Neighbour-Sum-2-Distinguishing Edge-Weightings: Doubling the 1-2-3 Conjecture

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Neighbour-Sum-2-Distinguishing Edge-Weightings: Doubling the 1-2-3 Conjecture[☆]

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Abstract

The 1-2-3 Conjecture asks whether every graph with no connected component isomorphic to K_2 can be 3-edge-weighted so that every two adjacent vertices u and v can be distinguished via the sum of their incident weights, that is the incident sums of u and v differ by at least 1.

We here investigate the consequences on the 1-2-3 Conjecture of requiring a stronger distinction condition. Namely, we consider two adjacent vertices distinguished when their incident sums differ by at least 2. As a guiding line, we conjecture that every graph with no connected component isomorphic to K_2 admits a 5-edge-weighting permitting to distinguish the adjacent vertices in this stronger way.

We verify this conjecture for several classes of graphs, including bipartite graphs and cubic graphs. We then consider algorithmic aspects, and show that it is NP-complete to determine the smallest k such that a given bipartite graph admits such a k-edge-weighting. In contrast, we show that the same problem can be solved in polynomial time for a given tree.

Keywords: 1-2-3 Conjecture; Difference-2 distinction; Bipartite graphs.

1. Introduction

Let G be a graph, and ω be an edge-weighting of G. For every vertex v, one can compute its *incident sum* $\sigma_\omega(v)$ (or simply $\sigma(v)$ when no ambiguity is possible) of weights by ω, being $\sigma(v) := \sum_{u \in N(v)} \omega(vu)$, where $N(v)$ denotes the set of neighbours of v. We call ω *neighbour-sum-distinguishing* if it yields a proper σ, i.e. we have $\sigma(u) \neq \sigma(v)$ for every edge uv of G. It can be observed that every connected graph different from K_2 admits a neighbour-sum-distinguishing edge-weighting. Graphs with no connected component isomorphic to K_2 are thus said *nice*, with respect to neighbour-sum-distinguishing edge-weightings. For a nice graph G, it thus makes sense to investigate the smallest k such that G admits a neighbour-sum-distinguishing k-edge-weighting. This smallest k is denoted by $\chi_\Sigma(G)$.

The 1-2-3 Conjecture, addressed by Karoński, Łuczak and Thomason [3], asks whether $\chi_\Sigma(G) \leq 3$ holds for every nice graph G.

1-2-3 Conjecture. *For every nice graph G, we have $\chi_\Sigma(G) \leq 3$.*

If true, the bound in the 1-2-3 Conjecture would be best possible, as attested for example by nice complete graphs and cycles with length not multiple of 4. More generally,

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it is NP-complete to decide whether $\chi_\Sigma(G) \leq 2$ holds for a given graph G, as first proved by Dudek and Wajc [1]. The same problem, however, can be handled in polynomial time when restricted to bipartite graphs, as recently shown by Thomassen, Wu and Zhang [6]. To date, the best result towards the 1-2-3 Conjecture is due to Kalkowski, Karoński and Pfender [2], who proved that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G.

When designing neighbour-sum-distinguishing edge-weightings, the goal is to make adjacent vertices distinguishable via their incident sums. In ordinary neighbour-sum-distinguishing edge-weightings, adjacent vertices are considered distinguished as soon as their incident sums are distinct. We here investigate edge-weightings that permit to distinguish the adjacent vertices in a stronger way. Namely, we require adjacent vertices to have incident sums differing by at least 2. An edge-weighting verifying this stronger requirement is said to be *neighbour-sum-2-distinguishing* throughout. As observed in upcoming Observation 2.1, a neighbour-sum-distinguishing k-edge-weighting can easily be turned into a neighbour-sum-2-distinguishing $2k$-edge-weighting. Moreover, since K_2 does clearly not admit any neighbour-sum-2-distinguishing edge-weighting, the notion of nice graphs for neighbour-sum-distinguishing edge-weightings and for neighbour-sum-2-distinguishing edge-weightings coincide. Again, we can thus wonder about the smallest k such that a given nice graph G admits a neighbour-sum-2-distinguishing k-edge-weighting, which we denote by $\chi_{\Sigma>1}(G)$.

Our main goal in this paper is to study how $\chi_{\Sigma>1}$ behaves in general, in particular for graphs for which the parameter χ_Σ is well understood. As noted in upcoming Observation 2.1, the 1-2-3 Conjecture, if true, would imply that $\chi_{\Sigma>1}(G) \leq 6$ holds for every nice graph G. One could thus naturally wonder about a 1-2-3-4-5-6 Conjecture for neighbour-sum-2-distinguishing edge-weightings. It actually turns out that we did not manage to exhibit nice graphs G with $\chi_{\Sigma>1}(G) = 6$. On the other hand, we prove, throughout this paper, that several common classes of nice graphs G verify $\chi_{\Sigma>1}(G) \leq 5$. We are thus tempted to address the following.

Conjecture 1.1. *For every nice graph G, we have $\chi_{\Sigma>1}(G) \leq 5$.*

We here give first evidence towards Conjecture 1.1. We start in Section 2 by raising connexions between neighbour-sum-distinguishing edge-weightings and neighbour-sum-2-distinguishing edge-weightings, from which we deduce first bounds on $\chi_{\Sigma>1}$. In Section 3, we then improve some of these bounds for some classes of nice bipartite graphs, and verify Conjecture 1.1 for all nice bipartite graphs. The algorithmic aspects are considered in Section 4, where we first prove that it is NP-complete to determine the exact value of $\chi_{\Sigma>1}$. This statement is showed to remain true even for bipartite graphs, which contrasts with the complexity of determining the exact value of χ_Σ for these graphs. We then show that determining the exact value of $\chi_{\Sigma>1}$ can be done in polynomial time for trees. Perspectives for future works are gathered in Section 5.

2. Preliminaries

We have the following relationship between neighbour-sum-distinguishing edge-weightings and neighbour-sum-2-distinguishing edge-weightings:

Observation 2.1. *For every nice graph G, we have $\chi_{\Sigma>1}(G) \leq 2\chi_\Sigma(G)$.*

Proof. Let ω be a neighbour-sum-distinguishing k-edge-weighting of G, where $k := \chi_\Sigma(G)$. Consider the $2k$-edge-weighting ω' of G where $\omega'(uv) = 2\omega(uv)$ for every edge uv of G.

We get $\sigma_{\omega'}(v) = 2\sigma_\omega(v)$ for every vertex v. Since ω is neighbour-sum-distinguishing, $\sigma_\omega(u)$ and $\sigma_\omega(v)$ differ by at least 1 for every edge uv, which yields that $\sigma_{\omega'}(u)$ and $\sigma_{\omega'}(v)$ differ by at least 2. So ω' is neighbour-sum-2-distinguishing. $\square$

Observation 2.1 already has several implications towards Conjecture 1.1. First, the 1-2-3 Conjecture, if true, would imply that $\chi_{\Sigma>1}(G) \leq 6$ holds for every nice graph G. Although we still do not know whether the 1-2-3 Conjecture is true, every partial result towards that conjecture can be adapted to Conjecture 1.1. In that line, perhaps the most interesting result to consider is the one due to Kalkowski, Karoński and Pfender [2], who proved that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G. In our context, this and Observation 2.1 yield the following, which shows that, although we have strengthened the distinction condition slightly, a constant number of weights is still sufficient to weight all nice graphs.

Corollary 2.2. *For every nice graph G, we have $\chi_{\Sigma>1}(G) \leq 10$.*

The bound in Corollary 2.2 is immediately improved for every graph G for which we know that $\chi_\Sigma(G) < 5$ holds. In particular, we have $\chi_{\Sigma>1}(G) \leq 6$ for every nice graph G verifying the 1-2-3 Conjecture, which is very close to Conjecture 1.1. Let us recall, in particular, that the 1-2-3 Conjecture was verified for nice bipartite graphs, 3-chromatic graphs, nice complete graphs, and regular graphs with sufficiently large degree. We here refer the reader to the survey [5] by Seamone, wherein all such results are gathered.

By multiplying all weights assigned by a neighbour-sum-distinguishing k-edge-weighting by a same integer α, we get another neighbour-sum-distinguishing αk-edge-weighting since each $\sigma(v)$ is multiplied by α. This of course does not have to be true if one decreases (or increases) all weights by a same α, since, here, the effect on each $\sigma(v)$ depends on $d(v)$. There are situations, however, where this can be done safely.

Observation 2.3. *Let ω be a neighbour-sum-distinguishing edge-weighting of a graph G. If we have $\sigma(u) < \sigma(v)$ (resp. $\sigma(u) > \sigma(v)$) for every two adjacent vertices u and v verifying $d(u) \geq d(v)$, then, by decreasing (resp. increasing) all edge weights by a same integer α, we get another neighbour-sum-distinguishing edge-weighting of G. Furthermore, if $\sigma(u)$ and $\sigma(v)$ were differing by at least x, then they still do.*

Due to the fact that, in the context of Conjecture 1.1, we focus on edge-weightings assigning strictly positive weights, when decreasing edge weights we should also make sure that none becomes null or negative. Observation 2.3 can nevertheless be used when the smallest edge weight value assigned by ω is known. As an illustration, we improve Observation 2.1 (and, thus, Corollary 2.2) for nice regular graphs.

Corollary 2.4. *For every nice regular graph G, we have $\chi_{\Sigma>1}(G) \leq 2\chi_\Sigma(G) - 1$.*

Proof. Let $k := \chi_\Sigma(G)$ and ω be a neighbour-sum-2-distinguishing $\{2, 4, \dots, 2k\}$-edge-weighting of G, which exists as attested in the proof of Observation 2.1. Since G is regular, according to Observation 2.3, when decreasing all edge weights by 1 we get another neighbour-sum-2-distinguishing edge-weighting ω' of G. Furthermore, since ω is a $\{2, 4, \dots, 2k\}$-edge-weighting, ω' is a $\{1, 3, \dots, 2k-1\}$-edge-weighting. $\square$

Corollary 2.4 notably implies that Conjecture 1.1 holds for nice complete graphs and 3-colourable regular graphs, as they verify the 1-2-3 Conjecture. In general, this decreases the bound in Corollary 2.2 down to 9 for nice regular graphs. More refined bounds also follow for regular graphs with larger degree, see [5].

3. Improved bounds for particular bipartite graphs

In this section, we focus on improving the bounds obtained in Section 2 for some nice bipartite graphs. It was proved by Thomassen, Wu and Zhang [6] that a bipartite graph G verifies $\chi_\Sigma(G) = 3$ if and only if G is an odd multicactus. *Odd multicacti* can be defined as follows. Start from a collection $C_1, \dots, C_m$ of $m \geq 1$ cycles whose lengths are at least 6 and congruent to 2 modulo 4, and colour the edges of the C_i's in a proper way using colours red and green. An odd multicactus is then any connected graph obtained by repeatedly applying the following operation: pick two connected components G_1 and G_2, and identify a green edge of G_1 with a green edge of G_2. Note that, in particular, every cycle whose length is congruent to 2 modulo 4 is an odd multicactus.

From Observation 2.1 and the previous remarks we directly get the following:

Corollary 3.1. *For every nice bipartite graph G, we have*

$$\chi_{\Sigma>1}(G) \leq \begin{cases} 4 & \text{if } G \text{ is not an odd multicactus,} \\ 6 & \text{otherwise.} \end{cases}$$

In what follows, we first refine the bounds in Corollary 3.1 for nice paths and cycles. We then prove that $\chi_{\Sigma>1}(G) \leq 5$ holds for every odd multicactus G, hence that nice bipartite graphs verify Conjecture 1.1.

3.1. Paths

We denote by P_ℓ the path of length ℓ. Therefore, the path $P_1 = K_2$ is not nice. In the next result, we determine the value of $\chi_{\Sigma>1}(P_\ell)$ for every $\ell \geq 2$.

Theorem 3.2. *For every path P_ℓ, $\ell \geq 2$, we have*

$$\chi_{\Sigma>1}(P_\ell) = \begin{cases} 2 & \text{if } \ell = 2, \\ 3 & \text{if } \ell > 2 \text{ and } \ell \equiv 0, 2, 3 \pmod 4, \\ 4 & \text{otherwise.} \end{cases}$$

Proof. Recall that $\chi_{\Sigma>1}(P_\ell) \leq 4$ holds for every $\ell \geq 2$, by Corollary 3.1. Moreover, since $\chi_{\Sigma>1}(G) = 1$ if and only if G is a graph such that the degrees of every two adjacent vertices differ by at least 2, we get $\chi_{\Sigma>1}(P_\ell) \geq 2$ for every $\ell \geq 2$.

Let $v_0, \dots, v_\ell$ denote the vertices of the path P_ℓ, with $v_i v_{i+1}$ being an edge for every i, $0 \leq i \leq \ell - 1$. We clearly have $\chi_{\Sigma>1}(P_2) = 2$ since the weighting ω given by $\omega(v_0v_1) = \omega(v_1v_2) = 2$ is neighbour-sum-2-distinguishing.

Suppose now that $\ell \geq 3$. Then P_ℓ has two adjacent vertices v_i and v_{i+1} with degree 2. For any 2-edge-weighting ω of P_ℓ, we have $\sigma(v_i) = \omega(v_{i-1}v_i) + \omega(v_iv_{i+1})$ and $\sigma(v_{i+1}) = \omega(v_iv_{i+1}) + \omega(v_{i+1}v_{i+2})$. Since $\omega(v_{i-1}v_i), \omega(v_{i+1}v_{i+2}) \in \{1, 2\}$, necessarily $\sigma(v_i)$ and $\sigma(v_{i+1})$ differ by at most 1, so that ω cannot be neighbour-sum-2-distinguishing.

If $\ell = 3$, then assigning successive edge weights $1, 3, 3$ to the edges of P_3 is neighbour-sum-2-distinguishing as it yields successive incident sums $1, 4, 6, 3$. So we may suppose from now on that $\ell \geq 4$. Under that assumption, note that a neighbour-sum-2-distinguishing 3-edge-weighting ω of P_ℓ cannot assign weight 2. Indeed, if an edge v_iv_{i+1} is assigned weight 2 by ω, then, because $\ell \geq 4$, either $v_{i-2}v_{i-1}$ or $v_{i+2}v_{i+3}$ is an edge, and that edge must be assigned weight 0 or 4 so that ω is neighbour-sum-2-distinguishing, which is not possible. We thus restrict our attention to $\{1, 3\}$-edge-weightings of P_ℓ. Note that when assigning weights from left to right, in order to get a neighbour-sum-2-distinguishing $\{1, 3\}$-edge-weighting, we must respect one of the two periodic patterns $1, 3, 3, 1, 1, 3, 3...$ or $3, 3, 1, 1, 3, 3, 1, 1....$ When applying any of these two patterns from left to right, we also have to find one of these two patterns when reading the edge weights from right to left, as otherwise the edge-weighting would not be neighbour-sum-2-distinguishing. When applying the first pattern, this property is fulfilled whenever $\ell \equiv 0, 3 \pmod 4$. When applying the second pattern, this property is fulfilled whenever $\ell \equiv 2, 3 \pmod 4$. We thus get $\chi_{\Sigma>1}(P_\ell) \leq 3$ if and only if $\ell \equiv 0, 2, 3 \pmod 4$, which concludes the proof. □

3.2. Cycles

For every $\ell \geq 3$, we denote by C_ℓ the cycle of length ℓ. In the next result, we determine the value of $\chi_{\Sigma>1}(C_\ell)$ for every cycle C_ℓ.

Theorem 3.3. *For every cycle C_ℓ, $\ell \geq 3$, we have*

$$\chi_{\Sigma>1}(C_\ell) = \begin{cases} 3 & \text{if } \ell \equiv 0 \pmod 4, \\ 5 & \text{otherwise.} \end{cases}$$

Proof. Observe first that since cycles verify the 1-2-3 Conjecture and are regular, Corollary 2.4 implies that $\chi_{\Sigma>1}(C_\ell) \leq 5$ holds for every cycle C_ℓ.

Let $v_0, \dots, v_{\ell-1}$ denote the vertices of the cycle C_ℓ, with $v_i v_{i+1}$ being an edge for every i, $0 \leq i \leq \ell - 1$ (here and in the following, all operations over the subscripts are understood modulo ℓ).

For any 1-edge-weighting ω of C_ℓ, $\sigma(v_i) = 2$ for every vertex v_i, and thus no such edge-weighting can be neighbour-sum-2-distinguishing. Similarly, for any 2-edge-weighting ω, $\sigma(v_i) \in \{2, 3, 4\}$ for every vertex v_i, which means that ω is neighbour-sum-2-distinguishing if and only if every two adjacent vertices have incident sums 2 and 4. But this is impossible, since $\sigma(v_i) = 2$ if and only if the two edges incident to v_i are weighted 1, while $\sigma(v_i) = 4$ if and only if the two edges incident to v_i are weighted 2. Therefore, $\chi_{\Sigma>1}(C_\ell) \geq 3$ for every cycle C_ℓ.

We first claim that no neighbour-sum-2-distinguishing 3-edge-weighting ω of C_ℓ can use weight 2. Indeed, if $\omega(v_i v_{i+1}) = 2$ for some edge $v_i v_{i+1}$, then, for $\sigma(v_i)$ and $\sigma(v_{i+1})$ to differ by at least 2, we must have $\omega(v_{i-1} v_i) = 1$ and $\omega(v_{i+1} v_{i+2}) = 3$, without loss of generality. We then get $\sigma(v_i) = 3$ and $\sigma(v_{i-1}) \in \{2, 3, 4\}$, so that ω cannot be neighbour-sum-2-distinguishing.

Therefore, a neighbour-sum-2-distinguishing 3-edge-weighting of C_ℓ can only use weights 1 and 3. In such a weighting, the edge weights must follow the pattern $1, 1, 3, 3, 1, 1, 3, 3...$ along the cycle, which is possible if and only if $\ell \equiv 0 \pmod 4$.

We now prove that $\chi_{\Sigma>1}(C_\ell) = 5$ whenever $\ell \not\equiv 0 \pmod 4$. According to the first observation in this proof, it suffices to prove that $\chi_{\Sigma>1}(C_\ell) > 4$ holds for every such cycle. Let ω be a neighbour-sum-2-distinguishing 4-edge-weighting of C_ℓ. We claim that we can produce, from ω, a neighbour-sum-2-distinguishing $\{1, 4\}$-edge-weighting ω' of C_ℓ, by decreasing all 2's and incrementing all 3's by one. More precisely, let ω' be the $\{1, 4\}$-edge-weighting of C_ℓ where, for every edge $v_i v_{i+1}$, we have

$$\omega'(v_i v_{i+1}) = \begin{cases} 1 & \text{if } \omega(v_i v_{i+1}) = 2, \\ 4 & \text{if } \omega(v_i v_{i+1}) = 3, \\ \omega(v_i v_{i+1}) & \text{otherwise.} \end{cases}$$

Let us prove that ω' is indeed neighbour-sum-2-distinguishing. Suppose that we build ω' sequentially, modifying the weights of the edges one by one. We thus construct a sequence of edge-weightings $\omega_0 = w, \omega_1, \dots, \omega_p = \omega'$, for some $p \geq 0$. We claim that for every j, $0 \leq j \leq p - 1$, ω_{j+1} is neighbour-sum-2-distinguishing whenever ω_j is neighbour-sum-2-distinguishing. Suppose that $\omega_j(v_i v_{i+1}) = 3$ (resp. 2) and $\omega_{j+1}(v_i v_{i+1}) = 4$ (resp.

1). Since $\sigma_{\omega_j}(v_i)$ and $\sigma_{\omega_j}(v_{i+1})$ differ by at least 2, $\sigma_{\omega_{j+1}}(v_i)$ and $\sigma_{\omega_{j+1}}(v_{i+1})$ also differ by at least 2. The only possible conflicts would thus concern vertices v_{i-1} and v_i, or v_{i+1} and v_{i+2}. Such a conflict arises if $\sigma_{\omega_{j+1}}(v_{i-1})$ and $\sigma_{\omega_{j+1}}(v_i)$, or $\sigma_{\omega_{j+1}}(v_{i+1})$ and $\sigma_{\omega_{j+1}}(v_{i+2})$, respectively, only differ by 1. But this would mean that $\sigma_{\omega_j}(v_{i-1}) = \sigma_{\omega_j}(v_i) + 2$ (resp. $\sigma_{\omega_j}(v_{i-1}) = \sigma_{\omega_j}(v_i) - 2$), or $\sigma_{\omega_j}(v_{i+2}) = \sigma_{\omega_j}(v_{i+1}) + 2$ (resp. $\sigma_{\omega_j}(v_{i+2}) = \sigma_{\omega_j}(v_{i+1}) - 2$), respectively, which implies $\omega_j(v_{i-2}v_{i-1}) = 5$ (resp. 0), or $\omega_j(v_{i+2}v_{i+3}) = 5$ (resp. 0), a contradiction in both cases. Hence, since ω is neighbour-sum-2-distinguishing, we get that ω' is also neighbour-sum-2-distinguishing.

Now, since C_ℓ is 2-regular, the incident sums induced by ω' range among $\{2, 5, 8\}$. Therefore, the edge weights assigned by ω' must follow the pattern $1, 1, 4, 4, 1, 1, 4, 4...$ along the cycle, which is possible if and only if $\ell \equiv 0 \pmod 4$, contradicting our assumption. □

3.3. Odd multicacti

Recall that $\chi_{\Sigma>1}(G) \leq 6$ holds for every odd multicactus G, according to Corollary 3.1. Some obvious odd multicacti G, such as cycles with length congruent to 2 modulo 4, verify $\chi_{\Sigma>1}(G) = 5$ (recall Theorem 3.3). In the next result, we prove that all odd multicacti verify Conjecture 1.1.

Observe first that connected multicacti can be defined inductively, as follows. Cycles of length at least 6 and congruent to 2 modulo 4, with edges coloured green and red alternatively, are multicacti. Consider now a multicactus G whose edges are coloured green and red, and let uv be a green edge of G. Then the graph obtained from G by identifying u and v with the end-vertices of a path of length at least 5 and congruent to 1 modulo 4, whose edges are alternatively coloured red, green, ..., red (from one end to the other), is a multicactus. This operation will be referred to as a *path attachment*. Note that, in any edge-coloured multicactus, the two ends of a green edge have the same degree.

Theorem 3.4. *For every odd multicactus G, we have $\chi_{\Sigma>1}(G) \leq 5$.*

Proof. We will prove a stronger statement, namely that every odd multicactus admits a neighbour-sum-distinguishing $\{1, 3, 5\}$-edge-weighting. The proof is by induction on the number of path attachments performed to construct G.

If no such path attachment was made, then G is a cycle C_{4k+2}, for some $k \geq 1$, and the $\{1, 3, 5\}$-edge-weighting obtained by applying the pattern $1, 1, 3, 3, 5, 5, 1, 1, 3, 3, 5, 5...$ cyclically is clearly a neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting of G.

Assume now that G is not a cycle. Then G must contain a green edge uv such that u and v are joined by exactly x paths $P_1, \dots, P_x$, $x \geq 1$, with length at least 5 and congruent to 1 modulo 4, and whose internal vertices have degree 2 in G. In other words, no green edge of the P_i's was used to make a path attachment. When removing all internal vertices of the P_i's from G, we get another connected odd multicactus G' in which both u and v have degree 2.

By the induction hypothesis, G' admits a neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting ω, which we would like to extend to the edges of the P_i's, in order to obtain a neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting of G. Let us denote by u' and v' the neighbours of u and v, respectively, different from v and u, respectively, in G'. When extending ω to the P_i's, we have to make sure that:

1. $\sigma(u)$ and $\sigma(u')$ still differ by at least 2;
2. $\sigma(v)$ and $\sigma(v')$ still differ by at least 2;
3. both $\sigma(u)$ and $\sigma(v)$ differ by at least 2 from the incident sums of their x neighbours in the P_i's;

4. $\sigma(u)$ and $\sigma(v)$ still differ by at least 2.

In order to respect the fourth condition above, we will edge-weight the P_i's in such a way that $\sigma(u)$ and $\sigma(v)$ are altered the same way, i.e. by a same integer. To that aim, we will $\{1, 3, 5\}$-edge-weight every P_i in such a way that its two end-edges are assigned the same weight. We say that such an extension is *uniform* (with respect to P_i).

Since the P_i's have length at least 5 and congruent to 1 modulo 4, there are many ways to $\{1, 3, 5\}$-edge-weight them in a neighbour-sum-2-distinguishing way, in particular when we relax the distinguishing condition for their end-vertices (that is, we do not require that the incident sums of the end-vertices are different from the incident sums of their neighbours). In particular, it is easy to check that, for such a path, following any of the patterns $\alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \alpha_4, \alpha_4...$ or $\alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \alpha_4, \alpha_4...$, where all α_i's belong to $\{1, 3, 5\}$ and every two successive α_i's are different, yields a (relaxed) neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting. In particular, when the length is at least 5, there is a satisfying $\{1, 3, 5\}$-weighting which is uniform for any same weight assigned to the two end-edges, and this remains true if one requires the first two edges to be weighted differently and the last two edges to be weighted with the same weight. That is, if the successive vertices of the path are $v_0, ..., v_\ell$, then, for any same weight $\alpha \in \{1, 3, 5\}$ assigned to $v_0v_1, v_{\ell-2}v_{\ell-1}, v_{\ell-1}v_\ell$, and any different weight $\alpha' \in \{1, 3, 5\} \setminus \{\alpha\}$ assigned to v_1v_2, it can be checked that there always exists a way for extending this pre-weighting to all edges, such that a (uniform and relaxed) neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting is obtained.

In order to extend ω to all the P_i's, we now consider four cases, depending on the number x of such paths attached to the edge uv. Recall that we always extend the weighting in a uniform way, to make sure that $\sigma(u)$ and $\sigma(v)$ are altered by a same integer α.

1. $x \geq 4$.
   In that case, when extending ω to the P_i's uniformly, we can increase both $\sigma(u)$ and $\sigma(v)$ by any value α among $\{x, x + 2, \ldots, 5x\}$. Since $\sigma(u)$ and $\sigma(v)$ should differ from $\sigma(u')$ and $\sigma(v')$, respectively, by at least 2, and since $\sigma(u')$ and $\sigma(v')$ may forbid at most six values for $\sigma(u)$ and $\sigma(v)$ (three consecutive for each), there exists $\alpha \in \{5x - 8, \ldots, 5x - 2, 5x\}$ such that $\sigma(u) + \alpha$ and $\sigma(v) + \alpha$ are not in conflict with $\sigma(u')$ and $\sigma(v')$, respectively.

   Since $x \geq 4$, we have $\alpha \geq 12$. Thereby we have $\sigma(u) + \alpha, \sigma(v) + \alpha \geq 14$, which means that, no matter how we extend ω to the P_i's (in a uniform way so that $\sigma(u)$ and $\sigma(v)$ are altered by precisely α), we cannot get any sum conflict involving u, v and the internal vertices of the P_i's (which are of degree 2, so their incident sums can have value at most 10). We thus assign, to each P_i, a same weight to its two end-edges, so that, in total, $\sigma(u)$ and $\sigma(v)$ are altered by precisely α. According to the remarks above, this pre-weighting of the P_i's can then be extended to a neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting of the P_i's, thus to a neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting of G.

2. $x = 3$.
   We apply the same strategy as in the previous case. Using the same arguments, there exists $\alpha \in \{7, 9, 11, 13, 15\}$ such that $\sigma(u) + \alpha$ and $\sigma(v) + \alpha$ are not in conflict with $\sigma(u')$ and $\sigma(v')$, respectively. If there is such an α with $\alpha \in \{11, 13, 15\}$, then we are done similarly as in the previous case since $\sigma(u) + \alpha, \sigma(v) + \alpha \geq 13$, meaning that we cannot get any sums conflict involving u, v and their neighbours in the P_i's.

Otherwise, we extend ω uniformly to the end-edges of the P_i's similarly as earlier, but with the further conditions that:

- the end-edges are all assigned weight 1 or 3 (more precisely, we assign weights $1, 3, 3$ to the end-edges of P_1, P_2, P_3, respectively, if $\alpha = 7$, or $3, 3, 3$ if $\alpha = 9$), and
- for every P_i, the second edge is assigned weight 1 while the second-to-last edge is assigned weight 3.

Such a pre-weighting of the P_i's can be extended to a neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting of the P_i's, according to similar arguments as previously. When doing so, we eventually get $\sigma(u), \sigma(v) \geq 9$, while the neighbours of u and v in the P_i's have incident sums at most 6. So this extension yields a neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting of G.

3. $x = 2$.
If $\omega(uv) \in \{1, 3\}$, then we raise that weight to 5. If this does not create any sum conflict between $\sigma(u)$ and $\sigma(u')$, or $\sigma(v)$ and $\sigma(v')$, then we proceed with the next step. Otherwise, similarly as in the previous case, and because there is currently a conflict, there exists $\alpha \in \{6, 8, 10\}$ such that $\sigma(u) + \alpha$ and $\sigma(v) + \alpha$ are not in conflict with $\sigma(u')$ and $\sigma(v')$, respectively. We then extend ω uniformly to the end-edges of the P_i's as previously. Since $\alpha \geq 6$ and $\omega(uv) = 5$, we eventually get $\sigma(u), \sigma(v) \geq 12$, while the internal vertices of the P_i's have incident sums at most 10.

   Assume now that $\omega(uv) = 5$. We again apply the same procedure. There must exist $\alpha \in \{2, 4, 6, 8, 10\}$ with the same properties as before. If $\alpha \in \{6, 8, 10\}$, then we are done by the same arguments as in the previous case. If $\alpha \in \{2, 4\}$, then we are done as well since, in that case, we get $\sigma(u) + \alpha, \sigma(v) + \alpha \geq 8$, while we can extend ω to the P_i's in such a way that the neighbours of u and v have incident sums at most 6 (just as in the case $x = 3$).

4. $x = 1$.
If $\omega(uv) = 1$, then we set $\omega(uv) = 3$. If no conflict arises, then we proceed with the next step. Otherwise, we apply the same procedure as in the previous case. If there exists $\alpha \in \{1, 3, 5\}$ with the desired properties, then we extend ω as follows. Note first that, because $\omega(uv) = 3$, we have, say, $\sigma(u) \geq 4$ and $\sigma(v) \geq 6$ by the induction hypothesis. We then extend ω to P_1 in the following way:

   - If $\alpha \in \{1, 3\}$, then we assign weight 1 to the first, second and last edges of P_1, and weight 3 to the second-to-last edge.
   - If $\alpha = 5$, then we assign weight 5 to the first and last edges of P_1, weight 1 to the second edge, and weight 3 to the second-to-last edge.

   In all cases, it can be checked that the pre-weighting can be extended to all edges of P_1, and that, by our choice of the first two weights and last two weights, that no sum conflict arises in G. Now, if no such α exists, then we set $\omega(uv) = 5$. If no conflict arises, then we proceed with the next step. Otherwise, there must now exist $\alpha \in \{1, 3, 5\}$ with the desired properties. Similar extension arguments then apply.

   So assume that $\omega(uv) \in \{3, 5\}$ and there is currently no sum conflict. Again, if there exists $\alpha \in \{1, 3, 5\}$ with the desired properties, then we are done. Otherwise, note that no conflict may arise when setting $\omega(uv) = 1$ (as otherwise there would exist an α when $\omega(uv) \in \{3, 5\}$), and there must exist $\alpha \in \{1, 3\}$ as required. Since $\omega(uv) = 1$, we have, say, $\sigma(u) \geq 2$ and $\sigma(v) \geq 4$. We then extend ω to the first edges of P_1 as follows:

- If $\sigma(u) = 2$ and $\sigma(v) \in \{4, 6\}$:
  - if $\alpha = 1$, then we assign weights 1 and 3 to the first and second edges of P_1, and weights 1 and 1 to the last and second-to-last edges;
  - if $\alpha = 3$, then we assign weights 3 and 5 to the first and second edges of P_1, and weights 3 and 1 to the last and second-to-last edges.
- If $\sigma(u) = 4$ and $\sigma(v) = 6$:
  - if $\alpha = 1$, then we assign weights 1 and 1 to the first and second edges of P_1, and weights 1 and 3 to the last and second-to-last edges;
  - if $\alpha = 3$, then we assign weights 3 and 1 to the first and second edges of P_1, and weights 3 and 3 to the last and second-to-last edges.

In all cases, it can be checked that the pre-weighting can be extended to the other edges of P_1 as required. Then no sum conflict arises in G, and we get a neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting.

This completes the proof. $\square$

4. Algorithmic aspects

In this section, we consider the hardness of determining the value of $\chi_{\Sigma>1}(G)$ for a given graph G. We first prove, in Subsection 4.1, that deciding whether $\chi_{\Sigma>1}(G) \leq 2$ holds for a given graph G is NP-complete, even when restricted to 3-degenerate planar bipartite graphs. In Subsection 4.2, we prove that, although $\chi_{\Sigma>1}(T)$ can take any value in $\{1, 2, 3, 4\}$ for a given tree T (recall Corollary 3.1), deciding the exact value of $\chi_{\Sigma>1}(T)$ can be done in polynomial time.

It is worth recalling that determining the value of $\chi_{\Sigma}(G)$ for a given graph G is NP-complete in general (Dudek and Wajc [1]), but can be done in polynomial time when restricted to bipartite graphs (Thomassen, Wu and Yang [6]). Hence our result in Subsection 4.1 shows another difference between the parameters χ_{Σ} and $\chi_{\Sigma>1}$.

4.1. General case

Before proceeding with the proof of the main result of this subsection, we first introduce gadgets that we will use to force some weights to be used by any neighbour-sum-2-distinguishing 2-edge-weighting. Each of these gadgets will have a *root vertex* of degree 1 being incident to a *root edge*. We here relax the notion of neighbour-sum-2-distinguishing 2-edge-weighting around the root; that is, we allow a neighbour-sum-2-distinguishing 2-edge-weighting to have adjacent incident sums differing by less than 2, but the incident sum of the root has to be involved in such a conflict. This is because our gadgets will be attached to other graphs via the root, so, in the properties we point out below, the incident sum of the root should not be regarded as fixed.

The gadgets we will construct are called (α, S)*-gadgets*, for some given $\alpha \in \{1, 2\}$ and $S \subset \mathbb{N}^*$. Every such gadget G will satisfy the two following properties:

1. the root edge of G is necessarily weighted α by any neighbour-sum-2-distinguishing 2-edge-weighting of G, and

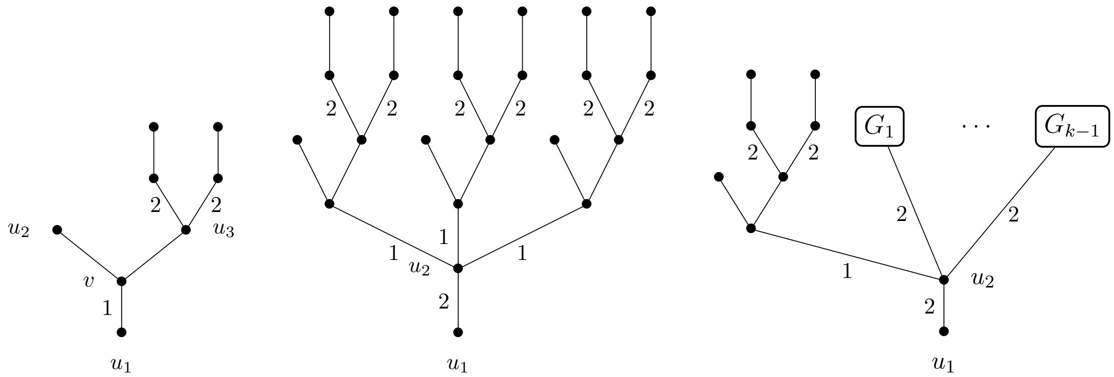

Figure 1: The $(1, \{3, 4\})$-gadget with root u_1 (left), the $(2, \{5\})$-gadget with root u_1 (middle), and the $(2, \{2k + 1\})$-gadget with root u_1 (right).

2. for any $s \in S$, there exists a neighbour-sum-2-distinguishing 2-edge-weighting of G where the (unique) neighbour of the root has incident sum s.

These gadgets will be used as follows. Let H be a graph and v be a vertex of H. Add to H an (α, S)-gadget G (for some α and S), and identify v with the root of G. Then, in any neighbour-sum-2-distinguishing 2-edge-weighting of H, v will necessarily receive weight α from the root edge of G, and v will be adjacent to a vertex whose incident sum belongs to S. This mechanism can be used both to force particular edge weights to appear around v, and to ensure that $\sigma(v)$ does not belong to a particular set (in particular when $|S| = 1$).

We now introduce the gadgets we will use (see Figure 1 for an illustration). Consider first a path $G := u_1u_2u_3$ of length 2. We claim that G is a $(2, \{3, 4\})$-gadget with root u_1. Indeed, in any neighbour-sum-2-distinguishing 2-edge-weighting ω of G (with the relaxation mentioned above), we have $\omega(u_1u_2) = 2$, while $\omega(u_2u_3)$ can have value either 1 or 2, in which cases we get $\sigma(u_2) = 3$ and $\sigma(u_2) = 4$, respectively.

Now consider a claw G with vertices v, u_1, u_2, u_3, where the u_i's are the leaves. Add two $(2, \{3, 4\})$-gadgets G_1 and G_2 to G, and identify u_3 and the roots of G_1 and G_2. We claim that G is a $(1, \{3, 4\})$-gadget with root u_1. In any neighbour-sum-2-distinguishing 2-edge-weighting ω of G, the vertex u_3 is incident to at least two edges with weight 2 (because of the gadgets G_1 and G_2), so that $\sigma(u_3) = 5$ (if $\omega(u_3v) = 1$) or $\sigma(u_3) = 6$ (otherwise). In both cases, we necessarily have $\omega(vu_2) = \omega(vu_1) = 1$, so that $\sigma(v)$ and $\sigma(u_3)$ differ by at least 2. We thus get $\sigma(v) = 3$ in the first case, and $\sigma(v) = 4$ in the second case.

We now describe how to obtain $(2, S)$-gadgets with $S := \{2k + 1\}$ for any $k \geq 2$. We first build a $(2, \{5\})$-gadget as follows. Start from $G := u_1u_2$ being the path of length 1, then add three $(1, \{3, 4\})$-gadgets G_1, G_2, G_3 to G, and identify u_2 and the roots of G_1, G_2 and G_3. We claim that G is a $(2, \{5\})$-gadget with root u_1. In any neighbour-sum-2-distinguishing 2-edge-weighting ω of G, the vertex u_2 is incident to at least three edges weighted 1, namely the root edges of the G_i's. Now, if $\omega(u_1u_2) = 1$, then $\sigma(u_2) = 4$, which creates sum conflicts with vertices from the G_i's. So we necessarily have $\omega(u_1u_2) = 2$, in which case $\sigma(u_2) = 5$, which is fine since the G_i's are $(1, \{3, 4\})$-gadgets.

We now turn to the general case. Let $2k+1 \geq 7$, and assume that we have constructed $(2, S')$-gadgets with $S' := \{2k' + 1\}$ for every k', $2 \leq k' < k$. Start from $G := u_1u_2$ being the path of length 1. Add $k - 1$ $(2, \{2k - 1\})$-gadgets $G_1, \dots, G_{k-1}$ to G, as well as one $(1, \{3, 4\})$-gadget G_0, and identify u_2 and the roots of $G_1, \dots, G_{k-1}$ and G_0. We claim that G is a $(2, \{2k + 1\})$-gadget with root u_1. In any neighbour-sum-2-distinguishing 2-edge-weighting ω of G, the G_i's force $\sigma(u_2)$ to have value at least $2k-1$. Depending on whether

$\omega(u_1u_2) = 1$ or $\omega(u_1u_2) = 2$, we thus have $\sigma(u_2) = 2k$ or $\sigma(u_2) = 2k + 1$, respectively. In the first case, we get sum conflicts between u_2 and its neighbours in $G_1, \dots, G_{k-1}$ since their incident sums differ by 1. Therefore, we necessarily have $\omega(u_1u_2) = 2$, so that $\sigma(u_2) = 2k + 1$, which produces no sum conflict in G.

Analogous $(1, S)$-gadgets with $S := \{2k + 1\}$, $k \geq 2$, will also be needed. A $(1, \{5\})$-gadget can be obtained as follows. Start from $G := u_1u_2$ being the path of length 1, add two $(2, \{7\})$-gadgets G_1 and G_2 to G, and identify u_2 and the roots of G_1 and G_2. In any neighbour-sum-2-distinguishing 2-edge-weighting ω of G, the G_i's force u_2 to be incident to at least two edges with weight 2. So we have $\sigma(u_2) = 5$ or $\sigma(u_2) = 6$ depending on whether $\omega(u_1u_2) = 1$ or $\omega(u_1u_2) = 2$, respectively. In the second case, however, we get sum conflicts between u_2 and its neighbours in G_1 and G_2. We thus necessarily have $\omega(u_1u_2) = 1$ and $\sigma(u_2) = 5$, which produces no sum conflict in G.

Let now $2k + 1 \geq 7$, and assume that we have constructed $(1, S')$-gadgets with $S' := \{2k' + 1\}$ for every k', $2 \leq k' < k$. Start from $G := u_1u_2$ being the path of length 1, add k $(2, \{2k + 3\})$-gadgets $G_1, \dots, G_k$ to G, and identify u_2 and the roots of $G_1, \dots, G_k$. We claim that G is a $(1, \{2k + 1\})$-gadget. In any neighbour-sum-2-distinguishing 2-edge-weighting ω of G, the G_i's force $\sigma(u_2)$ to be at least $2k$. Depending on whether $\omega(u_1u_2) = 1$ or $\omega(u_1u_2) = 2$, we thus have $\sigma(u_2) = 2k+1$ or $\sigma(u_2) = 2k+2$, respectively. In the second case, we get sum conflicts between u_2 and its neighbours in $G_1, \dots, G_k$, since their incident sums differ by 1. Therefore, we necessarily have $\omega(u_1u_2) = 1$ and $\sigma(u_2) = 2k + 1$, which produces no sum conflict in G.

Note that all the above-constructed gadgets are trees. With all these gadgets in hand, we now prove the main result of this section.

Theorem 4.1. *For a given* 3*-degenerate planar bipartite graph* G*, deciding whether* $\chi_{\Sigma>1}(G) \leq 2$ *holds is* NP*-complete.*

Proof. Since the problem is obviously in NP, we proceed with the proof of its NP-hardness. The proof is by reduction from 1-in-3 SAT. From a formula F, we construct a graph G such that F is satisfiable in a 1-in-3 way if and only if G admits a neighbour-sum-2-distinguishing 2-edge-weighting. Since the Monotone version of 1-in-3 SAT remains NP-complete (see e.g. [4]), we may assume that F has no negated variables. Also, we may assume that all clauses of F have three distinct variables, as otherwise F could be simplified. That is:

- if F has a clause $(x_{i_1} \vee x_{i_1} \vee x_{i_1})$, then F is not satisfiable in a 1-in-3 way;
- if F has a clause $(x_{i_1} \vee x_{i_1} \vee x_{i_2})$, then x_{i_2} and x_{i_1} are forced to *true* and *false*, respectively, by any truth assignment making F satisfied in a 1-in-3 way.

We denote by $x_1, \dots, x_n$ the variables of F, and by $C_1, \dots, C_m$ its clauses. The construction of G, which is clearly achieved in polynomial time, is as follows. We start by adding *variable gadgets* in the following way. For each variable x_i of F, we add to G a star V_i with root v_i and $2k_i$ leaves $u_{i,1}, \dots, u_{i,2k_i}$, where $2k_i$ is any even integer greater than 10. Next we add $(1, \{4k_i + 1\})$-, $(2, \{4k_i + 3\})$-, $(2, \{4k_i + 5\})$-, ..., $(2, \{6k_i - 3\})$- and $(2, \{6k_i - 3\})$-gadgets $G_1, \dots, G_{k_i}$ to G, and identify v_i and the roots of $G_1, \dots, G_{k_i}$. Because of the G_i's, in any neighbour-sum-2-distinguishing 2-edge-weighting ω of V_i, the value of $\sigma(v_i)$ lies between $4k_i - 1$ (when all $v_iu_{i,j}$'s are assigned weight 1) and $6k - 1$ (when all $v_iu_{i,j}$'s are assigned weight 2). Furthermore, $4k_i - 1$ and $6k_i - 1$ are both odd. Hence, we cannot have $\sigma(v_i) \in \{4k_i, \dots, 6k_i - 2\}$ as otherwise there would be a sum conflict involving v_i and one of its neighbours in the G_i's. Therefore, either all $\omega(v_iu_{i,j})$'s are equal to 1, or all $\omega(v_i u_{i,j})$'s are equal to 2. In what follows, we call the vertices $u_{i,1}, \dots, u_{i,2k_i}$ the *output vertices* of V_i, and the edges $v_i u_{i,1}, \dots, v_i u_{i,2k_i}$ the *output edges* of V_i,

We now modify G by considering the clauses of F. For each clause $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ of F, we add a *clause vertex* c_j to G. For each $V_{i_1}, V_{i_2}, V_{i_3}$, we then select one output vertex still having degree 1, and identify c_j and the three selected output vertices. Finally, we add a $(2, \{7\})$-gadget G_1 to G, as well as a $(2, \{11\})$-gadget G_2, and identify c_j and the roots of G_1 and G_2. In any neighbour-sum-2-distinguishing 2-edge-weighting ω of G, $\sigma(c_j)$ has thus value at least 4 (because of G_1 and G_2), and ranges in $\{7, \dots, 10\}$. However, $\sigma(c_j)$ cannot take any value among $\{7, 8, 10\}$ because of G_1 and G_2. So we necessarily have $\sigma(c_j) = 9$, which occurs only if exactly one of the three output edges originating from $V_{i_1}, V_{i_2}, V_{i_3}$ is assigned weight 1.

It can be checked that no unexpected sum conflicts (that is, different from those listed above) can arise, in particular thanks to our choice of the $2k_i$'s. We now claim that we have the desired equivalence. This directly follows from the following arguments:

- Assigning weight 1 (resp. 2) to an output edge $v_i c_j$ simulates the fact that variable x_i brings truth value *true* (resp. *false*) to C_j.
- Following that equivalence, the fact that, for any V_i, all output edges of V_i must be weighted 1 (resp. 2) simulates the fact that setting x_i to *true* (resp. *false*) by some truth assignment brings the same truth value to every clause containing x_i.
- The fact that, for every clause vertex c_j, exactly one incident output edge must be assigned weight 1 simulates the fact that a clause of F is considered satisfied if and only if it includes exactly one *true* variable.

To complete the proof of the theorem, it suffices to observe the following:

- The PLANAR version of MONOTONE 1-IN-3 SAT remains NP-complete (see [4]), so we may assume that F is a planar formula. Since every gadget is a tree, the construction above then yields a planar G.
- Since every gadget is a tree, the graph G is 3-degenerate.
- The only cycles in G are those of the subgraph induced by the v_i's and the c_j's. Since this subgraph is bipartite, so is G.

This concludes the proof. □

4.2. Tree case

In this section, we prove that the counterpart of Theorem 4.1 for trees is not true. That is, we prove that determining the value of $\chi_{\Sigma>1}(T)$ for a given tree T can be done in polynomial time. Recall that for a tree T, we always have $\chi_{\Sigma>1}(T) \leq 4$ (according to Corollary 3.1), while $\chi_{\Sigma>1}(T) = 1$ if and only if, for every two adjacent vertices u and v of T, the values of $d(u)$ and $d(v)$ differ by at least 2.

Theorem 4.2. *For a given tree T, determining $\chi_{\Sigma>1}(T)$ can be done in polynomial time.*

Proof. For any fixed $k \in \{1, 2, 3, 4\}$, we introduce below an algorithm that checks in polynomial time whether T admits a neighbour-sum-2-distinguishing k-edge-weighting. So, to determine $\chi_{\Sigma>1}(T)$, we can essentially run this algorithm successively with $k = 1, 2, 3, 4$. The first value of k for which the algorithm answers positively is the value of $\chi_{\Sigma>1}(T)$.

Designate a node r of T as being its *root*. This defines a root-to-leaf orientation of T in the usual way, where every non-root node v has a *father*, and every non-leaf node v has *children*. By the *descendants* of v, we refer to the nodes of T for which we find v when iterating the father relationship.

The subtree T_v of T rooted at v is the subtree whose nodes are v and all its descendants. This subtree T_v can itself be decomposed into several subtrees, in the following way. Assume that v has $d \geq 1$ descendants $u_1, \dots, u_d$, ordered following an arbitrary order (supposed to be fixed throughout the proof). Then T_v can be edge-decomposed into d subtrees $T_{v,1}, \dots, T_{v,d}$ being $T_{u_1} + vu_1, \dots, T_{u_d} + vu_d$, respectively, whose root, v, has degree precisely 1. Trees with this property are called *shrubs* throughout. Conversely, T_v is obtained by identifying the roots of the shrubs $T_{v,1}, \dots, T_{v,d}$. For every shrub, we call the edge incident to the root the *root edge*. The non-root end of the root edge is called the *subroot*.

We are now ready to describe our algorithm for deciding whether T admits a neighbour-sum-2-distinguishing k-edge-weighting. The rough ideas are the following. The tree T can be seen as a union of $d := d(r)$ shrubs $S_1, \dots, S_d$ whose roots were identified, resulting in r. A neighbour-sum-2-distinguishing k-edge-weighting of T is thus essentially the union of (relaxed, see below) neighbour-sum-2-distinguishing k-edge-weightings of the d shrubs attached to r, with the additional property that the resulting $\sigma(r)$ does not create any sum conflict. Therefore, in order to construct a neighbour-sum-2-distinguishing k-edge-weighting of T, it suffices to find convenient neighbour-sum-2-distinguishing k-edge-weightings of $S_1, \dots, S_d$ that can be "glued". So we need to know, for each shrub S_i and for every $\alpha \in \{1, \dots, k\}$, whether S_i admits a neighbour-sum-2-distinguishing k-edge-weighting where the root edge is assigned colour α, and, for such an edge-weighting of S_i, which possible incident sums can be obtained for the subroot.

More formally, for a shrub S with root v' and subroot v, we want to compute, for every weight $\alpha \in \{1, \dots, k\}$, the set $X_\alpha(v)$ of possible values of $\sigma(v)$ by a neighbour-sum-2-distinguishing k-edge-weighting of S assigning weight α to $v'v$. Note that a shrub might be a single edge, and may thus admit no neighbour-sum-2-distinguishing k-edge-weighting. In that special case, we relax the notion of neighbour-sum-2-distinguishing k-edge-weighting, and allow the root and the subroot to have the same incident sums.

Assume v has d children $u_1, \dots, u_d$, $d \geq 0$, and let $S_1, \dots, S_d$ denote the d shrubs attached to v in S. We claim that each $X_\alpha(v)$ can be computed in polynomial time from the sets

$$X_1(u_1), \dots, X_k(u_1), X_1(u_2), \dots, X_k(u_2), \dots, X_1(u_d), \dots, X_k(u_d),$$

computed by induction for the shrubs $S_1, \dots, S_d$. So, in a way, the sets $X_1(v), \dots, X_k(v)$ can be computed from smaller shrubs, and deduced successively towards the subroot of S. We prove this below.

The base case is when S is a single edge, that is, v has no child. If the edge $v'v$ is assigned any weight $\alpha \in \{1, \dots, k\}$ by a neighbour-sum-2-distinguishing k-edge-weighting, then $\sigma(v) = \alpha$. So, for such a shrub S, we have $X_\alpha(v) = \{\alpha\}$ for every $\alpha \in \{1, \dots, k\}$.

Suppose now that v has $d \geq 1$ children $u_1, \dots, u_d$, and, for each shrub S_i, $1 \leq i \leq d$, attached to v, and every $\alpha \in \{1, \dots, k\}$, the set $X_\alpha(u_i)$ has been computed by induction. We now want to compute the sets $X_1(v), \dots, X_k(v)$. Since $d(v) = d + 1$, by any k-edge-weighting of S, the sum $\sigma(v)$ can take up to $kd + k - d$ values, namely those among $\{d + 1, \dots, kd + k\}$. We repeatedly fix one of those sums x, and we determine whether x can be added to some of the sets $X_1(v), \dots, X_k(v)$.

Assume we want to determine whether x has to be added to $X_\alpha(v)$, where α is any value in $\{1, \dots, k\}$. Successively consider all partitions $x_1 + 2x_2 + \dots + kx_k$ of x into

$x_1 + \cdots + x_k = d + 1$ values among $\{1, \ldots, k\}$ only. Recall that we are focusing on computing $X_\alpha(v)$, so if $x_\alpha = 0$, then we can consider the next partition of x. Since x is linear in $|V(T)|$ and $k \leq 4$ is fixed, the number of such partitions to consider is polynomial in $|V(T)|$. Essentially, we have $x_1, \ldots, x_k \leq |V(T)|$, meaning that the number of such partitions is roughly $|V(T)|^{k-1}$. We now want to know if there is a neighbour-sum-2-distinguishing k-edge-weighting of S where x_i edges incident to v are assigned weight i, for every $i \in \{1, \ldots, k\}$. If such an edge-weighting exists, then for any S_i to which the weight β is assigned to the root edge, $X_\beta(u_i)$ contains some value not in $\{\alpha - 1, \alpha, \alpha + 1\}$.

Since we are focusing on $X_\alpha(v)$, one of the x_α weights α around v will be assigned to $v'v$. This leaves us with d other weights to assign bijectively to the vu_i's, with the constraint that if we assign a weight β to vu_i, then $X_\beta(u_i)$ should contain a value not among $\{\alpha - 1, \alpha, \alpha + 1\}$. If β can indeed by assigned to vu_i safely, then we call this a *valid assignment.* To find a correct assignation (if any exists), we build a compatibility bipartite graph C of the valid assignments, as follows. In one side of the bipartition of C, we put d vertices corresponding to the d weights we want to assign. In the other side, we put d vertices corresponding to the edges $vu_1, \ldots, vu_d$ of S. We then add an edge joining two vertices of C if assigning the corresponding weight to the corresponding edge of S is valid. Now, finding a satisfying assignment of the d weights to the root edges of the S_i's is equivalent to finding a perfect matching in C, which is known to be doable in polynomial time. If there indeed exists such a perfect matching of C, then we add x to $X_\alpha(v)$.

We now go back to T, with the root r having d children $u_1, \ldots, u_d$. For each S_i of the d shrubs $S_1, \ldots, S_d$ rooted at v, we can compute the sets $X_1(u_i), \ldots, X_k(u_i)$ as explained above. These sets memorize, in a compact way, all possible ways, in terms of incident sums and weights assigned to the root edges, to k-edge-weight the S_i's in a neighbour-sum-2-distinguishing way. Now, again, we can consider every potential incident sum x as $\sigma(r)$, every potential way to partition x into d integers among $\{1, \ldots, k\}$, and, building the compatibility bipartite graph as above, find, if it exists, a valid way to bijectively assign the d weights to the d root edges $vu_1, \ldots, vu_d$. If a valid assignment for a partition of some x exists, then T admits a neighbour-sum-2-distinguishing k-edge-weighting. Otherwise, it does not.

Concerning the complexity aspect, determining $X_\alpha(v)$ for a shrub of T with subroot v can be done in polynomial time. Recall that $k \leq 4$ is constant. The number of possible sums x as $\sigma(v)$ to consider is at most $k|V(T)|$. For each of these values of x, we consider up to $|V(T)|^{k-1}$ partitions into 1's, 2's, ..., and k's. Deciding whether there is a valid assignment for one of those partitions can be done in polynomial time, using for instance Edmonds' Blossom Algorithm for computing maximum matchings. The procedure above is almost the same when r is considered. By all these arguments, the whole procedure can be achieved in polynomial time. □

5. Conclusion

In this work, we have investigated the consequences on the 1-2-3 Conjecture of requiring adjacent vertices to be distinguishable in a stronger way, namely by asking their incident sums to differ by at least 2. We have addressed Conjecture 1.1, to which we did not manage to come up with any counterexample, as an equivalent of the 1-2-3 Conjecture in this context. As a main evidence that our conjecture might be true, we have pointed out some connexions between the 1-2-3 Conjecture and Conjecture 1.1, and verified the later one for nice bipartite graphs.

Several aspects related to Conjecture 1.1 remain unclear to us, and could thus be subject to further work. First, we do not fully understand how necessary the weights 2 and 4 are for our conjecture. In particular, most graphs for which we have verified Conjecture 1.1 actually admit neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weightings, as illustrated by our proof of Theorem 3.4. Also, the neighbour-sum-2-distinguishing edge-weightings constructed in the proof of Observations 2.1 and 2.3 use half of the weights only. Hence, we think the following stronger conjecture could be worth studying.

Conjecture 5.1. *Every nice graph admits a neighbour-sum-2-distinguishing $\{1, 3, 5\}$-edge-weighting.*

In the context of neighbour-sum-2-distinguishing edge-weightings, Conjecture 5.1 might be an equivalent to the 1-2-3 Conjecture more natural than Conjecture 1.1. Indeed, in the 1-2-3 Conjecture we aim at getting incident sums differing by at least 1 by using three successive weights $\alpha - 1, \alpha, \alpha + 1$ differing by 1. In Conjecture 5.1, we aim at getting incident sums differing by at least 2 by using three "successive" weights $\alpha - 2, \alpha, \alpha + 2$ differing by 2.

There are intriguing examples, though, such as nice paths P_ℓ of length congruent to 1 modulo 4 (for which $\chi_{\Sigma>1}(P_\ell) = 4$, recall Theorem 3.2), showing that the weights 2 and 4 are worth using in some cases. We thus believe this could be an interesting aspect to study further.

More directions for future works are also worth mentioning. Notably, we did not manage to improve the bounds given in Section 2 for many classes of graphs. Generally speaking, it does not seem obvious to us how to improve the bound in Corollary 2.2, and this would surely require new tools. Concerning particular classes of graphs, let us mention the case of subcubic graphs. Although we know that cubic graphs verify Conjecture 1.1, and even Conjecture 5.1 (recall Corollary 2.4), we did not manage to prove that nice subcubic graphs, in general, also do. We believe this would be an appealing first case to consider towards verifying Conjecture 1.1 for 3-chromatic graphs, which verify the 1-2-3 Conjecture.

More generally speaking, the notion of neighbour-sum-2-distinguishing edge-weighting could be generalized to neighbour-sum-d-distinguishing edge-weighting, for any $d \geq 3$, where one could require incident sums to differ by at least d. Many of the arguments and techniques used in this work could actually be generalized to deal with such a notion. For larger values of d, it is likely that more intriguing phenomenon arise.

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