Labeled embedding of (n, n − 2)-graphs in their complements

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LABELED EMBEDDING OF $(N, N-2)$-GRAPHS IN THEIR COMPLEMENTS

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Abstract

Graph packing generally deals with unlabeled graphs. In [4], the authors have introduced a new variant of the graph packing problem, called the *labeled packing of a graph.* This problem has recently been studied on trees [6] and cycles [4]. In this note, we present a lower bound on the labeled packing number of any $(n, n-2)$-graph into K_n. This result improves the bound given by Woźniak in [7].

Keywords: Packing of graphs, Labeled packing, Permutation.

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1. Context and definitions

Graph theoretical definitions

All graphs considered in this paper are finite, undirected, without loops or multiple edges. If T is a rooted tree of order n, we define an *end vertex* as a vertex which does not have any son, and a *leaf-parent* as a vertex whose all of its sons are end vertices.
Given a positive integer n, the graphs K_n, P_n and C_n will denote respectively the complete graph, the path and the cycle on n vertices. For a graph G, we will use $V(G)$ and $E(G)$ to denote its vertex and edge sets respectively. Given $V' \subset V$, the subgraph $G[V']$ denotes the subgraph of G induced by V', i.e., $E(G[V'])$ contains all the edges of E which have both extremities in V'. If a graph G has order n and size m, we say that G is an (n, m)-graph.
An independent set of G is a subset of vertices $X \subseteq V$, such that no two vertices in X are adjacent. An independent set is said to be maximal if no independent set properly contains it. An independent set of maximum cardinality is called a maximum independent set. For undefined terms, we refer the reader to [2]. A permutation σ is a one-to-one mapping of $\{1, \ldots, n\}$ into itself. We say that a permutation σ is *fixed-point-free* if $\sigma(x) \neq x$ for all x of $\{1, \ldots, n\}$.

The graph packing problem

The graph packing problem was introduced by Bollobás and Eldridge [1] and Sauer and Spencer [5] in the late 1970s. Let $G_1, \ldots, G_k$ be k graphs of order n. We say that there is a packing of $G_1, \ldots, G_k$ (into the complete graph K_n) if there exist bijections $\sigma_i : V(G_i) \longrightarrow V(K_n)$, where $1 \leq i \leq k$, such that $\sigma_i^*(E(G_i)) \cap \sigma_j^*(E(G_j)) = \emptyset$ for $i \neq j$, and here the map $\sigma_i^* : E(G_i) \longrightarrow E(K_n)$ is the one induced by σ_i. A packing of k copies of a graph G will be called a k-placement of G. A packing of two copies of G (*i.e.,* a 2-placement) is also called an embedding of G (into its complement $\overline{G}$). In other words, an embedding of a graph G is a permutation σ on $V(G)$ such that for each edge uv belonging to $E(G)$, its image $\sigma(u)\sigma(v)$ does not belong to $E(G)$.

In the literature, the question of the existence of an embedding of a given graph received a great attention (see the survey papers [8, 9]). In [3], full characterizations of all the $(n, n-1)$ and (n, n) embeddable graphs are given. The case of $(n, n-2)$-graphs was also solved independently in [1, 3, 5]. In particular, it is proved in [5] that any pair of $(n, n-2)$-graphs can be packed into K_n.

In [4], Duchêne *et al.* introduced and studied the graph packing problem for a vertex labeled graph. Roughly speaking, it consists of a graph packing which preserves the labels of the vertices. We give below the formal definition of this problem.

Definition [4]**.** Given a positive integer p, let G be a graph of order n and f be a mapping from $V(G)$ to the set $\{1, \ldots, p\}$. The mapping f is called a p-labeled-packing of k copies of G into K_n if there exist bijections $\sigma_i : V(G) \longrightarrow V(K_n)$ for $1 \leq i \leq k$, such that:

1. $\sigma_i^*(E(G)) \cap \sigma_j^*(E(G)) = \emptyset$ for all $i \neq j$.

2. For every vertex v of G, we have $f(v)=f(\sigma_1(v))=f(\sigma_2(v)) = \cdots = f(\sigma_k(v))$.

The maximum positive integer p for which G admits a *p-labeled-packing* of k copies of G is called the *labeled packing number* of k copies of G and is denoted by $\lambda^k(G)$. Throughout this paper, a labeled packing of two copies of G will be called a labeled embedding of G. It will be denoted by a pair (f, σ).

Remark that the existence of a packing of k copies of a graph G is a necessary condition for the existence of p-labeled-packing of k copies of G. Indeed, it suffices to choose $p = 1$. Therefore, the result of Sauer and Spencer [5] ensures the existence of a p-labeled packing for $(n, n-2)$-graphs. An estimation of the labeled packing number of such graphs is the main issue of the current paper.

The following result was proved in [4]. It gives an upper bound for the labeled packing number of two copies of a general graph.

Lemma 1 (Duchêne et al., 11)**.** *Let G be a graph of order n and let I be a maximum independent set of G. If there exists an embedding of G into K_n, then*

$$\lambda^2(G) \leq |I| + \lfloor \frac{n - |I|}{2} \rfloor$$

In [4], exact values of $\lambda^2(G)$ are given when G is a cycle or a path. In almost all cases, the upper bound of the above lemma is reached. More precisely, it is shown that for all $n \geq 6$,

$$\lambda^2(P_n) \in \{\lfloor \frac{3n}{4} \rfloor, \lfloor \frac{3n}{4} \rfloor + 1\}$$

$$\lambda^2(C_n) = \lfloor \frac{3n}{4} \rfloor$$

The case of trees is also considered [6], but only a lower bound is proposed.

2. Labeled embedding of graphs and permutations

In this section, we give a strong relationship between a labeled embedding and its permutation structure.

A permutation σ of a finite set can be written as the disjoint union of cycles (two cycles being disjoint if they do not have any common element). Here, a cycle $(a_1, \ldots, a_n)$ is a permutation sending a_i to a_{i+1} for $1 \leq i \leq n-1$ and a_n to a_1. This representation is called the *cyclic decomposition of* σ and is denoted by $C(\sigma)$. According to this definition, the cycles of length one correspond to fixed points of σ. For example, the cyclic decomposition of the permutation induced by the labeled embedding of T (in Figure 1) is: $\{(v_1)$, (v_2), (v_3), (v_4), (v_5), (v_6), (v_7), (v_8, v_{10}),(v_{11}, v_{13}), $(v_9, v_{12})\}$.

We now recall a fundamental property of labeled embeddings (see [4]). For any labeled embedding (f, σ) of a graph G, one can remark that the vertices of every cycle of $C(\sigma)$ share the same label. In other words, the labeled embedding number of G exactly corresponds to the maximum number of cycles induced by an embedding of G. It means that if G admits an embedding with k cycles, then $\lambda^2(G) \geq k$.

Although this correlation between labeled embeddings and the permutation's number of cycles was rencently stated, several studies can be found about the permutation structure of an embedding. In particular, the permutation structure of embeddings of $(n, n-2)$-graphs was investigated by Woźniak in [7]:

Theorem 2 (Woźniak, 94)**.** *Let G be a graph of order n, different from $K_3 \cup 2K_1$ and $K_4 \cup 4K_1$. If $|E(G)| \leq n-2$, then there exists a permutation σ on $V(G)$ such that σ_1, σ_2, σ_3 define a 3-placement of G. Moreover, σ has all its cycles of length 3, except for one of length one if $n \equiv 1 \bmod 3$ or two of length one if $n \equiv 2 \bmod 3$.*

According to our previous remarks, the above theorem induces the following result in the context of labeled embeddings.

Corollary 3. *Let G be a graph of order n, different from $K_3 \cup 2K_1$ and $K_4 \cup 4K_1$. If $|E(G)| \leq n-2$, then*

$$\lambda^2(G) \geq \lfloor \frac{n}{3} \rfloor + n \ mod\ 3$$

In the next section, we will show that the lower bound of Corollary 3 can be improved (including for the excluded graphs).

3. Main result

We first define the notion of *good permutation* for a graph.

Definition. Given a graph G, a permutation σ on $V(G)$ is said to be *good* if

- σ is an embedding of G,
- σ has at least $\lfloor \frac{2n}{3} \rfloor$ cycles,
- every cycle of σ is of order at most 2, *i.e., for every pair of distinct vertices u, v of G, if $\sigma(u) = v$, then $\sigma(v) = u$.*

The following lemma will be useful in a special case of our main result.

Lemma 4. *For $k > 0$, the graph $kC_3 \cup 2K_1$ admits a good permutation.*

Proof. According to the diagram below (Figure 1), first remark that $3C_3$ admits a good permutation. Indeed, the numbers inside the vertices correspond to a labeled embedding with 6 labels, with at most two vertices sharing the same label.

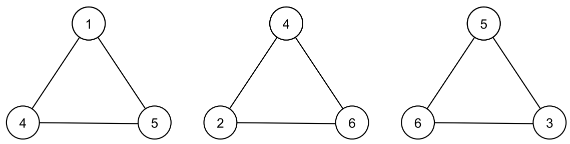

Figure 1. Good permutation for $3C_3$

Now let k be a positive integer and G be the graph $kC_3 \cup 2K_1$. Let u and t be the two isolated vertices of G. For $1 \leq i \leq k$, let $\{v_{i1}, v_{i2}, v_{i3}\}$ be the vertices of the i^{th} triangle C_3. For $k = 1$, consider the permutation σ where v_{11} is a fixed point, v_{12} and u are mutual images, as well as v_{13} and t. One can easily check that σ is good for G. For $k = 2$, Figure 2 shows a good permutation (more precisely, the corresponding labeled embedding).

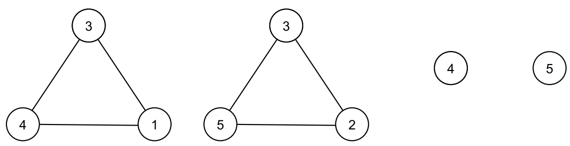

Figure 2. Good permutation for $2C_3 \cup 2K_1$

For $k = 3$, consider the permutation σ corresponding to the labeled embedding of Figure 1, and extend it to G by setting $\sigma(u) = u$ and $\sigma(t) = t$. Then σ remains good for G. For $k > 3$, we can now conclude to the existence of a good permutation for G by pairing good permutations of $3C_3$ with a good permutation of $rC_3 \cup 2K_1$ where r is in $\{1, 2, 3\}$. ∎

We now present a lower bound for the labeled embedding number of any $(n, n-2)$-graph.

Theorem 5. *Let $n \geq 2$ and G be an (n,m)-graph with $m \leq n-2$. The following inequality holds:*

$$\lambda^2(G) \geq \lfloor \frac{2n}{3} \rfloor$$

Proof. Let $n \geq 2$ and G be an (n,m)-graph with $m \leq n-2$. Without loss of generality, we can assume $|E(G)| = n-2$. We will show that G admits a good permutation by induction on n. If $n = 2, 3, 4$, then $G \in \{2K_1, 3K_1, K_1 \cup K_2, 2K_2, K_{1,2} \cup K_1\}$. In each case, one can quickly check that there exist good permutations with at least two cycles. The property still holds for $n = 5$, where $G \in \{K_3 \cup 2K_1, K_1 \cup K_{1,3}, K_2 \cup K_{1,2}, P_4 \cup K_1\}$. Good permutations with at least three cycles can be found.

Now let $n \geq 6$ and assume there exists a good permutation for every $(n', n'-2)$-graph of order $n' < n$ with $n' \geq 3$. Since G is an $(n, n-2)$-graph, at least two of its connected components are trees. Denote by T and H two trees of G of highest orders such that $|V(T)| \geq |V(H)|$. In what follows, we choose to consider T and H as rooted trees. We consider the following four cases:

Case 1: $|V(T)| \geq 3$ and $|V(H)| \geq 3$. Two subcases are considered as follows:
Subcase 1.1: T or H admits a leaf-parent of degree 2. By symmetry, we may assume that T admits a leaf-parent say x_1, of degree 2. Let x_0 and x_2 be the two vertices of T such that (x_0, x_1, x_2) is an induced path of T and x_2 is an end vertex. Let y_1 be an end vertex of H and y_0 its parent. Now consider the graph $G' = G - \{x_1, x_2, y_1\}$. Clearly, G' is an $(n-3, n-5)$-graph with $n-3 \geq 3$. Hence the induction hypothesis guarantees the existence of a good permutation σ' for G'. This permutation can be extended to a good permutation σ for G as follows:

$$\sigma(x_1) = \begin{cases} y_1 & \text{if } \sigma'(x_0) = x_0, \\ x_1 & \textit{otherwise}. \end{cases} \quad \sigma(x_2) = \begin{cases} x_2 & \text{if } \sigma'(x_0) = x_0, \\ y_1 & \textit{otherwise}. \end{cases}$$

$$\sigma(y_1) = \begin{cases} x_1 & \text{if } \sigma'(x_0) = x_0, \\ x_2 & \textit{otherwise}. \end{cases} \qquad \sigma(v) = \sigma'(v) \text{ if } v \in V(G')$$

Since the number of cycles of $\sigma_{|G-G'}$ equals two, and they all are of length at most 2, it ensures that σ is a good permutation for G.
Subcase 1.2: T and H do not have any leaf-parent of degree 2 and T is a star. Let ℓ be a leaf of H and v be its neighbour (of degree at least 3). Let u be the unique vertex of degree at least 3 in T. Now consider the graph $G' = G - T - \{\ell, v\}$. Clearly, G' is an (k, j)-graph with $k - 2 \geq j$ and $k \geq 2$. Hence the induction hypothesis guarantees the existence of a good permutation σ' for

G'. This permutation can be extended to a good permutation σ for G by setting $\sigma(u) = v$, $\sigma(v) = u$, $\sigma(\ell) = \ell$ and $\sigma(l_i) = l_i$ for all leaf l_i of T.

Subcase 1.3: T and H do not have any leaf-parent of degree 2 and T is not a star. Hence T admits two leaf parents, say u and t. Let $u_1, \ldots, u_k$ (resp. $t_1, \ldots, t_{k'}$) be the leaves adjacent to u (resp. t), with $k, k' \geq 2$. Let ℓ be a leaf of H and v be its neighbour (of degree at least 3). Let x be the non-leaf neighbour of u (it exists since T is not a star and $x = t$ if T is a bistar). See Figure 3 for a better view of these notations. Without loss of generality, we will assume $k \geq k'$.

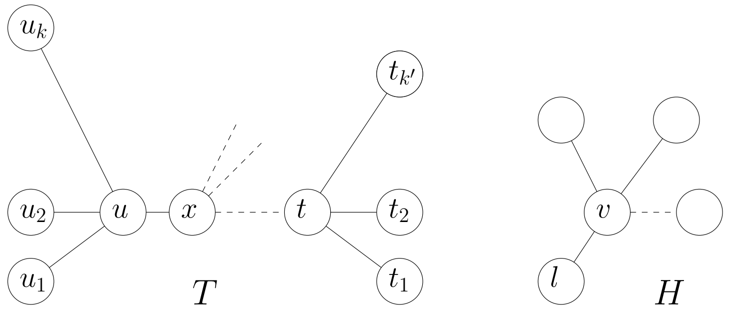

Figure 3. Subcase (1.3)

If $k \geq 3$, consider the graph $G' = G - \{u, u_1, \ldots, u_k, t_1, \ell\}$. Hence G' is a $(n-k-3, n-k-5)$-graph satisfying $|V(G')| \geq 3$ (since $|V(H)| \geq 4$), and the induction hypothesis can be applied. Let σ' be a good permutation for G'. We now define a permutation σ for G as follows:

$$\sigma(u) = \begin{cases} t_1 & \text{if } \sigma'(x) = v, \\ \ell & \textit{otherwise.} \end{cases} \qquad \sigma(t_1) = \begin{cases} u & \text{if } \sigma'(x) = v, \\ u_1 & \textit{otherwise.} \end{cases}$$

$$\sigma(u_1) = \begin{cases} \ell & \text{if } \sigma'(x) = v, \\ t_1 & \textit{otherwise.} \end{cases} \qquad \sigma(\ell) = \begin{cases} u_1 & \text{if } \sigma'(x) = v, \\ u & \textit{otherwise.} \end{cases}$$

$$\sigma(u_i)_{i>1} = u_i \qquad \sigma(v) = \sigma'(v) \text{ if } v \in V(G')$$

This permutation remains good as $(k+1)$ cycles of lengths 1 or 2 and $(k+3)$ vertices have been added to σ' et G' respectively.

If $k = k' = 2$, we first consider the case where $x = t$ (i.e., T is a bistar). Consider the graph $G' = G - \{u_1, t_1, \ell\}$ and proceed by induction as above. Since ut is an edge of T, we have $\sigma'(u) \neq \sigma'(t)$ and u and t cannot be fixed points simultaneously. Therefore, at least u_1 or t_1 can be set as a fixed point in σ and the two remaining vertices in $\{u_1, t_1, \ell\}$ can be images of each other (thus implying that σ is good).

In the case where $x \neq t$, consider the graph $G' = G - \{u, u_1, u_2, t_1, t_2, \ell\}$, and proceed as previously with the following permutation:

$$\sigma(u) = \begin{cases} t_1 & \text{if } \sigma'(t) = t, \\ t_1 & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ \ell & otherwise. \end{cases} \quad \sigma(t_1) = \begin{cases} u & \text{if } \sigma'(t) = t, \\ u & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ t_1 & otherwise. \end{cases}$$

$$\sigma(t_2) = \begin{cases} \ell & \text{if } \sigma'(t) = t, \\ t_2 & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ t_2 & otherwise. \end{cases} \quad \sigma(\ell) = \begin{cases} t_2 & \text{if } \sigma'(t) = t, \\ \ell & \text{if } \sigma'(t) \neq t \text{ and } \sigma'(x) = v, \\ u & otherwise. \end{cases}$$

$$\sigma(u_i)_{i=1,2} = u_i \qquad \sigma(v) = \sigma'(v) \text{ if } v \in V(G')$$

One can easily check that σ remains good since 6 vertices and at least 4 cycles have been added to σ'.

Case 2: $|V(T)| \geq 3$ and $H = K_1$.

Subcase 2.1: There exists a leaf-parent, say x, of degree at least 3. Let ℓ be one of its leaves, and y be the unique vertex of H. Now consider the graph $G' = G - \{x, \ell, y\}$, which satisfies $|E(G')| \leq |V(G')| - 2$. Hence it admits a good permutation σ' by induction hypothesis. A good permutation σ of G can thus be extended from G' by setting $\sigma(x) = y$, $\sigma(y) = x$ and $\sigma(\ell) = \ell$.

Subcase 2.2: All the vertices of T which are adjacent to leaves are of degree 2. Let x_0 be such a vertex (it exists since $|V(T)| \geq 3$), let ℓ_1 be its adjacent leaf, and x_1 its second neighbor. Now let ℓ_2 be a distinct leaf from ℓ_1 in T, and x_2 be its neighbor. If $x_1 \neq \ell_2$ and $x_1 \neq x_2$, then consider $G' = G - \{x_0, \ell_1, \ell_2\}$, which admits a good permutation σ' by induction hypothesis. Then set $\sigma_{|G'} = \sigma'$ and

- If $\sigma'(x_1) = x_1$:

$$\sigma(x_0) = \ell_2,\ \sigma(\ell_2) = x_0, \text{ and } \sigma(\ell_1) = \ell_1.$$

- If $\sigma'(x_1) \neq x_1$:

$$\sigma(x_0) = x_0,\ \sigma(\ell_2) = \ell_1, \text{ and } \sigma(\ell_1) = \ell_2.$$

One can now easily check that σ is good for G. If $x_1 = \ell_2$ or $x_1 = x_2$, then T is either a P_3 or a P_4. Since $n \geq 6$, it implies that G admits at least another connected component which is an $(n, n-1)$ or an (n, n) connected graph with $n \geq 2$ (since $|E(G)| \geq 4$, there exists at least one edge in $G - T$). In other words, this component is either a tree T', or a tree with an edge $T' \cup \{e\}$. Let ℓ_3 be a leaf in T'. Note that we do not care whether ℓ_3 is adjacent to e or not. By considering $G' = G - \{x_0, \ell_1, \ell_3\}$ together with the above permutation where ℓ_2

is replaced by ℓ_3, we find a good permutation for G.

Case 3: $|V(T)| = 2$. Let $T = (x_0, x_1)$ and let y be a vertex of degree 2 of G. Such a vertex exists since $n \geq 6$. Consider the graph $G' = G - \{x_0, x_1, y\}$. By induction hypothesis, there exists a good permutation for G', say σ'. We set $\sigma(x_0) = y$, $\sigma(y) = x_0$, $\sigma(x_1) = x_1$ and for every vertex $v \in V(G')$, $\sigma(v) = \sigma'(v)$, which defines a good permutation for G.

Case 4: $|V(T)| = 1$. In this case, G contains isolated vertices (at least two) and non-tree connected components. Two subcases are considered as follows:
Subcase 4.1: *G has a vertex, say x, of degree at least* 3. Let y and z be two isolated vertices of G. Consider the graph $G' = G - \{x, y, z\}$. The induction hypothesis guarantees the existence of a good permutation σ' for G'. By putting $\sigma(x) = y$, $\sigma(y) = x$, $\sigma(z) = z$ and for every vertex $v \in V(G')$, $\sigma(v) = \sigma'(v)$, we get a good permutation for G.
Subcase 4.2: *The complementary subcase to (4.1), i.e., G is the sum of two isolated vertices and an union of cycles.* This case is solved as follows:

(a) $G = kC_3 \cup 2K_1$ for some $k \geq 1$. Lemma 4 allows us to conclude.

(b) *G has at least one cycle, say H, of order at least* 4, *and one cycle, say Q, of order at least* 3: let (x_1, x_2, x_3) be an induced path of H, let x_4 be a vertex of Q and z, t be the two isolated vertices of G. Denote by x (resp. y) the neighbor of x_1 (resp. x_3) different from x_2. Note that we may have $x = y$ in the case $H = C_4$. See Figure 4 for a graphical depiction of these notations.

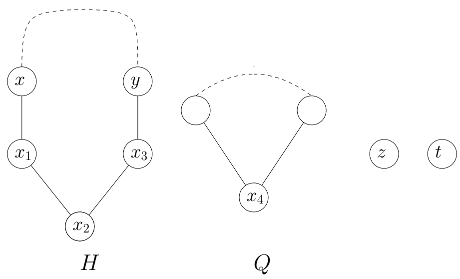

Figure 4. Case (4.2.b)

Consider the graph $G' = G - \{x_1, x_2, x_3, x_4, z, t\}$. Since $|V(G)| \geq 9$, we have $|V(G')| \geq 3$ and the induction hypothesis guarantees the existence of a good permutation σ' for G'. The permutation σ' can be extended to a good permutation σ of G by setting $\sigma(t) = t$, and

Hence $\sigma_{|G-G'}$ has four cycles of size at most 2, and σ is thus good for G.

(c) G is the sum of C_m (for some $m \geq 4$) and two isolated vertices. If $m < 8$, then Figure 5 shows labeled embeddings corresponding to good permutations.

$$\sigma(x_1) = \begin{cases} x_1 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_4 & \text{if } \sigma'(x) = x, \\ z & \mathit{otherwise}. \end{cases} \qquad \sigma(x_2) = \begin{cases} x_4 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_2 & \mathit{otherwise}. \end{cases}$$

$$\sigma(x_3) = \begin{cases} x_3 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ z & \text{if } \sigma'(x) = x, \\ x_4 & \mathit{otherwise}. \end{cases} \qquad \sigma(x_4) = \begin{cases} x_2 & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_1 & \text{if } \sigma'(x) = x, \\ x_3 & \mathit{otherwise}. \end{cases}$$

$$\sigma(z) = \begin{cases} z & \text{if } \sigma'(x) \neq x \text{ and } \sigma'(y) \neq y, \\ x_3 & \text{if } \sigma'(x) = x, \\ x_1 & \mathit{otherwise}. \end{cases}$$

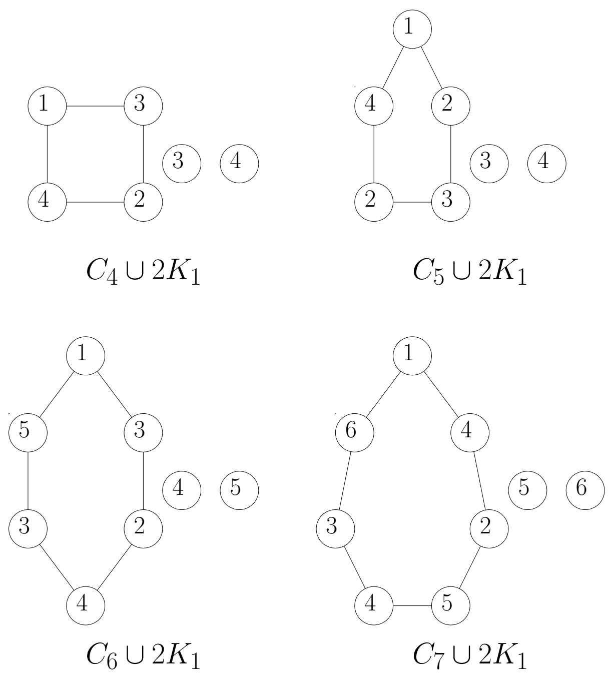

Figure 5. Case $C_m \cup 2K_1$ for $m = 4, \ldots, 7$

If $m \geq 8$, let $(x_1, \ldots, x_8)$ be a path of C_m. Let z, t be the two isolated vertices of G. We consider the graph $G' = G - \{x_2, x_3, x_6, x_7, z, t\}$ which admits a good permutation σ' by induction hypothesis. Since v_4 and v_5 are adjacent, at least one of them is not a fixed point under σ'. Without loss of generality, assume $\sigma'(x_4) \neq x_4$. The permutation σ' can be extended to a good permutation σ for G as follows: set x_3 and t as fixed points. If $\sigma'(x_5) \neq x_1$, we set $\sigma(x_2) = x_6$,

$\sigma(x_6) = x_2$, $\sigma(x_7) = z$, and $\sigma(z) = x_7$. Otherwise, we set $\sigma(x_2) = x_7$, $\sigma(x_7) = x_2$, $\sigma(x_6) = z$, and $\sigma(z) = x_6$. For the same reasons as in case $(4.2.b)$, this permutation is good for G.

■

CONCLUSION

Theorem 5 gives a first lower bound about the labeled embedding number of $(n, n-2)$-graphs. Yet, the computation of the exact value remains an open question, as this bound is not exact for many families of $(n, n-2)$-graphs. As an example, consider a cycle C_n without two edges. Its labeled packing number is at least the one of C_n, (i.e., $\lfloor 3n/4 \rfloor$). Yet, for any large value of n, we can find an $(n, n-2)$-graph for which the bound is tight. Indeed, consider G as an union of k disjoint triangles with $K_2 \cup K_1$. The size of a maximum independent set for this graph equals $k+2$. According to Lemma 1, we have that $\lambda_2(G) = 2k+2 = \lfloor 2n/3 \rfloor$.

In addition, we mention that this result can be used to study the labeled embedding of $(n, n-1)$-graphs. One can show for example that the same bound is valid for the union of cycles with a single tree.

REFERENCES

[1] B. Bollobás and S. E. Eldridge, *Packing of graphs and applications to computational complexity*, J. Comb. Theory (B) **25** (105–124) 1978.

[2] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications (McMillan, London; Elsevier, New York, 1976).

[3] D. Burns and S. Schuster, *Every $(p, p-2)$ graph is contained in its complement*, J. Graph Theory **1** (277–279) 1977.

[4] E. Duchêne, H. Kheddouci, R.J. Nowakowski and M.A. Tahraoui, *Labeled packing of graphs*, Australasian Journal of Combinatorics **57** (109–126) 2013.

[5] N. Sauer and J. Spencer, *Edge disjoint placement of graphs*, J. Combin. Theory Ser. B **25** (295–302) 1978.

[6] M. A. Tahraoui, E. Duchêne and H. Kheddouci, *Labeled 2-packings of trees*, Discrete Mathematics **338(5-6)** (816–824) 2015.

[7] M. Woźniak, *Embedding Graphs of Small Size*, Discrete Applied Mathematics **51(1-2)** (233–241) 1994.

[8] M. WoźniakPacking of graphs and permutations—a surveyDiscrete Mathematics 276 (1-3) 379–3912004

[9] H. P. YapPacking of graphs−a surveyDiscrete Mathematics72395–4041988