leather and Wood Industries, Varied Industries T21 Paper and Cardboard T22 Printing and Publishing T23 Rubber, Transformation of Plastics	Manufacturing
T24 Building Trade, Civil and Agricultural Engineering	Buildings
T25 Trade T29 Automobile Trade and Repair Services T30 Hotels, Catering	Trade
T31 Transport T32 Telecommunications and Mail	Transport and Telecommunications
T33 Business Services T34 Marketable Services to Private Individuals T35 Housing Rental and Leasing T36 Insurance	Services
T37 Financial Services	Services of Financial Institutions
T38 Non Marketable Services	Non Marketable Services

Table 1. Groups of sectors

	Agric.	Energy	Minerals	Manuf.	Buildg	Trade	Transp. Telecom.	Services	Financial Services	Non Market. Services	Total
Agric.	270 732	196	63	24 955	0	25 520	233	2 305	0	14 319	338 323
Energy	18 603	167 784	23 722	48 846	8 091	17 854	28 118	7 129	877	18 031	339 055
Minerals	1 962	2 303	83 346	72 775	60 063	1 880	493	810	0	2 757	226 389
Manuf.	50 722	13 485	10 610	439 871	74 100	20 307	13 867	59 304	3 437	53 288	738 991
Buildings	1 033	6 042	381	2 050	231	957	627	2 917	5 891	19 704	39 833
Trade	831	263	1 401	2 627	813	8 975	1 866	8 703	823	2 694	28 996
Transp. Telecom.	5 632	5 985	10 125	36 106	13 034	33 615	24 126	21 715	4 407	10 323	165 068
Services	18 792	12 857	9 866	83 142	48 570	30 480	15 907	103 334	12 802	35 896	371 646
Financial Services	1 038	568	829	5 826	5 940	2 074	636	1 796	3 812	410	22 929
Total	369 345	209 483	140 343	716 198	210 842	141 662	85 873	208 013	32 049	157 422	2 271 230

Table 2. Table for 1980

	Agric.	Energy	Minerals	Manuf.	Buildg	Trade	Transp. Telecom.	Services	Financial Services	Non Market. Services	Total
Agric.	317 085	95	21	25 573	0	29 232	258	2 891	0	20 815	395 970
Energy	22 142	127 202	17 172	55 509	9 316	26 834	36 397	11 705	1 196	27 164	334 637
Minerals	1 859	13 590	67 845	70 881	53 198	1 992	286	1 126	0	3 009	213 786
Manuf.	64 496	13 435	9 248	600 699	78 175	29 591	21 687	106 933	3 149	82 368	1 009 781
Buildings	1 238	7 566	300	2 446	213	1 289	763	5 092	11 543	28 037	58 487
Trade	879	302	942	2 717	641	11 656	2 549	12 557	423	3 515	36 181
Transp. Telecom.	8 199	7 071	9 439	64 480	15 330	60 468	51 078	58 113	7 949	21 596	303 723
Services	33 852	26 962	13 209	156 997	64 934	55 163	26 572	214 993	34 087	57 031	683 800
Financial Services	3 497	2 025	1 755	19 885	14 455	6 751	2 344	5 754	750 831	1 774	809 071
Total	453 247	198 248	119 931	999 187	236 262	222 976	141 934	419 164	809 178	245 309	3 845 436

Table 3. Table for 1996

	Agric.	Energy	Minerals	Manuf.	Buildg	Trade	Transp. Telecom.	Services	Financial Services	Non Market. Services	Total
Agric.	316 438	165	54	29 546	0	31 596	368	4 454	0	21 088	403 709
Energy	19 756	128 520	18 405	52 547	4 718	20 085	40 308	12 516	9 187	24 128	310 414
Minerals	2 134	1 807	66 226	80 177	35 872	2 166	724	1 456	0	3 778	192 206
Manuf.	67 507	12 945	10 317	593 023	54 156	28 629	24 913	130 483	45 123	89 365	988 954
Buildings	366	1 545	99	736	45	359	300	1 709	20 595	8 799	34 187
Trade	687	157	846	2 199	369	7 858	2 082	11 892	6 710	2 806	34 919
Transp. Telecom.	7 182	5 505	9 433	46 641	9 127	45 408	41 530	45 779	55 438	16 588	275 449
Services	20 416	10 075	7 831	91 498	28 976	35 077	23 328	185 591	137 197	49 139	568 712
Financial Services	20 003	7 895	11 672	113 728	62 859	42 337	16 544	57 217	724 646	9 956	1 046 854
Total	454 489	168 614	124 883	1 010 095	196 122	213 515	150 097	451 097	998 896	225 647	3 538 966

Table 4. $LS(\mathbf{Z}^{1980}, \mathbf{Z}^{1996})$

	Agric.	Energy	Minerals	Manuf.	Buildg	Trade	Transp. Telecom.	Services	Financial Services	Non Market. Services	Total
Agric.	271 086	112	24	21 666	0	23 380	189	1 991	0	15 950	334 398
Energy	20 762	164 986	21 539	51 581	11 039	23 540	29 203	8 842	758	22 831	355 081
Minerals	1 727	17 460	84 293	65 241	62 439	1 731	227	843	0	2 505	236 466
Manuf.	47 634	13 725	9 136	439 645	72 960	20 445	13 705	63 621	1 572	54 525	736 968
Buildings	933	7 889	302	1 827	203	909	492	3 092	5 883	18 943	40 473
Trade	746	354	1 069	2 284	687	9 249	1 850	8 580	243	2 672	27 734
Transp. Telecom.	4 589	5 474	7 067	35 763	10 842	31 661	24 461	26 202	3 008	10 834	159 901
Services	19 397	21 370	10 125	89 148	47 018	29 570	13 028	99 241	13 206	29 291	371 394
Financial Services	30	24	20	167	155	53	17	39	4 298	13	4 816
Total	366 904	231 394	133 575	707 322	205 343	140 538	83 172	212 451	28 968	157 564	2 267 231

Table 5. $LS(\mathbf{Z}^{1996}, \mathbf{Z}^{1980})$

	Agric.	Energy	Minerals	Manuf.	Buildg	Trade	Transp. Telecom.	Services	Financial Services	Non Market. Services	Total
Agric.	311 855	190	47	28 284	0	30 962	321	3 669	0	20 643	395 970
Energy	19 424	147 094	16 172	50 184	5 493	19 635	35 112	10 286	7 674	23 563	334 637
Minerals	2 382	2 347	66 056	86 924	47 410	2 404	716	1 359	0	4 189	213 786
Manuf.	66 919	14 938	9 139	571 017	63 570	28 218	21 880	108 111	38 001	87 988	1 009 781
Buildings	684	3 359	165	1 335	99	667	496	2 669	32 686	16 327	58 487
Trade	770	205	847	2 394	490	8 756	2 067	11 140	6 389	3 123	36 181
Transp. Telecom.	8 329	7 431	9 776	52 536	12 533	52 357	42 668	44 372	54 616	19 105	303 723
Services	25 931	14 896	8 889	112 886	43 581	44 299	26 251	197 028	148 046	61 992	683 800
Financial Services	16 953	7 789	8 840	93 626	63 084	35 678	12 423	40 532	521 766	8 381	809 071
Total	453 247	198 248	119 931	999 187	236 262	222 976	141 934	419 164	809 178	245 309	3 845 436

Table 6. $K(80,96)$ or $K\left(LS(80,96),96\right)$