CONVERGENCE OF EXPONENTIAL ATTRACTORS FOR A TIME SEMI-DISCRETE REACTION-DIFFUSION EQUATION

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Abstract. We consider a time semi-discretization of a generalized Allen-Cahn equation with time-step parameter τ. For every τ, we build an exponential attractor $\mathcal{M}_\tau$ of the discrete-in-time dynamical system. We prove that $\mathcal{M}_\tau$ converges to an exponential attractor $\mathcal{M}_0$ of the continuous-in-time dynamical system for the symmetric Hausdorff distance as τ tends to 0. We also provide an explicit estimate of this distance and we prove that the fractal dimension of $\mathcal{M}_\tau$ is bounded by a constant independent of τ. Our construction is based on the result of Efendiev, Miranville and Zelik concerning the continuity of exponential attractors under perturbation of the underlying semi-group. Their result has been applied in many situations, but here, for the first time, the perturbation is a discretization. Our method is applicable to a large class of dissipative problems.

Keywords: Allen-Cahn equation, backward Euler scheme, global attractor, exponential attractor.

1. Introduction

Understanding and predicting the asymptotic behaviour of systems arising from mechanics and physics is a fundamental issue. A key concept in the study of dissipative systems is the *global attractor*, a compact invariant set which attracts uniformly the bounded sets of the phase space (see [3, 19, 21, 25, 30] for reviews on this subject).

One major drawback of the global attractor is that the rate of attraction of the trajectories may be small, and consequently, it may be sensible to perturbations. In fact, global attractors are generally upper semi-continuous with respect to perturbations, but the lower semi-continuity property can be proved only in some particular cases (see e.g. [3, 24, 25, 30]). This includes of course perturbations which are obtained by time and/or space discretization of the governing equations [7, 29, 31, 32].

The notion of *exponential attractor* has been proposed in [8]: it is a compact positively invariant set which contains the global attractor, has finite fractal dimension and attracts exponentially the trajectories. Compared with the global attractor, an exponential attractor is expected to be more robust to perturbations; it can also capture important transient behaviours. We note however that, contrary to the global attractor, an exponential attractor is not necessarily unique, so that its construction relies upon an algorithm.

The continuity of exponential attractors was shown in [8] for classical Galerkin approximations, but only up to a time shift (see also [14, 16]). In [11], Efendiev, Miranville and Zelik proposed a construction of exponential attractors where the continuity holds without time shift (see also [12]). Moreover, the symmetric Hausdorff distance between the perturbed attractor and the unperturbed attractor is estimated. Their construction, which is based on a uniform "smoothing property" and an appropriate error estimate, is valid in Banach spaces; it has been adapted to many situations, including singular perturbations (see [15, 17, 24] and references therein). But, up to now, perturbations due to time or space discretization have not been considered (see however [1] for related robustness results for Galerkin approximations)

The purpose of this paper is to address the case of a time discretization on a model problem. The natural perturbation parameter is the time step $\tau > 0$; $\tau = 0$ corresponds to the continuous-in-time system. For every $\tau \geq 0$ small enough, we obtain an exponential attractor $\mathcal{M}_\tau$ for the corresponding system, and we prove that $\mathcal{M}_\tau$ converges to $\mathcal{M}_0$ for the symmetric Hausdorff distance, with an explicit estimate of this distance. Moreover, the fractal dimension of $\mathcal{M}_\tau$ is bounded by a constant, and $\mathcal{M}_\tau$ attracts the bounded sets of the phase space, uniformly with respect to τ.

We first state the continuity result in an abstract form, in a Banach setting (Theorem 2.5). The assumptions needed here may seem rather lengthy, but they should be interpreted as a methodology (see Remark 2.6 for details). The essential tool is the construction in [11].

The abstract result is applied to a dissipative reaction-diffusion equation with a polynomial nonlinearity on a bounded domain (see (3.1)). This example includes the famous Allen-Cahn [2] in any space dimension (also known as the Chafee-Infante equation [5]). The time semi-discrete problem is provided by the backward Euler scheme (see (4.1)).

For the continuous problem, existence of a global attractor which has finite fractal dimension is well-known (see e.g. [6, 20, 23, 30]), and exponential attractors have been constructed in [8, 9]. A fully discrete approximation of the problem was considered in [27], where the upper semi-continuity of the global attractor with respect to the discretization parameters was shown in one, two and three space dimension, with growth restriction on the nonlinearity in the latter case.

The proof is organized as follows. The estimates for the continuous problem are derived in Section 3, their discrete counterparts are derived in Section 4, and the error between the continuous and the discrete solution is estimated in Section 5. The main result is summarized in Theorem 6.1. We point out (Corollary 6.2) that this result implies a uniform bound on the dimension of the global attractors, which is generally very difficult to obtain (see e.g. Remark 3 in [13]).

We expect our method to be applicable to a large class of parabolic problems or damped wave equations. However, some difficulties specific to the time discretization may arise. For instance, it is not clear how to deal with the two-dimensional Navier-Stokes equation discretized by the backward Euler scheme, since in this case, obtaining a good definition of the discrete-in-time semi-group is already a difficulty [7]. We also note that it would be very interesting to obtain a result similar to Theorem 6.1, but for a space discretization instead of a time discretization.

2. The abstract setting

Throughout Section 2, H denotes a Banach space with norm $\|\cdot\|_H$. We recall that a continuous-in-time semi-group $\{S(t),\ t \in \mathbb{R}_+\}$ on H is a family of (nonlinear) operators such that $S(t)$ is a continuous operator from H into itself, for all $t \geq 0$, with $S(0) = Id$ (identity in H) and

$$S(t+s) = S(t) \circ S(s), \quad \forall s, t \geq 0.$$

A discrete-in-time semi-group $\{S(t),\ t \in \mathbb{N}\}$ on H is a family of (nonlinear) operators which satisfy these properties with $\mathbb{R}_+(=[0,+\infty))$ replaced by $\mathbb{N}$. A discrete-in-time semi-group will usually be denoted $\{S^n,\ n \in \mathbb{N}\}$, where $S(=S(1))$ is a continuous (nonlinear) operator from H into itself.

A (continuous or discrete) semi-group $\{S(t),\ t \geq 0\}$ defines a (continuous or discrete) dynamical system: if u_0 is the state of the dynamical system at time 0, then $u(t) = S(t)u_0$ is the state at time $t \geq 0$. The term "dynamical system" will sometimes be used instead of "semi-group".

We note that in the definitions above, the Banach space H can be replaced by a metric space (for instance a bounded subspace of H).

Definition 2.1 (Global attractor)**.** Let $\{S(t),\ t \geq 0\}$ be a continuous or discrete semi-group on H. A set $\mathcal{A} \subset H$ is called the global attractor of the dynamical system if the following three conditions are satisfied:

(1) $\mathcal{A}$ is compact in H;
(2) $\mathcal{A}$ is invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;
(3) $\mathcal{A}$ attracts all bounded sets in H, i.e., for every bounded set B in H,

$$\lim_{t \to +\infty} dist_H(S(t)B, \mathcal{A}) = 0.$$

Here, $dist_H$ denotes the non-symmetric Hausdorff semi-distance in H between two subsets, which is defined as

$$dist_H(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_H.$$

It is easy to see, thanks to the invariance and the attracting property, that the global attractor, when it exists, is unique [30].

Let $A \subset H$ be a subset of H. For $\varepsilon > 0$, we denote $N_\varepsilon(A, H)$ the minimum number of balls of H of radius $\varepsilon > 0$ which are necessary to cover A. The *fractal dimension of* A (see e.g. [8, 30]) is the number

$$d_F(A) = \limsup_{\varepsilon \to 0} \frac{\log(N_\varepsilon(A, H))}{\log(1/\varepsilon)}.$$

Definition 2.2 (Exponential attractor)**.** Let $\{S(t),\ t \geq 0\}$ be a continuous or discrete semi-group on H. A set $\mathcal{M} \subset H$ is an exponential attractor of the dynamical system if the following three conditions are satisfied:

(1) $\mathcal{M}$ is compact in H and has finite fractal dimension;
(2) $\mathcal{M}$ is positively invariant, i.e. $S(t)\mathcal{M} \subset \mathcal{M}$, for all $t \geq 0$;
(3) $\mathcal{M}$ attracts exponentially the bounded subsets of H in the following sense:

$$\forall B \subset H \text{ bounded}, \quad dist_H(S(t)B, \mathcal{M}) \leq \mathcal{Q}(\|B\|_H)e^{-\alpha t}, \quad t \geq 0,$$

where the positive constant α and the monotonic function $\mathcal{Q}$ are independent of B. Here, $\|B\|_H = \sup_{b \in B} \|b\|_H$.

The exponential attractor, if it exists, contains the global attractor (actually, the existence of an exponential attractor yields the existence of the global attractor, see [3, 24]).

Remark 2.3. If $\mathcal{B}$ is a closed bounded subset of H and if L is a (nonlinear) continuous operator from $\mathcal{B}$ into $\mathcal{B}$, we will say that a set $\mathcal{M}^d \subset \mathcal{B}$ is an *exponential attractor* for (the dynamical system generated by) the iterations of L if (1) $\mathcal{M}^d$ is compact and has finite fractal dimension in H, (2) $\mathcal{M}^d$ is positively invariant, i.e. $L\mathcal{M}^d \subset \mathcal{M}^d$, and (3) $\mathcal{M}^d$ attracts $\mathcal{B}$ exponentially, i.e.

$$dist_H(L^n\mathcal{B}, \mathcal{M}^d) \leq Ce^{-\alpha n}, \quad n \in \mathbb{N},$$

where C and $\alpha > 0$ are independent of n.

We first give conditions which ensure the existence of an exponential attractor for a continuous semi-group. These conditions could be weakened [24], but here, we have in mind our perturbation result (see Remark 2.6).

Theorem 2.4. *Let H and V be two Banach spaces such that V is compactly embedded into H. Let $\{S_0(t),\ t \in \mathbb{R}_+\}$ be a continuous semi-group on H. Suppose that the following four conditions are satisfied.*

H1 (Bounded absorbing set): *There exists a bounded set $\mathcal{B}$ in H such that*

$$\forall B \subset H \text{ bounded},\ \exists\ t(B) \geq 0,\ (t \geq t(B) \Rightarrow S_0(t)B \subset \mathcal{B}).$$

H2 (Smoothing property): *For all $T > 0$, for all $u_1, u_2 \in \mathcal{B}$,*

$$\|S_0(T)u_1 - S_0(T)u_2\|_V \leq c_1(T)\|u_1 - u_2\|_H,$$

where the function $c_1 : (0, +\infty) \to (0, +\infty)$ is continuous.

H3 (Hölder continuity in time): *For all $T > 0$, there exist $\beta = \beta(T) \in (0, 1]$ and a constant $c_2(T)$ such that for all $t_1, t_2 \in [0, T]$, for all $u \in \mathcal{B}$,*

$$\|S_0(t_1)u - S_0(t_2)u\|_H \leq c_2(T)|t_1 - t_2|^\beta. \tag{2.1}$$

H4 (Lipschitz continuity on bounded sets): *For all $T > 0$ and for all $B \subset H$ bounded, there exists $c_3(T, B)$ such that for all $t \in [0, T]$, for all $u_1, u_2 \in B$,*

$$\|S_0(t)u_1 - S_0(t)u_2\|_H \leq c_3(T, B)\|u_1 - u_2\|_H. \tag{2.2}$$

Then the continuous dynamical system $\{S_0(t),\ t \in \mathbb{R}_+\}$ possesses an exponential attractor $\mathcal{M}_0$.

Proof. We may assume, without loss of generality, that $\mathcal{B}$ is closed in H (otherwise, replace $\mathcal{B}$ by its closure in H). We choose $T_0 > t(\mathcal{B})$ (cf. assumption (H1)) and consider the continuous mapping $L_0 = S_0(T_0) : \mathcal{B} \to \mathcal{B}$. This mapping L_0 enjoys the smoothing property (assumption (H2) for $T = T_0$). By Proposition 1 in [10], the dynamical system generated by iterations of L_0 possesses an exponential attractor (cf. Remark 2.3) which attracts $\mathcal{B}$ exponentially, i.e.

$$dist_H(L_0^n\mathcal{B}, \mathcal{M}_0^d) \leq Ce^{-\alpha n},\ n \in \mathbb{N},$$

where C and $\alpha > 0$ only depend on $\mathcal{B}$. Next, we set

$$\mathcal{M}_0 = \bigcup_{t \in [0, T_0]} S_0(t)\mathcal{M}_0^d. \tag{2.3}$$

Assumptions (H3) and (H4) imply that the function $F(t, u) = S_0(t)u$ is β-Hölder continuous for the time variable and Lipschitz continuous for the phase variable on $[0, T_0] \times \mathcal{B}$. Since $\mathcal{M}_0 = F([0, T_0] \times \mathcal{M}_0^d)$, this shows that $\mathcal{M}_0$ is compact and has finite fractal dimension, with

$$dim_F(\mathcal{M}_0) \leq \frac{1}{\beta} + dim_F(\mathcal{M}_0^d). \tag{2.4}$$

A standard argument [8] shows that $\mathcal{M}_0$ is positively invariant, i.e. $S_0(t)\mathcal{M}_0 \subset \mathcal{M}_0$ for all $t \geq 0$, and that $\mathcal{M}_0$ attracts $\mathcal{B}$ exponentially, i.e.

$$dist_H(S_0(t)\mathcal{B}, \mathcal{M}_0) \leq C' e^{-\alpha' t},$$

where we can choose $C' = c_3(T_0, \mathcal{B}) C e^{\alpha}$ and $\alpha' = \alpha / T_0$.

For $R > 0$, consider now the closed ball $B_R = \{u \in H \ : \ \|u\|_H \leq R\}$. Since $\mathcal{B}$ is an absorbing set (assumption (H1)), there exists $t_R = t(B_R)$ such that $t \geq t_R$ implies $S_0(t)B_R \subset \mathcal{B}$. Thus, for $t \geq t_R$, we have

$$\begin{aligned} dist_H(S_0(t)B_R, \mathcal{M}_0) &= dist_H(S_0(t-t_R)S_0(t_R)B_R, \mathcal{M}_0) \\ &\leq C' e^{-\alpha'(t-t_R)}. \end{aligned} \tag{2.5}$$

For $t \in [0, t_R]$, we have

$$dist_H(S_0(t)B_R, \mathcal{M}_0) \leq dist_H\Big(\bigcup_{s \in [0,t_R]} S_0(s)B_R, \mathcal{M}_0\Big) =: Q_R, \tag{2.6}$$

with $Q_R < +\infty$ because the set $\bigcup_{s\in[0,t_R]} S_0(s)B_R$ is bounded, as well as $\mathcal{M}_0$. Indeed, using (H3) and (H4), we see that if $s \in [0, t_R]$ and $u \in B_R$, by choosing $u_1 \in \mathcal{B}$, we have

$$\begin{aligned} \|S_0(s)u\|_H &\leq \|S_0(s)u - S_0(s)u_1\|_H + \|S_0(s)u_1 - S_0(0)u_1\|_H + \|u_1\|_H \\ &\leq c_3(t_R, B_R \cup \mathcal{B})(R + \|\mathcal{B}\|_H) + c_2(t_R) t_R^{\beta} + \|\mathcal{B}\|_H. \end{aligned}$$

By setting $\mathcal{Q}(R) = e^{\alpha' t_R} \max(C', Q_R)$, from (2.5) and (2.6), we deduce

$$dist_H(S_0(t)B_R, \mathcal{M}_0) \leq \mathcal{Q}(R) e^{-\alpha' t}, \quad t \geq 0,$$

where $\alpha' > 0$ and the function $\mathcal{Q}$ are independent of R. We note that $\mathcal{Q}$ can easily be changed into a monotonic function by using that $B_R \subset B_{R'}$ if $R < R'$. This finishes the proof. □

We now state our perturbation result in a Banach setting. For a possible comparison with concrete situations, we note that the time t and the time step τ may have a unit, but the elements of H and the norms in H and V are dimensionless.

Theorem 2.5. *Let H and V be two Banach spaces such that V is compactly embedded into H. Let $\{S_0(t),\ t \in \mathbb{R}_+\}$ be a continuous semi-group on H which satisfies assumptions (H1)-(H4) from Theorem 2.4, and let $\{S_\tau^n,\ n \in \mathbb{N}\}$, $\tau \in (0, \tau_0]$ ($\tau_0 > 0$) be a family of discrete semi-groups on H. Suppose that the following five additional conditions are satisfied.*

- **H5 (Bounded absorbing set independent of τ):** *The bounded set $\mathcal{B} \subset H$ from assumption (H1) can be chosen such that, for all $B \subset H$ bounded, there is a time $t(B) \geq 0$ such that, for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}$ which satisfy $n\tau \geq t(B)$, we have $S_\tau^n B \subset \mathcal{B}$.*
- **H6 (Smoothing property, uniform with τ):** *For all $T \geq \tau_0$, there exist $\bar{\tau}_0(T) \in (0, \tau_0]$ and a constant $c_4(T)$ such that for all $\tau \in (0, \bar{\tau}_0(T)]$, for all $u_1, u_2 \in \mathcal{B}$,*

  $$\|S_\tau^{[T/\tau]} u_1 - S_\tau^{[T/\tau]} u_2\|_V \leq c_4(T) \|u_1 - u_2\|_H.$$

  Here, $[\cdot]$ is the floor function, i.e. for every real number x, $[x]$ is the largest integer less than or equal to x.

H7 (Finite time uniform error estimate): *For all $T > 0$, there exist $\gamma = \gamma(T) \in (0, 1]$ and a constant $c_5(T)$ such that, for all $\tau \in (0, \tau_0]$,*

$$\sup_{u \in \mathcal{B}, 0 \leq n\tau \leq T} \|S_\tau^n u - S_0(n\tau)u\|_H \leq c_5(T)\tau^\gamma.$$

H8 (Lipschitz continuity on bounded sets, uniform with τ): *For every time $T > 0$ and for all $B \subset H$ bounded, there exists $c_6(T, B)$ such that for all $\tau \in (0, \tau_0]$, for all $0 \leq n\tau \leq T$, for all $u_1, u_2 \in B$,*

$$\|S_\tau^n u_1 - S_\tau^n u_2\|_H \leq c_6(T, B)\|u_1 - u_2\|_H. \tag{2.7}$$

H9 (Bound on bounded sets, uniform with τ): *For all $T > 0$ and for all $B \subset H$ bounded, there exists $c_7(T, B)$ such that for all $\tau \in (0, \tau_0]$, for all $0 \leq n\tau \leq T$, for all $u \in B$,*

$$\|S_\tau^n u\|_H \leq c_7(T, B). \tag{2.8}$$

Then, for every $\tau \in (0, \tau_0']$, $\tau_0' > 0$ small enough, the discrete dynamical system associated to $\{S_\tau^n,\ n \in \mathbb{N}\}$ possesses an exponential attractor $\mathcal{M}_\tau$ on H, and the continuous dynamical system $\{S_0(t),\ t \in \mathbb{R}_+\}$ possesses an exponential attractor $\mathcal{M}_0$ such that:

(1) *the fractal dimension of $\mathcal{M}_\tau$ is bounded, uniformly with respect to $\tau \in [0, \tau_0']$,*

$$dim_F \mathcal{M}_\tau \leq c_8,$$

where c_8 is independent of τ;

(2) *$\mathcal{M}_\tau$ attracts the bounded sets of H, uniformly with respect to $\tau \in (0, \tau_0']$, i.e.*

$$\forall \tau \in (0, \tau_0'],\ \forall B \subset H \text{ bounded}, \quad dist_H(S_\tau^n B, \mathcal{M}_\tau) \leq \mathcal{Q}(\|B\|_H)e^{-c_9 n\tau},\ n \in \mathbb{N},$$

where the constant c_9 and the monotonic function $\mathcal{Q}$ are independent of τ;

(3) *the family $\{\mathcal{M}_\tau,\ \tau \in [0, \tau_0']\}$ is continuous at 0,*

$$dist_{sym}(\mathcal{M}_\tau, \mathcal{M}_0) \leq c_{10}\tau^{c_{11}},$$

where c_{10} and $c_{11} \in (0, 1)$ are independent of τ and $dist_{sym}$ denotes the symmetric Hausdorff distance between sets, defined by

$$dist_{sym}(A, B) := \max\left(dist_H(A, B), dist_H(B, A)\right).$$

Remark 2.6. Assumptions (H5), (H6) and (H8) are the discrete counterparts of assumptions (H1), (H2) and (H4) respectively (uniformly with respect to τ). Assumption (H9) is a weak discrete version of (H3). Assumption (H7) gives the relation between the continuous and the discrete semi-groups. We stress that the bounded set $\mathcal{B}$ is the same in (H1)-(H3) and (H5)-(H7).

Remark 2.7. 1. The constants c_i, $i = 8, \ldots, 11$ can be computed explicitly in terms of the physical parameters of the problem in concrete situations. In particular, from (2.4), (2.10), (2.13), and [10, 11] we find that, for all $\tau \in [0, \tau_0']$,

$$dim_F(\mathcal{M}_\tau) \leq \log_2[N_{1/(4c_4')}(B(0, 1; V), H)] + 1/\beta, \tag{2.9}$$

where $N_\varepsilon(B(0, 1; V), H)$ is the number of ball of radius ε in H which are necessary to cover the unit ball centered at 0 in V.
2. The continuity holds at $\tau = 0$ only.

Proof. As previously, we may assume that $\mathcal{B}$ is closed in H. We use assumption (H5) with $B = \mathcal{B}$, and we set $T_0 = t(\mathcal{B}) + \tau_0$. In particular, $S_0(T_0)\mathcal{B} \subset \mathcal{B}$. Moreover, for all $\tau \in (0, \tau_0]$, we have $[T_0/\tau]\tau \geq t(\mathcal{B})$, and so $S_\tau^{[T_0/\tau]}\mathcal{B} \subset \mathcal{B}$.

Let $\beta \in (0,1)$ be such that S_0 is β-Hölder continuous in time on $[0, T_0] \times \mathcal{B}$ (assumption (H3)). We may assume that assumption (H7) holds on $[0, T_0]$ with $\gamma \leq \beta$ (otherwise replace γ by β). We define $\tau_0' = \bar{\tau}_0(T_0) \in (0, \tau_0]$ (cf. (H6)). For $\tau \in (0, \tau_0']$, we set $\varepsilon = \varepsilon(\tau) = (\tau/\tau_0')^\gamma$ (or, equivalently, $\tau = \tau(\varepsilon) = \varepsilon^{1/\gamma}\tau_0'$), and we denote $L_\varepsilon = S_\tau^{[T_0/\tau]}$. We also set $L_0 = S_0(T_0)$. Note that ε, which is a renormalization of τ, belongs to $[0, 1]$. By assumptions (H2) and (H6), the family of operators $L_\varepsilon : \mathcal{B} \to \mathcal{B}$ satisfies a uniform smoothing property, i.e., for all $\varepsilon \in [0, 1]$, for all $u_1, u_2 \in \mathcal{B}$,

$$\|L_\varepsilon u_1 - L_\varepsilon u_2\|_V \leq c_4' \|u_1 - u_2\|_H, \tag{2.10}$$

where $c_4' = \max\{c_1(T_0), c_4(T_0)\}$ is independent of ε.

Let $\varepsilon \in (0, 1]$. For all $u \in \mathcal{B}$, we have, using the triangle inequality, assumptions (H7) and (H3),

$$\begin{aligned} \|L_\varepsilon u - L_0 u\|_H &\leq \|S_\tau^{[T_0/\tau]} u - S_0([T_0/\tau]\tau)u\|_H + \|S_0([T_0/\tau]\tau)u - S_0(T_0)u\|_H \\ &\leq c_5(T_0)\tau^\gamma + c_2(T_0)\tau^\beta, \end{aligned}$$

and so $\|L_\varepsilon u - L_0 u\|_H \leq c_5' \varepsilon$ where $c_5' = c_5(T_0)\tau_0'^\gamma + c_2(T_0)\tau_0'^\beta$ is independent of ε. Assumption (H8) shows that for all $u_1, u_2 \in \mathcal{B}$,

$$\|L_\varepsilon u_1 - L_\varepsilon u_2\|_H \leq c_6(T_0, \mathcal{B}) \|u_1 - u_2\|_H.$$

By induction, we find that for every integer $i \geq 1$, for all $u \in \mathcal{B}$,

$$\|L_\varepsilon^i u - L_0^i u\|_H \leq \left(\sum_{j=0}^{i-1} c_6(T_0, \mathcal{B})^j \right) c_5' \varepsilon. \tag{2.11}$$

Indeed, we have seen that the estimate (2.11) is satisfied for $i = 1$. Assume that it is satisfied for $i \geq 1$. Then

$$\begin{aligned} \|L_\varepsilon^{i+1} u - L_0^{i+1} u\|_H &\leq \|L_\varepsilon(L_\varepsilon^i u) - L_\varepsilon(L_0^i u)\|_H + \|L_\varepsilon(L_0^i u) - L_0(L_0^i u)\|_H \\ &\leq c_6(T_0, \mathcal{B}) \|L_\varepsilon^i u - L_0^i u\|_H + c_5' \varepsilon, \end{aligned}$$

and using the induction assumption (2.11) at step i, we find that (2.11) is satisfied at step $i + 1$. Estimate (2.11) implies that, for some constant $C = C(c_5', c_6(T_0, \mathcal{B}))$ independent of ε, we have, for all $\varepsilon \in [0, 1]$, for all $i \in \mathbb{N}$, for all $u \in \mathcal{B}$,

$$\|L_\varepsilon^i u - L_0^i u\|_H \leq C^i \varepsilon.$$

We may therefore apply the abstract result concerning the existence and convergence of exponential attractors, Theorem 4.4 in [11]. Namely, for every $\varepsilon \in [0, 1]$, there exists a set $\mathcal{M}_\varepsilon^d \subset \mathcal{B}$ which is an attractor for the iterations of L_ε (cf. Remark 2.3). Moreover,

(1) the fractal dimension of $\mathcal{M}_\varepsilon^d$ is bounded, uniformly with respect to ε,

$$dim_F \mathcal{M}_\varepsilon^d \leq C_1,$$

where C_1 is independent of ε;

(2) $\mathcal{M}_\varepsilon^d$ attracts $\mathcal{B}$ uniformly with respect to ε,
$$dist_H(L_\varepsilon^k\mathcal{B},\mathcal{M}_\varepsilon^d)\le C_2e^{-C_3k},\quad C_2>0,\ k\in\mathbb{N},$$
where C_2 and C_3 are independent of ε;

(3) the family $\{\mathcal{M}_\varepsilon^d,\ \varepsilon\in[0,1]\}$ is continuous at 0,
$$dist_{sym}(\mathcal{M}_\varepsilon^d,\mathcal{M}_0^d)\le C_4\varepsilon^{C_5},$$
where C_4 and $C_5\in(0,1)$ are independent of ε.

Now, for $\tau\in(0,\tau_0']$, we set
$$\mathcal{M}_\tau=\bigcup_{0\le n\tau\le T_0}S_\tau^n\mathcal{M}_{\varepsilon(\tau)}^d. \tag{2.12}$$
The attractor $\mathcal{M}_0$ is defined as previously by (2.3) (with T_0 and $\mathcal{M}_0^d$ as above). By arguing as in the continuous case (Theorem 2.4), we see that for every $\tau\in(0,\tau_0']$, $\mathcal{M}_\tau$ is an exponential attractor for the semi-group $\{S_\tau^n\ :\ n\in\mathbb{N}\}$. Indeed, for all n, S_τ^n is Lipschitz continuous on $\mathcal{B}$ (assumption (H8)), so $\mathcal{M}_\tau$ is compact and its fractal dimension satisfies
$$dim_F(\mathcal{M}_\tau)\le dim_F(\mathcal{M}_{\varepsilon(\tau)}^d)\le C_1. \tag{2.13}$$
Moreover, using $L_\varepsilon\mathcal{M}_\varepsilon^d=S_\tau^{[T_0/\tau]}\mathcal{M}_{\varepsilon(\tau)}^d\subset\mathcal{M}_{\varepsilon(\tau)}^d$, we see that
$$S_\tau\mathcal{M}_\tau=\bigcup_{1\le n\le[T_0/\tau]}S_\tau^n\mathcal{M}_{\varepsilon(\tau)}^d\cup S_\tau^{[T_0/\tau]+1}\mathcal{M}_{\varepsilon(\tau)}^d\subset\mathcal{M}_\tau.$$
Let us show that $\mathcal{M}_\tau$ attracts $\mathcal{B}$. For $n\in\mathbb{N}$, we write $n=k[T_0/\tau]+r$ with $k\in\mathbb{N}$ and $r\in\{0,\dots,[T_0/\tau]-1\}$. We have
$$dist_H(S_\tau^n\mathcal{B},\mathcal{M}_\tau)=dist_H(S_\tau^rL_{\varepsilon(\tau)}^k\mathcal{B},\mathcal{M}_\tau)\le dist_H(S_\tau^rL_{\varepsilon(\tau)}^k\mathcal{B},S_\tau^r\mathcal{M}_{\varepsilon(\tau)}^d).$$
Next, we use that S_τ^r is Lipschitz continuous on $\mathcal{B}$ (cf. (2.7)), and we obtain
$$dist_H(S_\tau^rL_{\varepsilon(\tau)}^k\mathcal{B},S_\tau^r\mathcal{M}_{\varepsilon(\tau)}^d)\le c_6(T_0,\mathcal{B})dist_H(L_{\varepsilon(\tau)}^k\mathcal{B},\mathcal{M}_{\varepsilon(\tau)}^d)\le c_6(T_0,\mathcal{B})C_2e^{-C_3k}.$$
Using $k>n/[T_0/\tau]-1\ge n\tau/T_0-1$, we note that $e^{-C_3k}\le e^{C_3}e^{-C_3n\tau/T_0}$, and we obtain
$$dist_H(S_\tau^n\mathcal{B},\mathcal{M}_\tau)\le C_2'e^{-C_3'n\tau},$$
where $C_2'=c_6(T_0,\mathcal{B})C_2e^{C_3}$ and $C_3'=C_3/T_0$ do not depend on τ.

For $R>0$, we consider now the closed ball $B_R=\{u\in H\ :\ \|u\|_H\le R\}$. Since $\mathcal{B}$ is an absorbing set (assumption (H5)), there exists $t_R=t(B_R)\ge\tau_0'$ such that for all $\tau\in(0,\tau_0']$, $n\tau\ge t_R-\tau_0'$ implies $S_\tau^nB_R\subset\mathcal{B}$. Let $\tau\in(0,\tau_0']$. Then, for $n\ge[t_R/\tau]$, since $S^{[t_R/\tau]}B_R\subset\mathcal{B}$, we can write
$$\begin{aligned}dist_H(S_\tau^nB_R,\mathcal{M}_\tau)&=dist_H(S_\tau^{n-[t_R/\tau]}S^{[t_R/\tau]}B_R,\mathcal{M}_\tau)\\&\le C_2'e^{-C_3'(n-[t_R/\tau])\tau}.\end{aligned} \tag{2.14}$$
Let now $n\in\{0,\dots,[t_R/\tau]\}$. Using (2.8), we see that for all $u\in B_R$,
$$\|S_\tau^nu\|_H\le c_7(t_R,B_R).$$
which shows that $S_\tau^nB_R\subset B_{g(R)}$ where $g(R)=c_7(t_R,B_R)$ is independent of τ. Using also $\mathcal{M}_\tau\supset\mathcal{M}_{\varepsilon(\tau)}^d$, we see that
$$dist_H(S_\tau^nB_R,\mathcal{M}_\tau)\le dist_H(B_{g(R)},\mathcal{M}_{\varepsilon(\tau)}^d). \tag{2.15}$$

Since $\mathcal{M}_{\varepsilon(\tau)} \subset \mathcal{B}$, by the triangle inequality, we have

$$dist_H(B_{g(R)}, \mathcal{M}^d_{\varepsilon(\tau)}) \leq g(R) + \|\mathcal{B}\|_H =: Q_R. \tag{2.16}$$

By setting $\mathcal{Q}(R) = e^{C_3' t_R} \max\{C_2', Q_R\}$, from (2.14)-(2.16), we deduce

$$dist_H(S^n_\tau B_R, \mathcal{M}_\tau) \leq \mathcal{Q}(R) e^{-C_3' n\tau}, \quad n \in \mathbb{N},$$

where C_3' and the function $\mathcal{Q}$ are independent of τ. The function $\mathcal{Q}$ can easily be changed into a monotonic function by using that $B_R \subset B_{R'}$ if $R < R'$. This yields conclusion (2) of Theorem 2.5.

It remains to prove conclusion (3). Let $\tau \in (0, \tau_0']$. Using the definitions of $\mathcal{M}_0$, $\mathcal{M}_\tau$ and the triangle inequality, we see that

$$dist_{sym}(\mathcal{M}_\tau, \mathcal{M}_0) \leq h_1(\tau) + h_2(\tau) + h_3(\tau),$$

where

$$\begin{aligned} h_1(\tau) &= dist_{sym}\Big(\bigcup_{0\leq n\tau\leq T_0} S^n_\tau \mathcal{M}^d_{\varepsilon(\tau)}, \bigcup_{0\leq n\tau\leq T_0} S_0(n\tau)\mathcal{M}^d_{\varepsilon(\tau)}\Big), \\ h_2(\tau) &= dist_{sym}\Big(\bigcup_{0\leq n\tau\leq T_0} S_0(n\tau)\mathcal{M}^d_{\varepsilon(\tau)}, \bigcup_{0\leq n\tau\leq T_0} S_0(n\tau)\mathcal{M}^d_0\Big), \\ h_3(\tau) &= dist_{sym}\Big(\bigcup_{0\leq n\tau\leq T_0} S_0(n\tau)\mathcal{M}^d_0, \bigcup_{0\leq t\leq T_0} S(t)\mathcal{M}^d_0\Big). \end{aligned}$$

By assumption (H7), we have $h_1(\tau) \leq c_5(T_0)\tau^\gamma$. Using that $S_0(n\tau)$ is Lipschitz continuous on $\mathcal{B}$ (assumption (H3)), we have

$$h_2(\tau) \leq c_3(T_0, \mathcal{B}) dist_{sym}(\mathcal{M}^d_{\varepsilon(\tau)}, \mathcal{M}^d_0) \leq c_3(T_0, \mathcal{B}) C_4 (\tau/\tau_0')^{C_5\gamma}.$$

Using that S_0 is β-Hölder continuous in time (assumption (H3)), we find that

$$h_3(\tau) \leq c_2(T_0)\tau^\beta.$$

Summing up, we have proved that

$$dist_{sym}(\mathcal{M}_\tau, \mathcal{M}_0) \leq c_{10}\tau^{c_{11}},$$

where c_{10} and $c_{11} = C_5\gamma \in (0,1)$ are independent of τ. This concludes the proof. ☐

3. The continuous problem

3.1. **The continuous semi-group S_0.**

We consider the following reaction-diffusion equation

$$\partial_t u - d\Delta u + g(u) = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \tag{3.1}$$

subject to homogeneous Dirichlet boundary conditions. Here, Ω is an open bounded subset of $\mathbb{R}^I$ ($I \geq 1$) with sufficiently smooth boundary, $d > 0$ is given, and g is a polynomial of odd degree with a positive leading coefficient,

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_{2p-1} > 0, \ p \geq 1.$$

Note that equation (3.1) is linear if and only if $p = 1$. When $g(s) = s^3 - s$ (in which case $p = 2$), equation (3.1) is known as the *Allen-Cahn equation.*

We supplement (3.1) with an initial condition

$$u(0) = u_0. \tag{3.2}$$

We set $H = L^2(\Omega)$ with norm $|\cdot|_H$ and scalar product $(\cdot,\cdot)_H$. We denote $V = H_0^1(\Omega)$ with norm $\|\cdot\|_V = |\nabla \cdot|_H$. For an nonempty interval I of $\mathbb{R}$ and for a Banach space E, we denote $C^0(I;E)$ the space of functions which are continuous on I with values in E; for $q \geq 1$, $L^q(I;E)$ is the usual Banach space of (classes of) functions endowed with the norm $v \mapsto (\int_I \|v(t)\|_E^q dt)^{1/q}$. The norm in $L^q(\Omega)$ is denoted $\|\cdot\|_{L^q}$.

The following existence and uniqueness result is well-known (see e.g. [22, 30]).

Theorem 3.1. *For $u_0 \in H$, there exists a unique solution u of* (3.1)-(3.2) *which satisfies $u \in C^0(\mathbb{R}_+;H)$ and $u \in L^2(0,T;V) \cap L^{2p}(0,T;L^{2p}(\Omega))$, for all $T > 0$. For all $t \geq 0$, the mapping $u_0 \mapsto u(t)$ is continuous in H. If, furthermore, $u_0 \in V$, then u belongs to $C^0([0,T];V) \cap L^2(0,T;H^2(\Omega))$, for all $T > 0$.*

This theorem is sufficient to define the continuous-in-time semi-group S_0:

$$S_0(t) : u_0 \in H \mapsto u(t) \in H.$$

3.2. **Some useful inequalities.** First recall the Poincaré inequality: there exists a constant $c_0 = c_0(I,\Omega)$ such that

$$|w|_H \leq c_0 \|w\|_V, \quad \forall w \in V. \tag{3.3}$$

More generally, thanks to the Sobolev imbeddings (see e.g. [18]), there is a constant $C_S(I,\Omega,q)$ such that

$$\|w\|_{L^q} \leq C_S(I,\Omega,q)\|w\|_V, \quad \forall w \in V, \tag{3.4}$$

for every $q \in [1,+\infty)$ if $I = 1$ or $I = 2$ and for every $q \in [1, 2I/(I-2)]$ if $I \geq 3$.

In our a priori estimates, we will consider two cases:

Case 1: $I \in \{1,2\}$ (no restriction on p), or $I = 3$ and $p \in \{1,2\}$, or $I \geq 4$ and $p = 1$;

Case 2: $I = 3$ and $p \geq 3$, or $I \geq 4$ and $p \geq 2$.

If Case 1 holds, then we have the continuous imbeddings

$$V \subset L^{4p-2}(\Omega) \subset L^{2p}(\Omega).$$

In this case, a bounded absorbing set in V will be sufficient. In Case 2, we will consider a bounded absorbing set in $V \cap L^q(\Omega)$ for q large enough, so that we will need estimates both in V and in $L^q(\Omega)$. We note that Case 1 includes the Allen-Cahn equation in space dimension 1, 2 and 3 ($g(s) = s^3 - s$ with $p = 2$).

Next, we collect a few inequalities related to g. Since $\sum_{j=1}^{2p-2} jb_j s^{j-1}$ is a polynomial of degree less than s^{2p-2}, there exists a constant $c_1' > 0$ such that

$$\left|\sum_{j=1}^{2p-2} jb_j s^{j-1}\right| \leq \frac{1}{2}(2p-1)b_{2p-1}s^{2p-2} + c_1', \quad \forall s \in \mathbb{R}.$$

Thus, $g'(s) = \sum_{j=1}^{2p-1} jb_j s^{j-1}$ satisfies

$$|g'(s)| \leq \frac{3}{2}(2p-1)b_{2p-1}s^{2p-2} + c_1', \quad \forall s \in \mathbb{R}, \tag{3.5}$$

and

$$\frac{2p-1}{2}b_{2p-1}s^{2p-2} - c_1' \leq g'(s) \leq \frac{3}{2}(2p-1)b_{2p-1}s^{2p-2} + c_1', \quad \forall s \in \mathbb{R}. \tag{3.6}$$

We note that by the mean value theorem we have, for all $s_1, s_2 \in \mathbb{R}$,

$$(g(s_1) - g(s_2))(s_1 - s_2) = g'(\xi_{s_1,s_2})(s_1 - s_2)^2 \geq -c_1'(s_1 - s_2)^2, \tag{3.7}$$

for some $\xi_{s_1,s_2} \in \mathbb{R}$. Let

$$G(s) = \int_0^s g(\sigma)d\sigma = \sum_{j=0}^{2p-1} b_j s^{j+1}/(j+1) \tag{3.8}$$

denote an anti-derivative of g. Using a similar argument, we have

$$\frac{1}{4p} b_{2p-1} s^{2p} - \hat{c}_1' \leq G(s) \leq \frac{3}{4p} b_{2p-1} s^{2p} + \hat{c}_1', \quad \forall s \in \mathbb{R}, \tag{3.9}$$

for some constant $\hat{c}_1' > 0$.

By a similar argument, for every $q \geq 2$, there exists a constant $c_q' > 0$ such that

$$\frac{1}{2} b_{2p-1} |s|^{2p+q-2} - c_q' \leq g(s)|s|^{q-2} s \leq \frac{3}{2} b_{2p-1} |s|^{2p+q-2} + c_q', \quad \forall s \in \mathbb{R}. \tag{3.10}$$

Lemma 3.2. *Let $w_1, w_2 \in V$ and $w_3 \in H$.*

If Case 1 holds and $\|w_i\|_V \leq R_1$ ($i = 1, 2$), then

$$\int_\Omega |g(w_1) - g(w_2)||w_3| dx \leq h_1(R_1) \|w_1 - w_2\|_V |w_3|_H, \tag{3.11}$$

where $h_1(R_1) = h_1(R_1, I, \Omega, p, b_{2p-1}, c_1')$ is monotonic in R_1.

If Case 2 holds and $\|w_i\|_{L^{I(2p-2)}} \leq R_2$ ($i = 1, 2$), then

$$\int_\Omega |g(w_1) - g(w_2)||w_3| dx \leq h_2(R_2) \|w_1 - w_2\|_V |w_3|_H, \tag{3.12}$$

where $h_2(R_2) = h_2(R_2, I, \Omega, p, b_{2p-1}, c_1')$ is monotonic in R_2.

Proof. First note that for all $s_1, s_2 \in \mathbb{R}$, we have

$$g(s_1) - g(s_2) = \int_0^1 g'(\sigma s_1 + (1-\sigma)s_2)(s_1 - s_2) d\sigma.$$

Using (3.5) and the convexity of the function $s \mapsto |s|^{2p-2}$, we find

$$|g(s_1) - g(s_2)| \leq \left[b_{2p-1}' \left(|s_1|^{2p-2} + |s_2|^{2p-2} \right) + c_1' \right] |s_1 - s_2|, \quad \forall s_1, s_2 \in \mathbb{R}, \tag{3.13}$$

where $b_{2p-1}' = \frac{3}{4}(2p-1) b_{2p-1}$.

Let $w_1, w_2 \in V$ and $w_3 \in H$. By (3.13), we have

$$\int_\Omega |g(w_1) - g(w_2)||w_3| dx \leq \int_\Omega [b_{2p-1}'(|w_1|^{2p-2} + |w_2|^{2p-2}) + c_1'] |w_1 - w_2||w_3| dx. \tag{3.14}$$

Assume first that Case 1 holds and that $\|w_i\|_V \leq R_1$ ($i = 1, 2$). If $p = 1$ (the linear case), then we may use the Cauchy-Schwarz inequality and the Poincaré inequality (3.3), and we find (3.11) with $h_1 = (2b_{2p-1}' + c_1')c_0$ (here, h_1 does not depend on R_1). If $p \geq 2$, we use that $V \subset L^{4p-2}(\Omega)$, i.e. (3.4) holds with $q = 4p-2$ and $\hat{C}_S = C_S(I, \Omega, 4p-2)$. In (3.14), we use Hölder's inequality with $|w_i|^{2p-2} \in L^{(4p-2)/(2p-2)}(\Omega)$ ($i = 1, 2$), $w_1 - w_2 \in L^{4p-2}(\Omega)$ and $w_3 \in L^2(\Omega)$. We obtain (3.11) with

$$h_1(R_1) = \left[2b_{2p-1}' \hat{C}_S^{2p-2} R_1^{2p-2} + c_1' |\Omega|^{(2p-2)/(4p-2)} \right] \hat{C}_S.$$

Assume now that Case 2 holds and that $\|w_i\|_{L^{I(2p-2)}} \leq R_2$ ($i = 1, 2$). We use that $V \subset L^{2^\star}(\Omega)$ with $2^\star = 2I/(I-2)$, i.e. (3.4) holds with $q = 2^\star$ and $C_S^\star = C_S(I, \Omega, 2^\star)$.

In (3.14), we use Hölder's inequality with $|w_i|^{2p-2} \in L^I(\Omega)$ ($i = 1, 2$), $w_1 - w_2 \in L^{2^\star}(\Omega)$, and $w_3 \in L^2(\Omega)$. We obtain (3.12) with

$$h_2(R_2) = \left[2b'_{2p-1}(C_S^\star)^{2p-2}R_2^2 + c'_1|\Omega|^{1/I}\right] C_S^\star.$$

This concludes the proof. $\square$

3.3. **A priori estimates for the solution.** In Section 3.3, u denotes a solution of (3.1)-(3.2). We denote $\rho_0^2 = c''_2 c_0^2/(2d)$, $\rho_0'^2 = 2\rho_0^2$ and, for every $q \geq 2$,

$$c''_q = qc'_q|\Omega| \quad \text{where} \quad |\Omega| = \int_\Omega 1dx. \tag{3.15}$$

Propositions 3.3 and 3.4 are proved in [30].

Proposition 3.3 (Absorbing set in H)**.** *If* $|u_0|_H \leq R$, *then*

$$|u(t)|_H \leq \rho'_0, \quad \forall t \geq t_0(R), \tag{3.16}$$

where

$$t_0(R) = \max\left\{\frac{c_0^2}{d}\log\left(\frac{R}{\rho_0}\right), 0\right\}. \tag{3.17}$$

In the remainder of the paper, $r > 0$ *denotes an arbitrary (but fixed) real number* (r has the same unit as t).

Proposition 3.4 (Absorbing set in V)**.** *If* $|u_0|_H \leq R$, *then*

$$\|u(t)\|_V \leq \rho_1, \quad t \geq t_1(R), \tag{3.18}$$

where

$$\rho_1^2 = \frac{\kappa}{r}\exp(2c'_1 r), \quad \kappa = \frac{1}{2d}(rc''_2 + \rho_0'^2), \quad t_1(R) = t_0(R) + r.$$

Lemma 3.5. *Let* $q \geq 2$. *The function* $y(t) = \int_\Omega |u(t)|^q dx$ *is locally integrable on* $(0, +\infty)$, *as well as* dy/dt. *Moreover,*

$$\frac{d}{dt}\int_\Omega |u|^q dx + \frac{q}{2}b_{2p-1}\int_\Omega |u|^{2p+q-2} \leq c''_q \text{ on } (0, +\infty). \tag{3.19}$$

In particular, if $u_0 \in L^q(\Omega)$, *then for all* $t \geq 0$, *we have*

$$\int_\Omega |u(t)|^q dx \leq \int_\Omega |u_0|^q + c''_q t. \tag{3.20}$$

Proof. By parabolic regularity [4, 26], $u \in C^0((0, +\infty); C_0(\Omega))$ (where $C_0(\Omega)$ is the space of continuous functions on $\overline{\Omega}$ which vanish on $\partial\Omega$) and $\partial_t u, \Delta u$ belong to $C^0((0, +\infty); L^2(\Omega))$. In particular, the functions y and dy/dt are locally integrable. We multiply (3.1) by $|u|^{q-2}u$ and integrate over Ω. We obtain, after some simple transformations,

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |u|^q dx + (q-1)d\int_\Omega |\nabla u|^2|u|^{q-2}dx + \int_\Omega g(u)|u|^{q-2}udx = 0.$$

Using (3.10), (3.15), we obtain (3.19). We infer (3.20) by integration on $[0, t]$. $\square$

For all $i \in \mathbb{N}$, we set $a_i = i(2p-2) + 2$. We introduce the sequence $(\rho'_i)_i$ defined recursively for $i \in \mathbb{N}$ by

$$\rho_{i+1}'^{a_{i+1}} = \frac{\kappa'_i}{r} + rc''_{a_{i+1}}, \quad \text{with} \quad \kappa'_i = \frac{2}{a_i b_{2p-1}}(rc''_{a_i} + \rho_i'^{a_i}).$$

Proposition 3.6 (Absorbing set in L^q)**.** *Let $i \in \mathbb{N}$. If $|u_0|_H \leq R$, then*

$$\|u(t)\|_{L^{a_i}} \leq \rho_i', \quad t \geq t_i(R), \tag{3.21}$$

where $t_i(R) = t_0(R) + ir$.

Proof. We proceed by induction. We first note that (3.21) is satisfied for $i = 0$ by (3.16). Let $i \geq 0$ and assume that (3.21) holds. We first apply (3.19) with $q = a_i$ and we integrate on $[t, t + r]$. This yields

$$\frac{a_i}{2} b_{2p-1} \int_t^{t+r} \|u(s)\|_{L^{a_{i+1}}}^{a_{i+1}} ds \leq r c_{a_i}'' + \|u(t)\|_{L^{a_i}}^{a_i}, \quad \forall t \geq 0.$$

Using the induction assumption (3.21), we see that

$$\int_t^{t+r} \|u(s)\|_{L^{a_{i+1}}}^{a_{i+1}} ds \leq \kappa_i', \quad \forall t \geq t_i(R). \tag{3.22}$$

Next, we apply (3.19) with $q = a_{i+1}$, and we use the uniform Gronwall lemma [30]. We obtain that (3.21) is satisfied at step $i + 1$. By induction, (3.21) is satisfied for every $i \in \mathbb{N}$. □

We introduce the monotonic function

$$Q_1(s) = \frac{3b_{2p-1}}{2dp} s^{2p} + \frac{4\hat{c}_1'|\Omega|}{d}. \tag{3.23}$$

Lemma 3.7. *If $u_0 \in V \cap L^{2p}(\Omega)$, then*

$$\|u(t)\|_V^2 + \frac{b_{2p-1}}{2dp} \|u(t)\|_{L^{2p}}^{2p} + \frac{2}{d} \int_0^t |\partial_t u|_H^2 ds \leq \|u_0\|_V^2 + Q_1(\|u_0\|_{L^{2p}}), \quad \forall t \geq 0, \tag{3.24}$$

where Q_1 is defined by (3.23). In particular, for all $t_1, t_2 \geq 0$, we have

$$|u(t_1) - u(t_2)|_H^2 \leq Q_2(\|u_0\|_V, \|u_0\|_{L^{2p}})|t_1 - t_2|, \tag{3.25}$$

for some function Q_2 which is monotonic in its arguments.

Proof. We multiply (3.1) by $\partial_t u$ and integrate over Ω. We obtain, after integration by parts,

$$|\partial_t u|_H^2 + \frac{d}{dt} \left(\frac{d}{2} \|u\|_V^2 + \int_\Omega G(u) dx \right) = 0. \tag{3.26}$$

Integrating on $[0, t]$ yields

$$\frac{d}{2} \|u(t)\|_V^2 + \int_\Omega G(u(t)) dx + \int_0^t |\partial_t u|_H^2 ds \leq \frac{d}{2} \|u_0\|_V^2 + \int_\Omega G(u_0) dx. \tag{3.27}$$

Estimate (3.24) follows from (3.9). Let now $t_2 \geq t_1 \geq 0$. Then

$$|u(t_2) - u(t_1)|_H^2 = \left| \int_{t_1}^{t_2} \partial_t u(s) ds \right|_H^2 \leq |t_2 - t_1| \int_{t_1}^{t_2} |\partial_t u(s)|_H^2 ds,$$

and (3.25) follows from (3.24) with

$$Q_2(\|u_0\|_V, \|u_0\|_{L^{2p}}) = \frac{d}{2} \left(\|u_0\|_V^2 + Q_1(\|u_0\|_{L^{2p}}) \right).$$

□

3.4. **Estimates for the difference of solutions.** In the following lemmas, u_1 and u_2 denote two solutions of (3.1), and $v(t) = u_1(t) - u_2(t)$ is their difference, which satisfies

$$\partial_t v - d\Delta v + g(u_1) - g(u_2) = 0 \quad \text{in } \Omega \times \mathbb{R}_+. \tag{3.28}$$

Lemma 3.8. *For all $t \geq 0$,*

$$|v(t)|_H^2 + 2d \int_0^t \|v\|_V^2 ds \leq |v(0)|_H^2 \exp(2c_1' t). \tag{3.29}$$

Proof. We multiply (3.28) by $2v$, integrate over Ω, and use (3.7). We obtain

$$\frac{d}{dt}|v|_H^2 + 2d\|v\|_V^2 \leq 2c_1'|v|_H^2.$$

The classical Gronwall lemma yields (3.29). □

We introduce the functions

$$\tilde{Q}_1(s) = \left(s^2 + Q_1(C_S(I, \Omega, 2p)s)\right)^{1/2}, \tag{3.30}$$

where C_S is the Sobolev constant (3.4), and

$$\tilde{h}_1(R_1) = \frac{1}{2d} h_1^2(\tilde{Q}_1(R_1)), \quad R_1 > 0, \tag{3.31}$$

where h_1 is defined by (3.11).

The following two lemmas show a smoothing property.

Lemma 3.9. *Assume that Case 1 holds. If $\|u_i(0)\|_V \leq R_1$ ($i = 1, 2$), then for all $t > 0$, we have*

$$\|v(t)\|_V^2 \leq C_1(t, R_1)|v(0)|^2, \tag{3.32}$$

where the function $C_1 : (0, +\infty)^2 \to \mathbb{R}_+$ is continuous.

Proof. Using $V \subset L^{2p}(\Omega)$, Lemma 3.7 and (3.4) with $q = 2p$, we first note that

$$\|u_i(t)\|_V \leq \tilde{Q}_1(R_1), \quad \forall t \geq 0,$$

for $i = 1, 2$. Next, we multiply (3.28) by $t\partial_t v$ and integrate over Ω. We obtain

$$t|\partial_t v|_H^2 + \frac{td}{2}\frac{d}{dt}\|v\|_V^2 = -t\int_\Omega [g(u_1) - g(u_2)]\partial_t v dx. \tag{3.33}$$

Using (3.11) and Young's inequality, we find

$$\frac{td}{2}\frac{d}{dt}(\|v\|_V^2) \leq \frac{t}{4} h_1^2(\tilde{Q}_1(R_1))\|v\|_V^2, \quad t \geq 0,$$

that is

$$\frac{d}{dt}\left(t\|v\|_V^2\right) \leq \tilde{h}_1(R_1)(t\|v\|_V^2) + \|v\|_V^2, \quad t \geq 0.$$

The classical Gronwall lemma yields

$$t\|v(t)\|_V^2 \leq \exp\left(\tilde{h}_1(R_1)t\right) \int_0^t \|v(s)\|_V^2 ds, \quad \forall t \geq 0.$$

We infer from (3.29) that

$$t\|v(t)\|_V^2 \leq \frac{1}{2d} \exp(2c_1' t) \exp\left(\tilde{h}_1(R_1)t\right) |v(0)|_H^2, \quad \forall t \geq 0,$$

i.e. estimate (3.32) holds. □

When Case 2 holds, we use the functions

$$\tilde{Q}_2(t, R_2) = \left(R_2^q + c_q'' t\right)^{1/q}, \quad t > 0,\ R_2 > 0, \tag{3.34}$$

with $q = I(2p-2)$, and

$$\tilde{h}_2(t, R_2) = \frac{1}{2d} h_2^2(\tilde{Q}_2(t, R_2)), \quad t > 0,\ R_2 > 0, \tag{3.35}$$

where h_2 is defined by (3.12).

Lemma 3.10. *Assume that Case 2 holds. If* $\|u_i(0)\|_{L^{I(2p-2)}} \leq R_2$ *(*$i = 1,\ 2$*), then for all* $t > 0$*, we have*

$$\|v(t)\|_V^2 \leq C_2(t, R_2) |v(0)|_H^2, \tag{3.36}$$

where the function $C_2 : (0, +\infty)^2 \to \mathbb{R}_+$ *is continuous.*

Proof. Let $T > 0$. Using (3.20) with $q = I(2p-2)$, we note that

$$\|u_i(t)\|_{L^{I(2p-2)}} \leq \tilde{Q}_2(T, R_2), \quad \forall t \in [0, T].$$

Arguing as in the proof of Lemma 3.9, and using (3.12), we obtain

$$\frac{d}{dt}\left(t\|v\|_V^2\right) \leq \tilde{h}_2(T, R_2)(t\|v\|_V^2) + \|v\|_V^2, \quad \forall t \in [0, T].$$

The classical Gronwall lemma and (3.29) yield

$$t\|v(t)\|_V^2 \leq \frac{1}{2d} \exp(2c_1' t) \exp \tilde{h}_2(T, R_2) t) |v(0)|_H^2, \quad \forall t \in [0, T].$$

Using this for $t = T$, we obtain (3.36). □

4. The time semi-discrete problem

4.1. **The discrete semi-group.**

For the time semi-discretization, we apply the backward Euler scheme to (3.1). Throughout this section, $\tau > 0$ denotes the time step. The scheme reads: let $u^0 \in H$ and for $n = 0,\ 1,\ 2,\ \ldots$, let $u^{n+1} \in V \cap L^{2p}(\Omega)$ solve

$$\frac{u^{n+1} - u^n}{\tau} - d\Delta u^{n+1} + g(u^{n+1}) = 0. \tag{4.1}$$

The following result shows that the discrete semi-group $S_\tau^n u_0 = u^n$ is well-defined.

Theorem 4.1. *Assume that* $\tau \leq 1/c_1'$*. Then for every* $u \in H$*, there exists a unique* $v = v_{\tau,u} \in V \cap L^{2p}(\Omega)$ *such that*

$$\frac{v - u}{\tau} - d\Delta v + g(v) = 0 \ \textit{in}\ V' + L^{2p/(2p-1)}(\Omega). \tag{4.2}$$

Moreover, the mapping $S_\tau : u \mapsto v_{\tau,u}$ *is Lipschitz continuous from* H *into* V*, with*

$$\|S_\tau u - S_\tau \hat{u}\|_V \leq \frac{c_0}{d\tau} |u - \hat{u}|_H, \quad \forall u, \hat{u} \in H. \tag{4.3}$$

As a consequence, S_τ is Lipschitz continuous from H into H, and from V into V. We note that the Lipschitz constant blows up as $\tau \to 0^+$.

Proof. Let $u \in H$. We can obtain v by minimizing the function

$$\mathcal{G}(w) = \frac{|w-u|_H^2}{2\tau} + \frac{d}{2}\|w\|_V^2 + \int_\Omega G(w)$$

in the space $V \cap L^{2p}(\Omega)$. Let now $\hat{u} \in H$, and consider a solution $\hat{v}$ of (4.2) associated to $\hat{u}$. Then the difference $w = v - \hat{v}$ satisfies

$$\frac{w}{\tau} - d\Delta w + g(v) - g(\hat{v}) = \frac{u-\hat{u}}{\tau} \tag{4.4}$$

We multiply by w, integrate over Ω, use (3.7) and the Cauchy-Schwarz inequality. We obtain

$$\frac{|w|_H^2}{\tau} + d\|w\|_V^2 - c_1'|w|_H^2 \leq \frac{1}{\tau}|u-\hat{u}|_H|w|_H.$$

Using $1/\tau \geq c_1'$ and the Poincaré inequality (3.3), we obtain

$$\|w\|_V \leq \frac{c_0}{d\tau}|u-\hat{u}|_H.$$

Since $w = v - \hat{v}$, this shows that v is unique and that (4.3) holds. □

In the remainder of the paper, we will assume that the time step τ satisfies at least $0 < \tau \leq 1/c_1'$.

4.2. **A priori estimates for the solution, uniform in τ.** We use the same notation as in Section 3. Throughout Section 4.2, (u^n) denotes a sequence in H which complies with (4.1). The following well-known identity will prove useful:

$$(a-b,a)_H = \frac{1}{2}(|a|_H^2 - |b|_H^2 + |a-b|_H^2), \quad a,b \in H. \tag{4.5}$$

Proposition 4.2 (Absorbing set in H)**.** *Assume that $\tau \leq c_0^2/(2d)$. If $|u^0|_H \leq R$, then for all $n \in \mathbb{N}$ such that $n\tau \geq 2t_0(R)$, we have*

$$|u^n|_H \leq \rho_0'. \tag{4.6}$$

Proof. We multiply (4.1) by u^{n+1} and integrate over Ω. We obtain

$$\frac{1}{\tau}(u^{n+1}-u^n, u^{n+1})_H + d\|u^{n+1}\|_V^2 + \int_\Omega g(u^{n+1})u^{n+1}dx = 0.$$

We use (4.5) and inequality (3.10) with $q = 2$. We find

$$\frac{1}{2\tau}(|u^{n+1}|_H^2 - |u^n|_H^2) + d\|u^{n+1}\|_V^2 \leq c_2'|\Omega|, \quad n \geq 0. \tag{4.7}$$

We infer from the Poincaré inequality (3.3) that

$$\left(1 + \frac{2d\tau}{c_0^2}\right)|u^{n+1}|_H^2 \leq |u^n|_H^2 + c_2''\tau, \quad n \geq 0.$$

Let $a = 1 + (2d\tau/c_0^2)$. By induction, we obtain that

$$|u^n|_H^2 \leq a^{-n}|u^0|_H^2 + c_2''\tau\frac{1-a^{-n}}{a-1}, \quad n \geq 0.$$

We note that $\exp(s/2) \leq 1 + s$, for all $s \in [0,1]$. Applying this to $s = 2d\tau/c_0^2 \leq 1$, we see that $a^{-1} \leq \exp(-d\tau/c_0^2)$, and we find

$$|u^n|_H^2 \leq \exp(-nd\tau/c_0^2)|u^0|_H^2 + \frac{c_2''c_0^2}{2d}(1-\exp(-nd\tau/c_0^2)), \quad n \geq 0.$$

This implies (4.6). □

We will use the following lemma from [28].

Lemma 4.3 (Discrete Uniform Gronwall Lemma). *Let $n_0, N \in \mathbb{N}$, $a_1, a_2, a_3, \tau, r' > 0$ and (d^n), (g^n), (h^n) be three sequences of nonnegative real numbers which satisfy*

$$\frac{d^{n+1} - d^n}{\tau} \le g^n d^n + h^n, \quad \forall n \ge n_0,$$

and

$$\tau \sum_{n=k_0}^{k_0+N} g^n \le a_1, \quad \tau \sum_{n=k_0}^{k_0+N} h^n \le a_2, \quad \tau \sum_{n=k_0}^{k_0+N} h^n \le a_2,$$

for all $k_0 \ge n_0$, with $r' = \tau N > 0$. Then

$$d^n \le \Big(a_2 + \frac{a_3}{r'}\Big) \exp(a_1), \quad \forall n \ge n_0 + N.$$

The following lemma will prove useful.

Lemma 4.4. *Assume that $\tau \le 1/(4c_1')$. Then*

$$\|u^{n+1}\|_V^2 + \|u^{n+1} - u^n\|_V^2 \le (1 + 4c_1'\tau)\|u^n\|_V^2, \quad n \ge 0. \tag{4.8}$$

Note that if $u^0 \notin V$, then (4.8) is valid for $n \ge 1$ only.

Proof. We multiply (4.1) by $-\Delta u^{n+1}$ and integrate over Ω. This yields

$$\frac{1}{\tau}(\nabla(u^{n+1} - u^n), \nabla u^{n+1})_H + d|\Delta u^{n+1}|_H^2 + \int_\Omega g'(u^{n+1})|\nabla u^{n+1}|^2 dx = 0.$$

Using (4.5) and (3.6), we obtain

$$\frac{1}{\tau}(\|u^{n+1}\|_V^2 - \|u^n\|_V^2 + \|u^{n+1} - u^n\|_V^2) \le 2c_1'\|u^{n+1}\|_V^2, \quad n \ge 0.$$

We note that

$$1 \le \frac{1}{1-s} \le 1 + 2s, \quad \forall s \in [0, 1/2], \tag{4.9}$$

and we apply this with $s = 2c_1'\tau$. We find (4.8). These formal computations are fully justified for smooth solutions, and (4.8) is valid in the general case by regularization (proceed as in Lemma 4.6 and use that S_τ is continuous from V into V). □

Proposition 4.5 (Absorbing set in V). *Assume that $\tau \le \tau_1$ where*

$$\tau_1 = \min\{c_0^2/(2d), r/2, 1/(4c_1')\} > 0.$$

If $|u^0|_H \le R$, then for all $n \in \mathbb{N}$ such that $n\tau \ge 2t_0(R) + 2r$, we have

$$\|u^n\|_V \le \hat{\rho}_1, \tag{4.10}$$

where

$$\hat{\rho}_1^2 = \frac{\hat{\kappa}_1}{r}\exp(8c_1'r), \quad \hat{\kappa}_1 = \frac{2c_2''r + \rho_0'^2}{d}.$$

Proof. Let $k_0, N \in \mathbb{N}\setminus\{0\}$. Summing (4.7) from $n = k_0 - 1$ to $k_0 + N - 1$, we obtain

$$|u^{k_0+N}|_H^2 + 2\tau d \sum_{n=k_0}^{k_0+N} \|u^n\|_V^2 \leq c_2''\tau(N+1) + |u^{k_0-1}|_H^2.$$

If $k_0\tau \geq 2t_0(R) + \tau$, from (4.6) we infer that

$$2\tau d \sum_{n=k_0}^{k_0+N} \|u^n\|_V^2 \leq c_2''\tau(N+1) + \rho_0'^2. \tag{4.11}$$

Let $n_0\tau \geq 2t_0(R) + \tau$ and $N = [r/\tau| \geq 2$. We set $r' = N\tau \in [r-\tau, r]$ and

$$\hat{\kappa}_1' = \frac{c_2''(r'+\tau) + \rho_0'^2}{2d}.$$

Using (4.11), (4.8) and Lemma 4.3, we obtain

$$\|u^n\|_V^2 \leq \frac{\hat{\kappa}_1'}{r'} \exp(4c_1'(r'+\tau)), \quad \forall n \geq n_0 + N.$$

This implies (4.10). $\square$

Lemma 4.6. *Let $q \geq 2$ and assume that $u^n \in L^q(\Omega)$ for some $n \geq 0$. Then u^{n+1} belongs to $L^{2p+q-2}(\Omega)$ and satisfies*

$$\frac{1}{\tau}\left(\|u^{n+1}\|_{L^q}^q - \|u^n\|_{L^q}^q\right) + \frac{q}{2} b_{2p-1} \|u^{n+1}\|_{L^{2p+q-2}}^{2p+q-2} \leq c_q''. \tag{4.12}$$

Proof. For simplicity, we denote $u = u^n$ and $v = u^{n+1} = S_\tau u$. First assume that $u \in C_c^\infty(\Omega)$. Then, by elliptic regularity, $v \in W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)$ for all $s > 0$, and in particular, $v \in L^\infty(\Omega)$. We multiply (4.2) by $|v|^{q-2}v$ and integrate over Ω. This yields

$$\frac{1}{\tau}(v-u, |v|^{q-2}v)_H + (q-1)d\int_\Omega |\nabla v|^2 |v|^{q-2} dx + \int_\Omega g(v)|v|^{q-2} v dx = 0. \tag{4.13}$$

Let $f(s) = |s|^q/q$. Then $f \in C^2(\mathbb{R})$ with $f''(s) \geq 0$ on $\mathbb{R}$, so by the Taylor-Lagrange formula, for all $s_1, s_2 \in \mathbb{R}$, we have

$$f(s_1) - f(s_2) = (s_1 - s_2)f'(s_2) + \frac{(s_1-s_2)^2}{2} f''(\xi_{s_1,s_2}) \geq (s_1 - s_2) f'(s_2).$$

We apply this with $s_1 = u(x)$, $s_2 = v(x)$ and we integrate over Ω. We obtain

$$\frac{1}{q}\int_\Omega |u|^q dx - \frac{1}{q}\int_\Omega |v|^q dx \geq \int_\Omega (u-v)|v|^{q-2} v dx.$$

We infer from (4.13) and (3.10) that (4.12) is true when $u \in C_c^\infty(\Omega)$. If $u \in L^q(\Omega)$, we obtain (4.12) by considering a sequence $u_\varepsilon \in C_c^\infty(\Omega)$ such that $u_\varepsilon \to u$ in $L^q(\Omega)$. $\square$

As a consequence, we have:

Lemma 4.7. *Let $q \geq 2$. If $u^0 \in L^q(\Omega)$, then*

$$\|u^n\|_{L^q}^q \leq \|u^0\|_{L^q}^q + n\tau c_q'', \quad \forall n \geq 0.$$

We introduce the sequence $(\hat{\rho}_i')$ defined by $\hat{\rho}_i' = \rho_0'$ and, for $i \geq 0$,

$$\hat{\rho}_i'^{a_{i+1}} = \frac{\hat{\kappa}_i'}{r} + 2c_{a_{i+1}}'' r \quad \text{with} \quad \hat{\kappa}_i' = \frac{4}{a_i b_{2p-1}}(2c_{a_i}'' r + \hat{\rho}_i'^{a_i}).$$

Proposition 4.8 (Absorbing set in L^q). *Assume that $\tau \le \min\{1/c_1', c_0^2/(2d), r/2\}$. If $|u^0|_H \le R$ and $i \in \mathbb{N}$, then for all $n \in \mathbb{N}$ such that $n\tau \ge \hat{t}_i(R)$, we have*

$$\|u^n\|_{L^{a_i}} \le \hat{\rho}_i', \tag{4.14}$$

where $\hat{t}_i(R) = 2t_0(R) + 3ir$ does not depend on τ.

Proof. We proceed by induction. We first note that (4.14) is satisfied for $i = 0$ by Proposition 4.2. Let $i \ge 0$ and assume that (4.14) holds for $n\tau \ge \hat{t}_i(R)$. Let $k_0\tau \ge \hat{t}_i(R) + \tau$ and $N = [r/\tau] \ge 2$. We apply Lemma 4.6 with $q = a_i$ and we add the resulting inequality from $n = k_0 - 1$ to $n = k_0 + N - 1$. This yields

$$\frac{a_i}{2} b_{2p-1}\tau \sum_{n=k_0}^{k_0+N} \|u^n\|_{L^{a_{i+1}}}^{a_{i+1}} \le c_q''\tau(N+1) + \|u^{k_0-1}\|_{L^{a_i}}^{a_i}.$$

We set $r' = N\tau \in [r-\tau, r]$. Using the induction assumption, we see that

$$\tau \sum_{n=k_0}^{k_0+N} \|u^n\|_{L^{a_i}}^{a_i} \le \hat{\kappa}_i'', \quad \hat{\kappa}_i'' = \frac{2}{a_i b_{2p-1}}(c_{a_i}''(r'+\tau) + \hat{\rho}_i'^{a_i}). \tag{4.15}$$

For $n \ge k_0 + 1$, we may apply Lemma 4.6 with $q = a_{i+1}$ and we obtain

$$\frac{1}{\tau}\left(\|u^{n+1}\|_{L^{a_{i+1}}}^{a_{i+1}} - \|u^n\|_{L^{a_{i+1}}}^{a_{i+1}}\right) \le c_{a_{i+1}}''. \tag{4.16}$$

Let $n_0\tau \ge \hat{t}_i(R) + \tau$. We infer from (4.15), (4.16) and Lemma 4.3 that

$$\|u^n\|_{L^{a_{i+1}}}^{a_{i+1}} \le \frac{\hat{\kappa}_i''}{r'} + c_{a_{i+1}}''(r'+\tau), \quad \forall n \ge n_0 + N + 1.$$

This implies (4.14). □

Lemma 4.9. *If $u^0 \in V \cap L^{2p}(\Omega)$, then*

$$\|u^n\|_V^2 + \frac{b_{2p-1}}{2dp}\|u^n\|_{L^{2p}}^{2p} + \frac{1}{d\tau}\sum_{k=0}^{n-1}|u^{k+1} - u^k|_H^2 \le \|u^0\|_V^2 + Q_1(\|u^0\|_{L^{2p}}), \quad \forall n \ge 0, \tag{4.17}$$

where Q_1 is the monotonic function independent of τ defined by (3.23).

Proof. We multiply (4.1) by $u^{n+1} - u^n$ and integrate over Ω. We obtain

$$\frac{|u^{n+1} - u^n|_H^2}{\tau} + d(\nabla u^{n+1}, \nabla(u^{n+1} - u^n))_H + \int_\Omega g(u^{n+1})(u^{n+1} - u^n)dx = 0, \tag{4.18}$$

for all $n \ge 0$. By the Taylor-Lagrange formula, for all $s_1, s_2 \in \mathbb{R}$

$$G(s_1) - G(s_2) = (s_1 - s_2)g(s_2) + \frac{(s_1 - s_2)^2}{2}g'(\xi_{s_1,s_2}),$$

for some $\xi_{s_1,s_2} \in \mathbb{R}$, and so, using (3.6), we have

$$G(s_1) - G(s_2) \ge (s_1 - s_2)g(s_2) - c_1'\frac{(s_1 - s_2)^2}{2}.$$

We choose $s_1 = u^n(x)$, $s_2 = u^{n+1}(x)$, and we integrate over Ω. We infer from the resulting inequality, from (4.18) and (4.5) that

$$\left(\frac{1}{\tau} - \frac{c_1'}{2}\right)|u^{n+1} - u^n|_H^2 + \frac{d}{2}(\|u^{n+1}\|_V^2 - \|u^n\|_V^2) + \int_\Omega G(u^{n+1})dx - \int_\Omega G(u^n)dx \le 0,$$

for all $n \geq 0$. By summation, we obtain

$$\frac{d}{2}\|u^n\|_V^2 + \int_\Omega G(u^n)dx + \frac{1}{2\tau}\sum_{k=0}^{n-1}|u^{k+1}-u^k|_H^2 \leq \frac{d}{2}\|u^0\|_V^2 + \int_\Omega G(u^0)dx,$$

for all $n \geq 0$, where we used that $1/\tau - c_1'/2 \geq 1/(2\tau)$. Using (3.9), we conclude that (4.17) holds with Q_1 defined by (3.23). □

4.3. **Estimates for the difference of solutions, uniform with τ.** Let (u^n) and $(\hat{u}^n)$ be two sequences generated by the scheme (4.1) and corresponding to the initial condition u^0 and $\hat{u}^0$ respectively. We denote $v^n = u^n - \hat{u}^n$ their difference, which satisfies

$$\frac{v^{n+1}-v^n}{\tau} - d\Delta v^{n+1} + g(u^{n+1}) - g(\hat{u}^{n+1}) = 0, \quad \forall n \geq 0. \tag{4.19}$$

Lemma 4.10. *Assume that $\tau \leq 1/(4c_1')$. Then*

$$|v^n|_H^2 + 2d\tau\sum_{k=0}^{n-1}\|v^{k+1}\|_V^2 \leq \exp(4c_1' n\tau)|v^0|_H^2, \quad \forall n \geq 0. \tag{4.20}$$

Proof. We multiply (4.19) by v^{n+1}, integrate over Ω, and we use (4.5) and (3.7). We find

$$\frac{1}{2\tau}(|v^{n+1}|_H^2 - |v^n|_H^2) + d\|v^{n+1}\|_V^2 \leq c_1'|v^{n+1}|_H^2, \quad \forall n \geq 0.$$

From (4.9), we infer that

$$|v^{n+1}|_H^2 + 2d\tau\|v^{n+1}\|_V^2 \leq (1+4c_1'\tau)|v^n|_H^2, \quad \forall n \geq 0.$$

We apply

$$1 + s \leq \exp(s), \quad \forall s \in \mathbb{R}, \tag{4.21}$$

to $s = 4c_1'\tau$ and we obtain (4.20) by induction. □

Proposition 4.11 (Bound on bounded sets)**.** *Assume that $\tau \leq 1/(4c_1')$. For all $T > 0$ and for all $R > 0$, there exists a constant $C(T, R)$ independent of τ such that $|u^0|_H \leq R$ and $0 \leq n\tau \leq T$ imply $|u^n|_H \leq C(T, R)$.*

Proof. Let $T > 0$, $R > 0$, $|u^0|_H \leq R$ and $0 \leq n\tau \leq T$. We choose $\hat{u}^0 = 0$. Using successively the triangle inequality, (4.20) and the Cauchy-Schwarz inequality, we find

$$\begin{aligned} |u^n|_H &\leq |u^n - \hat{u}^n|_H + |\hat{u}^n - \hat{u}^0|_H \\ &\leq \exp(2c_1'T)R + \sum_{k=0}^{n-1}|\hat{u}^{k+1}-\hat{u}^k|_H \\ &\leq \exp(2c_1'T)R + (n\tau)^{1/2}\left(\frac{1}{\tau}\sum_{k=0}^{n-1}|\hat{u}^{k+1}-\hat{u}^k|_H^2\right)^{1/2}. \end{aligned}$$

We conclude with (4.17) that

$$|u^n|_H \leq \exp(2c_1'T)R + \left(Td\tilde{Q}_1(0)\right)^{1/2},$$

and this proves the assertion. □

Proposition 4.12 (Smoothing property). *Assume that Case 1 holds and that*

$$\tau \leq \min\{1/c_1', 1/(2\tilde{h}_1(R_1))\}$$

where $\tilde{h}_1$ *(independent of* τ*) is defined by* (3.31). *If* $\|u^0\|_V \leq R_1$ *and* $\|\hat{u}^0\|_V \leq R_1$, *then for all* $n \geq 1$, *we have*

$$n\tau\|v^n\|_V^2 \leq \frac{1}{2d}\exp(4c_1'n\tau)\exp\left(2\tilde{h}_1(R_1)(n+1)\tau\right)|v^0|_H^2, \tag{4.22}$$

Proof. Lemma 4.9 shows that

$$\|u^n\|_V \leq \tilde{Q}_1(R_1) \quad \text{and} \quad \|\hat{u}^n\|_V \leq \tilde{Q}_1(R_1) \quad \forall n \geq 0,$$

where $\tilde{Q}_1$, defined by (3.30), is independent of τ. We multiply (4.19) by $(v^{n+1} - v^n)$ and we integrate over Ω. We find

$$\frac{|v^{n+1} - v^n|_H^2}{\tau} + d(\nabla v^{n+1}, \nabla(v^{n+1} - v^n))_H = -\int_\Omega [g(u^{n+1}) - g(\hat{u}^{n+1})](v^{n+1} - v^n)dx,$$

for all $n \geq 0$. We multiply this by n, we use (4.5) and (3.11), and we obtain

$$n\frac{|v^{n+1} - v^n|_H^2}{\tau} + \frac{nd}{2}(\|v^{n+1}\|_V^2 - \|v^n\|_V^2) \leq nh_1(\tilde{Q}_1(R_1))\|v^{n+1}\|_V|v^{n+1} - v^n|_H,$$

for all $n \geq 0$. Next, we use Young's inequality and the identity

$$n(\|v^{n+1}\|_V^2 - \|v^n\|_V^2) = ((n+1)\|v^{n+1}\|_V^2 - n\|v^n\|_V^2) - \|v^{n+1}\|_V^2,$$

and we find

$$(n+1)\|v^{n+1}\|_V^2 - n\|v^n\|_V^2 \leq \tilde{h}_1(R_1)\tau(n+1)\|v^{n+1}\|_V^2 + \|v^{n+1}\|_V^2,$$

for all $n \geq 0$. Let $\alpha_n = n\|v^n\|_V^2$. We infer from (4.9) that

$$\alpha_{n+1} \leq \left(1 + 2\tilde{h}_1(R_1)\tau\right)(\alpha_n + \|v^{n+1}\|_V^2), \quad \forall n \geq 0.$$

Using $\alpha_0 = 0$, we find by induction that

$$\alpha_n \leq \left(1 + 2\tilde{h}_1(R_1)\tau\right)^{n+1}\left(\sum_{k=0}^{n-1}\|v^{k+1}\|_V^2\right), \quad \forall n \geq 1.$$

We infer (4.22) from (4.20) and from (4.21) applied to $s = 2\tilde{h}_1(R_1)\tau$. □

Proposition 4.13 (Smoothing property). *Let* $T > 0$. *Assume that Case 2 holds and that* $\tau \leq min\{1/c_1', 1/(2\tilde{h}_2(T, R_2))\}$ *where* $\tilde{h}_2$ *(independent of* τ*) is defined by* (3.35). *If* $\|u^0\|_{L^{I(2p-2)}} \leq R_2$ *and* $\|\hat{u}^0\|_{L^{I(2p-2)}} \leq R_2$, *then for all* $1 \leq n \leq [T/\tau]$ *we have*

$$n\tau\|v^n\|_V^2 \leq \frac{1}{2d}\exp(4c_1'n\tau)\exp\left(2\tilde{h}_2(T, R_2)(n+1)\tau\right)|v^0|_H^2.$$

Proof. Let $T > 0$ and set $q = I(2p-2)$. Lemma 4.7 shows that

$$\|u^n\|_{L^q} \leq \tilde{Q}_2(T, R_2) \quad \text{and} \quad \|\hat{u}^n\|_{L^q} \leq \tilde{Q}_2(T, R_2), \quad \forall 0 \leq n \leq [T/\tau],$$

where $\tilde{Q}_2$ (independent of τ) is defined by (3.34). Arguing as in the proof of Proposition 4.12 and using (3.12), we find

$$(n+1)\|v^{n+1}\|_V^2 - n\|v^n\|_V^2 \leq \tilde{h}_2(T, R_2)\tau(n+1)\|v^{n+1}\|_V^2 + \|v^{n+1}\|_V^2,$$

for all $0 \leq n < [T/\tau]$. We conclude similarly. □

5. Finite time uniform error estimate

For the error estimate on a finite time interval, we follow the methodology in [31].

We assume that $\tau \in (0, 1/(4c_1'))$ and we consider a sequence (u^n) generated by (4.1). We first derive a very useful estimate. Using (4.21) with $s = 4c_1'\tau$, from (4.8) we infer that

$$\|u^n\|_V^2 + \sum_{k=0}^{n-1} \|u^{k+1} - u^k\|_V^2 \le \exp(4c_1' n\tau)\|u^0\|_V^2, \quad \forall n \ge 0. \tag{5.1}$$

To the sequence (u^n), we associate two functions $u_\tau, \bar{u}_\tau : \mathbb{R}_+ \to H$, namely

$$u_\tau(t) = u^n + \frac{t - n\tau}{\tau}(u^{n+1} - u^n), \quad t \in [n\tau, (n+1)\tau),$$

and

$$\bar{u}_\tau(t) = u^{n+1}, \quad t \in [n\tau, (n+1)\tau).$$

We assume that $u^0 \in V \cap L^{2p}(\Omega)$. Then, by definition, every u^n belongs to $V \cap L^{2p}(\Omega)$, so that $u_\tau \in C^0(\mathbb{R}_+; V \cap L^{2p}(\Omega))$, $\partial_t u_\tau \in L^\infty_{loc}(\mathbb{R}_+; V \cap L^{2p}(\Omega))$ and $\bar{u}_\tau \in L^\infty_{loc}(\mathbb{R}_+; V \cap L^{2p}(\Omega))$. The scheme (4.1) can be rewritten

$$\partial_t u_\tau - d\Delta \bar{u}_\tau + g(\bar{u}_\tau) = 0, \quad \text{a.e. } t \ge 0,$$

or equivalently,

$$\partial_t u_\tau - d\Delta u_\tau + g(u_\tau) = -d\Delta(u_\tau - \bar{u}_\tau) + [g(u_\tau) - g(\bar{u}_\tau)], \quad \text{a.e. } t \ge 0. \tag{5.2}$$

Let u denote a solution of (3.1)-(3.2) with $u_0 \in V \cap L^{2p}(\Omega)$, and set $e_\tau = u_\tau - u$. Subtracting (3.1) from (5.2), we find that

$$\partial_t e_\tau - d\Delta e_\tau + g(u_\tau) - g(u) = -d\Delta(u_\tau - \bar{u}_\tau) + [g(u_\tau) - g(\bar{u}_\tau)], \quad \text{a.e. } t \ge 0.$$

We multiply by e_τ, integrate over Ω and use (3.7). We obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|e_\tau|_H^2 + d\|e_\tau\|_V^2 &\le c_1'|e_\tau|_H^2 + d\|u_\tau - \bar{u}_\tau\|_V\|e_\tau\|_V \\ &\quad + \int_\Omega |g(u_\tau) - g(\bar{u}_\tau)||e_\tau|dx. \end{aligned} \tag{5.3}$$

Theorem 5.1. *Assume that Case 1 holds. For all $T > 0$ and $R_1 > 0$, there is a constant $C(T, R_1)$ independent of τ such that $u^0 = u_0$ and $\|u^0\|_V \le R_1$ imply*

$$\sup_{t \in [0, N\tau]} |e_\tau(t)|_H \le C(T, R_1)\tau^{1/2}.$$

Proof. From Lemma 4.9, we infer that $\|u^n\|_V \le \tilde{Q}(R_1)$ for all $n \ge 0$, where $\tilde{Q}_1$ (independent of τ) is defined by (3.30). Thus,

$$\|u_\tau(t)\|_V \le \tilde{Q}_1(R_1) \quad \text{and} \quad \|\bar{u}_\tau(t)\|_V \le \tilde{Q}_1(R_1), \quad \forall t \ge 0.$$

Using (5.3), (3.11) and Young's inequality, we infer that

$$\frac{d}{dt}|e_\tau|_H^2 \le C_3(R_1)|e_\tau|_H^2 + d\|u_\tau - \bar{u}_\tau\|_V^2, \quad \text{a.e. } t \ge 0,$$

for some constant $C_3(R_1)$ independent of τ. Let $T > 0$ and set $N = [T/\tau]$. Using $e_\tau(0) = 0$, the classical Gronwall lemma yields

$$|e_\tau|_H^2 \le d\exp(C_3(R_1)T)\int_0^{N\tau} \|u_\tau - \bar{u}_\tau\|_V^2 ds, \quad \forall t \in [0, N\tau]. \tag{5.4}$$

On $[n\tau,(n+1)\tau)$, we have $\|u_\tau - \bar{u}_\tau\|_V \le \|u^{n+1}-u^n\|_V$, so that

$$\int_0^{N\tau} \|u_\tau - \bar{u}_\tau\|_V^2 ds \le \tau \sum_{n=0}^{N-1} \|u^{n+1}-u^n\|_V^2.$$

This shows, using (5.1) and (5.4), that

$$|e_\tau|_H^2 \le \left[d\exp(C_3(R_1)T)\exp(4c_1'T)R_1^2\right]\tau, \quad \forall t\in[0,N\tau].$$

The proof is complete. □

Theorem 5.2. *Assume that Case 2 holds. For all $T>0$ and $R_1,R_2>0$ there is a constant $C(T,R_1,R_2)$ independent of τ such that $u^0=u_0$, $\|u^0\|_V\le R_1$ and $\|u^0\|_{L^{I(2p-2)}}^{I(2p-2)} \le R_2$ imply*

$$\sup_{t\in[0,N\tau]} |e_\tau(t)|_H \le C(T,R_1,R_2)\tau^{1/2}.$$

Proof. Let $T>0$, set $N=[T/\tau]$ and $q=I(2p-2)$. Lemma 4.7 shows that $\|u^n\|_{L^q}\le \tilde{Q}_2(T+1/c_1',R_2)$ for all $0\le n\le N+1$, where $\tilde{Q}_2$ (independent of τ) is defined by (3.34). Thus,

$$\|u_\tau(t)\|_V \le \tilde{Q}_2(T+1/c_1',R_2) \quad \text{and} \quad \|\bar{u}_\tau(t)\|_V \le \tilde{Q}_2(T+1/c_1',R_2), \quad \forall t\in[0,N\tau].$$

Using (5.3), (3.12) and Young's inequality, we obtain

$$\frac{d}{dt}|e_\tau|_H^2 \le C_4(T,R_2)|e_\tau|_H^2 + d\|u_\tau-\bar{u}_\tau\|_V^2, \quad \forall t\in[0,N\tau].$$

Using $e_\tau(0)=0$, the classical Gronwall lemma yields

$$|e_\tau|_H^2 \le d\exp(C_4(T,R_2)T)\int_0^{N\tau}\|u_\tau-\bar{u}_\tau\|_V^2 ds, \quad \forall t\in[0,N\tau]. \tag{5.5}$$

From (5.1) we infer that

$$|e_\tau|_H^2 \le [d\exp(C_4(T,R_2)T)\exp(4c_1'T)R_1^2]\tau, \quad \forall t\in[0,N\tau].$$

□

6. The convergence result

Theorem 6.1. *Let $H=L^2(\Omega)$ and $\tau_0=\min\{c_0^2/(2d),1/(4c_1')\}$. The continuous semi-group $\{S_0(t),\ t\in\mathbb{R}_+\}$ associated to (3.1) and the family of discrete semi-groups $\{S_\tau^n,\ n\in\mathbb{N}\}$, $\tau\in(0,\tau_0]$, associated to (4.1) satisfy the conclusions of Theorem 2.5 with $\tau_0'=\tau_0$.*

Proof. We apply Theorem 2.5 with $V=H_0^1(\Omega)$ which is compactly imbedded in H [18]. If Case 1 holds, we choose $r\ge 1/(2c_1')$ and we consider the set

$$\mathcal{B}=\{w\in H\ :\ \|w\|_V\le\hat{\rho}_1\},$$

which is absorbing in H. The estimates of Sections 3-5 show that assumptions (H1)-(H9) are satisfied with $\beta=\gamma=1/2$ and, in (H6), $\bar{\tau}(T)=\bar{\tau}=\min\{\tau_0,1/(2\tilde{h}_1(R_1))\}$, (cf. (3.31)) and the conclusions of Theorem 2.5 follow for some $\tau_0'\in(0,\tau_0]$ small enough.

For $\tau\in[\tau_0',\tau_0]$, we set $T_0=t(\mathcal{B})+\tau_0$ (cf. (H5)) and $\tilde{S}_\tau=S_\tau^{[T_0/\tau]}$, so that $\tilde{S}_\tau\mathcal{B}\subset\mathcal{B}$. By (4.3), $\tilde{S}_\tau$ satisfies a smoothing property on $\mathcal{B}$ with a Lipschitz constant bounded by a constant $\Lambda=\Lambda(T_0,\tau_0,\tau_0',c_0,d)$ independent of τ. Proposition 1 in [10] shows that the map $\tilde{S}_\tau : \mathcal{B} \to \mathcal{B}$ possesses an exponential attractor $\mathcal{M}_\tau^d$, i.e. a compact and positively invariant subset of $\mathcal{B}$ which has finite fractal dimension and which satisfies

$$dist_H(\tilde{S}_\tau^n \mathcal{B}, \mathcal{M}_\tau^d) \le 2\|\mathcal{B}\|_H 2^{-n}, \quad n \in \mathbb{N}.$$

Moreover,

$$dim_F(\mathcal{M}_\tau^d) \le \log_2 \left[N_{1/(4\Lambda)}(B(0,1;V),H)\right], \tag{6.1}$$

where $N_\varepsilon(B(0,1;V),H)$ is the minimal number of balls of radius ε in H which are necessary to cover the unit ball of center 0 in V. Next, we define $\mathcal{M}_\tau$ by the formula (2.12). We conclude as in the proof of Theorem 2.5 that $\mathcal{M}_\tau$ is an exponential attractor for S_τ in H, with fractal dimension bounded by a constant independent of $\tau \in [\tau_0', \tau_0]$ and which attracts the bounded sets of $\mathcal{B}$, uniformly with $\tau \in [\tau_0', \tau_0]$. This concludes the proof when Case 1 holds (the continuity holds only at $\tau = 0$).

If Case 2 holds, we choose $r \ge 1/(2c_1')$ and we consider the set

$$\mathcal{B} = \{w \in H \ : \ \|w\|_V \le \hat{\rho}_1 \text{ and } \|w\|_{L^{a_I}} \le \hat{\rho}_I'\},$$

which is absorbing in H. We note that $a_I = I(2p-2)+2 \ge I(2p-2)$ and $a_I \ge 2p$. Thus, assumptions (H1)-(H9) hold with $\beta = \gamma = 1/2$ and, in (H6),

$$\bar{\tau}_0(T) = \min\{\tau_0, 1/(2\tilde{h}_2(T, |\Omega|^{1/s}\hat{\rho}_I'))\} > 0,$$

where $1/s = 1/(I(2p-2)) - 1/(I(2p-2)+2)$. The conclusions of Theorem 2.5 hold for some $\tau_0' \in (0, \tau_0]$ small enough. For $\tau \in [\tau_0', \tau_0]$, we argue as previously. The proof is complete. □

Corollary 6.2. *For every $\tau \in [0, \tau_0]$, the semi-group $\{S_\tau(t),\ t \ge 0\}$ possesses a global attractor $\mathcal{A}_\tau$ in H which is bounded in V, compact and connected in H. Moreover, $dist_H(\mathcal{A}_\tau, \mathcal{A}_0\} \to 0$ as $\tau \to 0^+$, and the fractal dimension of $\mathcal{A}_\tau$ is bounded by a constant independent of τ.*

Proof. Existence of a connected global attractor is a consequence of Theorem 1.1 in [30]. Upper semi-continuity of $\mathcal{A}_\tau$ as $\tau \to 0^+$ is a consequence of assumptions (H1), (H5), (H7) and Proposition 1 in [31]. The upper bound on the fractal dimension follows from Theorem 6.1 and from the inclusion $\mathcal{A}_\tau \subset \mathcal{M}_\tau$. □

Remark 6.3. Our upper bound on the fractal dimension of the exponential attractors is quite crude (see (2.9), (6.1)). For the continuous problem, an upper bound which is optimal with respect to the physical parameters has been obtained in [9, Remark (i)]; it is based on L^∞-estimates (see also [8]).

Acknowledgements

The author is thankful to Alain Miranville for helpful discussions.

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