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# Tight Kernels for Covering with Points and Polynomials

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## Abstract

The POINT HYPERPLANE COVER problem in  $\mathbb{R}^d$  takes as input a set of  $n$  points in  $\mathbb{R}^d$  and a positive integer  $k$ . The objective is to cover all the given points with a set of at most  $k$  hyperplanes. The  $D$ -POLYNOMIAL POINTS COVER problem in  $\mathbb{R}^d$  takes as input a family  $\mathcal{F}$  of  $D$ -degree polynomials from a vector space  $\mathcal{R}$  in  $\mathbb{R}^d$ , and determines whether there is a set of at most  $k$  points in  $\mathbb{R}^d$  that hit all the polynomials in  $\mathcal{F}$ . Here, a point  $p$  is said to hit a polynomial  $f$  if  $f(p) = 0$ . For both problems, we exhibit tight kernels where  $k$  is the parameter. We also exhibit a tight kernel for the PROJECTIVE POINT HYPERPLANE COVER problem, where the hyperplanes that are allowed to cover the points must all contain a fixed point, and the fixed point cannot be included in the solution set of points.

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## 1 Introduction

A set system is a tuple  $(U, \mathcal{F})$  where  $U$  is a universe of  $n$  elements and  $\mathcal{F}$  is a family of  $m$  subsets of  $U$ . A set system is also referred to as a *hypergraph*, with the elements in the universe  $U$  named as *vertices* and the subsets in  $\mathcal{F}$  named as *hyperedges*. A hyperedge is said to *cover* a vertex if the vertex belongs to the hyperedge. Similarly a subfamily  $\mathcal{F}'$  of hyperedges is said to cover a subset  $V$  of vertices if for each vertex  $v \in V$  there is a hyperedge  $h \in \mathcal{F}'$  such that  $h$  covers  $v$ . A vertex is said to *hit* a hyperedge if the vertex belongs to the hyperedge, and a subset  $V$  of vertices is said to hit a subfamily  $\mathcal{F}'$  of hyperedges if for each hyperedge  $h \in \mathcal{F}'$  there is a vertex  $v \in V$  that belongs to  $h$ .

The SET COVER and HITTING SET problems are two of the most well-studied problems in computer science. For the SET COVER problem, the input is a set system  $(U, \mathcal{F})$  and a positive integer  $k$ . The objective is to determine whether there is a subfamily  $\mathcal{F}' \subseteq \mathcal{F}$  with at most  $k$  subsets, such that  $\mathcal{F}'$  covers all the elements in  $U$ . Such a family  $\mathcal{F}'$  is referred to as a *solution family* or a *covering family*. The HITTING SET problem can be thought of as a dual problem. Here, the input is the same as in SET COVER. However, now the objective is to determine whether there is a subset  $S \subseteq U$  of size at most  $k$ , such that for each hyperedge  $h \in \mathcal{F}$   $h \cap S \neq \emptyset$ . Such a set  $S$  is referred to as a *solution set*.

These problems are part of the original 21 NP-complete problems posed by Richard Karp [15]. However, the numerous applications for these problems inspires researchers to design algorithms to find solutions with reasonable efficiency, for different measures of



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efficiency. For SET COVER, the best approximation factor is  $\mathcal{O}(\log n)$  [20]. It was shown in [9] that  $\log n$  is the best possible approximation factor unless  $P=NP$ . Since HITTING SET is just a reformulation of the SET COVER problem, the same approximation factors hold. The  $d$ -HITTING SET problem, where the size of each subset in  $\mathcal{F}$  is exactly  $d$ , is known to be APX-hard [2], consequent to results obtained for the special case of the VERTEX COVER problem where  $d = 2$ .

SET COVER and HITTING SET have been studied in parameterized complexity. In parameterized complexity, we say that a problem is fixed parameter tractable (FPT) with respect to a parameter  $k$ , if there is an algorithm that takes an instance of size  $n$  of the problem, and solves the problem in  $f(k) \cdot n^{\mathcal{O}(1)}$  time, where  $f$  is a computable function. For a brief introduction to parameterized complexity please refer to the Preliminaries. For further details please refer to [11, 12, 7]. The  $d$ -HITTING SET problem, parameterized by the solution size  $k$ , is known to be FPT, with a tight  $\mathcal{O}(k^d)$ -sized kernel [8] under standard complexity theoretic assumptions. In this paper, unless otherwise mentioned, all variants of HITTING SET and SET COVER are parameterized by  $k$ . The SET COVER and the general HITTING SET problems are  $W[2]$ -hard, and are not expected to be FPT.

Interestingly, the instances of many real world applications of these two notoriously hard problems have inherent structure in them. With the hope of designing efficient algorithms for such instances by exploiting their structural information, numerous variants of SET COVER and HITTING SET have been studied. A very natural extension in this field of study is to assume geometric structure on the instances. In recent years, there has been a lot of attention to study geometric variants of both the problems.

The POINT LINE COVER is a example of a geometric variant of SET COVER. Here the universe is a set of points in  $\mathbb{R}^2$  and the hyperedges are the maximal sets of collinear points in the input. POINT LINE COVER is known to be FPT [17]. Kratsch et. al. showed in [16] that the problem has a tight polynomial kernel with  $\mathcal{O}(k^2)$  points. In [1], several generalizations of the POINT LINE COVER problem were studied - the universe is a set of points in a Euclidean space and the family of hyperedges are geometric structures like hyperplanes, spheres, curves, etc. Geometric variants of SET COVER have been studied in [4, 5].

The results in [16] also imply parameterized results for LINE POINT COVER, where the universe is a set of lines in  $\mathbb{R}^2$  and the objective is to find at most  $k$  points in  $\mathbb{R}^2$  to hit the universe of lines. This problem is FPT and has a tight kernel with  $\mathcal{O}(k^2)$  lines. Other geometric variants of the HITTING SET problem has been studied in parameterized complexity [10, 13, 14]. Bringmann et al. [6] studied the problem for set systems with bounded VC dimensions. They showed that there are set systems with VC dimension as low as 2, where both the HITTING SET, and consequently the SET COVER problem are  $W[1]$ -hard. This gives an interesting dichotomy, since they also show that when the VC dimension of the set system is 1, then the HITTING SET problem is in P.

In this paper, we consider two parameterized variants.

POINT HYPERPLANE COVER IN  $\mathbb{R}^d$

**Parameter:**  $k$

**Input:** A set  $\mathcal{P}$  of  $n$  points in  $\mathbb{R}^d$ , a positive integer  $k$ .

**Question:** Is there a family of hyperplanes in  $\mathbb{R}^d$  that cover all the points in  $\mathcal{P}$ ?

We also study the PROJECTIVE POINT HYPERPLANE COVER problem, where the family of hyperplanes allowed to cover the input set of points must pass through a fixed point in  $\mathbb{R}^d$ , and we are not allowed to include the fixed point in the solution set of points. Note that this problem is equivalent to that of covering points on a sphere with the great circles of the sphere, which has many applications in computational geometry.

*D*-POLYNOMIAL POINT COVER IN  $\mathbb{R}^d$  **Parameter:**  $k$

**Input:** A set  $\mathcal{F}$  of  $n$   $D$ -degree polynomials from a specified vector space  $\mathcal{R}$  of  $D$ -degree polynomials in  $\mathbb{R}^d$ , a positive integer  $k$ .

**Question:** Is there a set  $S$  of points in  $\mathbb{R}^d$  such that for each polynomial  $f \in \mathcal{F}$ , there is a point  $p \in S$  with  $f(p) = 0$ ?

Please refer to the Preliminaries for the definition of a  $D$ -degree polynomials. Both the problems are NP-hard, because of the NP-hardness of POINT LINE COVER [19]. Parameterized by  $k$ , we study the parameterized complexity of these problems.

### Our results

Extending the results of Kratsch et. al. [16], we show that POINT HYPERPLANE COVER and PROJECTIVE POINT HYPERPLANE COVER in  $\mathbb{R}^d$  have tight polynomial kernels with  $\mathcal{O}(k^d)$  and  $\mathcal{O}(k^{d-1})$  points, respectively. The highlight of this proof, given any positive integer  $n$ , is the construction of a set  $\mathcal{P}$  of  $n$  points in  $\mathbb{R}^d$  in general position such that the family of hyperplanes, defined by any  $d$  points from  $\mathcal{P}$ , do not have too many hyperplanes intersecting at a point outside  $\mathcal{P}$ . This is crucial for a many-one reduction from  $d$ -HITTING SET, that results in lower bounds on the size as well as the number of points in a kernel under complexity theoretic assumptions.

Our main contribution is to show tight polynomial bounds for kernel sizes for  $D$ -POLYNOMIAL POINT COVER in  $\mathbb{R}^t$ , for a large family of vector spaces  $\mathcal{R}$  of  $D$ -degree polynomials. For more details on the characterization of  $\mathcal{R}$ , please see the Preliminaries. The vector space of hyperplanes, spheres and ellipses are among natural vector spaces of polynomials that are covered by this characterization. Therefore, our techniques provide a general framework for proving tight kernels for covering problems, as one can get a tight bound for many families directly.

Our proof strategy is to use the Veronese mapping to transform the space of points and polynomials to a higher dimensional space, where the polynomials transform into hyperplanes and point-polynomial incidences are preserved. The upper bound on the kernel size comes directly from the Veronese mapping. For the lower bound, we show that points in general position with respect to polynomials transform to an equal-sized set of points in general position with respect to hyperplanes in the image space. Using this fact, we construct hard instances of  $D$ -POLYNOMIAL POINT COVER.

Note that the results of POINT HYPERPLANE COVER imply that HYPERPLANE POINT COVER in  $\mathbb{R}^d$  also has a tight kernel with  $\Omega(k^d)$  hyperplanes. Similarly, we can show that PROJECTIVE HYPERPLANE POINT COVER has a tight kernel with  $\Omega(k^{d-1})$  hyperplanes.

Due to paucity of space, we have moved some of the proofs to the Appendix.

## 2 Preliminaries

**Multivariate Polynomials.** Given a set  $\{X_1, X_2, \dots, X_d\}$  of variables a real multivariate polynomial on these variables is of the form  $P(X_1, \dots, X_d) = \sum_{i_1, i_2, \dots, i_d} a_{i_1 i_2 \dots i_d} \prod_{j \in [d]} X_j^{i_j}$  where  $[d] = \{1, \dots, d\}$  and  $a_{i_1 i_2 \dots i_d} \in \mathbb{R}$ . The set of all real multivariate polynomials in the variables  $X_1, \dots, X_d$  will be denoted by  $\mathbb{R}[X_1, X_2, \dots, X_d]$ . The degree of such a polynomial  $P(X_1, \dots, X_d)$  is defined as  $\deg(P) := \max\{i_1 + i_2 + \dots + i_d \mid a_{i_1 i_2 \dots i_d} \neq 0\}$ . A polynomial is said to be a  $D$ -degree polynomial if its degree is  $D$ .

In this paper, we are interested in the set/subsets of polynomials whose degree is bounded by  $D$ , for some  $D \in \mathbb{N}$ . In this context we define  $\text{Poly}_D[X_1, \dots, X_d] := \{f(X_1, \dots, X_d) \in$

$\mathbb{R}[X_1, X_2, \dots, X_d] \mid \deg(f) \leq D$ . Observe that  $\text{Poly}_D[X_1, \dots, X_d]$  is a vector space over  $\mathbb{R}$  with the following *monomials* as the basis  $\{X_1^{i_1} \dots X_d^{i_d} \mid 0 \leq \sum_{j=1}^d i_j \leq D\}$  and  $\left\{X_1^{i_1} \dots X_d^{i_d} \mid 0 \leq \sum_{j=1}^d i_j \leq D\right\} = \binom{D+d}{D}$ . For ease of notation, we define the vector  $X = (X_1, \dots, X_d)$ .

Given a polynomial  $f$  and a point  $p$ , the point *hits* the polynomial if  $f(p) = 0$ . In the same situation, the polynomial is said to *cover* the point.

► **Definition 1.** A vector space  $\mathcal{R}$  of polynomials in  $\mathbb{R}^d$  are said to be  $\alpha$ -good if for any positive integers  $b, m$  the following conditions hold:

1. In  $\mathcal{O}(1)$  time we can compute a set of  $b$  points in  $\mathbb{R}^d$  such that the set is in general position with respect to  $\mathcal{R}$ .
2. Given a  $d$ -dimensional  $m \times \dots \times m$  grid in  $\mathbb{R}^d$ , each polynomial in  $\mathcal{R}$  contains at most  $m^{d-\alpha}$  vertices of the grid, where  $\alpha > 0$ .

Hyperplanes, spheres, ellipses and many other natural vector spaces of polynomials are  $\alpha$ -good.

**General position in Geometry.** An  $i$ -flat in  $\mathbb{R}^d$  is the affine hull of  $i+1$  affinely independent points. The dimension of a (possibly infinite) set of points  $\mathcal{P}$ , denoted as  $\dim(\mathcal{P})$ , is the minimum  $i$  such that the entire set  $\mathcal{P}$  is contained in an  $i$ -flat of  $\mathbb{R}^d$  [17]. We use the term hyperplanes interchangeably for  $(d-1)$ -flats. A set  $\mathcal{P}$  of points in  $\mathbb{R}^d$  is said to be in general position with respect to hyperplanes, if for each  $i$ -flat,  $i \leq d-1$ , in  $\mathbb{R}^d$  there are at most  $i+1$  points from  $\mathcal{P}$  lying on the  $i$ -flat.

Consider, for  $i \leq d-1$ , a family  $\mathcal{F}$  of  $i$ -flats such that there is a point  $p$  that belongs to all the  $i$ -flats in  $\mathcal{F}$ . Then a set  $\mathcal{P} \in \mathbb{R}^d \setminus \{p\}$  of points is said to be in general position with respect to  $\mathcal{F}$  if each  $i$ -flat contains at most  $i$  points from  $\mathcal{P}$  lying on the  $i$ -flat. This is called general position in projective geometry.

Similarly, we can define the notion of general position (similarly, *projective* general position) with respect to multivariate polynomials. Let  $\mathcal{R}$  be a subvector space of  $\text{Poly}_D[X]$ , defined by a basis  $\{f_1(X), \dots, f_b(X), 1\}$  (or by a basis  $\{f_1(X), \dots, f_b(X)\}$  with  $\deg(f_i) > 0$ ). A subset of points are said to be in general position (or projective general position) with respect to the vector space  $\mathcal{R}$  of polynomials if no more than  $b$  points ( $b-1$  points) from the subset satisfy any equation of the form  $f(X) := \sum_{i=1}^b \lambda_i f_i(X) + \lambda_{b+1} = 0$  ( $f(X) := \sum_{i=1}^b \lambda_i f_i(X) = 0$ ), where all the  $\lambda_j \in \mathbb{R}$  and not all the  $\lambda_j$ 's can be zero simultaneously.

**Veronese mapping.** In this paper, one of our strategies for generalizing our results is to convert  $D$ -POINT POLYNOMIAL COVER in  $\mathbb{R}^d$  to POINT HYPERPLANE COVER (or PROJECTIVE POINT HYPERPLANE COVER) in  $\mathbb{R}^b$  by using a variant of Veronese mapping [18] from  $\mathbb{R}^d \rightarrow \mathbb{R}^b$ . The Veronese mapping of a vector space  $\mathcal{R}$  of  $D$ -degree polynomials, with a basis  $\{f_1(X), \dots, f_b(X), 1\}$  (or with a basis  $\{f_1(X), \dots, f_b(X)\}$  where  $\deg(f_i) > 0$ ), will be as the following:  $\Phi_{\mathcal{R}} : \mathbb{R}^d \rightarrow \mathbb{R}^b$ , where  $\Phi_{\mathcal{R}}(X) = (f_1(X), \dots, f_b(X))$  where  $X = (X_1, \dots, X_d)$ . Observe that if  $p = (p_1, \dots, p_d)$  satisfies the equation  $f(X) := \sum_{i=1}^b \lambda_i f_i(X) + \lambda_{b+1} = 0$  (or the equation  $f(X) := \sum_{i=1}^b \lambda_i f_i(X) = 0$ ) then  $\Phi_{\mathcal{R}}(p)$  will also satisfy the linear equation  $\sum_{j=1}^b \lambda_j Z_j + \lambda_{b+1} = 0$  (or the equation  $\sum_{j=1}^b \lambda_j Z_j = 0$ ), on the variable vector  $Z = (Z_1, \dots, Z_b)$ . In other words, for any set of points  $\mathcal{P}$  in  $\mathbb{R}^d$  and  $\mathcal{F}$ , the incidences between  $\mathcal{P}$  and  $\mathcal{R}$  and incidences between  $\Phi_{\mathcal{R}}(\mathcal{P})$  and hyperplanes in  $\mathbb{R}^b$  (or the hyperplanes passing through the origin in  $\mathbb{R}^b$ ) are preserved under the mapping  $\Phi_{\mathcal{R}}$ . Also, observe that there is a bijection between polynomials in  $\mathcal{R}$  and the hyperplanes in  $\mathbb{R}^b$  (or hyperplanes passing through the origin in  $\mathbb{R}^b$ ). This transformation from polynomials to hyperplanes is also referred to as *linearization*.

**Parameterized Complexity.** The instance of a parameterized problem/language is a pair containing the actual problem instance of size  $n$  and a positive integer called a parameter, usually represented as  $k$ . The problem is said to be in FPT if there exists an algorithm that solves the problem in  $f(k)n^{\mathcal{O}(1)}$  time, where  $f$  is a computable function. The problem is said to admit a  $g(k)$ -sized kernel, if there exists a polynomial time algorithm that converts the actual instance to a reduced instance of size  $g(k)$ , while preserving the answer. When  $g$  is a polynomial function, then the problem is said to admit a polynomial kernel. A reduction rule is a polynomial time procedure that changes a given instance  $I_1$  of a problem  $\Pi$  to another instance  $I_2$  of the same problem  $\Pi$ . We say that the reduction rule is *safe* when  $I_1$  is a YES instance of  $\Pi$  if and only if  $I_2$  is a YES instance. Readers are requested to refer [7] for more details on Parameterized Complexity.

**Lower bounds in Parameterized Algorithms.** There are several methods of showing lower bounds in parameterized complexity under standard assumptions in complexity theory. In this paper we require a lower bound technique given by Dell and Melkebeek [8]. This technique links kernelization to oracle protocols.

► **Definition 2.** Given a language  $L$ , an oracle communication protocol for  $L$  is a two-player communication protocol. The first player gets an input  $x$  and can only execute computations taking time polynomial in  $|x|$ . The second player is computationally unbounded, but does not know  $x$ . At the end of the protocol, the first player has to decide correctly whether  $x \in L$ . The cost of the protocol is the number of bits of communication from the first player to the second player.

► **Proposition 1.** [8] Let  $d \geq 2$  be an integer, and  $\epsilon$  be a positive real number. If  $\text{co-NP} \not\subseteq \text{NP/poly}$ , then there is no protocol of cost  $\mathcal{O}(n^{d-\epsilon})$  to decide whether a  $d$ -uniform hypergraph on  $n$  vertices has a  $d$ -hitting set of at most  $k$  vertices, even when the first player is co-nondeterministic.

Note that this implies that for any  $d \geq 2$  and any positive real number  $\epsilon$ , if  $\text{co-NP} \not\subseteq \text{NP/poly}$ , then there is no kernel of size  $\mathcal{O}(k^{d-\epsilon})$  for  $d$ -HITTING SET. In general, a lower bound for oracle communication protocols for a parameterized language  $L$  gives a lower bound for kernelization for  $L$ .

**Kernels: size vs number of elements.** In literature, a lower bound on the kernel means the lower bound on the size of the kernel, but not necessarily on the number of input elements in the kernel. Kratsch et. al [16] were one of the first to study lower bounds in terms of the number of input elements in the kernel. They used the results of Dell and Melkebeek [8] along with results in two dimensional geometry to build a new technique to show lower bounds for number of input elements in a kernel for a problem. In this paper, we have adhered to the general convention by saying that a kernel has a lower bound on its size if it has a lower bound on its representation in bits, while explicitly mentioning the cases where the kernel has a lower bound on the number of input elements.

### 3 Kernelization Lower bound for POINT HYPERPLANE COVER

In this Section, we show that POINT HYPERPLANE COVER in  $\mathbb{R}^d$  cannot have a kernel of size  $k^{d-\epsilon}$  if  $\text{co-NP} \not\subseteq \text{NP/poly}$ . We show by the standard technique of polynomial parameter transformation. For a fixed  $d$ , we reduce the  $d$ -HITTING SET problem to the problem of POINT HYPERPLANE COVER in  $\mathbb{R}^d$ . For our proof, first we state the folklore equivalence between POINT HYPERPLANE COVER in  $\mathbb{R}^d$  and HYPERPLANE POINT COVER in  $\mathbb{R}^d$ .

► **Lemma 3.** POINT HYPERPLANE COVER in  $\mathbb{R}^d$  and HYPERPLANE POINT COVER in  $\mathbb{R}^d$  are equivalent problems

From now on we will be showing lower bounds for HYPERPLANE POINT COVER. The proof strategy is the same as that in [16]. For this, we construct for each positive integer  $n$  and each  $d$ , a set of  $n$  points in  $\mathbb{R}^d$  with some special properties. This construction is more involved than in the case of POINT LINE COVER.

► **Lemma 4.** For every  $n \in \mathbb{Z}^+$ , there is a poly( $n$ ) time algorithm to construct a set  $\mathcal{P}$  of  $n$  points in  $\mathbb{R}^d$  that have the following properties:

- (1) The points are in general position.
- (2) Let  $\mathcal{H}$  be the family of hyperplanes defined by each set of  $d$  points from  $\mathcal{P}$ . The hyperplanes in the family  $\mathcal{H}$  are in general position, i.e., given  $r$  hyperplanes  $H_1, \dots, H_r$  in  $\mathcal{H}$  with  $r \leq d$  the dimension of the affine space  $\cap_{i=1}^r H_i$  is  $d - r$ .
- (3) For any point  $p$  in  $\mathbb{R}^d \setminus \mathcal{P}$ , there are at most  $d$  hyperplanes in  $\mathcal{H}$  that contain  $p$ .

**Proof.** The set  $\mathcal{P}$  is built inductively. When  $n = d$ , it is the base case and the construction follows trivially. Assume that for  $d \leq t < n$ , we have constructed a point set  $\mathcal{P}_t$  that satisfies the above conditions. As in [16], our goal will be to extend the point set  $\mathcal{P}_t$  by one point. We will show that points forbidden to be added to the set  $\mathcal{P}_t$  will lie on bounded number of hyperplanes and we will call these hyperplanes *forbidden hyperplanes*. Observe that the number of forbidden hyperplanes arising due to conditions (1) and (2) is bounded by  $O(t^d)$  and  $O(t^{d^2-1})$ .

Unlike the case when  $d = 2$ , it is harder to bound the number of forbidden hyperplanes due to condition (3). Let  $q \in \mathbb{R}^d$  be a point where the point set  $\mathcal{P}' = \mathcal{P}_t \cup \{q\}$  satisfies conditions (1) and (2), but not condition (3). We will call such a point  $q$  a *forbidden point*. Let  $\mathcal{H}'$  be the family of hyperplanes defined by each set of  $d$  points from  $\mathcal{P}'$ . Let  $H_1, \dots, H_{d+1}$  be a set of  $d + 1$  hyperplanes in  $\mathcal{H}'$  such that they intersect at point  $s$  with  $s \in \mathbb{R}^d \setminus \mathcal{P}'$ . Observe that since the point set  $\mathcal{P}_t$  satisfied all the three conditions,  $q$  will lie on at least 1 hyperplane and at most  $d$  hyperplanes from the family  $\{H_1, \dots, H_{d+1}\}$ . Suppose  $q$  was contained in  $d$  hyperplanes from the family, then  $q = s$  as  $\mathcal{P}'$  satisfies condition (2). Therefore, we assume that  $q$  lies in at least 1 and at most  $d - 1$  hyperplanes from the family  $\{H_1, \dots, H_{d+1}\}$ . Without loss of generality, assume that  $q$  lies on the hyperplanes  $\{H_1, \dots, H_r\}$ . Let  $A_{r-1}$  denote the  $(r - 1)$  dimensional affine plane  $\cap_{i=r+1}^{d+1} H_i$ . For  $j \in [r]$ , let the hyperplanes  $H_j$  be generated by the set  $\{q, p_1^j, \dots, p_{d-1}^j\} \subset \mathcal{P}'$ . The point  $s$  also belongs to  $A_{r-1}$ . Since we are interested in understanding where the forbidden point  $q$  can lie, we try to understand the inverse problem where  $A_{r-1}$ ,  $s$ , and points  $p_\ell^j$  (for all  $\ell \in [d - 1]$ ) are fixed and  $q$  is the variable point such that  $\cap_{i=1}^{d+1} H_i = s \in \mathbb{R}^d \setminus \mathcal{P}'$ . Using elementary Euclidean geometry, we get that  $q$  lies on a  $d - r$  dimensional affine plane passing through  $s$  and the slope of the affine plane depends only on  $A_{r-1}$  and the points  $p_\ell^j$ ,  $j \in [r]$  and  $\ell \in [d - 1]$ . This implies that as we vary  $s$  on  $A_{r-1}$  we will span a hyperplane containing  $A_{r-1}$  and which only depends on the points  $p_\ell^j$ ,  $j \in [r]$  and  $\ell \in [d - 1]$ . Therefore, once the hyperplanes  $H_{r+1}$  till  $H_{d+1}$  and the point set  $\{p_\ell^j | j \in [r], \ell \in [d - 1]\}$  are fixed, the point  $q$  will lie on a unique hyperplane. This implies that the number of forbidden hyperplanes due to condition (3) is bounded by  $O(t^{d^2+d-1})$ .

As we have an upper bound on the number of forbidden hyperplanes, we can now use the trick of Kratsch et. al. to generate points satisfying conditions (1) to (3) [16, Lemma 2.4]. In our case, we take a  $d$ -dimensional  $m \times \dots \times m$  grid with  $m = n^{d^2+d}$  and observe that the number of points from this  $d$ -dimensional grid that can lie on any hyperplane is bounded by  $m^{d-1}$ . ◀

Finally, we are ready to prove the main result.

► **Lemma 5.** HYPERPLANE POINT COVER in  $\mathbb{R}^d$  cannot have a kernel of size  $\mathcal{O}(k^{d-\epsilon})$  if  $\text{co-NP} \not\subseteq \text{NP}/\text{poly}$ .

**Proof.** We give a reduction from  $d$ -HITTING SET. Let  $(U, \mathcal{F}, k)$  be an instance of  $d$ -HITTING SET. First we reduce this instance to the following instance  $(U', \mathcal{F}', dk)$  where:

1. For each  $v \in U$  we make  $d$  copies  $\{v^1, v^2, \dots, v^d\}$ . We also refer to the set  $\{v^1, v^2, \dots, v^d\}$  as the row of  $v$ .
2.  $U' = U_1 \uplus U_2 \uplus \dots \uplus U_d$  such that for each  $i \in \{1, \dots, d\}$   $U_i = \{v^i | v \in U\}$ .
3.  $\mathcal{F} \subset \mathcal{F}'$
4. Assume that there is an arbitrary ordering on the vertices of  $U = \{v_1, v_2, \dots, v_n\}$ . For each  $f = \{v_{j_1}, v_{j_2}, \dots, v_{j_d}\} \in \mathcal{F}$ , and for each  $i_1, i_2, \dots, i_d \in \{1, \dots, d\}$ , we create a subset  $f_{i_1, i_2, \dots, i_d} = \{v_{j_1}^{i_1}, v_{j_2}^{i_2}, \dots, v_{j_d}^{i_d}\}$ . We put  $f_{i_1, i_2, \dots, i_d}$  in the set  $\mathcal{F}'$ .
5. For clarity of arguments in what follows, we give some more definitions. For each  $f = \{v_{j_1}, v_{j_2}, \dots, v_{j_d}\} \in \mathcal{F}$ , the subfamily of hyperedges  $\mathcal{F}_{v_{j_1}, v_{j_2}, \dots, v_{j_d}} = \{f_{i_1, i_2, \dots, i_d} = \{v_{j_1}^{i_1}, v_{j_2}^{i_2}, \dots, v_{j_d}^{i_d}\} | i_1, i_2, \dots, i_d \in \{1, \dots, d\}\}$  is called a subsystem of  $\{v_{j_1}, v_{j_2}, \dots, v_{j_d}\}$ . Also,  $\mathcal{F}_{v_{j_1}, v_{j_2}, \dots, v_{j_d}}$  is called a subsystem of  $f$ , for all hyperedges  $f \in \mathcal{F}_{v_{j_1}, v_{j_2}, \dots, v_{j_d}}$ . It follows from the previous definition that a row in a subsystem corresponds to the  $d$  copies of a vertex participating in the subsystem.

► **Claim 1** ( $\star^1$ ).  $(U, \mathcal{F}, k)$  is a YES instance of  $d$ -HITTING SET if and only if  $(U', \mathcal{F}', dk)$  is a YES instance of  $d$ -HITTING SET.

Next, we give a reduction from the instance  $(U', \mathcal{F}', dk)$  of  $d$ -HITTING SET to a instance of HYPERPLANE POINT COVER. The correctness of this reduction shows that there is a polynomial time reduction from  $d$ -HITTING SET to HYPERPLANE POINT COVER such that the parameter transformation is linear.

We construct the following instance of HYPERPLANE POINT COVER:

1. Using Lemma 4, we construct a set  $\mathcal{P}$  of  $dn$  points, same as the number of elements in the universe  $U'$ . We arbitrarily assign each element of  $U'$  to a unique point in  $\mathcal{P}$ .
2. For a hyperedge  $f \in \mathcal{F}'$ , let  $H_f$  be the hyperplane defined by the  $d$  points contained in  $f$ . The set  $\mathcal{H}$  is the family of such hyperplanes.

► **Claim 2.**  $(U, \mathcal{F}, k)$  is a YES instance of  $d$ -HITTING SET if and only if  $(\mathcal{H}, dk)$  is a YES instance of HYPERPLANE POINT COVER in  $\mathbb{R}^d$ .

Before we prove this claim, we need the following claim regarding a solution set with minimum number of points outside  $\mathcal{P}$ .

► **Claim 3** ( $\star$ ). Let  $Q$  be a minimum sized set of points that covers all the hyperplanes in  $\mathcal{H}$ . Also, assume that  $Q$  has the minimum possible points in  $Q \setminus \mathcal{P}$ . Moreover, let  $q \in Q \setminus \mathcal{P}$  that covers the minimum number of hyperedges uniquely. We assume that there is no other set  $Q'$  of the same size as  $Q$ , with  $|Q' \setminus \mathcal{P}| = |Q \setminus \mathcal{P}|$  and with a  $q' \in Q' \setminus \mathcal{P}$  that covers strictly less number of hyperedges uniquely in  $Q'$  than  $q$  does in  $Q$ . Then for any element  $v \in U' \setminus Q$ , at most  $d - 1$  hyperedges containing  $v$  can have no intersection with  $Q$ .

**Proof of Claim 2.** The forward direction is simple. A solution for  $(U, \mathcal{F}, k)$  gives a solution for  $(U', \mathcal{F}', dk)$  as described in Claim 1. By the correspondence of the vertices in  $U'$  with the points in  $\mathcal{P}$ , we obtain an equal sized solution set for  $(\mathcal{H}, dk)$ .

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<sup>1</sup> Results marked with a  $\star$  have their proofs in the Appendix.



On the other hand, suppose  $(\mathcal{H}, dk)$  be a YES instance. Let  $Q$  be a solution as described in Claim 3. We contradict the choice of  $Q$  by exhibiting a set  $Q'$  that is of the size at most that of  $Q$  and with strictly less bad points, or that there is a point  $q' \in Q'$  that covers strictly less number of hyperedges uniquely with respect to  $Q'$  than  $q$  does with respect to  $Q$ .

As argued in the proof of Claim 3, by the choice of  $Q$ , each point in  $Q \setminus \mathcal{P}$  can uniquely cover at least 2 and at most  $d$  hyperplanes in  $\mathcal{H}$ . Let  $q \in Q \setminus \mathcal{P}$  be a point that covers minimum number of hyperedges uniquely. Such a point is called a “bad” point. Let  $H_f \in \mathcal{H}$  be a hyperplane uniquely covered by  $q$ . By definition,  $H$  contains a set of  $d$  points from  $P$  and corresponds to a hyperedge  $f \in \mathcal{F}'$ . We call such a hyperplane and its corresponding hyperedge as a “bad” hyperplane and a “bad” hyperedge respectively. Similarly, the subsystem corresponding to the hyperedge  $f$  is also called a bad subsystem.

First, suppose the hyperedge  $f$  is  $f_{i_1, i_2, \dots, i_d} = \{v_{j_1}^{i_1}, v_{j_2}^{i_2}, \dots, v_{j_d}^{i_d}\}$ , for a subset  $\{v_{j_1}, v_{j_2}, \dots, v_{j_d}\} \subset U$ , and for a set  $\{i_1, i_2, \dots, i_d\} \in \{1, \dots, d\}$ . We show that from the set  $\hat{S} = \{v_{j_t}^\ell | t \in [d], \ell \in [d], \ell \neq i_t\}$  at most  $d-2$  vertices do not belong to  $Q$ . Consider the vertex  $v_{j_1}^{i_1} \in U'$ . From the set  $\{v_{j_t}^\ell | t \in \{2, \dots, d\}, \ell \in [d], \ell \neq i_t\}$  at most  $d-2$  vertices do not belong to  $Q$ . Otherwise,  $v_{j_1}^{i_1}$  is involved in more than  $d-1$  bad hyperedges, contrary to Claim 3. By pigeonhole principle, there is a  $v_{j_b}^{i_b}, b \neq 1$  such that all vertices in  $\{v_{j_t}^\ell | \ell \in [d], \ell \neq i_t\}$  belong to  $Q$ . This means that the vertices of  $\hat{S}$  that are not in  $Q$  are restricted to the set  $\{v_{j_t}^\ell | t \in [d] \setminus \{b\}, \ell \in [d], \ell \neq i_t\}$ . Again, this set can have at most  $d-2$  vertices missing from  $Q$  as otherwise the vertex  $v_{j_b}^{i_b}$  will be involved in more than  $d-1$  bad hyperedges, contradicting Claim 3.

Since we considered an arbitrary  $q \in Q \setminus \mathcal{P}$ , this implies that in a subsystem  $\mathcal{F}_1$  for any bad hyperedge  $f_1$ , there are two rows where except for the vertices contained in  $f_1$ , all other vertices are in  $Q$ . We call such rows as “covered” rows. Let one such covered row correspond to the vertex  $x \in U$  and let the copy of  $x$  in  $f$  be  $x^i$ . Let the other covered row correspond to a vertex  $z_1 \in U$ . Then, by Claim 3,  $x^i$  can belong to at most  $d-2$  other bad hyperedges  $\{f_2, f_3, \dots, f_{d-1}\}$ . Thus there are at most  $d-1$  bad subsystems  $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \dots, \mathcal{F}_{d-1}\}$  that correspond to the vertex  $x \in U$ . Each such subsystem has two covered rows, one of which is the row corresponding to  $x$ . For the subsystem  $\mathcal{F}_t, t \in [d-1]$  let the other covered row be corresponding to  $z_t \in U$ . We give a one-one map  $\phi$  from the vertices  $\{z_1, z_2, \dots, z_{d-1}\}$  to the copies  $\{x^j | j \in [d], j \neq i\}$ . Given  $j \in [d-1]$ , we delete  $\phi(z_j)$  from  $Q$  and include the copy of  $z_j$  that belongs to  $f_j$  into  $Q$ . This way we form a new subset  $Q'$ . First we show that  $Q'$  covers all hyperedges in  $\mathcal{F}'$ . The hyperedges that might not be covered by  $Q'$  are hyperedges that contain a vertex from  $\{x^j | j \in [d], j \neq i\}$ . Since for each  $t \in [d-1]$  all copies of  $z_t$  belong to  $Q'$ , all hyperedges of the subsystem  $\mathcal{F}_t$  are covered by  $Q'$ . In fact, no hyperedge in the subsystem  $\mathcal{F}_t$  is bad. Suppose there is a  $x^j, j \neq i$  and a hyperedge  $f = \{x^j, u_1, \dots, u_{d-1}\}$ , for some  $\{u_1, \dots, u_{d-1}\} \subset U'$ , such that  $Q'$  does not cover  $f$ . Then the hyperedge  $f' = \{x^i, u_1, \dots, u_{d-1}\}$  is either not covered or covered by a bad point from  $Q \setminus \mathcal{P}$ . As argued before, this hyperedge cannot belong to any subsystem  $\mathcal{F}_t, t \in [d-1]$ . This implies that there are at least  $d$  bad hyperedges containing  $x^i$  with respect to  $Q$ , which contradicts Claim 3. Thus, all hyperedges in  $\mathcal{F}'$  are covered by  $Q'$ .

Moreover, by the arguments above, all the bad hyperedges containing a copy of  $x \in U$ , including  $f_1$ , are no longer bad hyperedges with respect to  $Q'$ . On the other hand, any hyperedge not belonging to the subsystems  $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \dots, \mathcal{F}_{d-1}\}$  and that was covered by a point  $p$  in  $Q$  is still covered by  $p$  in  $Q'$ . Thus, the newly constructed  $Q'$  has the size at most that of  $Q$ , but where  $q$  uniquely covers strictly less number of bad hyperedges with respect to  $Q'$  than it did with respect to  $Q$ . Therefore, we contradict the choice of  $Q$ .

This implies, that there is a  $k$ -sized solution for  $(\mathcal{H}, dk)$  that only contains points from  $\mathcal{P}$ . Therefore, the corresponding vertices will be a solution for  $(U', \mathcal{F}, dk)$ . By Claim 1, this

implies that  $(U, \mathcal{F}, dk)$  is a YES instance.  $\blacktriangleleft$

Hence, by Claim 2, we show that there is a linear parameter transformation from  $d$ -HITTING SET to HYPERPLANE POINT COVER in  $\mathbb{R}^d$ . This implies that HYPERPLANE POINT COVER in  $\mathbb{R}^d$  cannot have a kernel of size  $\mathcal{O}(k^{d-\epsilon})$  if  $\text{co-NP} \not\subseteq \text{NP/poly}$ .  $\blacktriangleleft$

The following Corollary follows from Lemma 5 and Lemma 3.

► **Corollary 6.** POINT HYPERPLANE COVER in  $\mathbb{R}^d$  cannot have a kernel of size  $\mathcal{O}(k^{d-\epsilon})$  if  $\text{co-NP} \not\subseteq \text{NP/poly}$ .

### 3.1 Tight Kernels in Projective Geometry

In this section, we consider PROJECTIVE POINT HYPERPLANE COVER in  $\mathbb{R}^d$ , when we are only allowed to use hyperplanes passing through a fixed point, and we are not allowed to include this fixed point in the solution set of points. Without loss of generality, we can assume that that fixed point is at the origin. Also, if the point at the origin is part of the input for the set  $\mathcal{P}$  of points to be covered, then we can reduce the instance to that of covering all the points of  $\mathcal{P}$  except the point at the origin. This is because, even if there is one other point in  $\mathcal{P}$  this point needs to be covered by a hyperplane through the origin and this automatically ensures that the point at the origin is covered. Thus, we assume that the point at the origin is not in  $\mathcal{P}$ . We exhibit a tight kernelization for this problem.

We will now show the equivalence between PROJECTIVE POINT HYPERPLANE COVER problem in  $\mathbb{R}^d$  and POINT HYPERPLANE COVER problem in  $\mathbb{R}^{d-1}$ .

► **Lemma 7** ( $\star$ ). PROJECTIVE POINT HYPERPLANE COVER in  $\mathbb{R}^d$  is equivalent to POINT HYPERPLANE COVER in  $\mathbb{R}^{d-1}$ .

**Proof sketch.** The proof idea for this equivalence is to map the given point set  $\mathcal{P}$  in  $\mathbb{R}^d$  to a point set on a hyperplane  $H$  that does not pass through the origin  $o$  in  $\mathbb{R}^d$ . Note that  $H$  is isomorphic to  $\mathbb{R}^{d-1}$ . The equivalence is in showing that it is enough to cover the mapped point set with  $(d-2)$ -flats contained in  $H$  in order to extract a family of hyperplanes that pass through  $o$  and cover  $\mathcal{P}$ . The full proof is in the Appendix.  $\blacktriangleleft$

Thus, following from Lemma 7, we obtain tight kernels for PROJECTIVE POINT HYPERPLANE COVER in  $\mathbb{R}^d$ .

- **Lemma 8.** 1. PROJECTIVE POINT HYPERPLANE COVER in  $\mathbb{R}^d$  has a kernel of size  $\mathcal{O}(k^{d-1})$ .  
 2. PROJECTIVE HYPERPLANE POINT COVER in  $\mathbb{R}^d$  cannot have a kernel of size  $\mathcal{O}(k^{d-1-\epsilon})$  if  $\text{co-NP} \not\subseteq \text{NP/poly}$ .

## 4 Lower bounds on the number of elements in the kernel for POINT HYPERPLANE COVER

In this section, by the method suggested by Dell and Melkebeek [8], we can show a lower bound on the number of points of a polynomial kernel for POINT HYPERPLANE COVER in  $\mathbb{R}^d$ , for each fixed positive integer  $d$ .

In  $\mathbb{R}^d$ , we denote the coordinates of a point  $u$  as  $(u^1, u^2, \dots, u^d)$ . Given a set of points in  $\mathbb{R}^d$ , for each set  $\{u_1, u_2, \dots, u_d, u_{d+1}\}$  of  $d+1$  points we consider the following matrix:



We define orientation  $\langle u_1, u_2, \dots, u_d, u_{d+1} \rangle = \text{sgndet}M(\langle u_1, u_2, \dots, u_d, u_{d+1} \rangle)$ . Here,  $\text{det}$  is the determinant function and  $\text{sgn}$  is the sign function.

► **Observation 1** ( $\star$ ). orientation  $\langle u_1, u_2, \dots, u_d, u_{d+1} \rangle = 0$  if and only if the points are not in general position.

Given a set  $\mathcal{P}$  of points, and an arbitrary ordering  $\mathcal{P}'$  of  $\mathcal{P}$ , an *order type* of  $\mathcal{P}'$  is a function that maps each ordered set of  $d+1$  points to their orientation. Two sets of points  $\mathcal{P}$  and  $\mathcal{Q}$  are said to be combinatorially equivalent if there is an ordering  $\mathcal{P}'$  of  $\mathcal{P}$  and an ordering  $\mathcal{Q}'$  of  $\mathcal{Q}$  such that the order types of the two ordering are the same.

As proved in [16], two combinatorially equivalent instances are also equivalent with respect to the problem of POINT HYPERPLANE COVER in  $\mathbb{R}^d$ .

► **Observation 2.** Let  $\mathcal{P}$  and  $\mathcal{Q}$  be two sets of points that are combinatorially equivalent. Then  $(\mathcal{P}, k)$  is a YES instance of POINT HYPERPLANE COVER if and only if  $(\mathcal{Q}, k)$  is a YES instance of POINT HYPERPLANE COVER.

The proof of this observation is same as the proof of Lemma 2.1 in [16].

This leads us to the the next lemma, which follows from Theorem 3.2 in [16] and Theorem 4.1 of [3].

► **Lemma 9.** POINT HYPERPLANE COVER in  $\mathbb{R}^d$  cannot have a kernel with  $\mathcal{O}(k^{d-\epsilon})$  points if  $\text{co-NP} \not\subseteq \text{NP/poly}$ .

Since POINT HYPERPLANE COVER and HYPERPLANE POINT COVER are equivalent problems, we obtain the following corollary.

► **Corollary 10.** HYPERPLANE POINT COVER in  $\mathbb{R}^d$  cannot have a kernel with  $\mathcal{O}(k^{d-\epsilon})$  hyperplanes if  $\text{co-NP} \not\subseteq \text{NP/poly}$ .

We also obtain lower bounds on the number of points in the kernel for PROJECTIVE POINT HYPERPLANE COVER. This also implies a lower bound in the number of hyperplanes in a kernel for PROJECTIVE HYPERPLANE POINT COVER.

► **Corollary 11** ( $\star$ ). PROJECTIVE POINT HYPERPLANE COVER in  $\mathbb{R}^d$  cannot have a kernel with  $\mathcal{O}(k^{d-1-\epsilon})$  points if  $\text{co-NP} \not\subseteq \text{NP/poly}$ .

## 5 Covering Polynomials of bounded degree with Points

In this section, we consider the  $D$ -POLYNOMIAL POINT COVER problem and show that this problem is equivalent to HYPERPLANE POINT COVER in a higher dimensional space. We utilize this to give tight polynomial kernels for  $D$ -POLYNOMIAL POINT COVER, when the underlying vector space of polynomials is  $\alpha$ -good.

Recall that in  $D$ -POLYNOMIAL POINT COVER, a vector space  $\mathcal{R}$  of  $D$ -degree polynomials in  $\text{Poly}_D[X_1, X_2, \dots, X_d]$  are specified. The input is a set  $\mathcal{F}$  of  $n$  polynomials from  $\mathcal{R}$  and the objective is to find at most  $k$  points in  $\mathbb{R}^d$  that cover all the input polynomials.

We utilize the Veronese mapping from a vector space of  $D$ -degree polynomials to the subsystem of hyperplanes in Euclidean space  $\mathbb{R}^b$ . Such a mapping is a bijective mapping between the vector space of  $D$ -degree polynomials and the hyperplanes in  $\mathbb{R}^b$ . However, the mapping need not be an onto mapping from  $\mathbb{R}^d$  to  $\mathbb{R}^b$ . Let  $\text{Ver}_{\mathcal{R}}(\mathbb{R}^d)$  be the image of  $\mathbb{R}^d$  under the Veronese mapping  $\Phi_{\mathcal{R}}$ . Thus,  $\text{Ver}_{\mathcal{R}}(\mathbb{R}^d) \subseteq \mathbb{R}^b$ . We show that the HYPERPLANE POINT COVER problem for an  $\alpha$ -good vector space  $\mathcal{R}$  in  $\mathbb{R}^b$  when the solution set is restricted to belonging to  $\text{Ver}_{\mathcal{R}}(\mathbb{R}^d)$ , does not have a  $\mathcal{O}(k^{b-\epsilon})$  kernel unless  $\text{co-NP} \subseteq \text{NP/poly}$ .

Before this, we require a few results regarding the behaviour of points under the Veronese mapping.

First, we show that a set of  $n$  points in  $\mathbb{R}^d$  that are in general position with respect to  $\mathcal{R}$  are mapped to a set of  $n$  points in  $\mathbb{R}^b$  with respect to hyperplanes in  $\mathbb{R}^b$ .

► **Claim 4.** Let  $\mathcal{P}$  be a set of points in  $\mathbb{R}^d$ , and  $\mathcal{R}$  be a subspace of  $\text{Poly}_D[X_1, \dots, X_d]$  with a basis  $\{f_1(X), \dots, f_b(X), 1\}$  where  $X = (X_1, \dots, X_d)$ .

1. If the set  $\mathcal{P}$  is in general position with respect to the polynomial family  $\mathcal{R}$  then the image  $\Phi_{\mathcal{R}}(\mathcal{P})$ , under the Veronese mapping  $\Phi_{\mathcal{R}}$ , is a  $|\mathcal{P}|$ -sized set in general position with respect to hyperplanes in  $\mathbb{R}^b$ .
2. Let  $S = \{q_1, \dots, q_\ell\} \subseteq \Phi_{\mathcal{R}}(\mathcal{P})$  be in general position with respect to hyperplanes in  $\mathbb{R}^b$ . Then the set  $S' = \{p_1, \dots, p_\ell\}$ , where  $p_i \in \Phi_{\mathcal{R}}^{-1}(q_i) \cap \mathcal{P}$ , will be a  $|S|$ -sized set in general position with respect to  $\mathcal{R}$ .

**Proof. 1.** First, observe that if the map  $\Phi_{\mathcal{R}}$  is injective on  $\mathcal{P}$  then the result will directly follow. However, in general, the map  $\Phi_{\mathcal{R}}$  need not be an injective mapping on an arbitrary set of  $n$  points in  $\mathbb{R}^d$ . We show that  $\Phi_{\mathcal{R}}$  is injective when restricted to  $\mathcal{P}$  if  $\mathcal{P}$  is in general position with respect to  $\mathcal{R}$ . To reach a contradiction, let  $\Phi_{\mathcal{R}}(p_1) = \Phi_{\mathcal{R}}(p_2)$  where  $p_1, p_2 (\neq p_1) \in \mathcal{P}$ . Let  $S \subseteq \mathcal{P}$  be of size  $b + 1$  and  $p_1, p_2 \in \mathcal{P}$ . Observe that the set  $\Phi_{\mathcal{R}}(S)$  will have less than  $b + 1$  points and this will imply that there exists a hyperplane  $\sum_{i=1}^b \lambda_i Z_i + \lambda_{b+1} = 0$  on which the set  $\Phi_{\mathcal{R}}(S)$  will lie. But this implies that the polynomial  $\sum_{i=1}^b \lambda_i f_i(X) + \lambda_{b+1} = 0$  will be satisfied by all the points in  $S$ . Thus, we have reached a contradiction from the fact that the point set  $\mathcal{P}$  was in general position.

2. The second part of the Claim follows directly from the construction of the mapping  $\Phi_{\mathcal{R}}$ . ◀

Next, for each  $n$  we construct a set of  $n$  points that satisfy the conditions of Lemma 4 and belong to  $\text{Ver}_{\mathcal{R}}(\mathbb{R}^d)$ , where  $\mathcal{R}$  is  $\alpha$ -good. This construction is mainly done in  $\mathbb{R}^d$  and follows exactly along the lines of the proof of Lemma 4.

► **Lemma 12.** Let  $\mathcal{R}$  be a  $\alpha$ -good vector space of  $D$ -degree polynomials in  $\mathbb{R}^d$  and let the Veronese mapping  $\Phi_{\mathcal{R}}$  linearize  $\mathcal{R}$  into  $\mathbb{R}^b$ . Let  $\text{Ver}_{\mathcal{R}}(\mathbb{R}^d)$  be the image of  $\Phi_{\mathcal{R}}$ . Then for every  $n \in \mathbb{Z}^+$ , there is a  $\text{poly}(n)$  time algorithm to construct a set  $\mathcal{P}$  of  $n$  points in  $\mathbb{R}^b$  that have the following properties:

- (1) The points are in general position with respect to hyperplanes in  $\mathbb{R}^b$ .
- (2) Let  $\mathcal{H}$  be the family of hyperplanes defined by each set of  $b$  points from  $\mathcal{P}$ . The hyperplanes in the family  $\mathcal{H}$  are in general position, i.e., given  $r$  hyperplanes  $H_1, \dots, H_r$  in  $\mathcal{H}$  with  $r \leq b$  the dimension of the affine space  $\cap_{i=1}^r H_i$  is  $b - r$ .
- (3) For any point  $p$  in  $\mathbb{R}^b \setminus \mathcal{P}$ , there are at most  $b$  hyperplanes in  $\mathcal{H}$  that contain  $p$ .

**Proof.** As in the proof of Lemma 4, we will construct the set  $\mathcal{P}$  inductively. We start with a set  $\mathcal{P}_b$  of size  $b$  such that the set is in general position with respect to  $\mathcal{R}$ . This can be constructed in  $O(1)$  time as  $\mathcal{R}$  is  $\alpha$ -good. We then extend this set one point at a time using points from the grid (as in the proof of Lemma 4). Assume that for  $b \leq t < n$ , we have constructed a point set  $\mathcal{P}_t$  that satisfies the above conditions. The points forbidden to be added to the set  $\mathcal{P}_t$  will lie on bounded number of polynomials from  $\mathcal{R}$  and we will call these polynomials *forbidden polynomials*. The hyperplane that is in bijective correspondence with a forbidden polynomial under the Veronese mapping  $\Phi_{\mathcal{R}}$  is called a *forbidden hyperplane*. As in the proof of Lemma 4, we can show, using the Veronese mapping  $\Phi_{\mathcal{R}}$ , that the number of forbidden hyperplanes arising due to conditions (1), (2) and (3) is bounded by  $O(t^b)$ ,

$O(t^{d^2-1})$  and  $O(t^{b^2+b-1})$  respectively. This also gives a bound on the number of forbidden polynomials.

As we have an upper bound on the number of forbidden polynomials, we can now use the same trick to generate points satisfying conditions (1) to (3) as in the paper Lemma 4. In this case we take a  $d$ -dimensional  $m \times \dots \times m$  grid with  $m = n^{b^2+b}$  and use the fact given any polynomial from  $\mathcal{R}$  number of points of the grid hitting it is bounded by  $m^{d-\alpha}$ . This completes the proof.  $\blacktriangleleft$

This helps us to prove a kernel lower bound on the restricted version of HYPERPLANE POINT COVER described above.

**► Lemma 13.** *Let  $\mathcal{R}$  be an  $\alpha$ -good vector space of  $D$ -degree polynomials in  $\mathbb{R}^d$  and let the Veronese mapping  $\Phi_{\mathcal{R}}$  linearize  $\mathcal{R}$  into  $\mathbb{R}^b$ . Then HYPERPLANE POINT COVER, when the solution is restricted to belong to  $\text{Ver}_{\mathcal{R}}(\mathbb{R}^d) = \Phi_{\mathcal{R}}(\mathbb{R}^d)$ , cannot have a kernel of size  $\mathcal{O}(k^{b-\epsilon})$ , unless  $\text{co-NP} \subseteq \text{NP/poly}$ .*

**Proof.** The construction of  $n$  points in  $\text{Ver}_{\mathcal{R}}(\mathbb{R}^d)$  described in Lemma 12 has all the properties described in Lemma 4. The rest of the proof follows exactly as the proof of Lemma 5.  $\blacktriangleleft$

Finally, the following Theorem is derived from Lemma 13 by utilizing the Veronese mapping  $\Phi_{\mathcal{R}}$ .

**► Theorem 14.**  *$D$ -POLYNOMIAL POINT COVER for an  $\alpha$ -good vector space  $\mathcal{R}$  in  $\mathbb{R}^d$ , and having the Veronese mapping into  $\mathbb{R}^b$ , does not have a polynomial kernel of size  $\mathcal{O}(k^{b-\epsilon})$ , unless  $\text{co-NP} \subseteq \text{NP/poly}$ .*

**Proof.** To prove the tightness of a  $\mathcal{O}(k^b)$  kernel for  $D$ -POLYNOMIAL POINT COVER in  $\mathbb{R}^d$  with the Veronese mapping into  $\mathbb{R}^b$ , first we give an upper bound on the size of a kernel. Let the polynomials in an instance of  $D$ -POLYNOMIAL POINT COVER come from the vector space  $\mathcal{R}$ , as defined earlier. The Veronese mapping  $\Phi_{\mathcal{R}}$  is a reduction from  $D$ -POLYNOMIAL POINT COVER in  $\mathbb{R}^d$  to HYPERPLANE POINT COVER in  $\mathbb{R}^b$ . Thus, since HYPERPLANE POINT COVER in  $\mathbb{R}^b$  has a  $\mathcal{O}(k^b)$  kernel [17], so does  $D$ -POLYNOMIAL POINT COVER in  $\mathbb{R}^d$  with the Veronese mapping into  $\mathbb{R}^b$ .

To show the other direction, we use the Veronese mapping on the vector space  $\mathcal{R}$  of  $D$ -degree polynomials more carefully. Let the hyperplanes, to which the polynomials are mapped, be in  $\mathbb{R}^b$ . The mapping is a bijective function. Thus, in order to obtain the required result, we give a reduction from HYPERPLANE POINT COVER in  $\mathbb{R}^b$ , where the solution set of points come from  $\text{Ver}_{\mathcal{R}}(\mathbb{R}^d)$ . The reduction is simply the reverse function of the Veronese mapping. Suppose an instance  $(\mathcal{H}, k)$  of HYPERPLANE POINT COVER where the solution set of points belong to  $\text{Ver}_{\mathcal{R}}(\mathbb{R}^d)$  reduces to the instance  $(\mathcal{H}', k)$  of  $D$ -POLYNOMIAL POINT COVER. If  $(\mathcal{H}, k)$  is a YES instance, then there is a set  $S$  of at most  $k$  points in  $\mathbb{R}^b$  that covers all the hyperplanes in  $\mathcal{H}$ . Consider the set  $S'$  of points in  $\mathbb{R}^d$  by taking one preimage of each point in  $S$ . The set  $S'$  is exactly the same size as  $S$ , and therefore contains at most  $k$  points. Moreover, by definition of the Veronese mapping,  $S'$  covers all the polynomials in  $\mathcal{H}'$ . Therefore,  $(\mathcal{H}', k)$  is also a YES instance for  $D$ -POLYNOMIAL POINT COVER.

On the other hand, if  $(\mathcal{H}', k)$  is a YES instance of  $D$ -POLYNOMIAL POINT COVER, then there is a set  $S'$  of at most  $k$  points that cover all the polynomials in  $\mathcal{H}'$ . The image of  $S'$  under the Veronese mapping will be of size at most  $S'$  and will cover the family  $\mathcal{H}$ . Therefore,  $(\mathcal{H}, k)$  will be a YES instance of HYPERPLANE POINT COVER when the solution points can come only from  $\text{Ver}_{\mathcal{R}}(\mathbb{R}^d)$ . Thus, by Lemma 13, we conclude that  $D$ -POLYNOMIAL POINT COVER cannot have a kernel of size  $\mathcal{O}(k^{b-\epsilon})$  unless  $\text{co-NP} \subseteq \text{NP/poly}$ .  $\blacktriangleleft$

## 6 Conclusion

The  $D$ -POINT POLYNOMIAL COVER problem in  $\mathbb{R}^d$  requires a set of  $n$  points in  $\mathbb{R}^d$  to be covered by at most  $k$   $D$ -degree polynomials from a vector space  $\mathcal{R}$  of polynomials. Although polynomial kernels for  $D$ -POINT POLYNOMIAL COVER in  $\mathbb{R}^d$  can be exhibited, tight lower bounds for this problem are unknown. Similarly, a tight lower bound for the kernels for the general  $D$ -POLYNOMIAL POINT COVER in  $\mathbb{R}^d$  is open, as is a tight bound for the number of polynomials in the kernel. For these problems, it might be useful to understand the structural and topological properties of the image of a Veronese mapping for a vector space  $\mathcal{R}$  of  $D$ -degree polynomials in  $\mathbb{R}^d$ . It would be interesting to search for common structural properties of the Veronese mapping, over all vector spaces of  $D$ -degree polynomials.

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### Proof of Claim 1

**Proof.** In the forward direction, suppose  $S$  is a solution for  $d$ -HITTING SET in  $(U, \mathcal{F}, k)$ . We define  $S' = \{v_i | i \in \{1, \dots, d\}, v \in S\}$ . Suppose  $S'$  is not a solution for the instance  $(U', \mathcal{F}', dk)$ . Then by definition, there is a hyperedge  $f\{v_{j_1}, v_{j_2}, \dots, v_{j_d}\} \in \mathcal{F}$  and  $i_1, i_2, \dots, i_d \in \{1, \dots, d\}$ , such that  $f_{i_1, i_2, \dots, i_d} \in \mathcal{F}'$  is not hit by  $S'$ . This means, by the definition of  $S'$  that  $\{v_{j_1}, v_{j_2}, \dots, v_{j_d}\} \notin S$ . However, this implies that the hyperedge  $f$  is not hit by  $S$ , which is a contradiction.

In the backward direction, suppose  $S'$  is a solution for  $(U', \mathcal{F}', dk)$ . For each  $i \in \{1, 2, \dots, d\}$ , let  $S_i = \{v | v^i \in S'\}$ . By Pigeonhole Principle, there is one  $i \in \{1, \dots, d\}$  such that  $|S_i| \leq k$ . We show that  $S_i$  is a solution for the instance  $(U, \mathcal{F}, k)$ . Suppose  $S_i$  is not a solution for  $G$ . Then there is a hyperedge  $f \in \mathcal{F}$  that is not hit by  $S_i$ . This implies that the hyperedge  $f_{i_1, i_2, i_3, \dots, i_d} \in \mathcal{F}'$  with  $i_1 = i_2 = \dots = i_d = i$  is also not hit by  $S'$ , which is a contradiction.

Thus, the claim is proved. ◀

### Proof of Claim 3

**Proof.** Firstly, by the condition of minimality on  $Q$ , each point in  $Q \setminus \mathcal{P}$  must uniquely cover at least 2 hyperplanes in  $\mathcal{H}$ . Otherwise we could find a equal-sized solution  $Q'$  where  $|Q' \setminus \mathcal{P}| < |Q \setminus \mathcal{P}|$ , which is a contradiction.

Suppose that there is a vertex  $v \in U' \setminus Q$  such that at least a family  $\mathcal{H}'$  of  $d$  hyperedges in  $\mathcal{H}$  containing  $v$  can have no intersection with  $Q$ . These  $d$  hyperplanes are covered by a set  $Q'$  of points that are in  $Q \setminus \mathcal{P}$ . Suppose  $Q' = \{u_1, \dots, u_\ell\}$  such that the for each  $j \in \{1, \dots, \ell\}$ ,  $u_j$  uniquely covers  $c_j$  hyperplanes of  $\mathcal{H}'$ . By definition,  $\sum_j c_j = d$ . By the minimality condition of  $Q$  and property (2) of Lemma 4, each such point in  $Q' \subseteq Q \setminus \mathcal{P}$  uniquely covers between 2 to  $d$  hyperplanes of  $\mathcal{H}$ . Thus, for each  $j \in \{1, 2, \dots, \ell\}$  the vertex  $u_j$  covers at most  $d - c_j$  hyperplanes not in  $\mathcal{H}'$ . We call the family of all hyperplanes covered by vertices of  $Q'$  as  $\mathcal{H}''$ . This family has at most  $d(d-1) + d$  hyperplanes. We construct the following set  $\hat{Q}$ :

- All points of  $Q \setminus Q'$  are included in  $\hat{Q}$ . The point  $v$  is also included.
- For each  $j \in \{1, \dots, \ell\}$ , let  $\mathcal{H}_j$  be the subfamily of at most  $d - c_j$  hyperplanes that are uniquely covered by the vertex  $u_j$  and which are not in  $\mathcal{H}'$ . Starting from  $j = 1$ , we build a subfamily  $\mathcal{H}'_j$  and find a point  $u'_j$  corresponding to  $u_j$ . First all the hyperplanes in  $\mathcal{H}_j$  are added to  $\mathcal{H}'_j$ . Then, iterating a variable  $t$  from  $j + 1$  to  $\ell$ , we add the hyperplanes in  $\mathcal{H}_t$  till there are  $d$  hyperplanes or all hyperplanes in  $\bigcup_{t \geq j} \mathcal{H}_t$  have been added. Take a point in the intersection of  $\mathcal{H}'_j$  and name that point  $u'_j$ . We show that the last nonempty subfamily  $\mathcal{H}_t$  must be for  $t < \ell$ . Suppose not. Then by definition, when we consider the last point  $u_\ell$ , the number of hyperplanes in  $\mathcal{H}_\ell$  that are not yet covered by  $\{u'_1, \dots, u'_{\ell-1}\}$  are at most  $d - c_\ell - \sum_{j < \ell} c_j = d - \sum_{j \leq \ell} c_j = 0$ . Therefore,  $\{c'_1, c'_2, \dots, c'_{\ell-1}\}$  cover all the hyperplanes in  $\mathcal{H}'' \setminus \mathcal{H}'$ . By definition of  $\mathcal{H}'$ , the set  $\{v, c'_1, c'_2, \dots, c'_{\ell-1}\}$  covers all the hyperplanes in  $\mathcal{H}''$ .

By definition the size of  $\hat{Q}$  is at most that of  $Q$ . However, the number of vertices in  $\hat{Q} \setminus \mathcal{P}$  is strictly less than the number of vertices in  $Q \setminus \mathcal{P}$ . This is a contradiction to the definition of  $Q$ .

Hence, we have proven the claim. ◀



### Proof of Lemma 7

**Proof.** First, we reduce PROJECTIVE POINT HYPERPLANE COVER in  $\mathbb{R}^d$  to POINT HYPERPLANE COVER in  $\mathbb{R}^{d-1}$ . Suppose we are given an instance  $(\mathcal{P}, k)$  of PROJECTIVE POINT HYPERPLANE COVER in  $\mathbb{R}^d$ . Let  $\mathcal{P} = \{p_1, \dots, p_n\}$ , and  $o$  represent the point at the origin in  $\mathbb{R}^d$ . Since there are only  $n$  points in  $\mathcal{P}$ , there are finitely many hyperplanes in  $\mathbb{R}^d$  that contain  $o$  and some point from  $\mathcal{P}$ . Thus, there exists a hyperplane through  $o$  that does not contain any point of  $\mathcal{P}$ . Without loss of generality, we can assume that the hyperplane  $H := X_d = 0$  and  $\mathcal{P}$  are disjoint.

Define  $H_1 := X_d = 1$  and, for each  $p_i \in \mathcal{P}$  define  $\ell(p_i)$  as the line passing through the origin and  $p_i$ . Let  $p'_i = H_1 \cap \ell(p_i)$  and let  $\mathcal{P}' = \{p'_1, \dots, p'_n\}$ . Observe that  $\mathcal{P}' \subset H \equiv \mathbb{R}^{d-1}$ .

On one hand, if  $\mathcal{H}$  is a solution family of hyperplanes covering  $\mathcal{P}$ , then  $\mathcal{H}' = \{H' : H' = H \cap H_1, H \in \mathcal{H}\}$  is an equal sized covering family for  $\mathcal{P}'$  in  $\mathbb{R}^{d-1}$ . On the other hand, let  $\mathcal{H}'$  be a set of  $(d-2)$ -flats in  $H$  that cover  $\mathcal{P}'$ . For a  $(d-2)$ -flat  $H_{d-2} \in \mathcal{H}'$ ,  $\text{aff}(H_{d-2} \cup o)$  denotes the hyperplane in  $\mathbb{R}^d$  that passes through the origin  $o$  and containing the  $(d-2)$ -flat  $H_{d-2}$ . Suppose a point  $p'_i \in \mathcal{P}'$  is contained in  $H_{d-2}$ . Then, by definition, the line defined by  $p'_i$  and  $o$  also belongs to  $H_{d-2}$ . This implies that the point  $p_i \in \mathcal{P}$  also belongs to  $H_{d-2}$ . Thus, by construction, the set  $\mathcal{H} = \{\text{aff}(H_{d-2} \cup o) : H_{d-2} \in \mathcal{H}'\}$  is a set of hyperplanes of  $\mathbb{R}^d$  that pass through the origin and cover the point set  $\mathcal{P}$ . Notice that  $|\mathcal{H}| = |\mathcal{H}'|$ . Hence,  $(\mathcal{P}, k)$  is a Yes instance of PROJECTIVE POINT HYPERPLANE COVER in  $\mathbb{R}^d$  if and only if  $(\mathcal{P}', k)$  is a Yes instance of POINT HYPERPLANE COVER in  $\mathbb{R}^{d-1}$ .

In the reverse direction, we give a similar reduction from POINT HYPERPLANE COVER in  $\mathbb{R}^{d-1}$  to PROJECTIVE POINT HYPERPLANE COVER in  $\mathbb{R}^d$ . Suppose we are given an instance  $(\mathcal{P}', k)$  of POINT HYPERPLANE COVER in  $\mathbb{R}^{d-1}$ . Let  $\mathcal{P}'$  be a set of  $n$  points in  $\mathbb{R}^{d-1}$ . For a point  $p \in \mathcal{P}'$ , define  $\hat{p} = (p, 1)$ , meaning that the first  $d-1$  coordinates of  $\hat{p}$  and  $p$  are same while the last coordinate has value 1. Let  $\mathcal{P} = \{\hat{p} : p \in \mathcal{P}'\}$ . All the points in  $\mathcal{P}$  lie on the hyperplane  $H_1 : X_d = 1$ .

First, suppose  $\mathcal{H}'$  is a covering family for  $\mathcal{P}'$ . Define  $\mathcal{H} = \{\text{aff}(H' \cup o) : H' \in \mathcal{H}'\}$ .  $H' \in \mathcal{H}'$  is a  $(d-2)$ -flat in  $H_1$ : this is because for any point  $p$  contained in  $H'$  the point  $\hat{p}$  is contained in  $H_1$  by definition. Thus,  $\mathcal{H}$  is an equal sized covering family for  $\mathcal{P}$ . On the other hand, let  $\mathcal{H}$  be a set of hyperplanes that pass through  $o$  and cover  $\mathcal{P}$ . Consider the set of  $d-2$ -flats  $\mathcal{H}' = \{H' : H' = H \cap H_1, H \in \mathcal{H}\}$ . By definition,  $\mathcal{H}'$  covers all the points in  $\mathcal{P}'$  in  $\mathbb{R}^{d-1}$ . Also,  $|\mathcal{H}'| \leq |\mathcal{H}|$ . Therefore, solving the POINT HYPERPLANE COVER problem for  $\mathcal{P}'$  in  $\mathbb{R}^{d-1}$  is equivalent to solving PROJECTIVE POINT HYPERPLANE COVER problem for the point set  $\mathcal{P}$  in  $\mathbb{R}^d$ .

This completes the proof of equivalence.  $\blacktriangleleft$

### Proof of Observation 1

**Proof.** The  $\text{sgn}$  function is zero if and only if the determinant of the matrix is zero. By definition, the determinant of  $M(\langle u_1, u_2, \dots, u_d, u_{d+1} \rangle)$  is zero if and only if the rows are linearly dependent. This means that for some  $t \in \{1, \dots, d, d+1\}$ , the row  $t$  in  $M(\langle u_1, u_2, \dots, u_d, u_{d+1} \rangle)$  is a linear combination of the other rows. Deriving from the implication of this fact on the entries in the first column of  $M(\langle u_1, u_2, \dots, u_d, u_{d+1} \rangle)$ , this means that the point  $u_t$  is a convex combination of the other points. This gives us the following observation that the determinant of  $M(\langle u_1, u_2, \dots, u_d, u_{d+1} \rangle)$  is zero if and only if the  $d+1$  points are not in general position. This leads to the proof of the Observation.  $\blacktriangleleft$

**Proof of Corollary 11**

**Proof.** Consider two point sets  $\mathcal{P}$  and  $\mathcal{Q}$  containing a fixed point  $p$ . These point sets are *p-fixed-point combinatorially equivalent* if  $\mathcal{P}$  and  $\mathcal{Q}$  are combinatorially equivalent due to a bijective mapping  $\pi$  between an ordering  $\mathcal{P}'$  of  $\mathcal{P}$  and  $\mathcal{Q}'$  of  $\mathcal{Q}$  such that the label of the point  $p$  in  $\mathcal{P}'$  is mapped to the label of  $p$  in  $\mathcal{Q}'$  by  $\pi$ . The results in [3] also show that given a point  $p$ , the number of *p-fixed-point combinatorially distinct* order types arising from point sets that contain  $n$  points including the point  $p$  is  $n^{\mathcal{O}(n)}$ . The rest of the proof follows in the same way as the arguments for the lower bound on the number of points in the kernel of POINT HYPERPLANE COVER. ◀