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EXACT DISTANCE COLORING IN TREES

NICOLAS BOUSQUET, LOUIS ESPERET, ARARAT HARUTYUNYAN,
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Abstract. For an integer \( q \geq 2 \) and an even integer \( d \), consider the graph obtained from a large complete \( q \)-ary tree by connecting with an edge any two vertices at distance exactly \( d \) in the tree. This graph has clique number \( q + 1 \), and the purpose of this short note is to prove that its chromatic number is \( \Theta \left( \frac{d \log q}{\log d} \right) \). It was not known that the chromatic number of this graph grows with \( d \). As a simple corollary of our result, we give a negative answer to a problem of Van den Heuvel and Naserasr, asking whether there is a constant \( C \) such that for any odd integer \( d \), any planar graph can be colored with at most \( C \) colors such that any pair of vertices at distance exactly \( d \) have distinct colors. Finally, we study interval coloring of trees (where vertices at distance at least \( d \) and at most \( cd \), for some real \( c > 1 \), must be assigned distinct colors), giving a sharp upper bound in the case of bounded degree trees.

1. Introduction

Given a metric space \( X \) and some real \( d > 0 \), let \( \chi(X, d) \) be the minimum number of colors in a coloring of the elements of \( X \) such that any two elements at distance exactly \( d \) in \( X \) are assigned distinct colors. The classical Hadwiger-Nelson problem asks for the value of \( \chi(\mathbb{R}^2, 1) \), where \( \mathbb{R}^2 \) is the Euclidean plane. It is known that \( 5 \leq \chi(\mathbb{R}^2, 1) \leq 7 \) [1] and since the Euclidean plane \( \mathbb{R}^2 \) is invariant under homothety, \( \chi(\mathbb{R}^2, 1) = \chi(\mathbb{R}^2, d) \) for any real \( d > 0 \). Let \( \mathbb{H}^2 \) denote the hyperbolic plane. Kloekner [3] proved that \( \chi(\mathbb{H}^2, d) \) is at most linear in \( d \) (the multiplicative constant was recently improved by Parlier and Petit [6]), and observed that \( \chi(\mathbb{H}^2, d) \geq 4 \) for any \( d > 0 \). He raised the question of determining whether \( \chi(\mathbb{H}^2, d) \) grows with \( d \) or can be bounded independently of \( d \). As noticed by Kahle (see [3]), it is not known whether \( \chi(\mathbb{H}^2, d) \geq 5 \) for some real \( d > 0 \). Parlier and Petit [6] recently suggested to study infinite regular trees as a discrete analog of the hyperbolic plane. Note that any graph \( G \) can be considered as a metric space (whose elements are the vertices of \( G \) and whose metric is the graph distance in \( G \)), and in this context \( \chi(G, d) \) is precisely the minimum number of colors in a vertex coloring of \( G \) such that vertices at distance \( d \) apart are assigned different colors. Note that \( \chi(G, d) \) can be equivalently defined as the chromatic number of the exact \( d \)-th power of \( G \), that is, the graph with the same vertex-set as \( G \) in which two vertices are adjacent if and only if they are at distance exactly \( d \) in \( G \).

Let \( T_q \) denote the infinite \( q \)-regular tree. Parlier and Petit [6] observed that when \( d \) is odd, \( \chi(T_q, d) = 2 \) and proved that when \( d \) is even, \( q \leq \chi(T_q, d) \leq (d + 1)(q - 1) \). A

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similar upper bound can also be deduced from the results of Van den Heuvel, Kierstead, and Quiroz [2], while the lower bound is a direct consequence of the fact that when \( d \) is even, the clique number of the exact \( d \)-th power of \( T_q \) is \( q \) (note that it does not depend on \( d \)). In this short note, we prove that when \( q \geq 3 \) is fixed,

\[
\frac{d\log(q-1)}{4\log(d/2) + 4\log(q-1)} \leq \chi(T_q^d, d) \leq (2 + o(1))\frac{d\log(q-1)}{\log d},
\]

where the asymptotic \( o(1) \) is in terms of \( d \). A simple consequence of our main result is that for any even integer \( d \), the exact \( d \)-th power of a complete binary tree of depth \( d \) is of order \( \Theta(d/\log d) \) (while its clique number is equal to \( 3 \)).

The following problem (attributed to Van den Heuvel and Naserasr) was raised in [4] (see also [2] and [5]).

**Problem 1.1** (Problem 11.1 in [4]). Is there a constant \( C \) such that for every odd integer \( d \) and every planar graph \( G \) we have \( \chi(G, d) \leq C \)?

We will show that our result on large complete binary trees easily implies a negative answer to Problem 1.1. More precisely, we will prove that the graph \( U_3^d \) obtained from a complete binary tree of depth \( d \) by adding an edge between any two vertices with the same parent gives a negative answer to Problem 1.1 (in particular, for odd \( d \), the chromatic number of the exact \( d \)-th power of \( U_3^d \) grows as \( \Theta(d/\log d) \)). We will also prove that the exact \( d \)-th power of a specific subgraph \( Q_3^d \) of \( U_3^d \) grows as \( \Omega(\log d) \). Note that \( U_3^d \) and \( Q_3^d \) are outerplanar (and thus, planar) and chordal (see Figure 2).

Kloeckner [3] proposed the following variant of the original problem: For a metric space \( X \), an integer \( d \) and a real \( c > 1 \), we denote by \( \chi(X, [d, cd]) \) the smallest number of colors in a coloring of the elements of \( X \) such that any two elements of \( X \) at distance at least \( d \) and at most \( cd \) apart have distinct colors. Considering as above the natural metric space defined by the infinite \( q \)-regular tree \( T_q \), Parlier and Petit [6] proved that

\[
q(q-1)^{\lceil cd/2 \rceil - \lfloor d/2 \rfloor} \leq \chi(T_q^d, [d, cd]) \leq (q-1)^{\lceil cd/2 \rceil + 1}(\lfloor cd \rfloor + 1).
\]

We will show that \( \chi(T_q^d, [d, cd]) \leq \frac{q}{q-2} (q-1)^{\lfloor cd/2 \rfloor - d/2 + 1} + cd + 1 \), which implies that the lower bound of Parlier and Petit [6] (which directly follows from a clique size argument) is asymptotically sharp.

## 2. Exact Distance Coloring

Throughout the paper, we assume that the infinite \( q \)-regular tree \( T_q \) is rooted in some vertex \( r \). This naturally defines the children and descendants of a vertex and the parent and ancestors of a vertex distinct from \( r \). In particular, given a vertex \( u \), we define the ancestors \( u^0, u^1, \ldots \) of \( u \) inductively as follows: \( u^0 = u \) and for any \( i \) such that \( u^i \) is not the root, \( u^{i+1} \) is the parent of \( u^i \). With this notation, \( u^d \) can be equivalently defined as the ancestor of \( u \) at distance \( d \) from \( u \) (if such a vertex exists). For a given vertex \( u \) in \( T_q \), the depth of \( u \), denoted by \( \text{depth}(u) \), is the distance between \( u \) and \( r \) in \( T_q \). For a vertex \( v \) and an integer \( \ell \), we define \( L(v, \ell) \) as the set of descendants of \( v \) at distance exactly \( \ell \) from \( v \) in \( T_q \).
We first prove an upper bound on $\chi(T_q,d)$.

**Theorem 2.1.** For any integer $q \geq 3$, any even integer $d$, and any integer $k \geq 1$ such that $k(q - 1)^{k-1} \leq d$, we have $\chi(T_q,d) \leq (q - 1)^k + (q - 1)^{[k/2]} + \frac{d}{k} + 1$. In particular, $\chi(T_q,d) \leq d+q+1$, and when $q$ is fixed and $d$ tends to infinity, $\chi(T_q,d) \leq (2+o(1))\frac{d \log(q-1)}{\log d}$.

**Proof.** A vertex of $T_q$ distinct from $r$ and whose depth is a multiple of $k$ is said to be a special vertex. Let $v$ be a special vertex. Every special vertex $u$ distinct from $v$ such that $u^k = v^k$ is called a cousin of $v$. Note that $v$ has at most $q(q - 1)^{k-1} - 1$ cousins (at most $(q - 1)^k - 1$ if $v^k \neq r$). A special vertex $u$ is said to be a relative of $v$ if $u$ is either a cousin of $v$, or $u$ has the property that $u$ and $v$ have the same depth and are at distance at most $k$ apart in $T_q$. Two vertices $a,b$ at distance at most $k$ apart and at the same depth must satisfy $a^{[k/2]} = b^{[k/2]}$, and so the number of vertices $u$ such that $u$ and $v$ have the same depth and are at distance at most $k$ apart in $T_q$ is $(q - 1)^{[k/2]}$. It follows that if $v^k = r$, then $v$ has at most $(q - 1)^{k-1} - 1$ relatives and otherwise $v$ has at most $(q - 1)^k + (q - 1)^{[k/2]} - 1$ relatives.

The first step is to define a coloring $C$ of the special vertices of $T_q$. This will be used later to define the desired coloring of $T_q$, i.e. a coloring such that vertices of $T_q$ at distance $d$ apart are assigned distinct colors (in this second coloring, the special vertices will not retain their color from $C$).

We greedily assign a color $C(v)$ to each special vertex $v$ of $T_q$ as follows: we consider the vertices of $T_q$ in a breadth-first search starting at $r$, and for each special vertex $v$ we encounter, we assign to $v$ a color distinct from the colors already assigned to its relatives, and from the set of ancestors $v^i$ of $v$, where $2 \leq i \leq \frac{d}{k} + 1$ (there are at most $\frac{d}{k}$ such vertices). Note that if $v^k = r$, the number of colors forbidden for $v$ is at most $q(q - 1)^{k-1} - 1$ and if $v^k \neq r$ the number of colors forbidden for $v$ is at most $(q - 1)^k + (q - 1)^{[k/2]} + \frac{d}{k} - 1$. Since $(q - 1)^{k-1} \leq d$, in both cases $v$ has at most $(q - 1)^k + (q - 1)^{[k/2]} + \frac{d}{k} - 1$ forbidden colors, therefore we can obtain the coloring $C$ by using at most $(q - 1)^k + (q - 1)^{[k/2]} + \frac{d}{k}$ colors.

For any special vertex $v$, the set of descendants of $v$ at distance at least $d/2 - k$ and at most $d/2 - 1$ from $v$ is denoted by $K(v,k)$. We now define the desired coloring of $T_q$ as follows: for each special vertex $v$, all the vertices of $K(v,k)$ are assigned the color $C(v)$. Finally, all the vertices at distance at most $d/2 - 1$ from $r$ are colored with a single new color (note that any two vertices in this set lie at distance less than $d$ apart). The resulting vertex-coloring of $T_q$ is called $c$. Note that $c$ uses at most $(q - 1)^k + (q - 1)^{[k/2]} + \frac{d}{k} + 1$ colors, and indeed every vertex of $T_q$ gets exactly one color.

We now prove that vertices at distance $d$ apart in $T_q$ are assigned distinct colors in $c$. Assume for the sake of contradiction that two vertices $x$ and $y$ at distance $d$ apart were assigned the same color. Then the depth of both $x$ and $y$ is at least $d/2$. We can assume by symmetry that the difference $t$ between the depth of $x$ and the depth of $y$ is such that $0 \leq t \leq d$ since otherwise they would be at distance more than $d$. Let $u$ be the unique (special) vertex of $T_q$ such that $x \in K(u,k)$ and $v$ be the unique (special) vertex such that $y \in K(v,k)$. By the definition of our coloring $c$, we have $C(u) = C(v)$. Note that $u$ and $v$ are distinct; indeed, otherwise $x$ and $y$ would not be at distance $d$ in $T_q$. Assume first that
u and v have the same depth. Then since u and x (resp. v and y) are distance at least 
$d/2 − k$ apart, u and v are cousins (and thus, relatives), which contradicts the definition of
the vertex-coloring $C$. We may, therefore, assume that the depths of u and v are distinct.
Moreover, since u and v are special vertices, we may assume that their depths differ by at
least $k$. In particular, u lies deeper than v in $T_q$.

First assume that the depths of u and v differ by at least 2$k$. Then v is not an ancestor
of u in $T_q$. Indeed, for otherwise we would have $v = u^i$ for some integer $2 \leq i \leq \frac{d}{2} + 1$,
which would contradict the definition of $C$. This implies that the distance between x and
y is at least $d/2 − k + d − k + 2k + 2 = d + 2$, which is a contradiction. Therefore, we
can assume that the depths of u and v differ by precisely k. Since v is not a relative of u,
we have that $v \neq u^k$ and the distance between $u^k$ and v is more than k. Moreover, since u
and x (resp. v and y) are at distance at least $d/2 − k$ apart, this implies that the distance
between x and y is more than $d/2 − k + k + d/2 − k = d$, a contradiction.

Thus, c is a proper coloring.

By taking $k = 1$ we obtain a coloring c using at most $(q−1)^1 + (q−1)^{[1/2]} + \frac{d}{2} + 1 = q + d + 1$
colors, and by taking $k = \left\lceil \frac{1}{\log d} \right\rceil$, we obtain a coloring c using at most
$$\frac{d \log (q−1)}{\log d} + \sqrt{\frac{d \log (q−1)}{\log d}} + \frac{d \log (q−1)}{\log d} = (2 + o(1)) \frac{d \log (q−1)}{\log d}$$
colors.

For $k = 1$, the proof above can be optimized to show that $\chi(T_q, d) \leq q + \frac{d}{2}$ (by simply
noting that vertices at even depth and vertices at odd depth can be colored independently).
Since we are mostly interested in the asymptotic behaviour of $\chi(T_q, d)$ (which is of order
$O\left( \frac{d}{\log d} \right)$), we omit the details.

We now prove a simple lower bound on $\chi(T_q, d)$. Let $T_q^d$ be the rooted complete
$(q−1)$-ary tree of depth d, with root v. Note that each node has $q−1$ children, so this graph is
a subtree of $T_q$.

**Theorem 2.2.** For any integer $q \geq 3$ and any even d, $\chi(T_q^d, d) \geq \log_2(\frac{d}{4} + q − 1)$.

**Proof.** Consider any coloring of $T_q^d$ with colors 1, 2, . . . , $C$, such that vertices at distance
precisely d apart have distinct colors. For any vertex v at depth at most $\frac{d}{2} + 1$ in $T_q^d$, the
set of colors appearing in $L(v, \frac{d}{2} − 1)$ is denoted by $S_v$. Observe that if v and w have
the same parent, then $S_v$ and $S_w$ are disjoint since for any $x \in L(v, \frac{d}{2} − 1)$ and $y \in L(w, \frac{d}{2} − 1)$,
x and y are at distance d.

Fix some vertex u at depth at most $\frac{d}{2}$ in $T_q^d$ and some child v of u. We claim that:

**Claim 2.3.** For any integer $1 \leq k \leq \frac{\text{depth}(u)}{2}$, there is a color of $S_{u^{2k−1}}$ that does not appear
in $S_v$.

To see that Claim 2.3 holds, observe that in the subtree of $T_q^d$ rooted in $u^k$, there is a
vertex of $L(u^{2k−1}, \frac{d}{2} − 1)$ at distance d from all the elements of $L(v, \frac{d}{2} − 1)$. The color of
such a vertex does not appear in $S_v$, therefore Claim 2.3 holds.
In particular, Claim 2.3 implies that all the sets \( \{S_{u2^k-1} \mid 1 \leq k \leq d/4 \} \) and \( \{S_{w} \mid w \text{ is a child of } u \} \) are pairwise distinct. Since there are \( \frac{d}{4} + q - 1 \) such sets, we have \( \frac{d}{4} + q - 1 \leq 2^C \) and therefore \( C \geq \log_2(\frac{d}{4} + q - 1) \), as desired. \( \square \)

It was observed by Stéphan Thomassé that the proof of Theorem 2.2 only uses a small fraction of the graph \( T_d^q \). Consider for simplicity the case \( q = 3 \), and define \( P_{d3}^d \) as the graph obtained from a path \( P = v_0, v_1, \ldots, v_d \) on \( d \) edges, by adding, for each \( 1 \leq i \leq d \), a path on \( i \) edges ending at \( v_i \) (see Figure 1). This graph is an induced subgraph of \( T_d^q \) and the proof of Theorem 2.2 directly shows the following.

**Corollary 2.4.** For any even integer \( d \), \( \chi(P_{d3}^d, d) \geq \log_2(d + 8) - 2. \)

![Figure 1. The graph P_{d3}^d.](image)

The proof of Theorem 2.2 can be refined to prove the following better estimate for \( T_d^q \), showing that the upper bound of Theorem 2.1 is (asymptotically) tight within a constant multiplicative factor of 8.

**Theorem 2.5.** For any integer \( q \geq 3 \) and every even integer \( d \geq 2 \), \( \chi(T_d^q, d) \geq \frac{d \log(q-1)}{4 \log(d/2)+4 \log(q-1)} \).

**Proof.** Consider any coloring of \( T_d^q \) with colors \( 1, 2, \ldots, C \), such that vertices at distance precisely \( d \) apart have distinct colors. We perform a random walk \( v_0, v_1, \ldots, v_d \) in \( T_d^q \) as follows: we start with \( v_0 = r \), and for each \( i \geq 1 \), we choose a child of \( v_i \) uniformly at random and set it as \( v_{i+1} \). Note that the depth of each vertex \( v_i \) is precisely \( i \).

From now on we fix a color \( c \in \{1, \ldots, C\} \). For any vertex \( v \) of \( T_d^q \), the set of vertices contained in the subtree of \( T_d^q \) rooted in \( v \) is denoted by \( V_v \), and we set \( A_v = \{\text{depth}(u) \mid u \in V_v \text{ and } u \text{ has color } c\} \). When \( v = v_i \), for some integer \( 0 \leq i \leq d \), we write \( A_i \) instead of \( A_{v_i} \).

**Claim 2.6.** Assume that for some even integers \( i \) and \( j \) with \( 2 \leq i < j \leq d \), and for some vertex \( v \) at depth \( \frac{i+j-d}{2} \), the set \( A_v \) contains both \( i \) and \( j \). Then \( v \) has precisely one child \( u \) such that \( A_u \) contains \( i \) and \( j \), and moreover all the children \( w \) of \( v \) distinct from \( u \) are such that \( A_w \) contains neither \( i \) nor \( j \).

\(^1\)Stéphan Thomassé noticed that this can also be deduced from the fact that the vertices at depth at least \( \frac{d}{2} \) and at most \( d \) in the exact \( d \)-th power of \( P_3^d \) induce a shift graph.
To see that Claim 2.6 holds, simply note that \(\frac{i+j-d}{2} < i < j\) and if two vertices \(u_1, u_2\) colored \(c\) are respectively at depths \(i\) and \(j\), and their common ancestor is \(v\), then they are at distance \(d\) in \(T_q^d\) (which contradicts the fact that they were assigned the same color). Indeed, the distance of \(u_1\) to \(v\) is \(i - \frac{i+j-d}{2}\) and the distance of \(u_2\) to \(v\) is \(j - \frac{i+j-d}{2}\). This proves the claim.

We now define a family of graphs \((G_k)_{0 \leq k \leq d/2}\) as follows. For any \(0 \leq k \leq \frac{d}{2}\), the vertex-set \(V(G_k)\) of \(G_k\) is the set \(A_k \cap 2\mathbb{N} \cap (d/2, d]\), and two (distinct) even integers \(i, j \in A_k\) are adjacent in \(G_k\) if and only if \(\frac{i+j-d}{2} < k\). For each \(0 \leq k \leq \frac{d}{2}\) we define the energy \(E_k\) of \(G_k\) as follows: \(E_k = \sum_{i \in V(G_k)} (q-1)^{\deg(i)}\), where \(\deg(i)\) denotes the degree of the vertex \(i\) in \(G_k\).

Note that each graph \(G_k\) depends on the (random) choice of \(v_1, v_2, \ldots, v_k\).

**Claim 2.7.** For any \(0 \leq k \leq \frac{d}{2} - 1\), \(\mathbb{E}(E_{k+1}) \leq \mathbb{E}(E_k)\).

Assume that \(v_1, v_2, \ldots, v_k\) (and therefore also \(G_k\)) are fixed. Observe that \(G_{k+1}\) is obtained from \(G_k\) by possibly removing some vertices and adding some edges. Thus, \(E_{k+1}\) can be larger than \(E_k\) only if \(G_{k+1}\) contains edges that are not in \(G_k\). Therefore, it suffices to consider the contributions of those pairs of nonadjacent vertices in \(G_k\) which could become adjacent in \(G_{k+1}\) (since these correspond to pairs \(i, j\) with \(k = \frac{i+j-d}{2}\), these pairs are pairwise disjoint), and prove that these contributions are, in expectation, equal to 0. Fix a pair of even integers \(i < j\) in \(V(G_k)\) with \(k = \frac{i+j-d}{2}\) (and note that \(i\) and \(j\) are not adjacent in \(G_k\)). By Claim 2.6, either \(v_{k+1}\) is such that \(A_{k+1}\) contains \(i\) and \(j\) (this event occurs with probability \(\frac{1}{q-1}\), or \(A_{k+1}\) contains neither \(i\) nor \(j\) (with probability \(1 - \frac{1}{q-1}\)). As a consequence, for any \(i < j\) in \(V(G_k)\) with \(k = \frac{i+j-d}{2}\), with probability \(\frac{1}{q-1}\) we add the edge \(ij\) in \(G_{k+1}\) and with probability \(1 - \frac{1}{q-1}\) we remove vertices \(i\) and \(j\) from \(G_{k+1}\). This implies that for any \(i, j \in V(G_k), i < j\), with \(k = \frac{i+j-d}{2}\), with probability \(\frac{1}{q-1}\) we have contribution at most \((q-1)^{\deg(i)+1} + (q-1)^{\deg(j)+1} - (q-1)^{\deg(i)} - (q-1)^{\deg(j)} = (q-2)((q-1)^{\deg(i)} + (q-1)^{\deg(j)})\) to \(E_{k+1}\) (where \(\deg\) refers to the degree in \(G_k\)) and with probability \(1 - \frac{1}{q-1}\) we have a contribution of at most \(-(q-1)^{\deg(i)} - (q-1)^{\deg(j)}\) to \(E_{k+1}\). Thus, the expected contribution of such a pair \(i, j\) is at most \(\frac{1}{q-1}(q-2)((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) - \frac{q-2}{q-1}((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) = 0\).

Summing over all such pairs \(i, j\), we obtain \(\mathbb{E}(E_{k+1}) \leq \mathbb{E}(E_k)\). This proves Claim 2.7.

Since \(2 \leq i < j \leq d\), we have \(\frac{i+j-d}{2} \leq \frac{d}{2} - 1\), and in particular it follows that \(G_{d/2}\) is a (possibly empty) complete graph, whose number of vertices is denoted by \(\omega \geq 0\). Note that the energy \(E\) of a complete graph on \(\omega\) vertices is equal to \(\omega(q-1)^{\omega-1}\), while the energy \(E_0\) of \(G_0\) is equal to \(|A_0 \cap 2\mathbb{N} \cap (d/2, d]| \leq \frac{d}{4}\). For a vertex \(u \in L(r, \frac{d}{2})\), let \(\omega_u = |A_u \cap 2\mathbb{N} \cap (d/2, d]|\) (this is the number of distinct even depths at which a vertex colored \(c\) appears in the subtree of height \(\frac{d}{2}\) rooted in \(u\)). It follows from Claim 2.7 that the average of \(\omega_u(q-1)^{\omega_u-1}\), over all vertices \(u \in L(r, \frac{d}{2})\), is at most \(\frac{d}{4}\). Let \(a\) be the average of \(\omega_u\), over all vertices \(u \in L(r, \frac{d}{2})\). By Jensen’s inequality and the convexity of the function \(x \mapsto x(q-1)^{x-1}\) for \(x \geq 0\), we have that \(a(q-1)^{a-1} \leq \frac{d}{4}\), and thus \(a \leq \frac{\log(d/2)}{\log(q-1)} + 1\).
Note that \( a \) depends on the color \( c \) under consideration (to make this more explicit, let us now write \( a_c \) instead of \( a \)). Since there are \( \frac{d}{2} \) even depths between depth \( \frac{d}{2} \) and depth \( d \), there is a color \( c \in \{1, \ldots, C\} \) such that \( a_c \cdot C \geq \frac{d}{4} \) and thus, \( C \geq \frac{d}{4a_c} \geq \frac{d \log(q-1)}{4 \log(d/2) + 4 \log(q-1)} \), as desired.

We now explain how the results proved above give a negative answer to Problem 1.1. Let \( U_3^d \) (resp. \( Q_3^d \)) be obtained from \( T_3^d \) (resp. \( P_3^d \)) by adding an edge \( uv \) for any pair of vertices \( u, v \) having the same parent. Note that for any \( d \), \( U_3^d \) and \( Q_3^d \) are outerplanar (and thus, planar) and chordal, and \( Q_3^d \) has pathwidth 2 (\( U_3^d \) and \( Q_3^d \) are depicted in Figure 2) and the original copies of \( T_3^d \) and \( P_3^d \) are spanning trees of \( U_3^d \) and \( Q_3^d \), respectively. In the remainder of this section, whenever we write \( T_3^d \), we mean the original copy of \( T_3^d \) in \( U_3^d \).

![Figure 2](image)

The graphs \( U_3^3 \) (left) and \( Q_3^5 \) (right). The bold edges represent the original copies of \( T_3^3 \) and \( P_3^5 \), respectively.

Observe that for any two vertices \( u \) and \( v \) distinct from the root of \( T_3^d \), \( u \) and \( v \) are at distance \( d \) in \( T_3^d \) if and only if they are at distance \( d - 1 \) in \( U_3^d \) (since the depth of \( T_3^d \) is \( d \), the fact that \( u \) and \( v \) differ from the root and are at distance \( d \) apart implies that none of the two vertices is an ancestor of the other). The same property holds for \( Q_3^d \) and \( P_3^d \). As a consequence, for any odd integer \( d \), \( \chi(U_3^{d+1}, d) \) and \( \chi(T_3^{d+1}, d + 1) \) differ by at most one, and \( \chi(Q_3^{d+1}, d) \) and \( \chi(P_3^{d+1}, d + 1) \) also differ by at most one. Using this observation, we immediately obtain the following corollary of Theorem 2.5 and Corollary 2.4, which gives a negative answer to Problem 1.1.

**Corollary 2.8.** For any odd integer \( d \),

\[
\chi(U_3^{d+1}, d) \geq \frac{(d+1) \log(2)}{4 \log((d+1)/2) + 4 \log(2)} - 1 \quad \text{and} \quad \chi(Q_3^{d+1}, d) \geq \log_2(d + 8) - 3.
\]

The graphs \( U_3^{d+1} \) and its exact \( d \)-th power have \( n = 2^{d+2} \) vertices, and thus the chromatic number of the exact \( d \)-th power of \( U_3^{d+1} \) grows as \( \Omega\left(\frac{\log n}{\log \log n}\right) \). The graphs \( Q_3^{d+1} \) and its exact \( d \)-th power have \( n = \binom{d+2}{2} \) vertices, and thus the chromatic number of the exact \( d \)-th power of \( Q_3^{d+1} \) grows as \( \Omega(\log n) \). It is not difficult (using Theorem 2.1 for \( U_3^{d+1} \)) to show that these bounds are asymptotically tight.

It was recently proved by Quiroz [8] that if \( G \) is a chordal graph of clique number at most \( t \geq 2 \), and \( d \) is an odd number, then \( \chi(G, d) \leq \binom{t}{2}(d + 1) \). By Corollary 2.8, the
graph \( U^d_3 \) shows that this is asymptotically best possible (as \( d \) tends to infinity), up to a log \( d \) factor.

### 3. Interval coloring

For an integer \( d \) and a real \( c > 1 \), recall that \( \chi(T_q, [d, cd]) \) denotes the smallest number of colors in a coloring of the vertices of \( T_q \) such that any two vertices of \( T_q \) at distance at least \( d \) and at most \( cd \) apart have distinct colors. Parlier and Petit [6] proved that

\[
q(q - 1)^{|cd/2| - |d/2|} \leq \chi(T_q, [d, cd]) \leq (q - 1)^{|cd/2| + 1}(|cd| + 1).
\]

In this final section, we prove that their lower bound (which is proved by finding a set of \( \chi \) colors in which each pair of vertices at distance at least \( d \) and at most \( cd \) apart have distinct colors, as desired. □

**Theorem 3.1.** For any integers \( q \geq 3 \) and \( d \) and any real \( c > 1 \), \( \chi(T_q, [d, cd]) \leq \frac{q}{q-2}(q - 1)^{|cd/2| - d/2 + 1} + cd + 1 \).

**Proof.** The proof is similar to the proof of Theorem 2.1. We consider any ordering \( e_1, e_2, \ldots \) of the edges of \( T_q \) obtained from a breadth-first search starting at \( r \). Then, for any \( i = 1, 2, \ldots \) in order, we assign a color \( c(e_i) \) to the edge \( e_i \) as follows. Let \( e_i = uv \), with \( u \) being the parent of \( v \), and let \( \ell = |cd/2| - d/2 \). We assign to \( uv \) a color \( c(uv) \) distinct from the colors of all the edges \( xy \) (with \( x \) being the parent of \( y \)) such that \( x \) is at distance at most \( \ell \) from \( u^k \) (where \( k \) is the minimum of \( \ell \) and the depth of \( u \)), or \( x \) is an ancestor of \( u \) at distance at most \( cd \) from \( u \) (and \( y \) lies on the path from \( u \) to \( x \)). There are at most \( cd + \sum_{j=0}^{\ell} q(q - 1)^j \) such edges, so we can color all the edges following this procedure by using a total of at most \( \frac{q}{q-2}(q - 1)^{\ell+1} + cd \) colors.

As in the proof of Theorem 2.1, we now define our coloring of the vertices of \( T_q \) as follows: first color all the vertices at distance at most \( \frac{d}{2} - 1 \) from \( r \) with a new color that does not appear on any edge of \( T_q \), then for each vertex \( v \) with parent \( u \), we color all the vertices of \( L(v, \frac{d}{2} - 1) \) with color \( c(uv) \). In this vertex-coloring, at most \( \frac{q}{q-2}(q - 1)^{\ell+1} + cd \) colors are used.

Assume that two vertices \( s \) and \( t \), at distance at least \( d \) and at most \( cd \) apart, were assigned the same color. This implies that \( c(s^{d/2-1}s^{d/2}) = c(t^{d/2-1}t^{d/2}) \). Assume without loss of generality that the depth of \( s \) is at least the depth of \( t \), and consider first the case where \( t^{d/2-1} \) is an ancestor of \( s \). Then \( t^{d/2} \) is an ancestor of \( s^{d/2} \) at distance at most \( cd \) from \( s^{d/2} \) (and \( t^{d/2-1} \) lies on the path from \( s^{d/2} \) to \( t^{d/2} \)), which contradicts the definition of our edge-coloring \( c \). Thus, we can assume that \( t^{d/2-1} \) is not an ancestor of \( s \). This implies that \( t^{d/2-1}t^{d/2} \) lies on the path between \( s \) and \( t \), and therefore \( t^{d/2} \) is at distance at most \( \ell = |cd/2| - d/2 \) from the ancestor of \( s^{d/2} \) at distance \( \ell \) from \( s^{d/2} \) (or simply from \( r \), if the depth of \( s^{d/2} \) is at most \( \ell \)). Again, this contradicts the definition of our coloring \( c \). We obtained a coloring of the vertices of \( T_q \) with at most \( \frac{q}{q-2}(q - 1)^{\ell+1} + cd + 1 \) colors in which each pair of vertices at distance at least \( d \) and at most \( cd \) apart have distinct colors, as desired. □
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References