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MIXING RATE IN INFINITE MEASURE FOR 
$Z^d$-EXTENSION, APPLICATION TO THE PERIODIC SINAI BILLIARD

FRANÇOISE PÊNE

Abstract. We study the rate of mixing of observables of $Z^d$-extensions of probability preserving dynamical systems. We explain how this question is directly linked to the local limit theorem and establish a rate of mixing for general classes of observables of the $Z^2$-periodic Sinai billiard. We compare our approach with the induction method.

A measure preserving dynamical system $(X,f,\mu)$ is given by a measure space $(X,\mu)$ and a measurable $\mu$-preserving transformation $f$. Given such a dynamical system, the study of its mixing properties means the study of the behaviour of quantities of the following form:

$$\int_X u \cdot v \circ f^n \, d\mu \quad \text{as} \quad n \to +\infty. $$

(1)

When $\mu$ is a probability measure, $(X,f,\mu)$ is said to be mixing if, for every $u,v \in L^2(\mu)$, (1) converges to the product of integrals $\int_X u \, d\mu \int_X v \, d\mu$.

Assume from now on that $\mu$ is a $\sigma$-finite measure. As pointed out by [9], there is no reasonable generalization of mixing. Nevertheless, it makes sense to investigate the behaviour of (1). More precisely, we are interested in proving that (1) suitably normalized converges to $\int_X u \, d\mu \int_X v \, d\mu$. We call mixing rate the corresponding normalization.

Mixing rates (and refined estimates) in infinite measure have been studied by Thaler [19], Melbourne and Terhesiu [11], Gouëzel [7], Bruin and Terhesiu [3], Liverani and Terhesiu [10] for a wide family of dynamical systems including the Liverani-Saussol-Vaienti maps, etc. The method used by these authors is induction.

We emphasize here on the fact that, in the context of $Z^d$-extensions, such results are related to precised local limit theorems see [8, 17, 18]. In particular mixing properties for the periodic planar Sinai billiard have been established in [12, Prop. 4] and in [14, Prop. 4.1] for indicator functions of some bounded sets, with three different applications in [12, 13, 14]. We are interested here in stating such results for general functions (with full support). We will present our general approach and use it to establish a mixing rate for a general class of functions in the context of the $Z^2$-periodic Sinai billiard.

In some sense, these two approaches are converse one from the other. Indeed, whereas for the first mentioned method, the mixing rate follows from an estimate of the tail distribution of some return time; for the method
we use here, we first prove the mixing rate and can deduce from it the asymptotic behaviour of the tail distribution of the first return time (see [5, Thm. 1] and [14, Prop. 4.2]).

For both methods, the link between tail distribution return time and mixing is given by a renewal equation.

1. Mixing via induction

The strategy of the proof via induction consists:

a) to consider a set $Y \subset X$ of finite measure satisfying nice properties; in particular, $(\mu \{ \varphi > n \})$ is regularly varying, where $\varphi$ is the first return time to $Y$: $\varphi(y) := \inf \{ n \geq 1 : f^n y \in Y \}$.

b) to prove good estimates for $R_n : v \mapsto 1_Y L^n (1_{Y \cap \{ \varphi = n \}} v)$, where $L$ is the transfer operator of $u \mapsto u \circ f$, which is defined by $\int_X u \cdot \nu \, d\mu = \int_X u \circ f \cdot \nu \, d\mu$.

c) to deduce from the estimates of $R_n$ and from the renewal equation:

$$\forall n \geq 1, \quad T_n = \sum_{j=1}^n T_{n-j} R_j, \quad \text{with} \quad T_n : v \mapsto 1_Y L^n (1_Y v)$$

an estimate on $T_n$ of the following form:

$$T_n \sim \mu(Y \cap f^{-n} Y) \mathbb{E}_{\mu}[1_Y],$$

on some Banach space $B$ of functions $w : X \to \mathbb{C}$.

d) to deduce:

$$\int_X v \cdot T_n u \, d\mu = \int_X v \cdot L^n u \, d\mu = \int_X u \cdot v \circ f^n \, d\mu,$$

for every $u, v : X \to \mathbb{C}$ supported in $Y$ such that $v \in B$ and $w \mapsto \int_Y u \cdot w \, d\mu$ is in $B'$.

e) to go to the general situation (functions with full support in $X$) by considering the sets $A_k := f^{-k} Y \setminus \bigcup_{\ell=0}^{k-1} f^{-\ell} Y$.

2. $\mathbb{Z}^d$-extensions: Local limit theorem and mixing

We consider from now on the special case where $(X, f, \mu)$ is a $\mathbb{Z}^d$-extension of a probability preserving dynamical system $(\bar{X}, \bar{f}, \bar{\mu})$ by $\psi : \bar{X} \to \mathbb{Z}^d$, that is $X = \bar{X} \times \mathbb{Z}^d$, $f(x, k) = (\bar{f}(x), k + \psi(x))$ and $\mu = \bar{\mu} \otimes \lambda_d$, where $\lambda_d$ is the counting measure on $\mathbb{Z}^d$. Observe that $f^n(x, k) = (\bar{f}^n(x), k + S_n(x))$, with $S_n := \sum_{k=0}^{n-1} \psi \circ f^k$. We set $Y := X \times \{0\}$.

The crucial idea in this context is to consider a situation where $(S_n/a_n)_{n}$ converges in distribution to a stable random variable $B$ and the strategy is then:

a) to prove a local limit theorem (LLT):

$$\forall \ell \in \mathbb{Z}^d, \quad \bar{\mu}(S_n = \ell) = (\Phi_B(\ell/a_n) + o(1)) a_n^{-d}, \quad \text{as} \quad n \to +\infty,$$

where $\Phi_B$ is the density function of $B$, and more precisely a "spectral LLT":

$$Q_{n, \ell} := P^n (1_{S_n = \ell}) = \frac{\Phi_B(\ell/a_n) \mathbb{E}_{\mu}[\cdot] + \varepsilon_{n, \ell}}{a_n^d}, \quad \text{with} \quad \lim_{n \to +\infty} \sup_{\ell} \|\varepsilon_{n, \ell}\| = 0,$$

(2)
on some Banach space ($\mathcal{B}, \| \cdot \|$) of functions $w : \mathcal{X} \to \mathbb{C}$, where $P$ is the transfer operator of $\hat{f}$ (see [15, Lem. 2.6] for a proof of such a result in a general context).

The following identity makes a relation between $Q_{n,0}$ and the operator $T_n$ presented in the previous section:

$$(T_n v)(x, \ell) = (Q_{n,0}(v(\cdot, 0)))(x) 1_{\ell=0}.$$ 

Note that the LLT is already a decorrelation result since:

$$\int_X 1_Y \cdot 1_Y \circ f^n d\mu = \mu(Y \cap f^{-n}Y) = \bar{\mu}(S_n = 0).$$

b) to use (2) and the definition of $\mu$

d) to go from (2) to the study of $\bar{\mu}$

c) to generalize this as follows:

$$\int_X 1_{X \times \{k\}} u \cdot 1_{X \times \{\ell\}} \circ f^n d\mu = \int_X u(x, k) \cdot v(\hat{f}^n(x), \ell) 1_{\{S_n(\cdot) = \ell - k\}} d\bar{\mu}(x)$$

$$= \int_X v(\cdot, \ell) P^n (u(\cdot, k), 1_{\{S_n(\cdot) = \ell - k\}}) d\bar{\mu}(x)$$

$$= \Phi_B(0)a_n^{-d} \int_{X \times \{k\}} u d\mu \int_{X \times \{\ell\}} v d\mu + o(a_n^{-d}), \quad (3)$$

valid for every $k, \ell \in \mathbb{Z}^d$ and for every $u, v$ such that $u_{\ell} := u(\cdot, \ell) \in \mathcal{B}$ and such that $w \mapsto \int_X v_k(y) w(y) d\bar{\mu}(y)$ is in $\mathcal{B}'$, with $v_k(y) := v(y, k)$, since $\Phi_B$ is continuous.

c) to generalize this as follows:

$$\int_X u \cdot v \circ f^n d\mu = \sum_{\ell, m} \int_X 1_{X \times \{\ell\}} u_{\ell} \cdot (v_m 1_{X \times \{m\}}) \circ f^n d\mu$$

$$= a_n^{-d} \left( o(1) + \sum_{k, m} \Phi_B \left( \frac{k - m}{a_n} \right) \int_{X \times \{\ell\}} u d\mu \int_{X \times \{m\}} v d\mu \right)$$

$$= \Phi_B(0)a_n^{-d} \int_X u d\mu \int_X v d\mu + o(a_n^{-d}),$$

which holds true as soon as $\sum_{\ell} \|u_{\ell}\|_B < \infty$ and $\sum_{\ell} \|\mathbb{E}[v_{\ell} \cdot]\|_{\mathcal{B}'} < \infty$, since $\Phi_B$ is continuous and bounded, where we used again the notations $u_{\ell} := u(\cdot, \ell)$ and $v_m := v(\cdot, m)$.

d) to go from (2) to the study of $\bar{\mu}(\varphi > n)$, where $\varphi(x)$ is the first return time from $(x, 0)$ to $Y = \bar{X} \times \{0\}$, using the classical following renewal equation [6]:

$$1 = \sum_{j=0}^{n} 1_{\{\varphi > n - j\}} \circ \bar{f}^j 1_{\{S_j=0\}} \text{ on } \bar{X}$$

where $j$ plays the role of the last visit time to $Y$ before time $n$. Hence, applying $P^n$, this leads to:

$$1 = \sum_{j=0}^{n} U_{n-j} Q_{j,0} \quad \text{with } U_k := P^k \left( 1_{\{\varphi > k\}} \right).$$

This kind of properties has been used in [13] to study the asymptotic behaviour of the number of different obstacles visited by the Lorentz
process up to time $n$, in [14] to study some quantitative recurrence properties.

3. Example: Lorentz process

Consider a $\mathbb{Z}^2$-periodic configuration of obstacles in the plane: $O_i + \ell$, $i = 1, \ldots, I$, $\ell \in \mathbb{Z}^2$, with $I \geq 2$. We assume that the $O_i$ are convex open sets, with $C^3$-smooth boundary with non null curvature. We assume that the closures of any couple of distinct obstacles $O_i + \ell$ and $O_j + m$ are disjoint. The Lorentz process describes the displacement in $Q := \mathbb{R}^2 \setminus \bigcup_{\ell \in \mathbb{Z}^2} \bigcup_{i=1}^I O_i + \ell$ of a point particle moving with unit speed and with elastic reflection off the obstacles (i.e. reflected angle=incident angle). We assume that the horizon is finite, i.e. that each trajectory meets at least one obstacle.

We consider the dynamical system $(M, T, \nu)$ corresponding to the collision times, where $M$ is the set of reflected vectors, where $T : M \to M$ is the transformation mapping a reflected vector to the reflected vector at the next collision time and where $\nu$ is the invariant measure absolutely continuous with respect to the Lebesgue measure. For every $\ell \in \mathbb{Z}^2$, we write $C_\ell$ for the set of reflected vectors which are based on $\bigcup_{j=1}^I (O_j + \ell)$. Up to a renormalization of $\nu$, we assume that $\nu(C_0) = 1$. We call $C_\ell$ the $\ell$-cell.

It is well known that $(M, T, \nu)$ can be represented as the $\mathbb{Z}^2$-extension $(X, f, \mu)$ of $(\bar{X}, \bar{f}, \bar{\mu})$ by $\psi : \bar{X} \to \mathbb{Z}^2$, where $X = C_0$, $\bar{\mu} = \psi(C_0 \cap \cdot)$, where $\bar{f}$ and $\psi$ are such that $T(q, \bar{v}) = (q' + \psi(q, \bar{v}), \bar{v}')$ if $(q', \bar{v}') = \bar{f}(q, \bar{v})$ ($\bar{f}$ corresponds to $T$ quotiented by the equality of positions modulo $\mathbb{Z}^2$). Note that $S_n(x) := \sum_{k=0}^{n-1} \psi \circ \bar{f}^k(x)$ is the label of the cell in which the particle starting from configuration $x \in C_0$ is at the $n$-th reflection time.

The dynamical system $(\bar{X}, \bar{f}, \bar{\mu})$ is the Sinai billiard [16, 4]. Central limit theorems in this context have been established in [2, 1, 20]. In particular $(S_n/\sqrt{n})_n$ converges in distribution, with respect to $\bar{\mu}$ to a centered gaussian random variable $B$ with non-degenerate variance matrix $\Sigma$, so $\Phi_B(x) = e^{-\frac{(x-\bar{x})^2}{2}}/(2\pi \sqrt{\det \Sigma})$.

Let $R_0 \subset \bar{X}$ be the set of reflected vectors that are tangent to $\bigcup_{i=1}^I \partial O_i$. The billiard map $\bar{f}$ defines a $C^1$-diffeomorphism from $\bar{X} \setminus (R_0 \cup \bar{f}^{-1} R_0)$ onto $\bar{X} \setminus (R_0 \cup \bar{f} R_0)$. For any integers $k \leq k'$, we set $\xi_{k,k'}$ for the partition of $\bar{X} \setminus \bigcup_{j=k}^{k'} \bar{f}^{-j} R_0$ in connected components and $\xi_k := \bigvee_{j \geq k} \xi_{k,j}$. For any $\bar{u} : \bar{X} \to \mathbb{R}$ and $-\infty < k \leq k' \leq \infty$, we define the following local continuity modulus:

$$\omega_{k,k'}(\bar{u}, \bar{x}) := \sup_{\bar{y} \in \xi_{k,k'}(\bar{x})} |\bar{u}(\bar{x}) - \bar{u}(\bar{y})|.$$ 

The following result is established thanks to the use of the towers constructed by Young in [20].

**Proposition 3.1.** Let $p > 1$. There exists $c > 0$ such that, for any $k \geq 1$, for any measurable functions $\bar{u}, \bar{v} : \bar{X} \to \mathbb{R}$ such that $\bar{u}$ is $\xi^k_{-k,k}$-measurable and $\bar{v}$ is $\xi^k_{2-k,k}$-measurable, for every $n > 2k$ and for every $\ell \in \mathbb{Z}^2$,

$$\left| \mathbb{E}_\bar{\mu} \left[ \bar{u} 1_{\{S_n=\ell\}} \bar{v} \circ \bar{f}^k \right] - \frac{\Phi_B \left( \frac{\ell}{\sqrt{n-2k}} \right)}{n-2k} \int_{\bar{X}} \bar{u} \, d\bar{\mu} \int_{\bar{X}} \bar{v} \, d\bar{\mu} \right| \leq \frac{ck \|\bar{v}\|_p \|\bar{u}\|_\infty}{(n-2k)^\frac{1}{2}}.$$
Proof. The proof of this result is exactly the same as the proof of [14, prop 4.1], by replacing \( 1_A \) by \( \bar{u} \), \( 1_B \) by \( \bar{v} \), \( 1_A \) and \( 1_B \) by respectively \( \hat{u} \) and \( \hat{v} \) such that: \( \hat{u} \circ \hat{\pi} = \bar{u} \circ T^k \circ \hat{\pi} \) and \( \hat{v} \circ \hat{\pi} = \bar{v} \circ T^k \circ \hat{\pi} \). With the notations of [14], we have \( \sup_{t \in [-\pi, \pi]^2} \| P_k^k \bar{u} \| \leq c_0 \| \bar{u} \|_\infty \). So that \( \bar{\mu}(B)^{1/p} \) of [14, p. 865] is replaced by \( \| \bar{u} \|_\infty \| \bar{v} \|_p \).

For any \( u, v : M \rightarrow \mathbb{R} \) and \( k \in \mathbb{Z}^2 \), we set as previously: \( u_k := u(\cdot, k) \) and \( v_k := v(\cdot, k) \).

**Theorem 3.2.** Let \( p > 1 \) and \( u, v : X \rightarrow \mathbb{R} \) measurable such that

\[
\sum_{\ell \in \mathbb{Z}^2} (\| u_\ell \|_\infty + \| v_\ell \|_p) < \infty ,
\]

\[
\forall k \geq 1, \quad \sum_{\ell \in \mathbb{Z}^2} \| \omega^k_{-k}(v_\ell, \cdot) \|_p < \infty ,
\]

\[
\lim_{k \rightarrow +\infty} \sum_{\ell \in \mathbb{Z}^2} \left( \| \omega^k_{-k}(u_\ell, \cdot) \|_1 + \| \omega^k_{-k}(v_\ell, \cdot) \|_1 \right) = 0.
\]

Then

\[
\int_X u.v \circ f^n \, d\mu = \frac{\Phi_B(0)}{n} \int_X u \, d\mu \int_X v \, d\mu + o(n^{-1}).
\]

**Proof.** It is enough to prove the result for non-negative \( u, v \). We assume from now on that \( u, v \) take their values in \([0, +\infty)\). Let \( \ell \in \mathbb{Z}^2 \) and let \( k \) be a positive integer. We define \( u^{(k, \pm)}_\ell \) and \( v^{(k, \pm)}_\ell \):

\[
\begin{align*}
\quad u^{(k,-)}_\ell(\bar{x}) := & \quad \inf_{\bar{y} \in \xi^{k,-}_k(\bar{x})} u_\ell(\bar{y}), & \quad u^{(k,+)}_\ell(\bar{x}) := & \quad \sup_{\bar{y} \in \xi^{k,+}_k(\bar{x})} u_\ell(\bar{y})
\quad v^{(k,-)}_\ell(\bar{x}) := & \quad \inf_{\bar{y} \in \xi^{\infty,-}_k(\bar{x})} v_\ell(\bar{y}), & \quad v^{(k,+)}_\ell(\bar{x}) := & \quad \sup_{\bar{y} \in \xi^{\infty,+}_k(\bar{x})} v_\ell(\bar{y}).
\end{align*}
\]

Observe that

\[
\quad u^{(k,+)}_\ell - u^{(k,-)}_\ell \leq 2\omega^k_{-k}(u_\ell, \cdot)
\]

and that

\[
\quad v^{(k,+)}_\ell - v^{(k,-)}_\ell \leq 2\omega^\infty_{-k}(v_\ell, \cdot).
\]

We then consider \( u^{(k, \pm)}, v^{(k, \pm)} : X \rightarrow \mathbb{R} \) such that

\[
\forall \ell \in \mathbb{Z}^2, \quad u^{(k, \pm)}(\ell, \cdot) \equiv u^{(k, \pm)}_\ell(\cdot) \quad \text{and} \quad v^{(k, \pm)}(\ell, \cdot) \equiv v^{(k, \pm)}_\ell(\cdot).
\]

Note that

\[
\quad u^{(k,-)} \leq u \leq u^{(k,+)} \quad \text{and} \quad v^{(k,-)} \leq v \leq v^{(k,+)}
\]

and so

\[
\int_X u^{(k,-)} \circ f^n \, d\mu \leq \int_X u \circ f^n \, d\mu \leq \int_X u^{(k,+)} \circ f^n \, d\mu.
\]

We have

\[
\int_X u^{(k, \pm)} \circ f^n \, d\mu = \sum_{\ell, m \in \mathbb{Z}^2} \int_X u^{(k, \pm)}(\ell, \cdot)\mathbf{1}_{\{S_n = \ell \}} v^{(k, \pm)}_m \circ f^n \, d\mu.
\]
Applying Proposition 3.1 to the couples \((u^{(k,-)}_\ell, v^{(k,-)}_m)\) and \((u^{(k,+)}_\ell, v^{(k,+)}_m)\), for every \(\ell, m \in \mathbb{Z}^2\), we obtain that

\[
\left| \int_X u^{(k,\pm)}_\ell \cdot v^{(k,\pm)}_m \circ f^n \, d\mu - \sum_{\ell, m} \frac{\Phi_B \left( \frac{m-\ell}{\sqrt{n-2k}} \right)}{n-2k} \int_X u^{(k,\pm)}_\ell \, d\bar{\mu} \int_X v^{(k,\pm)}_m \, d\bar{\mu} \right| \leq \sum_{\ell, m \in \mathbb{Z}^2} c k \|v^\parallel m\|_p \|u^{(k,\pm)}_\ell\|_\infty \frac{n}{(n-2k)^2} = o(n^{-1}),
\]

due to (4) and (5). Hence

\[
\int_X u^{(k,\pm)}_\ell \cdot v^{(k,\pm)}_m \circ f^n \, d\mu = \frac{1}{n-2k} \sum_{\ell, m} \frac{\Phi_B \left( \frac{m-\ell}{\sqrt{n-2k}} \right)}{n-2k} \int_X u^{(k,\pm)}_\ell \, d\bar{\mu} \int_X v^{(k,\pm)}_m \, d\bar{\mu} + o(n^{-1}).
\]

But \(\Phi_B\) is continuous and bounded by \(\Phi_B(0)\). Hence, due to the Lebesgue dominated convergence theorem, we obtain

\[
\int_X u^{(k,\pm)}_\ell \cdot v^{(k,\pm)}_m \circ f^n \, d\mu = \frac{\Phi_B(0)}{n-2k} \sum_{\ell, m} \int_X u^{(k,\pm)}_\ell \, d\bar{\mu} \int_X v^{(k,\pm)}_m \, d\bar{\mu} + o(n^{-1})
= \frac{\Phi_B(0)}{n} \int_X u^{(k,\pm)}_\ell \, d\mu \int_X v^{(k,\pm)}_m \, d\mu + o(n^{-1}).
\]

Moreover (6), (8) and (9) imply that

\[
\lim_{k \to +\infty} \int_X |u^{(k,\pm)}_\ell - u| \, d\mu = \lim_{k \to +\infty} \int_X |v^{(k,\pm)}_m - v| \, d\mu = 0.
\]

We conclude by combining this with (11) and (12). \(\square\)

As a consequence we obtain the mixing for dynamically Lipschitz functions. Let \(\vartheta \in (0, 1)\). We set

\[
d_\vartheta(x, y) := |s(x, y)|,
\]

where \(s(x, y)\) is the maximum of the integers \(k > 0\) such that \(x\) and \(y\) lie in the same connected component of \(M \setminus \bigcup_{j=-k}^k T^{-j}S_0\), where \(S_0\) is the set of vectors of \(M\) tangent to \(\partial Q\). The function \(s(\cdot, \cdot)\) is called separation time.

We set

\[
L_\vartheta(u) := \sup_{x \neq y} \frac{|u(x) - u(y)|}{d_\vartheta(x, y)}
\]

for the Lipschtz constant of \(u\) with respect to \(d_\vartheta\).

It is worth noting that, for every \(\vartheta \in (0, 1]\), there exists \(\vartheta_0 > 0\) such that every \(\eta\)-Hölder function (both in position-speed) is dynamically Lipschitz continuous with respect to \(\vartheta\).

**Corollary 3.3.** Assume that \(u, v : M \to \mathbb{R}\) are bounded uniformly dynamically Hölder (in position and in speed) and that

\[
\sum_{\ell \in \mathbb{Z}^2} (\|u1_{C_\ell}\|_{\infty} + \|v1_{C_\ell}\|_{\infty}) < \infty, \quad (13)
\]

and

\[
\sum_{\ell \in \mathbb{Z}^2} (L_\vartheta(u1_{C_\ell}) + L_\vartheta(v1_{C_\ell})) < \infty. \quad (14)
\]
Then
\[ \int_X u.v \circ f^n \, d\mu = \frac{\Phi_B(0)}{n} \int_X u \, d\mu \int_X v \, d\mu + o(n^{-1}). \quad (15) \]

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References

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