

## Labeled embedding of $(n, n - 2)$ -graphs in their complements

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4 **LABELED EMBEDDING OF  $(N, N - 2)$ -GRAPHS IN THEIR**  
5 **COMPLEMENTS**

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21 **Abstract**

22 Graph packing generally deals with unlabeled graphs. In [4], the authors  
23 have introduced a new variant of the graph packing problem, called the  
24 *labeled packing of a graph*. This problem has recently been studied on trees  
25 [6] and cycles [4]. In this note, we present a lower bound on the labeled  
26 packing number of any  $(n, n - 2)$ -graph into  $K_n$ . This result improves the  
27 bound given by Woźniak in [7].

28 **Keywords:** Packing of graphs, Labeled packing, Permutation.

29 **2010 Mathematics Subject Classification:** 05C70.

30

## 1. CONTEXT AND DEFINITIONS

31 **Graph theoretical definitions**

32 All graphs considered in this paper are finite, undirected, without loops or  
 33 multiple edges. If  $T$  is a rooted tree of order  $n$ , we define an *end vertex* as a  
 34 vertex which does not have any son, and a *leaf-parent* as a vertex whose all of its  
 35 sons are end vertices.

36 Given a positive integer  $n$ , the graphs  $K_n$ ,  $P_n$  and  $C_n$  will denote respectively the  
 37 complete graph, the path and the cycle on  $n$  vertices. For a graph  $G$ , we will use  
 38  $V(G)$  and  $E(G)$  to denote its vertex and edge sets respectively. Given  $V' \subset V$ ,  
 39 the subgraph  $G[V']$  denotes the subgraph of  $G$  induced by  $V'$ , i.e.,  $E(G[V'])$   
 40 contains all the edges of  $E$  which have both extremities in  $V'$ . If a graph  $G$  has  
 41 order  $n$  and size  $m$ , we say that  $G$  is an  $(n, m)$ -graph.

42 An independent set of  $G$  is a subset of vertices  $X \subseteq V$ , such that no two vertices  
 43 in  $X$  are adjacent. An independent set is said to be maximal if no independent  
 44 set properly contains it. An independent set of maximum cardinality is called a  
 45 maximum independent set. For undefined terms, we refer the reader to [2]. A  
 46 permutation  $\sigma$  is a one-to-one mapping of  $\{1, \dots, n\}$  into itself. We say that a  
 47 permutation  $\sigma$  is *fixed-point-free* if  $\sigma(x) \neq x$  for all  $x$  of  $\{1, \dots, n\}$ .

48 **The graph packing problem**

49 The graph packing problem was introduced by Bollobás and Eldridge [1] and  
 50 Sauer and Spencer [5] in the late 1970s. Let  $G_1, \dots, G_k$  be  $k$  graphs of order  
 51  $n$ . We say that there is a packing of  $G_1, \dots, G_k$  (into the complete graph  $K_n$   
 52 ) if there exist bijections  $\sigma_i : V(G_i) \rightarrow V(K_n)$ , where  $1 \leq i \leq k$ , such that  
 53  $\sigma_i^*(E(G_i)) \cap \sigma_j^*(E(G_j)) = \emptyset$  for  $i \neq j$ , and here the map  $\sigma_i^* : E(G_i) \rightarrow E(K_n)$   
 54 is the one induced by  $\sigma_i$ . A packing of  $k$  copies of a graph  $G$  will be called a  
 55  $k$ -placement of  $G$ . A packing of two copies of  $G$  (i.e., a 2-placement) is also called  
 56 an embedding of  $G$  (into its complement  $\overline{G}$ ). In other words, an embedding of  
 57 a graph  $G$  is a permutation  $\sigma$  on  $V(G)$  such that for each edge  $uv$  belonging to  
 58  $E(G)$ , its image  $\sigma(u)\sigma(v)$  does not belong to  $E(G)$ .

59

60 In the literature, the question of the existence of an embedding of a given  
 61 graph received a great attention (see the survey papers [8, 9]). In [3], full char-  
 62 acterizations of all the  $(n, n-1)$  and  $(n, n)$  embeddable graphs are given. The  
 63 case of  $(n, n-2)$ -graphs was also solved independently in [1, 3, 5]. In particular,  
 64 it is proved in [5] that any pair of  $(n, n-2)$ -graphs can be packed into  $K_n$ .

65

66 In [4], Duchêne *et al.* introduced and studied the graph packing problem for  
 67 a vertex labeled graph. Roughly speaking, it consists of a graph packing which

68 preserves the labels of the vertices. We give below the formal definition of this  
 69 problem.

70 **Definition** [4]. Given a positive integer  $p$ , let  $G$  be a graph of order  $n$  and  $f$  be  
 71 a mapping from  $V(G)$  to the set  $\{1, \dots, p\}$ . The mapping  $f$  is called a  $p$ -labeled-  
 72 packing of  $k$  copies of  $G$  into  $K_n$  if there exist bijections  $\sigma_i : V(G) \rightarrow V(K_n)$   
 73 for  $1 \leq i \leq k$ , such that:

- 74 1.  $\sigma_i^*(E(G)) \cap \sigma_j^*(E(G)) = \emptyset$  for all  $i \neq j$ .
- 75 2. For every vertex  $v$  of  $G$ , we have  $f(v) = f(\sigma_1(v)) = f(\sigma_2(v)) = \dots = f(\sigma_k(v))$ .

76 The maximum positive integer  $p$  for which  $G$  admits a  $p$ -labeled-packing of  $k$   
 77 copies of  $G$  is called the *labeled packing number* of  $k$  copies of  $G$  and is denoted  
 78 by  $\lambda^k(G)$ . Throughout this paper, a labeled packing of two copies of  $G$  will be  
 79 called a labeled embedding of  $G$ . It will be denoted by a pair  $(f, \sigma)$ .

80 Remark that the existence of a packing of  $k$  copies of a graph  $G$  is a neces-  
 81 sary condition for the existence of  $p$ -labeled-packing of  $k$  copies of  $G$ . Indeed, it  
 82 suffices to choose  $p = 1$ . Therefore, the result of Sauer and Spencer [5] ensures  
 83 the existence of a  $p$ -labeled packing for  $(n, n - 2)$ -graphs. An estimation of the  
 84 labeled packing number of such graphs is the main issue of the current paper.

85

86 The following result was proved in [4]. It gives an upper bound for the labeled  
 87 packing number of two copies of a general graph.

**Lemma 1** (Duchêne et al., 11). *Let  $G$  be a graph of order  $n$  and let  $I$  be a maximum independent set of  $G$ . If there exists an embedding of  $G$  into  $K_n$ , then*

$$\lambda^2(G) \leq |I| + \lfloor \frac{n - |I|}{2} \rfloor$$

In [4], exact values of  $\lambda^2(G)$  are given when  $G$  is a cycle or a path. In almost all cases, the upper bound of the above lemma is reached. More precisely, it is shown that for all  $n \geq 6$ ,

$$\lambda^2(P_n) \in \{ \lfloor \frac{3n}{4} \rfloor, \lfloor \frac{3n}{4} \rfloor + 1 \}$$

$$\lambda^2(C_n) = \lfloor \frac{3n}{4} \rfloor$$

88 The case of trees is also considered [6], but only a lower bound is proposed.

89

## 2. LABELED EMBEDDING OF GRAPHS AND PERMUTATIONS

90 In this section, we give a strong relationship between a labeled embedding and  
91 its permutation structure.

92 A permutation  $\sigma$  of a finite set can be written as the disjoint union of cycles  
93 (two cycles being disjoint if they do not have any common element). Here, a  
94 cycle  $(a_1, \dots, a_n)$  is a permutation sending  $a_i$  to  $a_{i+1}$  for  $1 \leq i \leq n-1$  and  $a_n$  to  
95  $a_1$ . This representation is called the *cyclic decomposition of  $\sigma$*  and is denoted by  
96  $C(\sigma)$ . According to this definition, the cycles of length one correspond to fixed  
97 points of  $\sigma$ . For example, the cyclic decomposition of the permutation induced  
98 by the labeled embedding of  $T$  (in Figure 1) is:  $\{(v_1), (v_2), (v_3), (v_4), (v_5), (v_6),$   
99  $(v_7), (v_8, v_{10}), (v_{11}, v_{13}), (v_9, v_{12})\}$ .

100 We now recall a fundamental property of labeled embeddings (see [4]). For  
101 any labeled embedding  $(f, \sigma)$  of a graph  $G$ , one can remark that the vertices of  
102 every cycle of  $C(\sigma)$  share the same label. In other words, the labeled embedding  
103 number of  $G$  exactly corresponds to the maximum number of cycles induced by  
104 an embedding of  $G$ . It means that if  $G$  admits an embedding with  $k$  cycles, then  
105  $\lambda^2(G) \geq k$ .

106

107 Although this correlation between labeled embeddings and the permutation's  
108 number of cycles was recently stated, several studies can be found about the  
109 permutation structure of an embedding. In particular, the permutation structure  
110 of embeddings of  $(n, n-2)$ -graphs was investigated by Woźniak in [7]:

111 **Theorem 2** (Woźniak, 94). *Let  $G$  be a graph of order  $n$ , different from  $K_3 \cup 2K_1$   
112 and  $K_4 \cup 4K_1$ . If  $|E(G)| \leq n-2$ , then there exists a permutation  $\sigma$  on  $V(G)$   
113 such that  $\sigma_1, \sigma_2, \sigma_3$  define a 3-placement of  $G$ . Moreover,  $\sigma$  has all its cycles  
114 of length 3, except for one of length one if  $n \equiv 1 \pmod{3}$  or two of length one if  
115  $n \equiv 2 \pmod{3}$ .*

116 According to our previous remarks, the above theorem induces the following  
117 result in the context of labeled embeddings.

**Corollary 3.** *Let  $G$  be a graph of order  $n$ , different from  $K_3 \cup 2K_1$  and  $K_4 \cup 4K_1$ .  
If  $|E(G)| \leq n-2$ , then*

$$\lambda^2(G) \geq \lfloor \frac{n}{3} \rfloor + n \pmod{3}$$

118 In the next section, we will show that the lower bound of Corollary 3 can be  
119 improved (including for the excluded graphs).

120

## 3. MAIN RESULT

121 We first define the notion of *good permutation* for a graph.

122 **Definition.** Given a graph  $G$ , a permutation  $\sigma$  on  $V(G)$  is said to be *good* if

- 123 •  $\sigma$  is an embedding of  $G$ ,
- 124 •  $\sigma$  has at least  $\lfloor \frac{2n}{3} \rfloor$  cycles,
- 125 • every cycle of  $\sigma$  is of order at most 2, *i.e.*, for every pair of distinct vertices
- 126  $u, v$  of  $G$ , if  $\sigma(u) = v$ , then  $\sigma(v) = u$ .

127 The following lemma will be useful in a special case of our main result.

128 **Lemma 4.** For  $k > 0$ , the graph  $kC_3 \cup 2K_1$  admits a good permutation.

129 **Proof.** According to the diagram below (Figure 1), first remark that  $3C_3$  admits  
 130 a good permutation. Indeed, the numbers inside the vertices correspond to a  
 131 labeled embedding with 6 labels, with at most two vertices sharing the same  
 label.

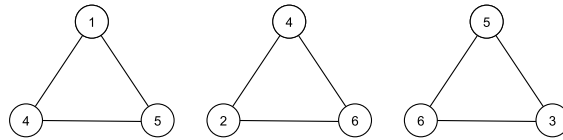


Figure 1. Good permutation for  $3C_3$

132

133 Now let  $k$  be a positive integer and  $G$  be the graph  $kC_3 \cup 2K_1$ . Let  $u$  and  $t$   
 134 be the two isolated vertices of  $G$ . For  $1 \leq i \leq k$ , let  $\{v_{i1}, v_{i2}, v_{i3}\}$  be the vertices  
 135 of the  $i^{th}$  triangle  $C_3$ . For  $k = 1$ , consider the permutation  $\sigma$  where  $v_{11}$  is a  
 136 fixed point,  $v_{12}$  and  $u$  are mutual images, as well as  $v_{13}$  and  $t$ . One can easily  
 137 check that  $\sigma$  is good for  $G$ . For  $k = 2$ , Figure 2 shows a good permutation (more  
 precisely, the corresponding labeled embedding).

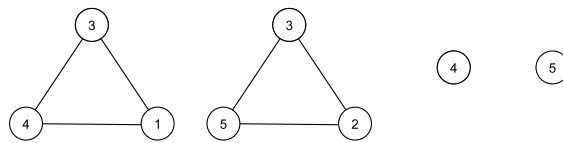


Figure 2. Good permutation for  $2C_3 \cup 2K_1$

138

139 For  $k = 3$ , consider the permutation  $\sigma$  corresponding to the labeled embed-  
 140 ding of Figure 1, and extend it to  $G$  by setting  $\sigma(u) = u$  and  $\sigma(t) = t$ . Then  $\sigma$   
 141 remains good for  $G$ . For  $k > 3$ , we can now conclude to the existence of a good  
 142 permutation for  $G$  by pairing good permutations of  $3C_3$  with a good permutation  
 143 of  $rC_3 \cup 2K_1$  where  $r$  is in  $\{1, 2, 3\}$ . ■

144 We now present a lower bound for the labeled embedding number of any  
145  $(n, n - 2)$ -graph.

**Theorem 5.** *Let  $n \geq 2$  and  $G$  be an  $(n, m)$ -graph with  $m \leq n - 2$ . The following inequality holds:*

$$\lambda^2(G) \geq \lfloor \frac{2n}{3} \rfloor$$

146 **Proof.** Let  $n \geq 2$  and  $G$  be an  $(n, m)$ -graph with  $m \leq n - 2$ . Without loss  
147 of generality, we can assume  $|E(G)| = n - 2$ . We will show that  $G$  admits a  
148 good permutation by induction on  $n$ . If  $n = 2, 3, 4$ , then  $G \in \{2K_1, 3K_1, K_1 \cup$   
149  $K_2, 2K_2, K_{1,2} \cup K_1\}$ . In each case, one can quickly check that there exist good  
150 permutations with at least two cycles. The property still holds for  $n = 5$ , where  
151  $G \in \{K_3 \cup 2K_1, K_1 \cup K_{1,3}, K_2 \cup K_{1,2}, P_4 \cup K_1\}$ . Good permutations with at least  
152 three cycles can be found.

153 Now let  $n \geq 6$  and assume there exists a good permutation for every  $(n', n' -$   
154  $2)$ -graph of order  $n' < n$  with  $n' \geq 3$ . Since  $G$  is an  $(n, n - 2)$ -graph, at least  
155 two of its connected components are trees. Denote by  $T$  and  $H$  two trees of  $G$  of  
156 highest orders such that  $|V(T)| \geq |V(H)|$ . In what follows, we choose to consider  
157  $T$  and  $H$  as rooted trees. We consider the following four cases:

158

159 **Case 1:**  $|V(T)| \geq 3$  and  $|V(H)| \geq 3$ . Two subcases are considered as follows:

160 **Subcase 1.1:**  $T$  or  $H$  admits a leaf-parent of degree 2. By symmetry, we may  
161 assume that  $T$  admits a leaf-parent say  $x_1$ , of degree 2. Let  $x_0$  and  $x_2$  be the  
162 two vertices of  $T$  such that  $(x_0, x_1, x_2)$  is an induced path of  $T$  and  $x_2$  is an end  
163 vertex. Let  $y_1$  be an end vertex of  $H$  and  $y_0$  its parent. Now consider the graph  
164  $G' = G - \{x_1, x_2, y_1\}$ . Clearly,  $G'$  is an  $(n - 3, n - 5)$ -graph with  $n - 3 \geq 3$ . Hence  
165 the induction hypothesis guarantees the existence of a good permutation  $\sigma'$  for  
166  $G'$ . This permutation can be extended to a good permutation  $\sigma$  for  $G$  as follows:

167

$$\sigma(x_1) = \begin{cases} y_1 & \text{if } \sigma'(x_0) = x_0, \\ x_1 & \text{otherwise.} \end{cases} \quad \sigma(x_2) = \begin{cases} x_2 & \text{if } \sigma'(x_0) = x_0, \\ y_1 & \text{otherwise.} \end{cases}$$

168

$$\sigma(y_1) = \begin{cases} x_1 & \text{if } \sigma'(x_0) = x_0, \\ x_2 & \text{otherwise.} \end{cases} \quad \sigma(v) = \sigma'(v) \text{ if } v \in V(G')$$

169

170 Since the number of cycles of  $\sigma|_{G-G'}$  equals two, and they all are of length  
171 at most 2, it ensures that  $\sigma$  is a good permutation for  $G$ .

172 **Subcase 1.2:**  $T$  and  $H$  do not have any leaf-parent of degree 2 and  $T$  is a  
173 star. Let  $\ell$  be a leaf of  $H$  and  $v$  be its neighbour (of degree at least 3). Let  $u$   
174 be the unique vertex of degree at least 3 in  $T$ . Now consider the graph  $G' =$   
175  $G - T - \{\ell, v\}$ . Clearly,  $G'$  is an  $(k, j)$ -graph with  $k - 2 \geq j$  and  $k \geq 2$ . Hence  
176 the induction hypothesis guarantees the existence of a good permutation  $\sigma'$  for

177  $G'$ . This permutation can be extended to a good permutation  $\sigma$  for  $G$  by setting  
 178  $\sigma(u) = v$ ,  $\sigma(v) = u$ ,  $\sigma(\ell) = \ell$  and  $\sigma(l_i) = l_i$  for all leaf  $l_i$  of  $T$ .

179 **Subcase 1.3:**  $T$  and  $H$  do not have any leaf-parent of degree 2 and  $T$  is not  
 180 a star. Hence  $T$  admits two leaf parents, say  $u$  and  $t$ . Let  $u_1, \dots, u_k$  (resp.  
 181  $t_1, \dots, t_{k'}$ ) be the leaves adjacent to  $u$  (resp.  $t$ ), with  $k, k' \geq 2$ . Let  $\ell$  be a leaf of  
 182  $H$  and  $v$  be its neighbour (of degree at least 3). Let  $x$  be the non-leaf neighbour  
 183 of  $u$  (it exists since  $T$  is not a star and  $x = t$  if  $T$  is a bistar). See Figure 3 for a  
 184 better view of these notations. Without loss of generality, we will assume  $k \geq k'$ .

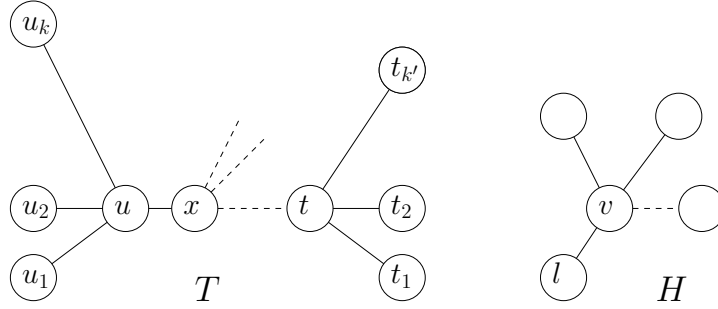


Figure 3. Subcase (1.3)

185 If  $k \geq 3$ , consider the graph  $G' = G - \{u, u_1, \dots, u_k, t_1, \ell\}$ . Hence  $G'$  is a  
 186  $(n - k - 3, n - k - 5)$ -graph satisfying  $|V(G')| \geq 3$  (since  $|V(H)| \geq 4$ ), and the  
 187 induction hypothesis can be applied. Let  $\sigma'$  be a good permutation for  $G'$ . We  
 188 now define a permutation  $\sigma$  for  $G$  as follows:

$$\sigma(u) = \begin{cases} t_1 & \text{if } \sigma'(x) = v, \\ \ell & \text{otherwise.} \end{cases} \quad \sigma(t_1) = \begin{cases} u & \text{if } \sigma'(x) = v, \\ u_1 & \text{otherwise.} \end{cases}$$

$$189 \quad \sigma(u_1) = \begin{cases} \ell & \text{if } \sigma'(x) = v, \\ t_1 & \text{otherwise.} \end{cases} \quad \sigma(\ell) = \begin{cases} u_1 & \text{if } \sigma'(x) = v, \\ u & \text{otherwise.} \end{cases}$$

$$190 \quad \sigma(u_i)_{i>1} = u_i \quad \sigma(v) = \sigma'(v) \text{ if } v \in V(G')$$

191 This permutation remains good as  $(k + 1)$  cycles of lengths 1 or 2 and  $(k + 3)$   
 192 vertices have been added to  $\sigma'$  at  $G'$  respectively.

193  
 194 If  $k = k' = 2$ , we first consider the case where  $x = t$  (i.e.,  $T$  is a bistar).  
 195 Consider the graph  $G' = G - \{u_1, t_1, \ell\}$  and proceed by induction as above. Since  
 196  $ut$  is an edge of  $T$ , we have  $\sigma'(u) \neq \sigma'(t)$  and  $u$  and  $t$  cannot be fixed points  
 197 simultaneously. Therefore, at least  $u_1$  or  $t_1$  can be set as a fixed point in  $\sigma$   
 198 and the two remaining vertices in  $\{u_1, t_1, \ell\}$  can be images of each other (thus  
 199 implying that  $\sigma$  is good).



200 In the case where  $x \neq t$ , consider the graph  $G' = G - \{u, u_1, u_2, t_1, t_2, \ell\}$ , and  
 201 proceed as previously with the following permutation:

$$\sigma(u) = \begin{cases} t_1 & \text{if } \sigma'(t) = t, \\ t_1 & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ \ell & \text{otherwise.} \end{cases} \quad \sigma(t_1) = \begin{cases} u & \text{if } \sigma'(t) = t, \\ u & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ t_1 & \text{otherwise.} \end{cases}$$

$$202 \quad \sigma(t_2) = \begin{cases} \ell & \text{if } \sigma'(t) = t, \\ t_2 & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ t_2 & \text{otherwise.} \end{cases} \quad \sigma(\ell) = \begin{cases} t_2 & \text{if } \sigma'(t) = t, \\ \ell & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ u & \text{otherwise.} \end{cases}$$

$$203 \quad \sigma(u_i)_{i=1,2} = u_i \qquad \sigma(v) = \sigma'(v) \text{ if } v \in V(G')$$

204 One can easily check that  $\sigma$  remains good since 6 vertices and at least 4  
 205 cycles have been added to  $\sigma'$ .

206 **Case 2:**  $|V(T)| \geq 3$  and  $H = K_1$ .

207 **Subcase 2.1:** There exists a leaf-parent, say  $x$ , of degree at least 3. Let  $\ell$  be  
 208 one of its leaves, and  $y$  be the unique vertex of  $H$ . Now consider the graph  
 209  $G' = G - \{x, \ell, y\}$ , which satisfies  $|E(G')| \leq |V(G')| - 2$ . Hence it admits a good  
 210 permutation  $\sigma'$  by induction hypothesis. A good permutation  $\sigma$  of  $G$  can thus be  
 211 extended from  $G'$  by setting  $\sigma(x) = y$ ,  $\sigma(y) = x$  and  $\sigma(\ell) = \ell$ .

212  
 213 **Subcase 2.2:** All the vertices of  $T$  which are adjacent to leaves are of degree 2.  
 214 Let  $x_0$  be such a vertex (it exists since  $|V(T)| \geq 3$ ), let  $\ell_1$  be its adjacent leaf,  
 215 and  $x_1$  its second neighbor. Now let  $\ell_2$  be a distinct leaf from  $\ell_1$  in  $T$ , and  $x_2$  be  
 216 its neighbor. If  $x_1 \neq \ell_2$  and  $x_1 \neq x_2$ , then consider  $G' = G - \{x_0, \ell_1, \ell_2\}$ , which  
 217 admits a good permutation  $\sigma'$  by induction hypothesis. Then set  $\sigma|_{G'} = \sigma'$  and

218 • If  $\sigma'(x_1) = x_1$ :

$$219 \quad \sigma(x_0) = \ell_2, \sigma(\ell_2) = x_0, \text{ and } \sigma(\ell_1) = \ell_1.$$

220 • If  $\sigma'(x_1) \neq x_1$ :

$$221 \quad \sigma(x_0) = x_0, \sigma(\ell_2) = \ell_1, \text{ and } \sigma(\ell_1) = \ell_2.$$

222 One can now easily check that  $\sigma$  is good for  $G$ . If  $x_1 = \ell_2$  or  $x_1 = x_2$ , then  $T$   
 223 is either a  $P_3$  or a  $P_4$ . Since  $n \geq 6$ , it implies that  $G$  admits at least another  
 224 connected component which is an  $(n, n-1)$  or an  $(n, n)$  connected graph with  
 225  $n \geq 2$  (since  $|E(G)| \geq 4$ , there exists at least one edge in  $G - T$ ). In other words,  
 226 this component is either a tree  $T'$ , or a tree with an edge  $T' \cup \{e\}$ . Let  $\ell_3$  be  
 227 a leaf in  $T'$ . Note that we do not care whether  $\ell_3$  is adjacent to  $e$  or not. By  
 228 considering  $G' = G - \{x_0, \ell_1, \ell_3\}$  together with the above permutation where  $\ell_2$

229 is replaced by  $\ell_3$ , we find a good permutation for  $G$ .

230

231 **Case 3:**  $|V(T)| = 2$ . Let  $T = (x_0, x_1)$  and let  $y$  be a vertex of degree 2 of  $G$ .  
 232 Such a vertex exists since  $n \geq 6$ . Consider the graph  $G' = G - \{x_0, x_1, y\}$ . By  
 233 induction hypothesis, there exists a good permutation for  $G'$ , say  $\sigma'$ . We set  
 234  $\sigma(x_0) = y, \sigma(y) = x_0, \sigma(x_1) = x_1$  and for every vertex  $v \in V(G')$ ,  $\sigma(v) = \sigma'(v)$ ,  
 235 which defines a good permutation for  $G$ .

236

237 **Case 4:**  $|V(T)| = 1$ . In this case,  $G$  contains isolated vertices (at least two) and  
 238 non-tree connected components. Two subcases are considered as follows:

239 **Subcase 4.1:**  $G$  has a vertex, say  $x$ , of degree at least 3. Let  $y$  and  $z$  be two  
 240 isolated vertices of  $G$ . Consider the graph  $G' = G - \{x, y, z\}$ . The induction  
 241 hypothesis guarantees the existence of a good permutation  $\sigma'$  for  $G'$ . By putting  
 242  $\sigma(x) = y, \sigma(y) = x, \sigma(z) = z$  and for every vertex  $v \in V(G')$ ,  $\sigma(v) = \sigma'(v)$ , we  
 243 get a good permutation for  $G$ .

244 **Subcase 4.2:** *The complementary subcase to (4.1), i.e.,  $G$  is the sum of two*  
 245 *isolated vertices and an union of cycles.* This case is solved as follows:

246 (a)  $G = kC_3 \cup 2K_1$  for some  $k \geq 1$ . Lemma 4 allows us to conclude.

247 (b)  $G$  has at least one cycle, say  $H$ , of order at least 4, and one cycle, say  $Q$ ,  
 248 of order at least 3: let  $(x_1, x_2, x_3)$  be an induced path of  $H$ , let  $x_4$  be a vertex of  
 249  $Q$  and  $z, t$  be the two isolated vertices of  $G$ . Denote by  $x$  (resp.  $y$ ) the neighbor  
 250 of  $x_1$  (resp.  $x_3$ ) different from  $x_2$ . Note that we may have  $x = y$  in the case  
 251  $H = C_4$ . See Figure 4 for a graphical depiction of these notations.

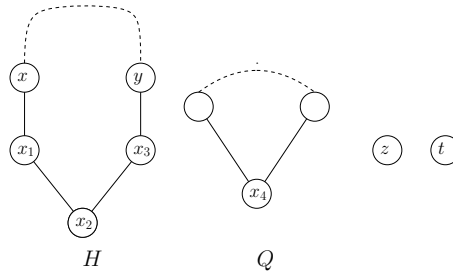


Figure 4. Case (4.2.b)

252 Consider the graph  $G' = G - \{x_1, x_2, x_3, x_4, z, t\}$ . Since  $|V(G)| \geq 9$ , we  
 253 have  $|V(G')| \geq 3$  and the induction hypothesis guarantees the existence of a  
 254 good permutation  $\sigma'$  for  $G'$ . The permutation  $\sigma'$  can be extended to a good  
 255 permutation  $\sigma$  of  $G$  by setting  $\sigma(t) = t$ , and

256 Hence  $\sigma|_{G-G'}$  has four cycles of size at most 2, and  $\sigma$  is thus good for  $G$ .

257 (c)  $G$  is the sum of  $C_m$  (for some  $m \geq 4$ ) and two isolated vertices. If  $m < 8$ ,  
 258 then Figure 5 shows labeled embeddings corresponding to good permutations.

$$\sigma(x_1) = \begin{cases} x_1 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_4 & \text{if } \sigma'(x) = x, \\ z & \text{otherwise.} \end{cases} \quad \sigma(x_2) = \begin{cases} x_4 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_2 & \text{otherwise.} \end{cases}$$

$$\sigma(x_3) = \begin{cases} x_3 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ z & \text{if } \sigma'(x) = x, \\ x_4 & \text{otherwise.} \end{cases} \quad \sigma(x_4) = \begin{cases} x_2 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_1 & \text{if } \sigma'(x) = x, \\ x_3 & \text{otherwise.} \end{cases}$$

$$\sigma(z) = \begin{cases} z & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_3 & \text{if } \sigma'(x) = x, \\ x_1 & \text{otherwise.} \end{cases}$$

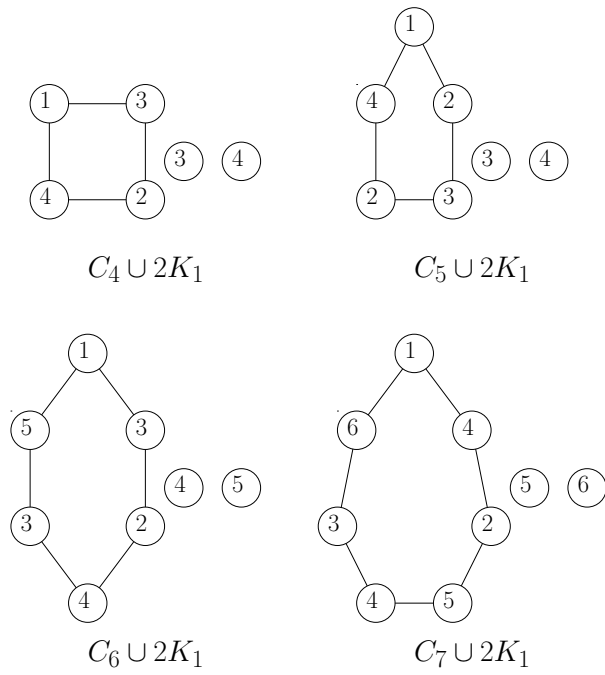


Figure 5. Case  $C_m \cup 2K_1$  for  $m = 4, \dots, 7$

259 If  $m \geq 8$ , let  $(x_1, \dots, x_8)$  be a path of  $C_m$ . Let  $z, t$  be the two isolated vertices  
 260 of  $G$ . We consider the graph  $G' = G - \{x_2, x_3, x_6, x_7, z, t\}$  which admits a good  
 261 permutation  $\sigma'$  by induction hypothesis. Since  $v_4$  and  $v_5$  are adjacent, at least  
 262 one of them is not a fixed point under  $\sigma'$ . Without loss of generality, assume  
 263  $\sigma'(x_4) \neq x_4$ . The permutation  $\sigma'$  can be extended to a good permutation  $\sigma$  for  
 264  $G$  as follows: set  $x_3$  and  $t$  as fixed points. If  $\sigma'(x_5) \neq x_1$ , we set  $\sigma(x_2) = x_6$ ,

265  $\sigma(x_6) = x_2$ ,  $\sigma(x_7) = z$ , and  $\sigma(z) = x_7$ . Otherwise, we set  $\sigma(x_2) = x_7$ ,  $\sigma(x_7) =$   
 266  $x_2$ ,  $\sigma(x_6) = z$ , and  $\sigma(z) = x_6$ . For the same reasons as in case (4.2.b), this  
 267 permutation is good for  $G$ .

268



269 CONCLUSION

270 Theorem 5 gives a first lower bound about the labeled embedding number of  
 271  $(n, n - 2)$ -graphs. Yet, the computation of the exact value remains an open  
 272 question, as this bound is not exact for many families of  $(n, n - 2)$ -graphs. As an  
 273 example, consider a cycle  $C_n$  without two edges. Its labeled packing number is  
 274 at least the one of  $C_n$ , (i.e.,  $\lfloor 3n/4 \rfloor$ ). Yet, for any large value of  $n$ , we can find an  
 275  $(n, n - 2)$ -graph for which the bound is tight. Indeed, consider  $G$  as an union of  $k$   
 276 disjoint triangles with  $K_2 \cup K_1$ . The size of a maximum independent set for this  
 277 graph equals  $k+2$ . According to Lemma 1, we have that  $\lambda_2(G) = 2k+2 = \lfloor 2n/3 \rfloor$ .

278 In addition, we mention that this result can be used to study the labeled  
 279 embedding of  $(n, n - 1)$ -graphs. One can show for example that the same bound  
 280 is valid for the union of cycles with a single tree.

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