Local Gradient Estimates for Second-Order Nonlinear Elliptic and Parabolic Equations by the Weak Bernstein's Method

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Abstract

In the theory of second-order, nonlinear elliptic and parabolic equations, obtaining local or global gradient bounds is often a key step for proving the existence of solutions but it may be even more useful in many applications, for example to singular perturbations problems. The classical Bernstein's method is the well-known tool to obtain these bounds but, in most cases, it has the defect of providing only a priori estimates. The "weak Bernstein's method", based on viscosity solutions' theory, is an alternative way to prove the global Lipschitz regularity of solutions together with some estimates but it is not so easy to perform in the case of local bounds. The aim of this paper is to provide an extension of the "weak Bernstein's method" which allows to prove local gradient bounds with reasonnable technicalities.

The classical Bernstein's method is a well-known tool for obtaining gradient estimates for solutions of second-order, elliptic and parabolic equations

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(cf. Gilbarg and Trudinger[4] and Lions[5]). The underlying idea is very simple: if $u : \Omega \subset \mathbb{R}^N \to R$ is a smooth solution of

$$-\Delta u = 0 \quad \text{in } \Omega \,,$$

where Δ denotes the Laplacian in $\mathbb{R}^N$, then $w := |Du|^2$ satisfies

$$-\Delta w \leq 0 \quad \text{in } \Omega \,.$$

The gradient bounded is deduced from this property by using the Maximum Principle if one knows that Du is bounded on $\partial\Omega$ and this bound on the boundary is usually the consequence of the existence of barriers functions.

Of course this strategy can be used for far more general equations but it has a clear defect: in order to justify the above computations, the solution has to be C^3 and, in general, the classical Bernstein's method just provides an *a priori estimates* and one has to find a suitable approximation of the equation to actually prove the gradient bound.

In 1990, this difficulty was partially overcomed by the Weak Bernstein's method whose idea is even simpler: if one looks at the maximum of the function

$$(x, y) \mapsto u(x) - u(y) - L|x - y| \quad \text{in } \overline{\Omega} \times \overline{\Omega} \,,$$

and if one can prove that it is achieved only for $x = y$ for L large enough, then $|Du| \leq L$. Surprisingly, as it is explained in the introduction of [1], the computations and structure conditions which are needed to obtain this bound are the same (or almost the same with tiny differences) as for the classical Bernstein's method. Of course, the main advantage of the Weak Bernstein's method is that it does not require u to be smooth since there is no differentiation of u and it can even be used in the framework of viscosity solutions.

Problem solved? Not completely because the Weak Bernstein's method is not of an easy use if one looks for local bounds instead of global bounds. In fact, in order to get such local gradient bounds, the only possible way seems to multiply the solution by a cut-off function and to look for a gradient bound for this new function. Unfortunately, this new function satisfies a rather complicated equation where the derivatives of the cut-off function appear at different places and the computations are rather technical. The classical Bernstein's method faces also similar difficulties but, at least in some cases, succeeds in providing these local bounds in a not too complicated way.

The aim of this article is to describe a slight improvement of the Weak Bernstein's method which allows to obtain local gradient bounds in a simpler way, "simpler" meaning that the technicalities are as reduced as possible, although some are unavoidable. This improvement is based on an idea of P. Cardaliaguet [2] which dramatically simplifies a matrix analysis which is keystone in [1] but also allows this extension to local bounds.

To present our result, we consider second-order, possibly degenerate, elliptic equations which we write in the general form

$$F(x,u,Du,D^2u) = 0 \quad \text{in } \Omega\ , \tag{1}$$

where Ω is a domain of $\mathbb{R}^N$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \times \to \mathbb{R}$ is a locally Lipschitz continuous function, $\mathcal{S}^N$ denotes the space of $N \times N$ symmetric matrices, the solution u is a real-valued function defined on Ω, Du, D^2u denote respectively its gradient and Hessian matrix. We assume that F satisfies the (degenerate) ellipticity condition : for any $(x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ and for any $X, Y \in \mathcal{S}^N$,

$$F(x,r,p,X) \leq F(x,r,p,Y) \quad \text{if } X \geq Y.$$

Among all these general equations, we focus in particular on the following one

$$-\Delta u + |Du|^m = f(x) \quad \text{in } \Omega\ , \tag{2}$$

where $m > 1$ and $f \in W^{1,\infty}_{loc}(\Omega)$, which is a particular case for which the classical Bernstein's method provides local bound in a rather easy way, while it is not the case for the Weak Bernstein's method.

1 Some preliminary results

In this section, we are going to construct the functions we use in the proof of our main result. To do so, we introduce $\mathcal{K}$ which is the class of increasing functions $\chi : (0,+\infty) \to [1,+\infty)$ such that $\chi(t) \leq K(\chi)t^\alpha$ for some $\alpha < 1$ and some constant $K(\chi) > 0$ and

$$\int_1^{+\infty} \frac{dt}{t\chi(t)} < +\infty.$$

The first ingredient we use below is a smooth function $\varphi : [0,1[\to \mathbb{R}$ such that $\varphi(0) = 0$, $\varphi'(0) = 1 \leq \varphi'(t)$ for any $t \in [0,1[$ with $\varphi(t) \to +\infty$ as $t \to 1^-$

and $\varphi''(t) \leq K_1\varphi'(t)\chi(\varphi'(t))$ for some constant $K_1 > 0$. In fact the existence of such function is classical using that

$$\int_1^{\varphi'(t)} \frac{ds}{s\chi(s)} = K_1 t\,,$$

and by choosing $K_1 = \int_1^{+\infty} \frac{ds}{s\chi(s)}$ we already see that $\varphi'(t) \to +\infty$ as $t \to 1^-$. Moreover

$$\int_{\varphi'(t)}^{+\infty} \frac{ds}{s\chi(s)} = K_1(1-t)\ ,$$

and therefore

$$K_1(1-t) \geq [K(\chi)]^{-1} \int_{\varphi'(t)}^{+\infty} \frac{ds}{s^{1+\alpha}} = [K(\chi)\alpha]^{-1}\varphi'(t)^{-\alpha}\ .$$

This means that

$$\varphi'(t) \geq \left(\frac{K_1(1-t)}{[K(\chi)\alpha]^{-1}}\right)^{-1/\alpha}\ ,$$

and therefore $\varphi'(t)$ is not integrable at 1 since $1/\alpha > 1$.

On the other hand, given $x_0 \in \mathbb{R}^N$ and $R > 0$, we use below a smooth function $C : B(x_0, 3R/4) \to \mathbb{R}$ is a smooth function such that $C(z) = 1$ on $B(x_0, R/4)$, $C(z) \geq 1$ in $B(x_0, 3R/4)$ and $C(z) \to +\infty$ when $z \to \partial B(x_0, 3R/4)$ and with

$$\frac{|D^2C(x)|}{C(x)}, \frac{|DC(x)|^2}{[C(x)]^2} \leq K_2(R)[\chi(C(x))]^2\ ,$$

where χ is a function in the class $\mathcal{K}$. If C_1 is a function which satisfies the above properties for $R = 1$, we see that we can choose C as

$$C(x) = C_1(\frac{x}{R})\ ,$$

and therefore $K_2(R)$ behaves like $R^{-2}K_2(1)$.

To build C, we first solve

$$\psi''(t) = K_3\psi(t)[\chi(\psi(t))]^2,\ \psi(0) = 1,\ \psi'(0) = 0\ .$$

Multiplying the equation by $2\psi'(t)$, we obtain that

$$\psi'(t) = F(\psi(t))\ ,$$

where

$$[F(\tau)]^2 = 2K_3 \int_1^\tau s[\chi(s)]^2 ds \; .$$

Again we look for a function ψ such that $\psi(t) \to +\infty$ as $t \to 1^-$ and to do so, the following condition should hold

$$\int_1^{+\infty} \frac{d\tau}{F(\tau)} < +\infty \; .$$

But

$$[F(\tau)]^2 \geq 2K_3 \int_{\tau/2}^\tau s[\chi(s)]^2 ds \geq \; 2K_3[\tau/2\chi(\tau/2)]^2,$$

and since $\tau \mapsto \chi(\tau/2)$ is in $\mathcal{K}$, we have the result for F and for ψ by choosing appropriately the constant K_2.

Moreover

$$[F(\tau)]^2 \leq 2K_3(\tau-1)\tau[\chi(\tau)]^2 \leq 2K_3[\tau\chi(\tau)]^2 \; ,$$

and therefore

$$\psi'(t) \leq (2K_3)^{1/2}\psi(t)\chi(\psi(t)) \; .$$

With ψ the construction of C is easy, we may choose

$$C(x) := \psi(\frac{4(|x-x_0|-R/2)}{R} \; ,$$

for $|x-x_0| \geq R/2$ and we extend it properly in all the ball $B(x_0, 3R/4)$.

2 The Main Result

Our result is the following

Theorem 2.1 *Assume that F is a locally Lipschitz function in $\Omega\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}^N\times\to\mathbb{R}$ which satisfies : $F(x,r,p,M)$ is Lipschitz continuous in M and*

$$F_M(x,r,p,M) \leq 0 \text{ and } F_r(x,r,p,M) \geq 0 \quad \text{a.e. in } \Omega\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}^N\times\to\mathbb{R} \; ,$$

and let $u \in C(\Omega)$ be a solution of (1).

(i) **(Uniformly elliptic equation : estimates which are independant of the oscillation of u)** *Assume that there exists a function $\chi \in \mathcal{K}$, $\eta > 0$ small enough such that, for any $K > 0$, there exists $L = L(\eta, K)$ large enough such that*

$$-F_M \cdot M^2 \geq \eta + (2+\eta)|F_x||p|\chi(|p|) + \eta|F_p \cdot p| + K||F_M||_\infty(|p|\chi^2(|p|))^2) \text{ a.e.},$$

in the set $\{(x, r, p, M);\ |F(x, r, p, M))| \leq \eta|p|\chi^2(|p|)\ ,\ |p| \geq L\}$. If $B(x_0, R) \subset \Omega$ then u is Lipschitz continuous in $B(0, R/2)$ and $|Du| \leq \bar{L}$ in $B(0, R/2)$ where $\bar{L}$ is given by $L(\eta, K)$ where $K = K(R)$.

(ii) **(Uniformly elliptic equation : estimates depending the oscillation of u)** *Assume that there exists a function $\chi \in \mathcal{K}$, $\eta > 0$ small enough such that, for any $K > 0$, there exists $\bar{L} = \bar{L}(\eta, R, K)$ large enough such that*

$$-F_M \cdot M^2 \geq \eta + (1+\eta)|F_x||p| + \eta|F_p \cdot p| + K||F_M||_\infty(|p|\chi(|p|))^2) \text{ a.e.},$$

in the set $\{(x, r, p, M);\ |F(x, r, p, M))| \leq \eta|p|\chi(|p|)\ ,\ |p| \geq \bar{L}\}$. If $B(x_0, R) \subset \Omega$ then u is Lipschitz continuous in $B(0, R/2)$ and $|Du| \leq \bar{L}$ in $B(0, R/2)$ where $\bar{L}$ is given by $L(\eta, K)$ where $K = K(R)$.

(iii) **(Non-uniformly elliptic equation : estimates depending the oscillation of u)** *Assume that there exists a function $\chi \in \mathcal{K}$, $\eta > 0$ small enough such that, for any $K > 0$, there exists $\bar{L} = \bar{L}(\eta, R, K)$ large enough such that*

$$F_x \cdot p + F_u|p|^2 - \frac{1}{1+\eta} F_M \cdot M^2 \geq \eta + \eta(|F_x \cdot p| + F_u|p|^2 + F_p \cdot p) + K||F_M||_\infty(|p|\chi(|p|))^2) \text{ a.e.},$$

in the set $\{(x, r, p, M);\ |F(x, r, p, M))| \leq \eta|p|\chi(|p|)\ ,\ |p| \geq \bar{L}\}$. If $B(x_0, R) \subset \Omega$ then u is Lipschitz continuous in $B(0, R/2)$ and $|Du| \leq \bar{L}$ in $B(0, R/2)$ where $\bar{L}$ is given by $L(\eta, K)$ where $K = K(R)$.

As an application we can consider Equation (2): the idea is to choose $\chi(t) = t^\alpha$ with $\gamma := 1 + 2\alpha < m$. The most important point is that the constraint $|F(x, r, p, M))| \leq \eta|p|^\gamma$ if $\gamma < m$, $\eta \leq 1/2$ and $|p|$ large enough (depending only on γ) implies

$$\mathrm{Tr}(M) \geq \frac{1}{2}|p|^m - ||f||_{L^\infty(B(0,R)}\ ,$$

but, by Cauchy-Schwarz inequality

$$\mathrm{Tr}(M) \leq C(N)[\mathrm{Tr}(M^2)]^{1/2} .$$

Therefore the term $F_M \cdot M^2$ behaves like $|p|^{2m}$ and clear dominates the terms $||F_M||_\infty |p|^{2\gamma}$ since $\gamma < m$, while the terms $|F_x|(1+\gamma)|p|^\gamma$ and $F_p \cdot p$ grows like $|p|^\gamma$ and $|p|^m$ respectively, therefore we have the gradient bound and the classical case ($m = 1$) can be also treated under the assumptions of (ii).

In this example, it is also clear that we can replace the term $|Du|^m$ by a term $H(Du)$ where H satisfies: there exists $\chi \in \mathcal{K}$ such that

$$\frac{H(p)}{|p|\chi^2(|p|)} \to +\infty \quad \text{as } |p| \to +\infty .$$

In the case of non-uniformly elliptic equation, the gradient bound comes necessarely from the $F_u|p|^2$-term. For the equation

$$-\mathrm{Tr}(A(x)D^2u) + |Du|^m = f(x) \quad \text{in } \Omega , \tag{3}$$

where $m > 1$ and A, f are locally bounded and Lipschitz continuous, a change of variable is necessary (we also may assume that $A(x)$ is a symmetric matric for any x). Assuming (without loss of generality) that $u \geq 1$ at least in the ball $\overline{B(0,R)}$, we can use the change $u = \exp(v)$. The equation satisfied by v is

$$-\mathrm{Tr}(A(x)D^2v) + A(x)Dv \cdot Dv + \exp((m-1)v)|Dv|^m = \exp(-v)f(x) \quad \text{in } \Omega ,$$

The computation of the different terms gives

$$F_v(x,v,p,M) = (m-1)\exp((m-1)v)|p|^m + \exp(-v)f(x) ,$$

$$F_x(x,v,p,M) = -\mathrm{Tr}(A_x(x)M) + A_x(x)p \cdot p - \exp(-v)f_x(x)$$

$$F_p(x,v,p,M) = 2A(x)p + \exp((m-1)v)|p|^{m-2}p ,$$

$$-F_M(x,v,p,M)M^2 = \mathrm{Tr}(A(x)M^2) .$$

If we assume that $A(x) = \sigma(x).\sigma^T(x)$ for some bounded, Lipschitz continuous function σ, where $\sigma^T(x)$ denotes the transpose matrix of $\sigma(x)$, then Cauchy-Schwarz inequality implies

$$|\mathrm{Tr}(A_x(x) \cdot pM)| \leq \frac{1}{1+\eta}\mathrm{Tr}(A(x)M^2) + O((|\sigma_x||p|)^2) ;$$

This control of the first term in $F_x(x,v,p,M$ is the only use of the $-F_M(x,v,p,M)M^2$-term.

Therefore the $F_v(x,v,p,M)|p|^2$-term which behaves like $|p|^{m+2}$ if $m>1$, has to control the terms

$$(A_x(x)\cdot p)(p\cdot p) = O(|p|^3)\ ,\ -\exp(-v)f_x(x)\cdot = O(|p|)\ ,\ 2A(x)p\cdot p = O(|p|^2)\ ,$$

and the term $\exp((m-1)v)|p|^m$ which appear in $F_p\cdot p$ but multiplied by η that we can take small enough.

The most important constraint is to control the term $K||F_M||_\infty(|p|\chi(|p|))^2)$ but by choosing $\chi(t)=t^\alpha$ for $\alpha>0$ small enough, this term behaves as $|p|^{2+2\alpha}$ and is controlled by $|p|^{m+2}$ since $m>1$. Therefore Theorem 2.1 (iii) applies.

3 Proof of Theorem 2.1

We start by proving (i) : the aim is to prove that, for any $x\in B(x_0,R/2)$, $D^+u(x)$ is bounded with an explicit bound. This will provide the desired gradient bound.

To do so, we consider on

$$\Gamma_L := \{(x,y)\in B(x_0,3R/4)\times B(x_0,R) : LC(x)(|x-y|+\alpha)<1\}$$

the following function

$$\chi(x,y) = u(x)-u(y)-\varphi(LC(x)(|x-y|+\alpha))\ ,$$

where

- $L\geq 1$ is a constant which is our future gradient bound (and therefore which has to be choosen large enough),
- the functions φ and C are built in Section 1,
- $\alpha>0$ is a small constant devoted to tend to 0.

We remark that the above function achieves it maximum in the open set Γ_L: indeed, if $(x,y)\in\Gamma_L$, we have $LC(x)\alpha<1$ and therefore $x\in\overline{B(0,R')}$ for some $R'<3R/4$ and $LC(x)|x-y|<1$ which implies $|x-y|<L^{-1}$ and for $L>4/R$ this implies $y\in\overline{B(0,R'+R/4)}$ and $R'+R/4<R$. Therefore, clearly $\chi(x,y)\to+\infty$ if $(x,y)\to\partial\Gamma_L$.

Next we argue by contradiction: if, for some large L, this maximum is achieved for any α at $(\bar{x}, \bar{y})$ with $\bar{x} = \bar{y}$, then $\chi(\bar{x}, \bar{x}) = -\varphi(LC(\bar{x})\alpha)$ and therefore necessarely $\bar{x} \in B(x_0, R/4)$ by the maximality property and the form of C. Moreover, for any x, y

$$u(x) - u(y) - \varphi(LC(x)(|x-y| + \alpha)) \leq -\varphi(L\alpha) ,$$

and if this is true, for a fixed L, this implies that, for any x, y

$$u(x) - u(y) - \varphi(LC(x)|x-y|) \leq 0 .$$

Choosing $x \in B(x_0, R/4)$, we have

$$u(y) - u(x) \geq -\varphi(L|x-y|) ,$$

and this inequality implies that any element in $D^+u(x)$ has a norm which is less than L, which we wanted to prove.

Otherwise, this means that, for any fixed L, the maximum point $(\bar{x}, \bar{y})$ of χ, satisfies $\bar{x} \neq \bar{y}$ for α small enough and we are going to prove that this is a contradiction for an L large enough but independant of α.

For the sake of simplicity of notations, we denote by (x, y) a maximum point of χ and we set $t = LC(x)(|x-y| + \alpha)$ and

$$p = \varphi'(t)LC(x)\frac{(x-y)}{(|x-y| + \alpha)} \ , \ q = \varphi'(t)L.D_xC(x)(|x-y| + \alpha) .$$

We have $(p + q, X) \in D^{2,+}u(x)$, $(p, Y) \in D^{2,-}u(y)$ and the viscosity inequalities

$$F(x, u(x), p + q, X) \leq 0 \ , \ F(y, u(y), p, Y) \geq 0 ,$$

with, for any $r, s \in \mathbb{R}^N$

$$Xr \cdot r - Ys \cdot s \leq \gamma_1|r - s|^2 + \gamma_2|r - s||r| + \gamma_3|r|^2 ,$$

where

$$\gamma_1 = \frac{\varphi'(t)LC(x)}{(|x-y| + \alpha)} + \varphi''(t)(LC(x))^2 ,$$

$$\gamma_2 = \varphi'(t)L.|D_xC(x)| + \varphi''(t)L^2.|D_xC(x)|.C(x).(|x-y| + \alpha) ,$$

$$\gamma_3 = \varphi'(t)\frac{|D^2C(x)|}{C(x)}t + \varphi''(t)\frac{|D_xC(x)|^2}{[C(x)]^2}t^2 ,$$

By easy manipulations, it is easy to see that

$$\gamma_2 \leq \gamma_1 \frac{|D_x C(x)|}{C(x)}(|x-y|+\alpha) \leq \gamma_1 K_2^{1/2} \chi(C(x))(|x-y|+\alpha) \; ,$$

$$\gamma_3 \leq 2\gamma_1 K_2 [\chi(C(x))]^2 (|x-y|+\alpha)^2 \; .$$

And, by Cauchy-Schwarz inequality, we deduce that, for any $\nu > 0$,

$$Xr\cdot r - Ys\cdot s \leq (1+\frac{\nu}{2})\gamma_1 |r-s|^2 + B(R,\nu)\gamma_1[\chi(C(x))]^2(|x-y|+\alpha)^2|r|^2 \; .$$

where $B(R,\nu) = (2+(2\nu)^{-1})K_2$ depends on R through K_2 and therefore is a $O(R^{-2})$ if ν is fixed.

Coming back to p and q, we also have

$$|q| = |p|\frac{|D_x C(x)|}{C(x)}(|x-y|+\alpha) \leq |p|\frac{|D_x C(x)|}{L[C(x)]^2} \leq O((RL)^{-1})|p| \; ,$$

since $LC(x)(|x-y|+\alpha) \leq 1$, $C \geq 1$ everywhere and since $\frac{|D_x C(x)|}{C(x)}$ is a $O(R^{-1})$. In order to have simpler formulas, we denote below by ϖ_1 any quantity which is a $O((RL)^{-1})$.

Now we arrive at the key point of the proof: by the above matrices inequality, choosing $r=0$, we have $-Y \leq (1+\frac{\nu}{2})I_N$ where I_N is the identity matrix in $\mathbb{R}^N$. Therefore the matrix $I_N + [(1+\nu)\gamma_1]^{-1}Y$ is invertible and rewriting the matrices inequality as

$$Xr\cdot r \leq Ys\cdot s + (1+\nu)\gamma_1|r-s|^2 + B(R,\nu)\gamma_1[\chi(C(x))]^2(|x-y|+\alpha)^2|r|^2 \; ,$$

we can take the infimum in s in the right-hand side and we end up with

$$X \leq Y(I_N + \frac{1}{(1+\nu)\gamma_1}Y)^{-1} + B(R,\nu)\gamma_1[\chi(C(x))]^2(|x-y|+\alpha)^2.I_N \; .$$

Setting $\tilde{Y} := Y(I_N + \frac{1}{(1+\nu)\gamma_1}Y)^{-1}$, this implies that we have $(p+q, \tilde{Y} + 3\gamma_1[\chi(C(x))]^2(|x-y|+\alpha)^2.I_N) \in D^{2,+}u(x)$, $(p,Y) \in D^{2,-}u(y)$ and, using the Lipschitz continuity of F in M, the viscosity inequalities

$$F(x,u(x),p+q,\tilde{Y}) \leq ||F_M||_\infty B(R,\nu)\gamma_1[\chi(C(x))]^2|x-y|^2 \; , \; F(y,u(y),p,Y) \geq 0 \; .$$

Next we introduce the function

$$g(\tau) := F(X(\tau),U(\tau),P(\tau),Z(\tau)) - \tau||F_M||_\infty B(R,\nu)\gamma_1[\chi(C(x))]^2(|x-y|+\alpha)^2 \; ,$$

where

$$X(\tau) = \tau x + (1-\tau)y \ , \ U(\tau) = \tau u(x) + (1-\tau)u(y) \ , \ P(\tau) = p + \tau q \ ,$$

$$Z(\tau) = Y(I_N + \frac{\tau}{(1+\nu)\gamma_1} Y)^{-1} \ .$$

From now on, in order to simplify the exposure, we are going to argue as if F were C^1: the case when F is just locally Lipschitz continuous follows from tedious but standard approximation arguments.

We have $g(0) \geq 0$ and $g(1) \leq 0$: if we can show that the C^1 function g satisfies $g'(\tau) > 0$ if $g(\tau) = 0$, we would have a contradiction. Therefore we compute

$$\begin{aligned} g'(\tau) &= F_x \cdot (x-y) + F_u(u(x) - u(y)) + F_p \cdot q + F_M \cdot Z'(\tau) \\ &\quad -||F_M||_\infty B(R,\nu)\gamma_1[\chi(C(x))]^2(|x-y|+\alpha)^2 \ , \end{aligned}$$

and using that $F_u \geq 0$, $Z'(\tau) = -((1+\nu)\gamma_1)^{-1}[Z(\tau)]^2$ and the estimates on p, q, we are lead to

$$\begin{aligned} g'(\tau) &\geq ((1+\nu)\gamma_1)^{-1} \left\{ F_x \cdot (1+\nu)\gamma_1(x-y) + \varpi_1 F_p \cdot P(\tau) - F_M \cdot [Z(\tau)]^2 \right. \\ &\quad \left. -(1+\nu)B(R,\nu)||F_M||_\infty(\gamma_1)^2[\chi(C(x))]^2(|x-y|+\alpha)^2 \right\} \ . \end{aligned}$$

Now we estimate $\gamma_1|x-y|$ using that $LC(x)(|x-y|+\alpha) \leq 1$ and the properties of φ

$$\begin{aligned} \gamma_1|x-y| &\leq \gamma_1(|x-y|+\alpha) \\ &\leq \varphi'(t)LC(x) + \varphi''(t)(LC(x))^2(|x-y|+\alpha) \\ &\leq |P(\tau)|(1+\varpi_1+o_\alpha(1)) + \varphi'(t)\chi(\varphi'(t))LC(x) \\ &\leq |P(\tau)|(1+\varpi_1+o_\alpha(1)) + \chi(\varphi'(t))|P(\tau)|(1+\varpi_1+o_\alpha(1)) \\ &\leq |P(\tau)|\chi(|P(\tau)|)(2+\varpi_1+o_\alpha(1)) \ . \end{aligned}$$

Indeed $|P(\tau)| = \varphi'(t)LC(x)(1+\varpi_1\tau)$ and recalling that $\varpi_1 = O((RL)^{-1})$, we can choose L large enough in order that $LC(x)(1+\varpi_1\tau) \geq 1$, allowing to use the inequality $\chi(\varphi'(t)) \leq \chi(|P(\tau)|)$. In the same way, one can choose L large enough to have $\chi(C(x)) \leq \chi(|P(\tau)|)$ and therefore

$$\gamma_1\chi(C(x))(|x-y|+\alpha) \leq |P(\tau)|[\chi(|P(\tau)|)]^2(2+\varpi_1+o_\alpha(1)) \ ,$$

and

$$\gamma_1[\chi(C(x))]^2(|x-y|+\alpha)^2 \leq |P(\tau)|\chi^2(|P(\tau)|)(1+\varpi_1+o_\alpha(1)).\chi(C(x))(|x-y|+\alpha) \ ;$$

but $(|x-y|+\alpha) \leq (LC(x))^{-1}$ and $\chi(C(x))(C(x))^{-1}$ is bounded, therefore

$$\gamma_1[\chi(C(x))]^2(|x-y|+\alpha)^2 \leq \varpi_1|P(\tau)|[\chi(|P(\tau)|)]^2 \; .$$

We end up with

$$\begin{aligned} g'(\tau) \;\; \geq \;\; & ((1+\nu)\gamma_1)^{-1}\{-(1+\nu)|F_x||P(\tau)|\chi(|P(\tau)|)(2+\varpi_1+o_\alpha(1)) \\ & +\varpi_1 F_p \cdot P(\tau) - F_M \cdot [Z(\tau)]^2 \\ & -(1+\nu)B(R,\nu)||F_M||_\infty[|P(\tau)|\chi^2(|P(\tau)|)]^2(2+\varpi_1+o_\alpha(1))\} \; . \end{aligned}$$

On the other hand, the constraint $g(\tau)=0$ implies

$$|F(X(\tau),U(\tau),P(\tau),Z(\tau))| \leq B(R,\nu)\varpi_1|P(\tau)|\chi^2(|P(\tau)|) \; .$$

The conclusion follows by choosing $2(1+\nu) < 2+\eta$ (for example $\nu = \eta/3$) and applying the assumption on F for L large enough in order that $\varpi_1 < \eta$, and taking $L \geq \bar{L}$ depending on $K = B(R,\nu)\varpi_1$ (which behaves like $O(R^{-2})$) and α small enough for which we have a contradiction.

Now we turn to the proof of (ii) where we choose $\varphi(t) = t$ and

$$\Gamma'_L := \{(x,y) \in B(x_0, 3R/4) \times B(x_0,R) : LC(x)(|x-y|+\alpha) \leq osc_R(u)\} \; .$$

The proof follows the same arguments, except that the fact that $\varphi''(t) \equiv 0$ allows different estimates on the γ_i, $i = 1,2,3$ since $\varphi''(t) \equiv 0$ implies that several terms do not exist anymore. Denoting by ϖ_2 any quantity of the form $O(osc_R(u)(RL)^{-1})$, we have

$$p = LC(x)\frac{(x-y)}{(|x-y|+\alpha)} \; , \; |q| = L.|D_xC(x)|(|x-y|+\alpha) = \varpi_2|p| \; ,$$

$$\gamma_1 = \frac{LC(x)}{(|x-y|+\alpha)} \; , \; \gamma_2 = L.|D_xC(x)| \; , \; \gamma_3 = L|D^2C(x)|(|x-y|+\alpha) \; .$$

And we still have the same estimates on $\gamma_1, \gamma_2, \gamma_3$

$$\gamma_2 = \gamma_1\frac{|D_xC(x)|}{C(x)}(|x-y|+\alpha) \leq \gamma_1\chi(C(x))(|x-y|+\alpha) \; ,$$

$$\gamma_3 \leq \gamma_1[\chi(C(x))]^2(|x-y|+\alpha)^2 \; .$$

The proof is then done in the same way as in the first case with the computation of $g'(\tau)$ and the estimates

$$\begin{aligned} g'(\tau) &\geq ((1+\nu)\gamma_1)^{-1}\left\{F_x\cdot(1+\nu)\gamma_1(x-y)+\varpi_2 F_p\cdot P(\tau)-F_M\cdot[Z(\tau)]^2\right.\\ &\left.\quad -(1+\nu)B(R,\nu)||F_M||_\infty(\gamma_1)^2[\chi(C(x))]^2(|x-y|+\alpha)^2\right\}\ . \end{aligned}$$

But here

$$\gamma_1(x-y)=p=P(\tau)(1+\varpi_2)\ ,$$

and in the same way,

$$\gamma_1\chi(C(x))(|x-y|+\alpha)\leq|P(\tau)|\chi(|P(\tau)|)(1+\varpi_2+o_\alpha(1))\ .$$

We end up with

$$\begin{aligned} g'(\tau) &\geq ((1+\nu)\gamma_1)^{-1}\{F_x\cdot P(\tau)(1+\varpi_2)\\ &\quad +\varpi_2F_p\cdot P(\tau)-F_M\cdot[Z(\tau)]^2\\ &\quad -(1+\nu)B(R,\nu)||F_M||_\infty[|P(\tau)|\chi(|P(\tau)|)]^2(1+\varpi_2+o_\alpha(1))\}\ . \end{aligned}$$

On the other hand, the constraint $g(\tau)=0$ implies

$$|F(X(\tau),U(\tau),P(\tau),Z(\tau))|\leq B(R,\nu)\varpi_2|P(\tau)|\chi(|P(\tau)|)\ . \qquad (4)$$

The conclusion follows as in the first case by applying the assumption on F for L large enough in order that $\varpi_2<\eta$, and taking $L\geq\bar{L}$ (depending on $K=B(R,\nu)\varpi_2=O(R^{-2})$ and α small enough for which we have a contradiction.

For the proof of (iii), we keep the same test-function and the same set Γ'_L but since we are not expecting the gradient bound to come from the same term in $g'(\tau)$, we are going to change the strategy in our computation of $g'(\tau)$ by keeping the F_u-term. Using that $F_u\geq 0$ and

$$u(x)-u(y)\geq LC(x)(|x-y|+\alpha)=\frac{|p|^2}{\gamma_1}\ ,$$

we obtain

$$\begin{aligned} g'(\tau) &= F_x\cdot(x-y)+F_u(u(x)-u(y))+F_p\cdot q+F_M\cdot Z'(\tau)\\ &\quad -||F_M||_\infty B(R,\nu)\gamma_1[\chi(C(x))]^2(|x-y|+\alpha)^2\ ,\\ &= (\gamma_1)^{-1}\left\{F_x\cdot p+F_u|p|^2+\varpi_2F_p\cdot P(\tau)-\frac{1}{1+\eta}F_M\cdot[Z(\tau)]^2\right.\\ &\quad\left.-B(R,\nu)||F_M||_\infty(\gamma_1)^2[\chi(C(x))]^2(|x-y|+\alpha)^2\right\}\ . \end{aligned}$$

This computation is close to the one given in [1] if there is no localization term ($C \equiv 1$).

Since $p = P(\tau)(1+\varpi_2)$ and using anagolous estimates as above, we are lead to

$$\begin{aligned} g'(\tau) \quad \geq \quad & (\gamma_1)^{-1} \Big\{ F_x \cdot P(\tau) + F_u |P(\tau)|^2 - \frac{1}{1+\eta} F_M \cdot [Z(\tau)]^2 \\ & + \varpi_2 \left(F_x \cdot P(\tau) + F_u |P(\tau)|^2 + F_p \cdot P(\tau) \right) \\ & - B(R,\nu) ||F_M||_\infty [|P(\tau)| \chi(|P(\tau)|)]^2 (1 + \varpi_2 + o_\alpha(1)) \Big\} \ . \end{aligned}$$

On the other hand, the constraint $g(\tau) = 0$ still implies (4) and we also conclude by choosing L large enough.

4 The parabolic case

In this section, we consider evolution equation under the general form

$$u_t + F(x,t,u,Du,D^2u) = 0 \quad \text{in } \Omega \times (0,T) \ , \tag{5}$$

and the aim is to provide a local gradient bound where "local" means both local in space and time. As a consequence, we will have to provide a localization also in time and a second main difference is that we will not be able to use that the equation holds since the u_t-term has no property in general and therefore the assumptions on F have to concern any x, t, r, p, M and not only those for which $F(x,t,r,p,M)$ is close to 0.

Theorem 4.1 (Estimates for non-uniformly parabolic equations : estimates depending the oscillation of u)
Assume that F is a locally Lipschitz function in $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \times \to \mathbb{R}$ which satisfies : $F(x,r,p,M)$ is Lipschitz continuous in M and

$$F_M(x,t,r,p,M) \leq 0 \quad a.e. \ in \ \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \times \to \mathbb{R} \ ,$$

and let $u \in C(\Omega \times (0,T))$ be a solution of (5). Assume that there exists a function $\chi \in \mathcal{K}$, $\eta > 0$ small enough such that, for any $K > 0$, there exists $\bar{L} = \bar{L}(\eta, R, K)$ large enough such that, in the set $\{(x,t,r,p,M);\ ,\ |p| \geq \bar{L}\}$, $F_u \geq 0$ and

$$\begin{aligned} F_x \cdot p + F_u |p|^2 - \frac{1}{1+\eta} F_M \cdot M^2 \quad \geq \quad & \eta + \eta(|F_x \cdot p| + F_u |p|^2 + F_p \cdot p + |p|^2 \chi(|p|)) \\ & + K ||F_M||_\infty (|p| \chi(|p|))^2) \ a.e.. \end{aligned}$$

If $B(x_0, R) \subset \Omega$ then u is Lipschitz continuous in $B(0, R/2)$ and $|Du| \leq \bar{L}$ in $B(0, R/2)$ where $\bar{L}$ is given by $L(\eta, K)$ where $K = K(R)$.

It is worth pointing out that the assumptions of Theorem 4.1 are rather close to the one of Theorem 2.1 (iii) and the same computations provide a gradient bound for the evolution equation

$$u_t - \mathrm{Tr}(A(x)D^2u) + |Du|^m = f(x) \quad \text{in } \Omega \, , \tag{6}$$

if $m > 1$.

Proof of Theorem 4.1 : We argue as in the proof of Theorem 2.1 (iii), except that here $L = L(t)$ with $L(t) \to +\infty$ as $t \to 0^+$. We still choose $\varphi(t) = t$ and

$$\Gamma'_L := \{(x, y, t) \in B(x_0, 3R/4) \times B(x_0, R) \times (0, T) : L(t)C(x)(|x-y|+\alpha) \leq osc_R(u)\} \, .$$

We consider maximum point $(x, y, t) \in \Gamma'_L$ of

$$u(x, t) - u(y, t) - L(t)C(x)(|x - y| + \alpha) \, ,$$

and if $x \neq y$ we are lead to the viscosity inequalities

$$a + F(x, t, u(x, t), p + q, X) \leq 0 \, , \; b + F(y, t, u(y, t), p, Y) \geq 0 \, ,$$

where $(a, p+q, X) \in D^{2,+}u(x, t)$, $(p, Y) \in D^{2,-}u(y, t)$ and $a-b \geq L'(t)C(x)(|x-y| + \alpha)$.

Subtracting these inequalities, we have

$$L'(t)C(x)(|x - y| + \alpha) + F(x, u(x), p + q, X) - F(y, u(y), p, Y) \leq 0 \, ,$$

and if we set

$$\begin{aligned} g(\tau) \; := \; & F(X(\tau), U(\tau), P(\tau), Z(\tau)) \\ & +\tau \left(L'(t)C(x)(|x - y| + \alpha) - ||F_M||_\infty B(R, \nu)\gamma_1[\chi(C(x))]^2(|x - y| + \alpha)^2] \right) \, , \end{aligned}$$

but here we have to show that $g'(\tau) > 0$ for any $\tau \in (0, 1)$, which will be a contradiction with $g(1) - g(0) \leq 0$.

The computation of $g'(\tau)$ and the estimates are done as above : denoting by ϖ_3 any quantity which is a $O(osc_R(u)(RL(t))^{-1})$ we obtain

$$\begin{aligned} g'(\tau) \ \geq \ & ((1+\nu)\gamma_1)^{-1}\left\{F_x\cdot(1+\nu)\gamma_1(x-y)+\varpi_3F_p\cdot P(\tau)-F_M\cdot[Z(\tau)]^2\right. \\ & -(1+\nu)B(R,\nu)||F_M||_\infty(\gamma_1)^2[\chi(C(x))]^2(|x-y|+\alpha)^2 \\ & \left.+(1+\nu)\gamma_1L'(t)C(x)(|x-y|+\alpha)\right\} \ . \end{aligned}$$

But

$$(1+\nu)\gamma_1L'(t)C(x)(|x-y|+\alpha)=(1+\nu)L(t)L'(t)[C(x)]^2 \ ,$$

and we end up with

$$\begin{aligned} g'(\tau) \ \geq \ & ((1+\nu)\gamma_1)^{-1}\left\{F_x\cdot P(\tau)(1+\varpi_3)+\varpi_3F_p\cdot P(\tau)-F_M\cdot[Z(\tau)]^2\right. \\ & -(1+\nu)B(R,\nu)||F_M||_\infty[|P(\tau)|\chi(|P(\tau)|)]^2(1+\varpi_3+o_\alpha(1)) \\ & \left.(1+\nu)L(t)L'(t)[C(x)]^2\right\} \ . \end{aligned}$$

In order to conclude, we have to choose L as the solution of the ode

$$L'(t)=k_TL(t)\chi(L(t)) \ , L(T)=L_T \text{ (large enough)} \ .$$

By choosing properly k, we have $L(0^+)=+\infty$ and

$$(1+\nu)L(t)L'(t)[C(x)]^2\geq-(1+\nu)k_T|P(\tau)|^2\chi(|P(\tau)|) \ .$$

Notice that $k_T\to0$ as $L_T\to+\infty$.

The conclusion follows as above by applying the assumption on F for L_T large enough in order that $\varpi_3<\eta$ and $(1+\nu)k_T\leq\eta$, and taking $L_T\geq\bar{L}$ and α small enough for which we have a contradiction.

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