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Cutting degree of meanders

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Abstract. We study the cutting problems of meanders using 2-Motzkin words. These words uniquely define elevated peakless Motzkin paths, which under specific conditions correspond to meanders. A procedure for the determination of the set of meanders with a given sequence of cutting degrees, or with a given cutting degree, is presented by using proper conditions.

Keywords: Dyck path, Grand Dyck path, 2-Motzkin word, elevated peakless Motzkin path.

1 Introduction

The cutting problem of meanders has been introduced by the authors in [9]. Using meandric polygons, we have also enumerated the set of meanders with given numbers of cuttings, i.e., the numbers of arcs of the meandric curve lying above and below the horizontal line at a given point.

In the present paper, we complete the cutting problem of meanders. At first, we represent a meander or meandric system using Motzkin words, and we specify the subset of meandric Motzkin words, through a necessary and sufficient condition. Additionally, we determine the matching nested sets of a meandric Motzkin word. Afterwards, we establish a bijection between meandric Motzkin words and elevated peakless Motzkin paths. We calculate the cardinality of this class, as well as the number of these paths, passing through a given point. Finally, we confront the two main cutting problems of meanders, as we determine the set of meanders with a given sequence of cutting degrees, or with a given cutting degree.

The two cutting problems can be considered as data-mining problems, since we extract the subset of Motzkin words which are meandric, or the subset of Motzkin paths which are peakless and elevated. Specifically in the field of Bioinformatics, the already mentioned problems have many facets in relation with the M-folding problem and RNA prediction problem, based on the fact that closed RNA structures can be viewed as mathematical objects obtained by abstracting topologically non-relevant properties of planar folding of single-stranded nucleic

acids. In the literature, many approximation data-mining algorithms had been constructed, concerning the enumeration of sequential terms of RNA secondary structures, through the bijection with alternative representations such as energy models, plane trees and Motzkin numbers [11], k-noncrossing set of partitions [3], set of nested arcs [3], Motzkin paths and Dyck paths [5].

The following definitions and notation refer to notions that are necessary for the development of the paper. A set S of disjoint pairs of $[2n]$ such that $\bigcup_{\{a,b\}\in S}\{a,b\} = [2n]$ and for any $\{a,b\}$, $\{c,d\}$ we never have $a < c < b < d$, is called *nested set* of pairs on $[2n]$. We denote the set of all nested sets of pairs on $[2n]$ by N_{2n}. Two nested sets $S_1, S_2 \in N_{2n}$ define a permutation σ on $[2n]$, such that $\sigma(2i-1) = j$ iff $\{2i-1, j\} \in S_1 \quad$ and $\quad \sigma(2i) = j$ iff $\{2i, j\} \in S_2$, for every $i \in [n]$. The sets S_1, S_2 are *k-matching* if and only if σ has k cycles. In the case where $k = 1$, S_1, S_2 are simply called *matching*.

A word $v \in \{a, \bar{a}\}^*$ is called *Dyck word* if $|v|_a = |v|_{\bar{a}}$ and for every factorization $v = pq$ we have $|p|_a \geq |p|_{\bar{a}}$ where $|v|_a$, $|p|_a$ (resp. $|v|_{\bar{a}}$, $|p|_{\bar{a}}$) denote the number of occurrences of a (resp. $\bar{a}$) in the words v and p. A word $v \in \{a, \bar{a}\}^*$, such that $|v|_a = |v|_{\bar{a}}$ is called *Grand Dyck word*. We denote by D_{2n} the set of all Dyck words of length $2n$ and it is well known that the cardinality of D_{2n} is equal to the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

2 Meanders

A *(closed) meander of order n* is a closed self avoiding curve, crossing a horizontal line $2n$ times [4]. We denote by M_{2n} the set of meanders of order n [7].

We can extend the definition of meanders to *meandric systems* with k components, by allowing configurations with k disconnected meanders [4].

In this paper, every meander of M_{2n} is represented by a word $w \in \{a, \bar{a}, b, \bar{b}\}^*$, which is called *meandric Motzkin word* [8]. For example, the meander of Fig. 1 corresponds to the word $w = a\ a\ \bar{b}\ b\ a\ b\ \bar{a}\ \bar{a}\ \bar{b}\ \bar{a}$.

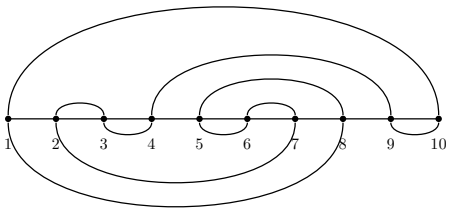

Fig. 1. A meander with $w = a\ a\ \bar{b}\ b\ a\ b\ \bar{a}\ \bar{a}\ \bar{b}\ \bar{a}$.

In order to formally introduce the meandric Motzkin words, following [8], we recall that for every pair $v = v_1 v_2 \cdots v_\ell$ and $v' = v'_1 v'_2 \cdots v'_\ell$ of words in $\{a, \bar{a}\}^*$ of length ℓ, we define $v \circ v'$ to be the word $w = w_1 w_2 \cdots w_\ell \in \{a, \bar{a}, b, \bar{b}\}^*$, with

$$w_i = v_i \circ v'_i, i \in [\ell], \quad \text{where } a \circ a = a, \bar{a} \circ \bar{a} = \bar{a}, a \circ \bar{a} = b, \bar{a} \circ a = \bar{b}.$$

Note that the words v and v' can be uniquely recovered from w using the following relations

$$v_i = \begin{cases} a, & \text{if } w_i = a \text{ or } b, \\ \bar{a}, & \text{if } w_i = \bar{a} \text{ or } \bar{b}, \end{cases} \qquad \text{and} \qquad v'_i = \begin{cases} a, & \text{if } w_i = a \text{ or } \bar{b}, \\ \bar{a}, & \text{if } w_i = \bar{a} \text{ or } b, \end{cases}$$

for $i \in [\ell]$, and we call v, v' *associates* of w.

We define the set of the meandric Motzkin words of length $2n$ as the set

$$\hat{W}_{2n} = \{w : w = v \circ v', v, v' \in D_{2n}\}.$$

Notice that from the word $v \circ v'$ we immediately obtain the word $v' \circ v$, by interchanging the letters b and $\bar{b}$. So, in order to generate the set $\hat{W}_{2n}$ it is actually enough to construct half of its elements.

The set $\hat{W}_{2n}$ can be partitioned into the classes W^k_{2n}, $k \in [n]$, where $W_{2n} = W^1_{2n}$. Each class W^k_{2n} contains the meandric Motzkin words with $v, v' \in D_{2n}$, such that their corresponding nested sets are k-matching.

It has been proved in [8] that there exists a bijection between the sets M_{2n} and W_{2n}. According to this bijection, every meander μ can be considered as an ordered pair (U, L) of nested sets of pairs, where U (resp. L) is defined by the arcs of the meandric curve above (resp. below) the horizontal line. Since U, L are encoded by two Dyck words, say v and v', then the meandric Motzkin word $w = v \circ v'$ encodes the meander μ.

In [8], it was also proved (see Proposition 2.1) that every word w of $\hat{W}_{2n}$ is a 2-Motzkin word of $\{a, \bar{a}, b, \bar{b}\}^*$, with $|w|_b = |w|_{\bar{b}}$. More specifically, if $w \in \hat{W}_{2n}$, then by deleting the letters $b, \bar{b}$ (resp. $a, \bar{a}$) in w, a Dyck word (resp. Grand Dyck word) is obtained, or equivalently, we have that $|w|_a = |w|_{\bar{a}}$, $|w|_b = |w|_{\bar{b}}$ and $|p|_a \geq |p|_{\bar{a}}$, for every factorization $w = pq$. Consequently, every word $w \in \hat{W}_{2n}$ can be considered as the shuffle of a Dyck word in $\{a, \bar{a}\}^*$ and a Grand Dyck word in $\{b, \bar{b}\}^*$, where the first and last letter of w are a and $\bar{a}$ respectively. Obviously, not every word w satisfying the above conditions encodes a meander, or a meandric system. As an example, consider the word $a\ \bar{b}\ \bar{b}\ b\ b\ \bar{a}$. The next result states the necessary and sufficient condition for a word to be a meandric Motzkin word.

Proposition 1 *For a word $w \in \{a, \bar{a}, b, \bar{b}\}^*$ of length $2n$, we have that $w \in \hat{W}_{2n}$ if and only if $|w|_a - |w|_{\bar{a}} = |w|_b - |w|_{\bar{b}} = 0$ and $|p|_a - |p|_{\bar{a}} \geq \big||p|_b - |p|_{\bar{b}}\big|$, for every factorization $w = pq$.*

Proof. Let $w \in \{a, \bar{a}, b, \bar{b}\}^*$ and let $w = pq$ be a factorization of w. Then, the associates v, v' of w, such that $w = v \circ v'$, are uniquely defined and, furthermore, the factorization of w uniquely defines the factorizations $v = rs$ and $v' = r's'$, such that $p = r \circ r'$ and $q = s \circ s'$. By definition, we have the following relations

$$|r|_a = |p|_a + |p|_b, \quad |r|_{\bar{a}} = |p|_{\bar{a}} + |p|_{\bar{b}}, \quad |r'|_a = |p|_a + |p|_{\bar{b}}, \quad |r'|_{\bar{a}} = |p|_{\bar{a}} + |p|_b,$$

which lead to

$$|r|_a - |r|_{\bar{a}} = |p|_a - |p|_{\bar{a}} + |p|_b - |p|_{\bar{b}} \quad \text{and} \quad |r'|_a - |r'|_{\bar{a}} = |p|_a - |p|_{\bar{a}} - (|p|_b - |p|_{\bar{b}}). \quad (1)$$

Note that the above relations also hold, when we set $p = w$, and in this case we have $r = v$ and $r' = v'$.

If $w \in \hat{W}_{2n}$, then $v, v' \in D_{2n}$, so that $|r|_a - |r|_{\bar{a}} \geq 0$, $|r'|_a - |r'|_{\bar{a}} \geq 0$ and $|v|_a - |v|_{\bar{a}} = |v'|_a - |v'|_{\bar{a}} = 0$. In this case, from relations (1), we have that $|p|_a - |p|_{\bar{a}} \geq |p|_{\bar{b}} - |p|_b$ and $|p|_a - |p|_{\bar{a}} \geq |p|_b - |p|_{\bar{b}}$, or equivalently $|p|_a - |p|_{\bar{a}} \geq \big||p|_b - |p|_{\bar{b}}\big|$, while applying relations (1), for $p = w$, we obtain that $|w|_a - |w|_{\bar{a}} = |w|_{\bar{b}} - |w|_b$ and $|w|_a - |w|_{\bar{a}} = |w|_b - |w|_{\bar{b}}$, or equivalently $|w|_a - |w|_{\bar{a}} = |w|_b - |w|_{\bar{b}} = 0$.

Conversely, if $|w|_a - |w|_{\bar{a}} = |w|_b - |w|_{\bar{b}} = 0$ and $|p|_a - |p|_{\bar{a}} \geq \big||p|_b - |p|_{\bar{b}}\big|$, then relations (1) imply that $|r|_a - |r|_{\bar{a}} \geq 0$ and $|r'|_a - |r'|_{\bar{a}} \geq 0$, while applying relations (1) for $p = w$, we obtain that $|v|_a - |v|_{\bar{a}} = |v'|_a - |v'|_{\bar{a}} = 0$. From the above, we deduce that $v, v' \in D_{2n}$, so that $w \in \hat{W}_{2n}$. □

The process of determining the matching nested sets U, L that correspond to a meander $w \in W_{2n}$ can be simplified by the following easily verified proposition:

Proposition 2 *Let U, L be the matching nested sets that correspond to a meander $w \in W_{2n}$. Then, every pair of U (resp. L) corresponds to a subsequence of w having one of the following forms: $a\bar{a}, a\bar{b}, b\bar{a}, b\bar{b}$ (resp. $a\bar{a}, ab, \bar{b}\bar{a}, \bar{b}b$).*

According to Proposition 2, the short pairs of U (i.e., pairs of the form $\{i, i+1\}$) are immediately determined by identifying the positions of the occurrences of the digrams $a\bar{a}, a\bar{b}, b\bar{a}, b\bar{b}$ in w. Then, by deleting these digrams from w, we obtain a new word $w' \in \{a, \bar{a}, b, \bar{b}\}^*$. We repeat the same procedure for w', until all pairs are determined. The pairs of L are obtained analogously.

For example, for the meander $w = aa\bar{b}ba\bar{b}\bar{a}\bar{a}\bar{b}\bar{a}$ of Fig. 1, the nested set U is obtained as follows:

$$\overset{a}{1}\overbrace{\overset{a}{2}\overset{\bar{b}}{3}}\overset{b}{4}\overset{a}{5}\overbrace{\overset{b}{6}\overset{\bar{a}}{7}}\overset{\bar{a}}{8}\overset{\bar{b}}{9}\overset{\bar{a}}{10} \longrightarrow \overset{a}{1}\overset{b}{4}\overbrace{\overset{a}{5}\overset{\bar{a}}{8}}\overset{\bar{b}}{9}\overset{\bar{a}}{10} \longrightarrow \overset{a}{1}\overbrace{\overset{b}{4}\overset{\bar{b}}{9}}\overset{\bar{a}}{10} \longrightarrow \overbrace{\overset{a}{1}\overset{\bar{a}}{10}}$$

Thus, $U = \{\{1, 10\}, \{2, 3\}, \{4, 9\}, \{5, 8\}, \{6, 7\}\}$. Analogously, we obtain that $L = \{\{1, 8\}, \{2, 7\}, \{3, 4\}, \{5, 6\}, \{9, 10\}\}$.

Examples

1. The meandric Motzkin word $w = a\ b\ b\ b\ \bar{b}\ a\ \bar{b}\ \bar{a}\ \bar{b}\ \bar{a}$, does not belong to the set $\hat{W}_{10}$. This is verified by the following table:

	a	b	b	b	$\bar{b}$	a	$\bar{b}$	$\bar{a}$	$\bar{b}$	$\bar{a}$
$\lvert p\rvert_a - \lvert p\rvert_{\bar{a}}$	1	1	1							
$\big\lvert\lvert p\rvert_b - \lvert p\rvert_{\bar{b}}\big\rvert$	0	1	2							

2. The meandric Motzkin word $w = a\ b\ \bar{b}\ a\ b\ \bar{a}\ \bar{b}\ b\ \bar{b}\ \bar{a}$, belongs to the set $w \in W_{10}$. This is verified by the following table

	a	b	$\bar{b}$	a	b	$\bar{a}$	$\bar{b}$	b	$\bar{b}$	$\bar{a}$
$\lvert p\rvert_a - \lvert p\rvert_{\bar{a}}$	1	1	1	2	2	1	1	1	1	0
$\big\lvert\lvert p\rvert_b - \lvert p\rvert_{\bar{b}}\big\rvert$	0	1	0	0	1	1	0	1	0	0

and by the fact that its corresponding nested sets

$U = \{\{1,10\},\{2,3\},\{4,7\},\{5,6\},\{8,9\}\}, L = \{\{1,2\},\{3,6\},\{4,5\},\{7,8\},\{9,10\}\},$

which are obtained using Proposition 2, are matching.

3. The meandric Motzkin word $w = a\ a\ b\ b\ \bar{b}\ \bar{b}\ \bar{a}\ b\ \bar{b}\ \bar{a}$, belongs to the set $w \in W_{10}^2$. This is verified by the following table

	a	a	b	b	$\bar{b}$	$\bar{b}$	$\bar{a}$	b	$\bar{b}$	$\bar{a}$
$\lvert p\rvert_a - \lvert p\rvert_{\bar{a}}$	1	2	2	2	2	2	1	1	1	0
$\big\lvert \lvert p\rvert_b - \lvert p\rvert_{\bar{b}} \big\rvert$	0	0	1	2	1	0	0	1	0	0

and by the fact that its corresponding nested sets

$U = \{\{1,10\},\{2,7\},\{3,6\},\{4,5\},\{8,9\}\}, L = \{\{1,4\},\{2,3\},\{5,8\},\{6,7\},\{9,10\}\},$

which are obtained using Proposition 2, are 2-matching.

3 Motzkin paths

We recall that a *Motzkin path* (resp. 2-*Motzkin path*) is a lattice path in the first quadrant beginning at the origin, ending at the x-axis and consisting of the steps $a = (1,1)$ as the up-step, $\bar{a} = (1,-1)$ as the down-step and $b = (1,0)$ as the horizontal step (resp. a, $\bar{a}$ and two kinds of horizontal steps denoted by b and $\bar{b}$). Every Motzkin path (resp. 2-*Motzkin path*) is encoded by its sequence of steps, i.e., a word $y \in \{a,\bar{a},b\}^*$ (resp. $y \in \{a,\bar{a},b,\bar{b}\}^*$). The set of all Motzkin paths (resp. 2-Motzkin paths) of length n is enumerated by the n-th Motzkin number $\mathrm{M}_n = \sum_{k\geq 0} \binom{n}{2k} C_k$ [2], (resp. the number C_{n+1} [1]).

An *elevated peakless Motzkin path*, is a Motzkin path that never touches the x-axis except at its first and last points and it has no peaks (i.e., no occurrences $a\bar{a}$). For example, the path of Fig. 2 is an elevated peakless Motzkin path of length 10 with $y = a\ a\ b\ b\ a\ b\ \bar{a}\ \bar{a}\ b\ \bar{a}$. We denote by P_{2n} the set of all elevated

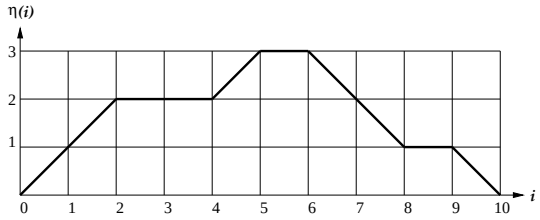

Fig. 2. The elevated peakless Motzkin path with $y = a\ a\ b\ b\ a\ b\ \bar{a}\ \bar{a}\ b\ \bar{a}$.

peakless Motzkin paths of length $2n$.

Given that (see [10], A004148) the number of peakless Motzkin paths of length n is equal to $a(n) = \sum_{k\geq\lceil\frac{n+1}{2}\rceil} \frac{1}{k}\binom{k}{n-k}\binom{k}{n-k+1}$, we easily deduce that the number of paths of P_{2n} is equal to $a(2n-2)$. Thus, we have the following result.

Proposition 3 *The cardinality of P_{2n} is given by*

$$\sum_{k\geq\lceil\frac{2n-1}{2}\rceil}\frac{1}{k}\binom{k}{2n-k-2}\binom{k}{2n-k-1}.$$

A list of terms of the above sequence are presented in [10], $A203019$.

The set P_{2n} can also be encoded in $\{0,1,2,\ldots,n\}$ by the heights $\eta(i)$ of the vertices $(i,\eta(i))$, $i=0,1,\ldots,2n$ of the path. For the heights $\eta(i)$ we have the following properties:

(i) $\eta(0)=\eta(2n)=0$, $\eta(1)=\eta(2n-1)=1$.
(ii) $1\leq\eta(i)\leq\min\{i,2n-i\}$, $i\in[2n-1]$.
(iii) $|\eta(i)-\eta(i-1)|=0$ or 1, $i\in[2n]$.
(iv) There exists no $i\in[2n-2]$, such that $\eta(i)=\eta(i+2)=\eta(i+1)-1$.

Indeed, the first property holds since each path of P_{2n} begins at $(0,0)$, ends at $(2n,0)$ and it is elevated. The second property is deduced by the fact that the difference of the number of occurrences of a's and $\bar{a}$'s at the i-step is at most i, if $i\leq n$ or $2n-i$, if $i\geq n$. The third property is obvious since at each step we add $+1$, -1 or 0 to the height of the previous point. The last property is an immediate consequence of the fact that the path is peakless.

By omitting the initial and final terms $\eta(0)=\eta(2n)=0$, we introduce the finite sequences $\eta=(\eta(i))_{i\in[2n-1]}$, and their set will be denoted by H_{2n-1}.

For example, for the path of Fig. 2, we obtain the sequence

$$\eta=1\ 2\ 2\ 2\ 3\ 3\ 2\ 1\ 1.$$

In general, to every path $y\in P_{2n}$ corresponds a sequence $\eta\in H_{2n-1}$ with

$$\eta(1)=\eta(2n-1)=1\quad\text{and}\quad\eta(i)=\begin{cases}\eta(i-1)+1, & \text{if } y_i=a,\\ \eta(i-1), & \text{if } y_i=b,\\ \eta(i-1)-1, & \text{if } y_i=\bar{a},\end{cases}\quad 2\leq i\leq 2n-2.$$

The converse also holds, since to every sequence $\eta\in H_{2n-1}$ corresponds a path $y\in P_{2n}$ with

$$y_1=a, y_{2n}=\bar{a}\quad\text{and}\quad y_i=\begin{cases}a, & \text{if } \eta(i)>\eta(i-1),\\ b, & \text{if } \eta(i)=\eta(i-1),\\ \bar{a}, & \text{if } \eta(i)<\eta(i-1),\end{cases}\quad 2\leq i\leq 2n-1.$$

So, we have arrived at the following proposition.

Proposition 4 *There exists a bijection between the sets P_{2n} and H_{2n-1}.*

From the above bijection between the sets P_{2n} and H_{2n-1}, we deduce that the cardinality of the set H_{2n-1} is given by the formula of Proposition 3.

The paths of P_{2n} can be obtained by their corresponding sequences of H_{2n-1} following the above properties of the heights $\eta(i)$.

This is done with the aid of a suitable tree of height $2n-1$, in which each vertex is labeled by an integer of $[n]$, so that:

(i) if a vertex labeled by ξ belongs to the e^{th} level, then $\xi \leq \min\{e, 2n-e\}$,
(ii) to every vertex with label ξ correspond at most three children, with labels $\xi-1$, ξ, $\xi+1$, on $[n]$.
(iii) at every path of the tree, we can not find three consecutive vertices with labels ξ, $\xi+1$, ξ on $[n]$.

Our next effort concerns the number $a_{i,k}$ of paths of P_{2n} passing through the point (i,k). We already know (see [10], A097724) that the number of left factors of peakless Motzkin paths, having length n and endpoint height k is equal to $T(n,k) = (k+1)\sum_{j\geq\lceil\frac{n-k+1}{2}\rceil}\frac{1}{j}\binom{j}{n-k-j}\binom{j+k}{n+1-j}$.

The number of left factors for the paths of P_{2n} starting from $(0,0)$ and ending at (i,k) is given by $T(i-1,k-1) = k\sum_{j\geq\lceil\frac{i-k+1}{2}\rceil}\frac{1}{j}\binom{j}{i-k-j}\binom{j+k-1}{i-j}$.

Furthermore, the number of right factors for the paths of P_{2n} starting from (i,k) and ending at $(2n,0)$ is equal to the number of left factors for the paths of P_{2n} starting from $(0,0)$ and ending at $(2n-i,k)$, i.e., $T(2n-i-1,k-1) = k\sum_{j\geq\lceil\frac{2n-i-k+1}{2}\rceil}\frac{1}{j}\binom{j}{2n-i-k-j}\binom{j+k-1}{2n-i-j}$. Thus, we have the following result.

Proposition 5 *The number of paths of P_{2n} passing through the point (i,k) is given by the product* $k^2\sum_{j\geq\lceil\frac{i-k+1}{2}\rceil}\frac{1}{j}\binom{j}{i-k-j}\binom{j+k-1}{i-j}\sum_{j\geq\lceil\frac{2n-i-k+1}{2}\rceil}\frac{1}{j}\binom{j}{2n-i-k-j}\binom{j+k-1}{2n-i-j}$.

The paths of P_{2n} passing through the point (i,k) are obtained by the subtree of the tree of H_{2n-1} with $\eta(i) = k$.

We denote by $P_{2n}^{(2)}$ the set of elevated peakless 2-Motzkin paths y of length $2n$, with $|y|_b = |y|_{\bar{b}}$. Since the length of every path $w \in P_{2n}$ and the number $|w|_a + |w|_{\bar{a}}$ are even, we conclude that $|w|_b$ is also even. Therefore, every path $y \in P_{2n}^{(2)}$, with $|y|_b = |y|_{\bar{b}} = \nu$, is uniquely obtained from a path $w \in P_{2n}$, with $|w|_b = 2\nu$, by converting half of the b steps of w into $\bar{b}$ steps. Conversely, $w \in P_{2n}$ generates $\binom{2\nu}{\nu}$ paths of $P_{2n}^{(2)}$, which cannot be generated by any other path of P_{2n}.

Furthermore, the number of peakless Motzkin paths of length $2n$ with k a steps and 2ν horizontal steps is given by the formula (sequence $A089732$ in [10]) $T(2n, n-\nu) = \binom{2n-k}{k}\binom{2n-k}{k+1}\frac{1}{2n-k} = \binom{n+\nu}{2\nu}\binom{n+\nu}{2\nu-1}\frac{1}{n+\nu}$, where $T(2n,n-\nu)$ is also the number of paths in P_{2n} with 2ν horizontal steps, for $n,\nu > 0$.

Finally, since P_{2n} is partitioned with respect to the number 2ν of b steps, where $1 \leq \nu \leq n$, using the previous formula, we have the following result.

Proposition 6 *The cardinality of $P_{2n}^{(2)}$ is given by* $\sum_{\nu=1}^{n}\binom{2\nu}{\nu}\binom{n+\nu}{2\nu}\binom{n+\nu}{2\nu-1}\frac{1}{n+\nu}$.

4 Cutting problem

For every meander of M_{2n} and for any $i \in [2n-1]$ we consider the vertical line passing through the middle point of the segment $(i, i+1)$, which we call as the *i-line* [9]. The number of arcs of the meandric curve, which are intersected by the i-line is called *cutting degree* $\epsilon(i)$ of the meander at i.

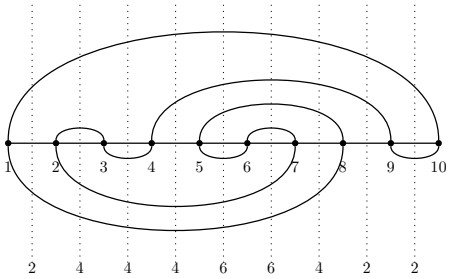

Fig. 3. A meander with $\epsilon = 2\ 4\ 4\ 4\ 6\ 6\ 4\ 2\ 2$.

For example, for the meander of Fig. 1, we have $\epsilon = 244466422$ (see Fig. 3).
The cutting degrees for a meander $w \in W_{2n}$ are obviously deduced by

$$\epsilon(1) = \epsilon(2n-1) = 2 \quad \text{and} \quad \epsilon(i) = \begin{cases} \epsilon(i-1) + 2, & \text{if } w_i = a, \\ \epsilon(i-1), & \text{if } w_i = b \text{ or } \bar{b}, \\ \epsilon(i-1) - 2,, & \text{if } w_i = \bar{a}, \end{cases}$$

for $i = 2, 3, \ldots, 2n-2$.

We denote by E_{2n-1} the set of the sequences $\epsilon = (\epsilon(i))_{i\in[2n-1]}$ satisfying

$$\epsilon(1) = \epsilon(2n-1) = 2 \quad \text{and} \quad |\epsilon(i) - \epsilon(i+1)| \in \{0, 2, -2\}, i \in [2n-2].$$

From the previous, we have that

1. To every meander $w \in W_{2n}$ corresponds a unique finite sequence $\epsilon \in E_{2n-1}$ but obviously, the converse does not hold.
2. The sequences ϵ and η, corresponding to a meander of M_{2n}, satisfy $\epsilon(i) = 2\eta(i)$, $i \in [2n-1]$.
3. For the cutting degree, the following properties hold
   (i) $2 \leq \epsilon(i) \leq 2\min\{i, 2n-i\}$, $i \in [2n-1]$.
   (ii) $|\epsilon(i+1) - \epsilon(i)| \in \{0, 2\}$, $i \in [2n-1]$.
   (iii) There exists no $i \in [2n-2]$, such that $\epsilon(i) = \epsilon(i+2) = \epsilon(i+1) - 2$.

The main cutting problems of meanders are to determine:

a) The subset $M_{2n}(\epsilon)$ of M_{2n} with given sequence of cutting degrees ϵ.
b) The subset $M_{2n}(i, \epsilon(i))$ of M_{2n} with given cutting degree $\epsilon(i)$ at i.

For the first problem, i.e. to determine the set $W_{2n}(\epsilon)$ we have that if ϵ is deduced by a given meander then $W_{2n}(\epsilon) \neq \emptyset$, while if ϵ is deduced by a given $\eta \in H_{2n-1}$ then $W_{2n}(\epsilon)$ could be empty.

So we start by finding the Motzkin word $y \in P_{2n}$ which corresponds to the sequence $\eta \in H_{2n-1}$, where $\eta(i) = \frac{\epsilon(i)}{2}$, for $i \in [2n-1]$.

From y we determine all the 2-Motzkin words $\omega \in \{a, \bar{a}, b, \bar{b}\}^*$ with the following properties:

The letter h is replaced by the letters $b, \bar{b}$ such that $|\omega|_b = |\omega|_{\bar{b}}$.

Obviously, if $|y|_b = 2\nu$ there exist $\binom{2\nu}{\nu}$ such words ω. Those words can be obtained, if we properly place the b's and $\bar{b}$'s in the positions of the b's, with the help of a binary tree $B_{2\nu}$ of height 2ν. In this tree, each left child is labeled by b and each right child is labeled by $\bar{b}$, given that the equal number of b's and $\bar{b}$'s is ν. Thus, the number of possible meanders belongs to the set $\{0, 2, 4, \ldots, \binom{2\nu}{\nu}\}$. We notice that the paths of the right subtree of $B_{2\nu}$ can be produced by replacing the b's with $\bar{b}$'s at the paths of the left subtree.

For example, for the Motzkin word $y = a\ a\ b\ b\ a\ b\ \bar{a}\ \bar{a}\ b\ \bar{a}$ of Fig. 2, we have $\nu = 2$ and the following $\binom{4}{2} = 6$ words: $a\ a\ b\ b\ a\ \bar{b}\ \bar{a}\ \bar{a}\ \bar{b}\ \bar{a}$, $a\ a\ b\ \bar{b}\ a\ b\ \bar{a}\ \bar{a}\ \bar{b}\ \bar{a}$, $a\ a\ b\ \bar{b}\ a\ \bar{b}\ \bar{a}\ \bar{a}\ b\ \bar{a}$, $a\ a\ \bar{b}\ b\ a\ b\ \bar{a}\ \bar{a}\ b\ \bar{a}$, $a\ a\ \bar{b}\ b\ a\ \bar{b}\ \bar{a}\ \bar{a}\ b\ \bar{a}$, $a\ a\ \bar{b}\ b\ a\ b\ \bar{a}\ \bar{a}\ \bar{b}\ \bar{a}$, which are obtained with the aid of the tree B_4.

According to the above, the problem can be confronted by the following procedure:

Step 1. Find the Motzkin word y which corresponds to a given sequence $\eta \in H_{2n-1}$.
Step 2. Find the tree $B_{2\nu}$, where $\nu = \frac{1}{2}|y|_b$.
Step 3. Find, using y and $B_{2\nu}$, the 2-Motzkin words $\omega = \omega_1\omega_2\cdots\omega_{2n}$ of $\{a, \bar{a}, b, \bar{b}\}^*$ for which:

$$\text{(i) } \omega_i = \begin{cases} a, & \text{if } y_i = u, \\ b \text{ or } \bar{b}, & \text{if } y_i = h, \\ \bar{a}, & \text{if } y_i = d, \end{cases} \quad \text{for } i \in [2n]. \qquad \text{(ii) } |\omega|_b = |\omega|_{\bar{b}} = \nu.$$

Step 4. Find the associate words r, r' for each ω and check if there are Dyck words.
Step 5. Find from the remaining ω, the ones that satisfy the matching property for the corresponding nested sets.

Examples. The following examples illustrate the above procedure.

1) For $\eta = 1\ 1\ 1\ 1\ 1\ 2\ 2\ 1\ 1$ we have $y = a\ b\ b\ b\ b\ a\ b\ \bar{a}\ b\ \bar{a}$ and $\nu = 3$. From the left subtree of B_6, we derive $\frac{1}{2}\binom{6}{3} = 10$ words.
We can easily observe that none of these words satisfies the condition of the fourth step. Hence, for $\epsilon = 2\ 2\ 2\ 2\ 2\ 4\ 4\ 2\ 2$, we have that $W_{10}(\epsilon) = \emptyset$.
2) For $\eta = 1\ 1\ 1\ 2\ 2\ 1\ 1\ 1\ 1$ we have that $y = a\ b\ b\ a\ b\ \bar{a}\ b\ b\ b\ \bar{a}$ and $\nu = 3$, hence from the left subtree of B_6, we derive 10 words, only four of them satisfying the conditions of the fourth and fifth step.
Hence, for $\epsilon = 2\ 2\ 2\ 4\ 4\ 2\ 2\ 2\ 2$, we have that

$$\begin{aligned} W_{10}(\epsilon) = \{&a\ b\ \bar{b}\ a\ b\ \bar{a}\ \bar{b}\ b\ \bar{b}\ \bar{a},\ a\ b\ \bar{b}\ a\ b\ \bar{a}\ \bar{b}\ \bar{b}\ b\ \bar{a},\ a\ b\ \bar{b}\ a\ \bar{b}\ \bar{a}\ b\ b\ \bar{b}\ \bar{a},\\ &a\ b\ \bar{b}\ a\ \bar{b}\ \bar{a}\ b\ \bar{b}\ b\ \bar{a},\ a\ \bar{b}\ b\ a\ \bar{b}\ \bar{a}\ b\ \bar{b}\ b\ \bar{a},\ a\ \bar{b}\ b\ a\ \bar{b}\ \bar{a}\ b\ b\ \bar{b}\ \bar{a},\\ &a\ \bar{b}\ b\ a\ b\ \bar{a}\ \bar{b}\ \bar{b}\ b\ \bar{a},\ a\ \bar{b}\ b\ a\ b\ \bar{a}\ \bar{b}\ b\ \bar{b}\ \bar{a}\}. \end{aligned}$$

3) For the meander of Fig. 1 we have $\eta = 1\ 2\ 2\ 2\ 3\ 3\ 2\ 1\ 1$, $y = a\ a\ b\ b\ a\ b\ \bar{a}\ \bar{a}\ b\ \bar{a}$, and $\nu = 2$, hence from the left subtree of B_4, we derive $\frac{1}{2}\binom{4}{2} = 3$ words. We can easily observe that all these words satisfy the conditions of the fourth and fifth steps. Hence, for $\epsilon = 2\ 4\ 4\ 4\ 6\ 6\ 4\ 2\ 2$, we have that

$$\begin{aligned} W_{10}(\epsilon) = \{&a\ a\ b\ b\ a\ \bar{b}\ \bar{a}\ \bar{a}\ \bar{b}\ \bar{a},\ a\ a\ b\ \bar{b}\ a\ b\ \bar{a}\ \bar{a}\ \bar{b}\ \bar{a},\ a\ a\ b\ \bar{b}\ a\ \bar{b}\ \bar{a}\ \bar{a}\ b\ \bar{a},\\ &a\ a\ \bar{b}\ \bar{b}\ a\ b\ \bar{a}\ \bar{a}\ b\ \bar{a},\ a\ a\ \bar{b}\ b\ a\ \bar{b}\ \bar{a}\ \bar{a}\ b\ \bar{a},\ a\ a\ \bar{b}\ b\ a\ b\ \bar{a}\ \bar{a}\ \bar{b}\ \bar{a}\}. \end{aligned}$$

We will now solve the second problem, i.e., we will determine the elements of the set $W_{2n}(i, \epsilon(i))$.

For the existence of such meanders, the conditions that $\epsilon(i)$ is even and $2 \leq \epsilon(i) \leq 2\min\{i, 2n-i\}$, for $i = 2, 3, \ldots, 2n-2$, must hold.

If the cutting degree at i for a meander is $\epsilon(i)$, then its corresponding path of P_{2n} passes through the point $(i, \eta(i)) = \left(i, \frac{\epsilon(i)}{2}\right)$.

It is natural to find the paths of P_{2n} passing through the point $(i, \eta(i))$ and to check which of them satisfy the matching property.

From Proposition 5 we know the number of paths of P_{2n} passing through the point $(i, \eta(i)) = \left(i, \frac{\epsilon(i)}{2}\right)$. These paths of P_{2n} can be determined by the subtree of the tree of H_{2n-1} presented in Section 2.

For example, for $n = 3$ and $\epsilon(4) = 2$ the number of paths of P_6 passing through the point $(4, 1)$ is equal to 2. Using the corresponding subtree, we obtain the sequences 1 1 1 1 1 and 1 2 2 1 1 hence,

$$
\begin{aligned}
&W_6(4;2) = W_6(2\ 2\ 2\ 2\ 2) \cup W_6(2\ 4\ 4\ 2\ 2) = \\
&= \{a\ \bar{b}\ b\ \bar{b}\ b\ \bar{a}, a\ \bar{b}\ b\ b\ \bar{b}\ \bar{a}, a\ b\ \bar{b}\ \bar{b}\ b\ \bar{a}, a\ b\bar{b}\ b\ \bar{b}\ \bar{a}\} \cup \{a\ a\ \bar{b}\ \bar{a}\ b\ \bar{a}, a\ a\ b\ \bar{a}\ \bar{b}\ \bar{a}\} \\
&= \{a\ \bar{b}\ b\ \bar{b}\ b\ \bar{a},\ a\ \bar{b}\ b\ b\ \bar{b}\ \bar{a},\ a\ b\ \bar{b}\ \bar{b}\ b\ \bar{a}, a\ b\ \bar{b}\ b\ \bar{b}\ \bar{a},\ a\ a\ \bar{b}\ \bar{a}\ b\ \bar{a},\ a\ a\ b\ \bar{a}\ \bar{b}\ \bar{a}\}
\end{aligned}
$$

Using the above algorithms, we have obtained the following enumeration results: A192927, A207851, A203019, A208357, A208358 in [10].

References

1. M.-P. Delest and G. Viennot, Algebraic languages and polyominoes enumeration, *Theoret. Comput. Sci.* **34** (1984), 169–206.
2. R. Donaghey and L.W. Shapiro, Motzkin numbers, *J. Combin. Theory, Ser. A* **23** (1977), 291–301.
3. T. Jiang, G. H. Lin, B. Ma, K. Zhang, A general edit distance between RNA structures, *Journal of Computational Biology*, **9(2)** (2002), 371–388.
4. S. Lando and A. Zvonkin, Plane and projective meanders, *Theoret. Comput. Sci.* **117** (1993), 227–241.
5. M. E. Nebel, Combinatorial Properties of RNA Secondary Structures, *Journal of Computational Biology*, **9** (2001), 541–573.
6. A. Panayotopoulos and P. Tsikouras, The multimatching property of nested sets, *Math. & Sci. Hum.* **149** (2000), 23–30.
7. A. Panayotopoulos and P. Tsikouras, Properties of meanders, *JCMCC* **46** (2003), 181–190.
8. A. Panayotopoulos and P. Tsikouras, Meanders and Motzkin words, *J. Integer Seq.* **7** (2004), Article 04.1.2.
9. A. Panayotopoulos and P. Vlamos, Meandric polygons, *Ars Combinatoria* **87** (2008), 147–159.
10. N. Sloane, *The Online-Encyclopedia of Integer Sequences*, published electronically at www.research.att.com/ njas/sequences/.
11. P. R. Stein and M. S. Waterman, On some new sequences generalizing the Catalan and Motzkin numbers, *Discrete Math.*, **26** (1978), 261–272.