Journal of Integrable Systems (2017) **2**, 1–18
doi: 10.1093/integr/xyx010

Discrete CMC surfaces in $\mathbb{R}^3$ and discrete minimal surfaces in $\mathbb{S}^3$: a discrete Lawson correspondence

ALEXANDER I. BOBENKO
Institut für Mathematik, Technische Universität Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany

AND

PASCAL ROMON†
Université Paris-Est, LAMA (UMR 8050), UPEC, UPEMLV, F-77454, Marne-la-Vallée, France
†Corresponding author. Email: pascal.romon@u-pem.fr

Communicated by: Prof. Maciej Dunajski

[Received on 3 May 2017; editorial decision on 25 August 2017; accepted on 28 August 2017]

The main result of this article is a discrete Lawson correspondence between discrete CMC surfaces in $\mathbb{R}^3$ and discrete minimal surfaces in $\mathbb{S}^3$. This is a correspondence between two discrete isothermic surfaces. We show that this correspondence is an isometry in the following sense: it preserves the metric coefficients introduced previously by Bobenko and Suris for isothermic nets. Exactly as in the smooth case, this is a correspondence between nets with the same Lax matrices, and the immersion formulas also coincide with the smooth case.

Keywords: discrete surfaces; integrable systems; constant mean curvature; minimal surface; discrete isometry.

1. Introduction

The Lawson correspondence states [1] that for any minimal surface in $\mathbb{S}^3$ there exists an isometric constant mean curvature surface in $\mathbb{R}^3$. It is an important tool for the investigation and construction of CMC surfaces. In particular, it was a crucial tool for the classification of trinoids in [2] and for the numerical construction of examples of CMC surfaces with higher topology in [3]. For the last purpose, it was important to integrate it once and to formulate it terms of the corresponding frames [4] (see also Theorem 3).

Although discrete CMC surfaces in $\mathbb{R}^3$ have been known for a longtime already [5], as well as discrete minimal surfaces in $\mathbb{S}^3$ [6, 7], the discrete Lawson correspondence has remained a challenge. The main problem was to define a proper discrete analogue of isometry.

The main result of this article is a discrete Lawson correspondence between discrete CMC surfaces in $\mathbb{R}^3$ and discrete minimal surfaces in $\mathbb{S}^3$ formulated in Theorem 7. This is a correspondence between two discrete isothermic surfaces. We show that it is an isometry in the following sense: it preserves the metric coefficients for isothermic nets introduced previously in [8]. Exactly as in the smooth case, this is a correspondence between nets with the same Lax matrices, and the immersion formulas also coincide with the smooth case. As necessary intermediary results, we show that commuting Lax pairs generate discrete CMC and minimal surfaces via an immersion formula as in the smooth case (Theorems 4 and 5), and, conversely, that all discrete CMC and minimal surfaces are generated by Lax pairs (Theorem 6).

Another approach to discrete isothermic surfaces in spaceforms via conserved quantities and discrete line bundles has been proposed in [7, 9], encompassing constant mean curvature nets as a special case. A Calapso transformation is defined therein, which generalizes the Lawson correspondence between the relevant spaceforms and preserves geometric quantities. In contrast, our more pedestrian method focuses on the immersion formulas which are proven to be identical as in the smooth case, and provide a more explicit definition of the correspondence and its metric invariance. Still, both definitions agree, as we prove in Remark 2.

We shall consider in this article only meshes with quadrilateral planar faces, known as quad-nets or Q-nets for short (also called PQ-meshes), whose theoretical properties mimic those of their smooth counterparts. In the particular case of nets indexed by $\mathbb{Z}^2$, indices play the same role as coordinates of an immersion, and specific choices of Q-nets correspond to specific parametrizations of surfaces.

Throughout the text, we will use the shift notation to describe the local geometry: when F is a net, $F = F_0 = F(0,0)$ will denote a base point, while F_1, F_2, F_{12} will stand for $F(1,0), F(0,1), F(1,1)$, so that indices $1,2$ correspond to shift in the first and second variables, respectively. The same holds for any vertex-based function. The edges of the face (F, F_1, F_{12}, F_2) are labelled $(0,1)$, $(1,12)$, $(12,2)$ and $(2,0)$ and the values of an edge-based function u will be denoted by u_{01}, $u_{1,12}$, etc. If (i,j) is a pair of indices corresponding to an edge, $\mathrm{d}\varphi_{ij}$ is by definition $\varphi_j - \varphi_i$.

2. The smooth theory

2.1 *Constant mean curvature surfaces in $\mathbb{R}^3$*

We recall here a well-known (see for example [5, 10] for more details) description of CMC surfaces in $\mathbb{R}^3$ in terms of loop groups and quaternionic frames. The normalizations used in the present article coincide with the normalizations in [5].

In the sequel, we identify the Euclidean three space $\mathbb{R}^3$ with imaginary quaternions Im $\mathbb{H}$, and the standard imaginary quaternions with an orthonormal basis of $\mathbb{R}^3$, and use the following matrix representation:

$$\mathbb{i} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \mathbb{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathbb{k} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

This results in the following matrix representation of vectors in $\mathbb{R}^3$:

$$X = (X_1, X_2, X_3) \ \longleftrightarrow \ \begin{pmatrix} -iX_3 & -iX_1 - X_2 \\ -iX_1 + X_2 & iX_3 \end{pmatrix}. \tag{2}$$

Umbilic free CMC-surfaces are isothermic (conformal curvature line parametrization). Let $(x,y) \mapsto F(x,y)$ be a CMC isothermically parametrized surface. Without loss of generality, one can normalize the mean curvature $H = 1$ and the Hopf differential $Q = \langle F_{xx} - F_{yy} + 2iF_{xy}, N\rangle = \frac{1}{2}$. Let e^u be the corresponding conformal metric: $\langle dF, dF\rangle = e^u(dx^2 + dy^2)$. It satisfies the elliptic sinh-Gordon equation

$$u_{xx} + u_{yy} + \sinh u = 0. \tag{3}$$

The quaternionic frame Φ is defined as a solution of the system

$$\Phi_x = U\Phi, \qquad \Phi_y = V\Phi, \tag{4}$$

where

$$U = \frac{1}{2}\begin{pmatrix} -\frac{i}{2}u_y & -\lambda e^{-u/2} - \frac{1}{\lambda}e^{u/2} \\ \lambda e^{u/2} + \frac{1}{\lambda}e^{-u/2} & \frac{i}{2}u_y \end{pmatrix},$$
$$V = \frac{1}{2}\begin{pmatrix} \frac{i}{2}u_x & -i\lambda e^{-u/2} + \frac{i}{\lambda}e^{u/2} \\ i\lambda e^{u/2} - \frac{i}{\lambda}e^{-u/2} & -\frac{i}{2}u_x \end{pmatrix}. \tag{5}$$

The matrices (5) belong to the loop algebra

$$g_H[\lambda] = \{\xi : S^1 \to su(2) : \xi(-\lambda) = \sigma_3\xi(\lambda)\sigma_3\}, \text{ where } \sigma_3 = i\mathbb{k} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and Φ in (4) lies in the corresponding loop group

$$G_H[\lambda] = \{\phi : S^1 \to SU(2) : \phi(-\lambda) = \sigma_3\phi(\lambda)\sigma_3\}. \tag{6}$$

Here S^1 is the set $|\lambda| = 1$.

The system (5) is the Lax representation for (3), where the parameter λ is called the spectral parameter.

THEOREM 1 The formulas

$$\hat{N} = -\check{N} = -\Phi^{-1}\mathbb{k}\Phi, \quad \begin{cases} \hat{F} & = -\Phi^{-1}\dfrac{\partial\Phi}{\partial\gamma} - \dfrac{1}{2}\hat{N} \\ \check{F} & = -\Phi^{-1}\dfrac{\partial\Phi}{\partial\gamma} + \dfrac{1}{2}\hat{N} = \hat{F} + \hat{N} \end{cases}, \tag{7}$$

where $\lambda = e^{i\gamma}$, describe two parallel surfaces $\hat{F}, \check{F}$ with constant mean curvature $H = 1$ and their Gauss maps $\hat{N}, \check{N}$. Variation of γ is an isometry, and the corresponding one parameter family of CMC surfaces is called the *associated family*. For $\gamma = 0$, i.e. $\lambda = 1$, the parametrizations of $\hat{F}$ and $\check{F}$ are isothermic.

2.2 *Constant mean curvature and minimal surfaces in* $\mathbb{S}^3$

The same Lax pair yields a CMC net in $\mathbb{S}^3$ through the immersion formula obtained in [10]. We identify S^3 with unitary quaternions

$$X = (X_1, X_2, X_3, X_4) \longleftrightarrow \begin{pmatrix} -iX_3 + X_4 & -iX_1 - X_2 \\ -iX_1 + X_2 & iX_3 + X_4 \end{pmatrix}. \tag{8}$$

If we gauge the frame into $\Psi = \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix}\Phi = \exp\left(-\frac{\gamma}{2}\mathbb{k}\right)\Phi$, then for any pair $\lambda_1 = e^{i\gamma_1}, \lambda_2 = e^{i\gamma_2}$ in the unit circle,

$$F = \Psi(\lambda_1)^{-1}\Psi(\lambda_2), \quad N = -\Psi(\lambda_1)^{-1}\mathbb{k}\Psi(\lambda_2)$$

are an orthogonal pair of vectors in $\mathbb{S}^3$. They describe a surface F with constant mean curvature $H = \cot(\gamma_1 - \gamma_2)$ and its Gauss map N. In terms of the original frame the formulas look as follows:

$$F = \Phi(\lambda_1)^{-1} M \Phi(\lambda_2),\ N = -\Phi(\lambda_1)^{-1} \Bbbk M \Phi(\lambda_2), \tag{9}$$

where $M = \exp(\frac{\gamma_1-\gamma_2}{2}\Bbbk)$.

THEOREM 2 Let $\Phi(\lambda)$ be a solution of (4). Formulas (9) describe a surface F with constant mean curvature $H = \cot(\gamma_1-\gamma_2)$ and its Gauss map N. The parametrization $F(x,y)$ is isothermic if and only if $\lambda_2 = \pm\lambda_1^{-1}$ ($\gamma_1 + \gamma_2 \equiv 0 \mod \pi$). In particular, for $\gamma_1 = -\gamma_2 = \frac{\pi}{4}$ one obtains an isothermically parametrized minimal surface F with the conformal metric e^{-u} and the Gauss map N, which is Christoffel dual of F. Equivalently they can be treated as an isothermically parametrized minimal surface N with the Gauss map F and the conformal metric e^u.

2.3 *The Lawson correspondence*

As we have indicated already in Theorem 2, CMC surfaces in $\mathbb{R}^3$ and minimal surfaces in $\mathbb{S}^3$ corresponding to the same Lax pair are isometric. This correspondence can be lifted to the frames without referring to the Lax representation. The corresponding formulas were obtained in [4]. We will derive them from the immersion formulas (7) and (9).

THEOREM 3 Let F be an isothermically parametrized minimal surface in $\mathbb{S}^3$ with the Gauss map N. Then there exist surfaces $\hat{F}$ and $\check{F}$ in $\mathbb{R}^3$ with constant mean curvature $H = 1$ isometric to F and N respectively. They and their Gauss maps $\hat{N}$ and $\check{N}$ are given by the following formulas:

$$\begin{aligned} d\hat{F} &= F^{-1} * dF, \quad \hat{N} = F^{-1}N = -NF^{-1} \\ d\check{F} &= N^{-1} * dN, \quad \check{N} = -F^{-1}N = NF^{-1}. \end{aligned} \tag{10}$$

Here $*$ is the Hodge star defined by $(*f_x = -f_y, *f_y = f_x)$. The parametrization of surfaces $\hat{F}, \check{F}$ given by (10) inherited from the isothermic parametrization of F and N is not isothermic. Formulas (10) give the surfaces from the associated family (7) corresponding to $\lambda = \lambda_2 = e^{-i\frac{\pi}{4}}$.

Surfaces F, N in $\mathbb{S}^3$ and surfaces $\hat{F}, \check{F}$ in $\mathbb{R}^3$ with the Gauss maps $\hat{N}, \check{N}$ are described by the formulas (9) and (7) with the same Lax matrices (5).

Proof. Follows from direct computation. In the minimal surface case (9) becomes

$$F = \Phi(\lambda_1)^{-1} M \Phi(\lambda_2),\ N = \Phi(\lambda_1)^{-1} M^{-1} \Phi(\lambda_2),\ M = \begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}. \tag{11}$$

Formulas for the Gauss maps follow immediately:

$$F^{-1}N = \Phi^{-1}(\lambda_2) M^{-2} \Phi(\lambda_2) = \hat{N}(\lambda_2).$$

Computations for $d\hat{F}$ and $d\check{F}$ are slightly more involved,

$$d\hat{F} = -d(\Phi)^{-1} \frac{\partial \Phi}{\partial \gamma} - \Phi^{-1} \frac{\partial d\Phi}{\partial \gamma} = \Phi^{-1} \left(-\frac{\partial}{\partial \gamma}(Udx + Vdy) + \frac{1}{2}[\Bbbk, Udx + Vdy] \right) \Phi.$$

Calculating at $\lambda_2 = e^{-i\frac{\pi}{4}}$, we get

$$d\hat{F} = e^{-u/2}\Phi^{-1}(\lambda_2)\left(\begin{pmatrix} 0 & e^{i\frac{\pi}{4}} \\ -e^{-i\frac{\pi}{4}} & 0 \end{pmatrix} dx + \begin{pmatrix} 0 & -e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & 0 \end{pmatrix} dy\right) \Phi(\lambda_2). \tag{12}$$

On the other hand from the formulas for surfaces in $\mathbb{S}^3$, we obtain

$$dF = \Phi^{-1}(\lambda_1)\left((U(\lambda_1)dx - V(\lambda_1)dy)M + M(U(\lambda_2)dx + M(\lambda_2)dy)\right)\Phi(\lambda_2),$$

which implies

$$* \, dF = \Phi^{-1}(\lambda_1)\left((V(\lambda_1)M - MV(\lambda_2))dx + (-U(\lambda_1)M + MU(\lambda_2))dy\right)\Phi(\lambda_2),$$
$$F^{-1} * F = \Phi^{-1}(\lambda_2)\left((M^{-1}V(\lambda_1)M - V(\lambda_2))dx + (-M^{-1}U(\lambda_1)M + U(\lambda_2))dy\right)\Phi(\lambda_2)$$

A direct computation shows that the last expression coincides with (12). The identity for $d\check{F}$ follows in the same way. □

3. Discrete CMC and minimal surfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$

We will now define the discrete analogues of constant mean curvature and minimal surfaces, following the Steiner formula approach of [11] and [6].

Let (F, N) be a pair of edge-parallel maps from $\mathbb{Z}^2$ to $\mathbb{R}^4$, with planar faces, where either

- F lies in $\mathbb{R}^3$ and N takes values in $\mathbb{S}^2$, or,
- F and N lie in $\mathbb{S}^3$ and $F \perp N$ at each vertex.

The map N is treated as the Gauss map of F. Since N is planar and constrained to a sphere, its faces are circular, and so are those of F, by parallelism. Note that circular implies planar.

The area $\mathcal{A}(f)$ of a planar face f being a quadratic form in its coordinates, we define the mixed area $\mathcal{A}(f, f')$ of two edge-parallel faces to be the polar form applied to f, f':

$$\mathcal{A}(f, f') = \frac{1}{4}(\mathcal{A}(f + f') - \mathcal{A}(f - f'))$$

where $\mathcal{A}(f) = \mathcal{A}(f, f)$. This allows us to write a Steiner formula for the area of the parallel face $f + \varepsilon f'$ as

$$\mathcal{A}(f + \varepsilon f') = \mathcal{A}(f) + 2\varepsilon\mathcal{A}(f, f') + \varepsilon^2\mathcal{A}(f').$$

Applying this formula to the mesh pair (F, N) on the face f, we identify the mean and Gaussian curvature by $\mathcal{A}(F(f) + \varepsilon N(f)) = (1 - 2\varepsilon H_f + \varepsilon^2 K_f)\mathcal{A}(F(f))$, so that

$$H_f = -\frac{\mathcal{A}(F(f), N(f))}{\mathcal{A}(F(f))} \text{ and } K_f = \frac{\mathcal{A}(N(f))}{\mathcal{A}(F(f))}.$$

DEFINITION 1 A circular Q-net (F, N), with F, N as above, is of *constant mean curvature* $H \neq 0$ (CMC) if $H_f = H$ on all faces f. It is *minimal* if H_f vanishes identically.

Such a net is automatically *Koenigs* (see [8]), i.e. it possesses a Christoffel dual F^* such that (i) F^* is edge-parallel to F and (ii) $A(F, F^*)$ vanishes identically. Indeed, if (F, N) has constant mean curvature H (resp. is minimal), then $F^* = F + \frac{1}{H}N$ (resp. $F^* = N$) is the dual. Being Koenigs and circular is equivalent for F to be *discrete isothermic*. Discrete isothermic nets were originally defined in [12] as nets with factorizable cross ratios, i.e. the cross ratio $\mathrm{cr}(F, F_1, F_{12}, F_2)$ is of the form A/B, with A depending on the first coordinate and B on the second. Such functions A, B are called *edge labellings* and are uniquely defined up to a common factor.

In [8, 13], a discrete analogue of conformal metric was introduced for discrete isothermic surfaces. It was shown that Koenigs nets possess a function $s : \mathbb{Z}^2 \to \mathbb{R}^+$ defined at vertices, called the (*discrete conformal*) *metric coefficient*. Consider black and white sublattices of $\mathbb{Z}^2$ so that every elementary quad contains two vertices of each displaced diagonally.

The conformal factor s is defined up to a so called black-white rescaling: $s \mapsto \lambda s$ at black points and $s \mapsto \mu s$ at white points. In particular, s relates the net to its Christoffel dual:

$$F_i^* - F^* = \frac{1}{s_i s}(F_i - F), \quad i = 1, 2. \tag{13}$$

Moreover, for discrete isothermic nets the edge labelling[1] is linked to the discrete conformal factor s and the edge lengths as follows (see [8, 13]):

$$A = \frac{\|F_1 - F\|^2}{s s_1}, \; B = \frac{\|F_2 - F\|^2}{s s_2}. \tag{14}$$

One can approximate smooth isothermic surfaces by discrete isothermic surfaces [14]. Probably this is also the case with minimal and CMC surfaces, although this is not yet proven.

4. Loop group description

Here following [5], we present the loop group description of discrete CMC surfaces in $\mathbb{R}^3$. We will show also that discrete CMC surfaces in $\mathbb{S}^3$ are described by the same discrete Lax representation and the immersion formula (9) of the smooth case.

4.1 *Discretization in the loop group*

As in the smooth case, we consider a frame $\Phi : \mathbb{Z}^2 \to G_H[\lambda]$. The discrete Lax pair $\mathcal{U}(\lambda) = \Phi_1(\lambda)\Phi(\lambda)^{-1}$, $\mathcal{V}(\lambda) = \Phi_2(\lambda)\Phi(\lambda)^{-1}$ are maps from the edges into the loop group. By analogy with the smooth immersions $\mathcal{U}(\lambda)$, $\mathcal{V}(\lambda)$ are defined of the following form: on each edge,

$$\begin{aligned} \mathcal{U}(\lambda) &= \frac{1}{\alpha(\lambda)} \begin{pmatrix} a & -\lambda u - \lambda^{-1} u^{-1} \\ \lambda u^{-1} + \lambda^{-1} u & \bar{a} \end{pmatrix}, \\ \mathcal{V}(\lambda) &= \frac{1}{\beta(\lambda)} \begin{pmatrix} b & -i\lambda v + i\lambda^{-1} v^{-1} \\ i\lambda v^{-1} - i\lambda^{-1} v & \bar{b} \end{pmatrix}, \end{aligned} \tag{15}$$

[1] Edge labellings are unique up to global multiplication of A and B by a constant. The choice mentioned here is canonical.

where complex valued a, b and real valued u, v do not depend on λ, u, v are positive and $\alpha(\lambda)$ and $\beta(\lambda)$ are real such that the determinants are equal to 1:

$$\alpha(\lambda)^2 = |a|^2 + \lambda^2 + \lambda^{-2} + u^2 + u^{-2}, \quad \beta(\lambda)^2 = |b|^2 - \lambda^2 - \lambda^{-2} + v^2 + v^{-2}. \tag{16}$$

Furthermore, α and β on the opposite edges coincide, i.e. they are edge labelling for the first and second indices respectively.

We will now focus on a single quad (F, F_1, F_{12}, F_2), and let $\mathcal{U}, \mathcal{V}$ be the Lax matrices associated to the edges (F, F_1) and (F, F_2) respectively; for the sake of simplicity, we will mark with a prime the corresponding quantities on the opposite edges (F_2, F_{12}) and (F_1, F_{12}): $\mathcal{U}'$, $\mathcal{V}'$, a', u', etc. In particular $\Phi_{12} = \mathcal{U}'\Phi_2$, $\Phi_{12} = \mathcal{V}'\Phi_1$. Note that $\alpha' = \alpha$ and $\beta' = \beta$.

The Lax pair satisfies

$$\mathcal{V}'(\lambda)\,\mathcal{U}(\lambda) = \mathcal{U}'(\lambda)\,\mathcal{V}(\lambda) \tag{17}$$

on any quad and gives rise to a frame $\Phi(\lambda) : \mathbb{Z}^2 \to G_H[\lambda]$.

This commutation property yields

$$uu' = vv' \tag{18}$$

$$b'a - ba' = i(u'v + uv' - u'^{-1}v^{-1} - u^{-1}v'^{-1}) \tag{19}$$

$$\bar{b}u' - b'u = i(\bar{a}v' - a'v) \tag{20}$$

$$\bar{b}u'^{-1} - b'u^{-1} = i(a'v^{-1} - \bar{a}v'^{-1}) \tag{21}$$

As noticed in [5, (4.23)], equation (18) is equivalent to the existence of a vertex function w such that

$$u = ww_1, \quad u' = w_2w_{12}, \quad v = ww_2, \quad v' = w_1w_{12} \tag{22}$$

The function w turns out to be essentially the discrete conformal metric s, as we will show in the next section.

4.2 *Discrete CMC nets in Euclidean three space*

Let $\Phi(\lambda)$ be a frame defined from commuting Lax pairs as above, and let $\lambda = e^{i\gamma} \in \mathbb{S}^1$ be a spectral parameter. We define two nets $\hat{F}, \check{F}$ and a unit Gauss map $\hat{N}$ as follows:

$$\hat{N} = -\Phi^{-1}\mathbb{k}\Phi, \quad \begin{cases} \hat{F} & = -\Phi^{-1}\dfrac{\partial\Phi}{\partial\gamma}_{|\gamma=0} - \dfrac{1}{2}\hat{N} \\ \check{F} & = -\Phi^{-1}\dfrac{\partial\Phi}{\partial\gamma}_{|\gamma=0} + \dfrac{1}{2}\hat{N} = \hat{F} + \hat{N}, \end{cases} \tag{23}$$

where all the matrices are evaluated at $\gamma = 0$ (i.e. $\lambda = 1$).

THEOREM 4 [5]
The pair $(\hat{F}, \hat{N})$ given by (23) is a CMC net in $\mathbb{R}^3$ with $H = 1$. On any quad, the discrete conformal metric s is given by

$$ss_1 = -u^2 \text{ and } ss_2 = v^2, \tag{24}$$

and the cross ratio $\mathrm{cr}(\hat{F}, \hat{F}_1, \hat{F}_{12}, \hat{F}_2)$ is equal to $-\beta(1)^2/\alpha(1)^2$. The edge lengths are equal

$$\|\hat{F}_1 - \hat{F}\|^2 = \frac{4u^2}{\alpha(1)^2} = -\frac{4ss_1}{|a|^2 + 2 - ss_1 - s^{-1}s_1^{-1}},$$
$$\|\hat{F}_2 - \hat{F}\|^2 = \frac{4v^2}{\beta(1)^2} = \frac{4ss_2}{|b|^2 - 2 + ss_2 + s^{-1}s_2^{-1}}.$$

$\check{F}$ and $\hat{F}$ are Christoffel dual discrete isothermic nets.

Proof. Note that [5] use a slightly different, albeit equivalent notation. For the sake of completeness and compatibility with the spherical case, we shall sketch the proof here using our notations. Note: all λ-dependent quantities are evaluated at $\lambda = 1$, and we shall not write the variable λ for greater legibility, so α stands for $\alpha(1)$, etc.

Evaluating the edge vectors, one obtains

$$\hat{F}_1 - \hat{F} = -\Phi^{-1}\mathcal{U}^{-1}\left(\dot{\mathcal{U}} - \frac{1}{2}[\mathbb{k}, \mathcal{U}]\right)\Phi, \qquad \hat{F}_2 - \hat{F} = -\Phi^{-1}\mathcal{V}^{-1}\left(\dot{\mathcal{V}} - \frac{1}{2}[\mathbb{k}, \mathcal{V}]\right)\Phi, \tag{25}$$

$$\hat{N}_1 - \hat{N} = -\Phi^{-1}\mathcal{U}^{-1}[\mathbb{k}, \mathcal{U}]\Phi, \qquad \hat{N}_2 - \hat{N} = -\Phi^{-1}\mathcal{V}^{-1}[\mathbb{k}, \mathcal{V}]\Phi, \tag{26}$$

where $\dot{\mathcal{U}}$ the derivative at $\gamma = 0$, and similar formulas for $\check{F}$. Because α and β have an extremum at $\lambda = 1$, $\dot{\mathcal{U}} = \frac{u-u^{-1}}{\alpha}\mathbb{i}$ and $\dot{\mathcal{V}} = -\frac{v+v^{-1}}{\beta}\mathbb{j}$. Since $[\mathbb{k}, \mathcal{U}] = -\frac{2(u+u^{-1})}{\alpha}\mathbb{i}$ and $[\mathbb{k}, \mathcal{V}] = \frac{2(v-v^{-1})}{\beta}\mathbb{j}$, parallelism between the nets $\hat{F}$, $\hat{N}$ (and thus $\check{F}$) is clear, as are the lengths of the edges. We also derive the proportionality factors

$$\mathrm{d}\check{F}_{01} = -u^{-2}\,\mathrm{d}\hat{F}_{01}, \quad \mathrm{d}\check{F}_{02} = v^{-2}\,\mathrm{d}\hat{F}_{02},$$

and using (22)

$$\mathrm{d}\check{F}_{01} = -\frac{\mathrm{d}\hat{F}_{01}}{w^2 w_1^2}, \quad \mathrm{d}\check{F}_{02} = \frac{\mathrm{d}\hat{F}_{02}}{w^2 w_2^2}.$$

That proves that $\hat{F}$ is Koenigs with Christoffel dual $\check{F}$ and a discrete conformal metric given by $s = \pm w^2$ (see [8] for details). The sign can be chosen constant in one direction and alternates in other direction.

The claim about the cross ratios is proven by direct computation using the quaternionic formulas for cross ratios (see [5]). □

As noted in the proof, the choice of $\lambda = 1$ for the spectral parameter in (23) is crucial. Other values of the spectral parameter, as in the smooth case (see Theorem 1), will not satisfy the parallelism condition between the edges, nor the planarity of the faces, so that these nets are neither circular nor Koenigs. Such nets retain interesting properties, in particular they are *edge-constraint nets*, as defined in [15]. Edge-constraint nets are quadrilateral nets F with non-necessarily planar faces, and vertex normals N

such that, on any edge $e=(FF_i)$, the average of the normals at the endpoints is orthogonal to the edge: $F_i - F \perp N + N_i$.

For edge-constraint nets the mean curvature can also be defined via the Steiner formula. For that purpose, the area functional for non-planar quads is defined by projecting along the direction orthogonal to the diagonals of the quad (N, N_1, N_{12}, N_2) of the normals. It has been shown in [15] that all surfaces of the *associated family* (i.e. those defined by (23) with unitary λ) possess constant mean curvature. We refer the reader to this article for more details.

4.3 *Discrete CMC and minimal nets in the three sphere*

As in the case of smooth CMC surfaces, the same Lax pair leads yields a discrete CMC surface in $\mathbb{S}^3$ through the immersion formula (9). We gauge again the frame into $\Psi = \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2}\end{pmatrix}\Phi = \exp\left(-\frac{\gamma}{2}\mathbb{k}\right)\Phi$, and define, for any pair $\lambda_1 = e^{i\gamma_1}, \lambda_2 = e^{i\gamma_2}$ in the unit circle,

$$F = \Psi(\lambda_1)^{-1}\Psi(\lambda_2), \quad N = -\Psi(\lambda_1)^{-1}\mathbb{k}\Psi(\lambda_2)$$

which are an orthogonal pair of vectors in $\mathbb{S}^3$. Equivalently,

$$F = \Phi(\lambda_1)^{-1} M \Phi(\lambda_2),\ N = -\Phi(\lambda_1)^{-1}\mathbb{k} M \Phi(\lambda_2), \tag{27}$$

where $M = \begin{pmatrix} e^{i\frac{\gamma_2-\gamma_1}{2}} & 0 \\ 0 & e^{-i\frac{\gamma_2-\gamma_1}{2}}\end{pmatrix} = \exp(\frac{\gamma_1-\gamma_2}{2}\mathbb{k})$.

THEOREM 5 The pair (F, N) given by (27) is a discrete isothermic CMC surface in $\mathbb{S}^3$ if, and only if, $\lambda_2 = \pm\lambda_1^{-1}$ $(\gamma_1+\gamma_2 \equiv 0 \mod \pi)$. Its mean curvature is equal to

$$H = \frac{\operatorname{Re}\lambda_1^2}{\operatorname{Im}\lambda_1^2} = \cot(2\gamma_1) = \cot(\gamma_1-\gamma_2)\,.$$

Its Christoffel dual is $F^* = F + \frac{1}{H}N$ if $H \neq 0$, and $F^* = N$ if $H = 0$. On any quad the discrete conformal metric s of F is given by

$$\begin{aligned} ss_1 &= -u^2\sqrt{\frac{H^2}{1+H^2}} \text{ and } ss_2 = v^2\sqrt{\frac{H^2}{1+H^2}} && \text{if } H \neq 0, \\ ss_1 &= -u^2 \text{ and } ss_2 = v^2 && \text{if } H = 0, \end{aligned} \tag{28}$$

and the cross ratio is equal to $-\beta^2/\alpha^2$ evaluated at λ_1. The edge lengths satisfy, when $H \neq 0$,

$$\begin{aligned} \|\,\mathrm{d}F_{01}\|^2 &= \frac{4u^2\sin^2(2\gamma_1)}{\alpha^2} = \frac{4|ss_1|}{\alpha^2 H\sqrt{1+H^2}} = \frac{4|ss_1|}{\left(|a|^2+u^2+u^{-2}+\sqrt{\frac{H^2}{1+H^2}}\right)H\sqrt{1+H^2}}, \\ \|\,\mathrm{d}F_{02}\|^2 &= \frac{4v^2\sin^2(2\gamma_1)}{\beta^2} = \frac{4|ss_2|}{\beta^2 H\sqrt{1+H^2}} = \frac{4|ss_2|}{\left(|b|^2+v^2+v^{-2}-\sqrt{\frac{H^2}{1+H^2}}\right)H\sqrt{1+H^2}}, \end{aligned} \tag{29}$$

and for minimal nets,

$$\| \mathrm{d}F_{01} \|^2 = \frac{4u^2}{\alpha^2} = -\frac{4ss_1}{|a|^2 - ss_1 - s^{-1}s_1^{-1}}, \ \| \mathrm{d}F_{02} \|^2 = \frac{4v^2}{\beta^2} = \frac{4ss_2}{|a|^2 + ss_2 + s^{-1}s_2^{-1}}. \tag{30}$$

Proof. It is given by direct computation. Let us check that the edges of F and N are parallel.

$$F_1 - F = \Phi(\lambda_1)^{-1}\mathcal{U}(\lambda_1)^{-1}(M\mathcal{U}(\lambda_2) - \mathcal{U}(\lambda_1)M)\Phi(\lambda_2),$$
$$N_1 - N = -\Phi(\lambda_1)^{-1}\mathcal{U}(\lambda_1)^{-1}(\mathbb{k}M\mathcal{U}(\lambda_2) - \mathcal{U}(\lambda_1)M\mathbb{k})\Phi(\lambda_2).$$

Note that, given two unitary matrices U_1, U_2 in $SU(2)$, $U_2 - U_1$ is a real multiple of $\mathbb{k}U_2 - U_1\mathbb{k}$ iff their diagonals coincide. Applying it to $U_1 = \mathcal{U}(\lambda_1)M$, $U_2 = M\mathcal{U}(\lambda_2)$, we conclude in particular that $\alpha(\lambda_1) = \alpha(\lambda_2)$; so by (16), $\operatorname{Re}\lambda_2^2 = \operatorname{Re}\lambda_1^2$. The case $\lambda_2 = \pm\lambda_1$ leads to constant maps. The case $\lambda_2 = \pm\lambda_1^{-1}$ leads to non-trivial discrete surfaces. Let us focus on $\lambda_2 = \lambda_1^{-1}$, i.e. $\gamma_2 + \gamma_1 = 0$. We have then

$$M\mathcal{U}(\lambda_2) - \mathcal{U}(\lambda_1)M = -\frac{2u\sin(2\gamma_1)}{\alpha}\mathbb{i} \text{ and } -\mathbb{k}M\mathcal{U}(\lambda_2) + \mathcal{U}(\lambda_1)M\mathbb{k} = \frac{2}{\alpha}(u^{-1} + u\cos(2\gamma_1))\mathbb{i},$$

and similarly

$$M\mathcal{V}(\lambda_2) - \mathcal{V}(\lambda_1)M = \frac{2v}{\beta}\sin(2\gamma_1)\mathbb{j} \text{ and } -\mathbb{k}M\mathcal{V}(\lambda_2) + \mathcal{V}(\lambda_1)M\mathbb{k} = \frac{2}{\beta}(v^{-1} - v\cos(2\gamma_1))\mathbb{j}$$

which proves the parallelism of the corresponding edges. If we set $F^* = F + \frac{1}{H}N$, with $H = \cot(2\gamma_1) \neq 0$, then

$$\mathrm{d}F^*_{01} = \mathrm{d}F_{01} + \tan(2\gamma_1)\,\mathrm{d}N_{01} = -\frac{u^{-2}}{\cos(2\gamma_1)}\,\mathrm{d}F_{01} = -u^{-2}\sqrt{\frac{1+H^2}{H^2}}\,\mathrm{d}F_{01},$$
$$\mathrm{d}F^*_{02} = \mathrm{d}F_{02} + \tan(2\gamma_1)\,\mathrm{d}N_{02} = \frac{v^{-2}}{\cos(2\gamma_1)}\,\mathrm{d}F_{02} = v^{-2}\sqrt{\frac{1+H^2}{H^2}}\,\mathrm{d}F_{02}.$$

We infer the existence of a discrete conformal metric s, as in the proof of Theorem 4. Note that

$$s = \pm w^2\sqrt[4]{\frac{H^2}{1+H^2}}.$$

When $\gamma_1 = \pi/4$, $H = 0$; we set $F^* = N$ and obtain

$$\mathrm{d}N_{01} = -u^{-2}\,\mathrm{d}F_{01} \text{ and } \mathrm{d}N_{02} = v^{-2}\,\mathrm{d}F_{02}.$$

For all values of H, this proves that F^* is a Christoffel dual of F, and therefore (F, N) has constant mean curvature H.

To compute quaternionic cross ratio we assume that $\Phi(\lambda_1) = \mathbb{1}$ and use $\mathcal{U}'\mathcal{V} = \mathcal{V}'\mathcal{U}$:

$$\begin{aligned}
\mathrm{cr}(F, F_1, F_{12}, F_2) &= (F - F_1)(F_1 - F_{12})^{-1}(F_{12} - F_2)(F_2 - F)^{-1} \\
&= (\mathcal{U}(\lambda_1)^{-1} M \mathcal{U}(\lambda_2) - M)\mathcal{U}(\lambda_2)^{-1}(\mathcal{V}'(\lambda_1)^{-1} M \mathcal{V}'(\lambda_2) - M)^{-1} \\
&= \left(-\frac{2u\sin(2\gamma_1)}{\alpha}\mathbb{i}\right)\left(\frac{2v'}{\beta}\sin(2\gamma_1)\mathbb{j}\right)^{-1}\left(-\frac{2u'\sin(2\gamma_1)}{\alpha}\mathbb{i}\right)\left(\frac{2v}{\beta}\sin(2\gamma_1)\mathbb{j}\right)^{-1} \\
&= -\frac{\beta^2}{\alpha^2}\frac{uu'}{vv'}\mathbb{1} = -\frac{\beta^2}{\alpha^2}\mathbb{1}.
\end{aligned}$$

Formulas (29, 30) follow directly from the quaternionic formulas for the corresponding edges derived above. □

5. From discrete minimal and CMC surfaces to the Lax pair

We have seen that the (same) frame Φ integrating the Lax pair in (15), gives rise to CMC or minimal quad-nets in $\mathbb{R}^3$ and $\mathbb{S}^3$. We shall now prove the converse.

THEOREM 6 For any Q-net of constant mean curvature in $\mathbb{R}^3$ or $\mathbb{S}^3$, or minimal in $\mathbb{S}^3$, there exists a Lax pair satisfying (15), such that the immersion formula (23) or (27) holds.

We will prove this theorem in several steps. First we identify geometric quantities s, A, B and algebraic quantities u, v, α, β appearing in the corresponding descriptions (13, 14) and (15, 16). We will further denote the edges by $\mathrm{d}\hat{F}_{0i} = F_i - F$.

Let us start with the case of discrete CMC surfaces $\hat{F}$ with $H = 1$. Its Christoffel dual is given by $\check{F} = \hat{F}^* = \hat{F} + \hat{N}$. The corresponding edges are related by (13), where s is the conformal metric coefficient of $\hat{F}$. We assume that s alternates its sign along the first direction and does not change the sign along the second direction, i.e. $s_1 s < 0$, $s_2 s > 0$. Geometrically this means that the quads $(\hat{F}, \hat{F}_1, \check{F}_1, \check{F})$ are crossing trapezoids, and the quads $(\hat{F}, \hat{F}_2, \check{F}_2, \check{F})$ are embedded trapezoids.

Comparing the analytic formulas of Theorem 4 with the edge length formulas (14), we obtain the following identification:

$$u^2 = -s_1 s, \quad v^2 = s_2 s$$
$$\alpha(1) = \frac{2}{\sqrt{-A}}, \quad \beta(1) = \frac{2}{\sqrt{B}}. \tag{31}$$

Let us compute the angle between an edge $\mathrm{d}\hat{F}_{0i}$ of a discrete CMC surface and its unit normal $\hat{N}$. Since $\mathrm{d}\hat{F}_{01}$ and $\mathrm{d}\hat{F}^*_{01}$ are parallel (though in opposite directions) and the other sides have length 1, one derives

$$\langle \mathrm{d}\hat{F}_{01}, \hat{N}\rangle = \frac{1}{2}\left(\|\,\mathrm{d}\hat{F}_{01}\| + \|\,\mathrm{d}\hat{F}^*_{01}\|\right)\|\,\mathrm{d}\hat{F}_{01}\| = \frac{1}{2}A(s_1 s - 1), \tag{32}$$

and similarly,

$$\langle \mathrm{d}\hat{F}_{02}, \hat{N}\rangle = \frac{1}{2}\left(\|\,\mathrm{d}\hat{F}_{02}\| - \|\,\mathrm{d}\hat{F}^*_{02}\|\right)\|\,\mathrm{d}\hat{F}_{02}\| = \frac{1}{2}B(s_2 s - 1). \tag{33}$$

Here we have used (13, 14).

We will use the following technical lemma, which can be easily checked.

LEMMA 1 The commutation property (17) for all λ is equivalent to the condition $uu' = vv'$ (equation (18)), together with the commutation property for matrices and its derivative evaluated *only* at $\lambda = 1$:

$$\mathcal{V}'(1)\mathcal{U}(1) = \mathcal{U}'(1)\mathcal{V}(1) \text{ and } \dot{\mathcal{V}}'(1)\mathcal{U}(1) + \mathcal{V}'(1)\dot{\mathcal{U}}(1) = \dot{\mathcal{U}}'(1)\mathcal{V}(1) + \mathcal{U}'(1)\dot{\mathcal{V}}(1).$$

It is also equivalent to $uu' = vv'$ together with the commutation property evaluated at two values λ, λ^{-1}, provided $\lambda \notin \pm 1, \pm i$.

The following lemmas prove the existence and uniqueness of the Lax pair for a given quad of a discrete CMC surface.

LEMMA 2 ($\mathbb{R}^3$ version) Let $Q = (\hat{F}, \hat{F}_1, \hat{F}_{12}, \hat{F}_2)$ be a CMC-1 quad in $\mathbb{R}^3$ with Gauss map $(\hat{N}, \hat{N}_1, \hat{N}_{12}, \hat{N}_2)$. Let Φ be any frame for $\hat{N}$, namely $\hat{N} = -\Phi^{-1}\mathbb{k}\Phi$ (such a frame at the point $\hat{F}$ is determined up to $U(1)$ action). Then there exist $\mathcal{U}(\lambda), \mathcal{V}(\lambda), \mathcal{U}'(\lambda), \mathcal{V}'(\lambda)$ satisfying (17), and thus generating the quad, together with its Gauss map by formulas (23).

Proof. In the following, $\mathcal{U}$, α, etc. will denote the usual quantities evaluated at $\lambda = 1$, and we will write specifically $\mathcal{U}(\lambda)$ when considering the loop.

As we have demonstrated above (geometric) CMC-1 quad Q in $\mathbb{R}^3$, together with its Gauss map determines $\alpha, \beta, u, v, u', v'$ so only a, b remain to be found.[2] The Lax matrix $\mathcal{U}$ can be determined uniquely from Φ, through equation (25):

$$\Phi \,\mathrm{d}\hat{F}_{01} \Phi^{-1} = -\frac{2u}{\alpha}\mathbb{i}\mathcal{U} = \frac{2iu}{\alpha^2}\begin{pmatrix} u + u^{-1} & \bar{a} \\ a & -(u + u^{-1}) \end{pmatrix}.$$

Indeed, the length of the edge is $|2u/\alpha|$ by (31), so that a is here only to fix the directions in the $(\mathbb{i}, \mathbb{j})$ plane, provided the coordinate along the $(-\mathbb{k})$ axis (i.e. the projection of $\mathrm{d}\hat{F}_{01}$ along $\hat{N}$) is

$$\frac{2u}{\alpha}\frac{u + u^{-1}}{\alpha}.$$

This fact follows from (32). So that a is now fixed, though it depends on the Φ gauge. Similarly, $\mathrm{d}\hat{F}_{02}$ determines $\mathcal{V}$ and b.

Let us now check that Φ_1 and Φ_2 defined through $\mathcal{U}$ and $\mathcal{V}$ are frames for $\hat{N}_1$ and $\hat{N}_2$ respectively. By property of the Koenigs net, $\hat{N}_1 - \hat{N} = -(1 + u^{-2})\,\mathrm{d}\hat{F}_{01}$, and, conjugating by Φ, we need to check that

$$-\mathcal{U}^{-1}\mathbb{k}\mathcal{U} + \mathbb{k} = -\frac{2}{\alpha}\frac{u^2 + 1}{u}\mathbb{i}\mathcal{U}$$

which holds due to the specific form of $\mathcal{U}$, and similarly for $\mathcal{V}$. From Φ_1 and Φ_2, we derive $\mathcal{U}', \mathcal{V}'$ as above, with their specific form.

[2] Although $|a|$ and $|b|$ are given by (16).

Having defined all their coefficients, the Lax matrices are fully determined, and there remains only to check the commutation property (17) for all λ. This can be done using Lemma 1.

By (31) $uu' = vv'$ holds. Comparing the cross ratio written in terms of $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U}'$, $\mathcal{V}'$ and its known value $-\beta^2/\alpha^2$ proves the commutation for $\lambda = 1$ (reverse the proof of Theorem 4). This shows also that $\hat{N}_{12} = -\Phi_{12}^{-1}\mathbb{k}\Phi_{12}$, where $\Phi_{12} = \mathcal{V}'\mathcal{U}\Phi = \mathcal{U}'\mathcal{V}\Phi$. The derived commutation property is a consequence of the additive commutation relation

$$\mathrm{d}\hat{F}_{01} + \mathrm{d}\hat{F}_{1,12} = \mathrm{d}\hat{F}_{02} + \mathrm{d}\hat{F}_{2,12}\,.$$

For simplicity, we apply it to $G = \hat{F} + \frac{1}{2}\hat{N} = -\Phi^{-1}\frac{\partial\Phi}{\partial\gamma}_{|\gamma=0}$, which is a well-defined quad, and hence closes:

$$\begin{aligned} 0 &= \Phi(\mathrm{d}G_{01} + \mathrm{d}G_{1,12} - \mathrm{d}G_{02} - \mathrm{d}G_{2,12})\Phi^{-1} \\ &= \mathcal{V}^{-1}\dot{\mathcal{V}} + \mathcal{V}^{-1}\mathcal{U}'^{-1}\dot{\mathcal{U}}'\mathcal{V} - \mathcal{U}^{-1}\dot{\mathcal{U}} - \mathcal{U}^{-1}\mathcal{V}'^{-1}\dot{\mathcal{V}}'\mathcal{U} \\ &= \mathcal{U}^{-1}\mathcal{V}'^{-1}(\mathcal{U}'\dot{\mathcal{V}} + \dot{\mathcal{U}}'\mathcal{V} - \mathcal{V}'\dot{\mathcal{U}} - \dot{\mathcal{V}}'\mathcal{U}), \end{aligned}$$

where we use twice the commutation at order zero.

We have thus determined the Lax matrices (uniquely, once a frame Φ at $\hat{F}$ is set). The CMC-1 quad they generate is the one we started from. Note that, although we have started as usual with the lower left vertex, this choice plays no role, and we might have fixed Φ_1, Φ_2 or Φ_{12} and recovered the rest similarly. □

LEMMA 3 ($\mathbb{S}^3$ version) Let $Q = (F, F_1, F_{12}, F_2)$ be a CMC or minimal quad in $\mathbb{S}^3$ with Gauss map (N, N_1, N_{12}, N_2). Let (ϕ, ϕ') be any frame at the vertex F, i.e. any couple in $SU(2)$ such that $F = \phi'^{-1}M\phi$ and $N = -\phi'^{-1}\mathbb{k}M\phi$ (such a pair is determined up to $U(1)$ action). Then there exist $\mathcal{U}(\lambda), \mathcal{V}(\lambda), \mathcal{U}'(\lambda), \mathcal{V}'(\lambda)$ satisfying (17), and thus generating the quad, together with its Gauss map by formulas (27).

Proof. The proof goes along the same lines as in Lemma 2, with a few specificities; in particular, whereas in $\mathbb{R}^3$ scaling allows us to freely set the mean curvature to 1, in $\mathbb{S}^3$, we use the mean curvature to determine $\lambda_1 = e^{i\gamma_1}, \lambda_2 = e^{i\gamma_2}$ as in theorem 5: $\gamma_1 = -\gamma_2 = \frac{1}{2}\mathrm{arccot} H$ ($\gamma_1 = \pi/4$ if $H = 0$), γ_1 being taken in $[0, \pi/2]$. A (geometric) CMC-H quad Q in $\mathbb{S}^3$, together with its Gauss map, comes with a discrete conformal metric s and canonical edge labellings A, B, such that the edge lengths in both directions are Ass_1 and Bss_2 respectively. This allows us to determine u, v by

$$\begin{gathered} u^2 = -ss_1\sqrt{\frac{1+H^2}{H^2}} = -\frac{ss_1}{\cos(2\gamma_1)}, \quad v^2 = ss_2\sqrt{\frac{1+H^2}{H^2}} = \frac{ss_2}{\cos(2\gamma_1)} \text{ if } H \neq 0, \\ u^2 = -ss_1, \quad v^2 = ss_2 \quad \text{if } H = 0, \end{gathered} \tag{34}$$

and similarly u', v'. We set positive α, β such that

$$\begin{gathered} \alpha^2 = -\frac{4\sin^2(2\gamma_1)}{A\cos(2\gamma_1)}, \quad \beta^2 = \frac{4\sin^2(2\gamma_1)}{B\cos(2\gamma_1)} \text{ if } H \neq 0, \\ \alpha^2 = -\frac{4}{A}, \quad \beta^2 = \frac{4}{B} \text{ if } H = 0. \end{gathered} \tag{35}$$

In the following, we will favour the notations in terms of γ_1:

$$\phi' \, \mathrm{d}F_{01}\phi^{-1} = \mathcal{U}(\lambda_1)^{-1}M\mathcal{U}(\lambda_2) - M = \mathcal{U}(\lambda_1)^{-1}\left(-\frac{2u\sin(2\gamma_1)}{\alpha}\mathbb{i}\right)$$
$$= \frac{2u\sin(2\gamma_1)}{\alpha}\frac{1}{\alpha}\begin{pmatrix} i(\lambda_1 u + \lambda_1^{-1}u^{-1}) & \bar{a} \\ a & -i(\lambda_1 u^{-1} + \lambda_1^{-1}u) \end{pmatrix}$$

We recognize the length of the edge, which proves incidentally that the right-hand side matrix is unimodular, and so is $\mathcal{U}(\lambda_1)$. As in $\mathbb{R}^3$, a is given by the $(\mathbb{i}, \mathbb{j})$ component, provided the $(\mathbb{1}, \mathbb{k})$ component is correct. The latter is equivalent to the following two conditions:

1. the angle θ between $\mathrm{d}F_{01}$ and N satisfies

$$\cos\theta = \frac{\langle \mathrm{d}F_{01}, N\rangle}{\|\,\mathrm{d}F_{01}\|} = \frac{1}{\alpha}(u^{-1} + \cos(2\gamma_1)u)\,, \tag{36}$$

2. the angle χ between $\mathrm{d}F_{01}$ and F satisfies

$$\cos\chi = \frac{\langle \mathrm{d}F_{01}, F\rangle}{\|\,\mathrm{d}F_{01}\|} = -\frac{u\sin(2\gamma_1)}{\alpha}\,. \tag{37}$$

Geometric derivation of $\cos\theta$ is analogous to (32). In particular in the CMC case $H = \cot(2\gamma_1) \neq 0$ the dual isothermic surface of F is $F + \frac{1}{H}N$, and (32) is modified to

$$\frac{2}{H}\cos\theta = \|\,\mathrm{d}F_{01}\| + \|\,\mathrm{d}F_{01}^*\| = \|\,\mathrm{d}F_{01}\|(1 + \frac{1}{s_1 s}) = \sqrt{s_1 sA}(1 + \frac{1}{s_1 s}).$$

Substituting (34) and (35) we arrive at (36).

Identity (37) follows directly from $\|F\| = \|F_1\| = 1$:

$$\cos\chi = \langle F, \mathrm{d}F_{01}\rangle = -\frac{1}{2}\|\,\mathrm{d}F_{01}\|^2 = -\frac{u\sin(2\gamma_1)}{\alpha}.$$

Along the other coordinate, we have v and non-crossing trapezoids, but the reasoning is analogous, and fixes b.

Setting $\Phi(\lambda_1) = \phi'$ and $\Phi(\lambda_2) = \phi$, we check that $N_1 = -\Phi_1(\lambda_1)^{-1}\mathbb{k}M\Phi_1(\lambda_2)$, where $\Phi_1(\lambda_i) = \mathcal{U}(\lambda_i)\Phi(\lambda_i)$. Indeed, this amounts to reversing the Proof in Theorem 5. We have the analogous result for N_2, which allows us to compute a', b' and therefore $\mathcal{U}'(\lambda)$ and $\mathcal{V}'(\lambda)$.

To prove (17) we reverse the calculation in the Proof of Theorem 5, which shows that $\mathcal{V}'(\lambda_1)\mathcal{U}(\lambda_1) = \mathcal{U}'(\lambda_1)\mathcal{V}(\lambda_1)$.

To prove the analogous result at λ_2, we consider the symmetric (see (8)) surface F^{-1} with the normal N^{-1}. This exchanges λ_1 with λ_2 while preserving all the metric and affine properties. We conclude with Lemma 1. □

Proof of the theorem.
The same strategy works for the Euclidean and spherical nets, so we will describe it in the Euclidean case. We start by constructing the Lax pair on one quad $Q = (\hat{F}, \hat{F}_1, \hat{F}_{12}, \hat{F}_2)$. For any choice of compatible frame Φ, i.e. $\hat{N} = -\Phi^{-1}\Bbbk\Phi$ at the base vertex, we can find a (unique) Lax pair generating Q, according to the Lemmas 2 or 3. This in turn determines a Lax pair on adjacent faces, which generates the corresponding quads, and therefore the whole Q-net.

This reasoning holds provided that we do not obtain a contradiction when the Gauss map is given at more than one vertex. For example, once $Q = (\hat{F}, \hat{F}_1, \hat{F}_{12}, \hat{F}_2)$ has been constructed, we may construct the adjacent quad $Q_1 = (\hat{F}_1, \hat{F}_{11}, \hat{F}_{112}, \hat{F}_{12})$ by starting with $\hat{N}_1$ or with $\hat{N}_{12}$. However the two quads constructed this way must agree, because $\hat{N}_1$ fully determines $\hat{N}_{12}$ (knowing the corresponding edge), see Lemma 2. □

As a consequence, we have the following integrability result

COROLLARY 1 A discrete CMC Q-net in $\mathbb{S}^3$ or $\mathbb{R}^3$ is determined, uniquely up to congruence, by its mean curvature H, its discrete conformal metric s and its edge labellings A, B.

6. The discrete Lawson correspondence

The results shown above allow us to define a discrete Lawson correspondence between Q-nets in $\mathbb{S}^3$ and $\mathbb{R}^3$.

THEOREM 7 Let F be a minimal Q-net in $\mathbb{S}^3$ with discrete conformal metric s. Then there exists a constant mean curvature Q-net $\hat{F}$ in $\mathbb{R}^3$ with constant mean curvature $H = 1$ and the same discrete conformal metric s. More generally, this correspondence takes any Q-net of constant mean curvature H lying in the sphere of curvature κ to a Q-net of constant mean curvature H' in the sphere of curvature κ', provided $H^2 + \kappa = H'^2 + \kappa'$. The Euclidean case corresponds to $\kappa' = 0$. Additionally, the two Q-nets F and $\hat{F}$ are given by formulas (27) and (23) with the same Lax matrices $\mathcal{U}(\lambda)$ and $\mathcal{V}(\lambda)$.

Proof. The construction is a direct consequence of the previous theorems, but to see it clearly, we shall state a slightly modified version of Theorems 5 and 6.

Let us start with the claim about Q-nets in spheres with different curvatures. Any Lax pair as defined in (15), and any choice $\lambda_1 = e^{i\gamma_1}$ of the spectral parameter gives rises to an constant mean curvature Q-net (F^{γ_1}, N) in the sphere of curvature $\kappa = \sin^2(2\gamma_1)$, obtained as a a scaled up version of the spherical net F defined in Theorem 5:

$$F^{\gamma_1} = \frac{1}{\sin(2\gamma_1)} F = \frac{1}{\sin(2\gamma_1)} \Phi(\lambda_1)^{-1} M \Phi(\lambda_1^{-1}),$$

and N remains the same. The mean curvature is $H = \cos(2\gamma_1)$. If $\gamma_1 = \pi/4$, this is the minimal Q-net in $\mathbb{S}^3$.

By the same reasoning according to Theorems 5 and 6 there exists infinitely many other CMC Q-nets $(F^{\gamma_1'}, N)$ of constant mean curvature $H' = \cos(2\gamma_1')$ in the sphere of curvature $\kappa' = \sin^2(2\gamma_1')$ with the same coefficients u, v in the Lax pair.

One checks easily that

$$H'^2 + \kappa' = \cos^2(2\gamma_1') + \sin^2(2\gamma_1') = 1 = H^2 + \kappa.$$

The most general case, where $H^2 + \kappa \neq 1$ is obtained from the latter by direct scaling. Finally, since the discrete conformal metric coefficient is defined geometrically up to general factor (see Section 3), the coincidence of the Lax matrix coefficients u, v is equivalent to the preservation of the discrete conformal metric coefficient s.

The case of $\kappa' = 0$ is dealt in exactly the same way, except that the immersion formula in Theorem 4 is used for $\mathbb{R}^3$. We note now that the factorization formulas (24) and (28) for minimal nets in $\mathbb{S}^3$ and CMC-1 nets in $\mathbb{R}^3$ coincide. The coincidence of the coefficients u, v implies that the discrete conformal metric factors s determined from these formulas are identical, in the same way as, in the smooth case, both surfaces are isometric. $\square$

REMARK 1 Furthermore, the Q-net in $\mathbb{R}^3$ is the limit of the spherical CMC nets, when the radius of the sphere increases to infinity, keeping the point $\mathbb{1}$ fixed. Indeed, let γ_1 tend to zero. Then $H = \cos(2\gamma_1)$ goes to 1, and

$$\begin{aligned} F - \mathbb{1} &= \Phi(\lambda_1)^{-1} M \Phi(\lambda_1^{-1}) = (\Phi + \gamma_1 \dot{\Phi} + o(\gamma_1))^{-1}(\mathbb{1} + \gamma_1 \mathbb{k} + o(\gamma_1))(\Phi - \gamma_1 \dot{\Phi} + o(\gamma_1)) - \mathbb{1} \\ &= \gamma_1(-2\Phi^{-1}\dot{\Phi} + \Phi^{-1}\mathbb{k}\Phi) + o(\gamma_1) \end{aligned}$$

so

$$\frac{1}{\sin(2\gamma_1)}(F - \mathbb{1}) \sim \frac{\gamma_1}{2\gamma_1}\left(-2\Phi^{-1}\dot{\Phi} + \Phi^{-1}\mathbb{k}\Phi\right) = -\Phi^{-1}\dot{\Phi} + \frac{1}{2}\Phi^{-1}\mathbb{k}\Phi = \hat{F}\,.$$

REMARK 2 At last, let us remark that this definition of the Lawson correspondence coincides with the seemingly very different one proposed in [7, 9], based on the Calapso transform and the conserved quantities formalism. We will not go into the details of the latter, but we will show that both definitions agree. Indeed, and despite their very different formulations, it suffices to show that the mean curvature, the discrete conformal metric and the labellings change in the same way. We conclude with Corollary 1.

Let $H = \cos(2\gamma_1)$ and $H' = \cos(2\gamma_1')$ be the two corresponding mean curvatures as defined above. Since u, v are common to both immersions, and $ss_1 = -u^2 H$ (resp. $ss_2 = v^2 H$), then $r = sH^{-1/2}$ takes the same values for both surfaces. The Reader may check that this vertex function r is the one used in [9, Sections 3 and 4.2], which is invariant under the Calapso transform. The edge labellings are $-\alpha(\gamma_1)^{-2}, \beta(\gamma_1)^{-2}$ and $-\alpha(\gamma_1)^{-2}, \beta(\gamma_1)^{-2}$, up to a multiplicative constant c. From (16), we see that

$$\alpha\left(\gamma_1'\right)^2 - \alpha(\gamma_1)^2 = \cos\left(2\gamma_1'\right) - \cos(2\gamma_1) = H' - H,$$

and similarly $\beta(\gamma_1')^2 - \beta(\gamma_1)^2 = H - H'$. Choosing $c = -1$, the edge labelling satisfy

$$a_{01}' = \frac{1}{\alpha'^2} = \frac{1}{\alpha^2 + H' - H} = \frac{1}{\frac{1}{a_{01}} + H' - H} = \frac{a_{01}}{1 + (H' - H)a_{01}},$$

and similarly $a_{02}' = \frac{a_{02}}{1 - a_{02}(H - H')}$, which is again the prescribed behaviour of the Calapso transform. Therefore both correspondences agree.

Conclusion and open questions

In this article, we have established a discrete Lawson isometry between discrete isothermic minimal surfaces in $\mathbb{S}^3$ and discrete isothermic CMC surfaces in $\mathbb{R}^3$. The isometry is understood in the sense that both corresponding isothermic surfaces have the same discrete conformal metric coefficient. It is appealing to lift this correspondence to the level of frames as in the smooth case (Theorem 3). However the isothermic parametrizations in (10) do not correspond: an isothermic surface in $\mathbb{S}^3$ corresponds to a CMC surface from the associated family.

One way to reach that, and an important achievement by itself, would be to introduce geometrically a discrete metric for the associated families in $\mathbb{R}^3$ and $\mathbb{S}^3$ that generalizes the conformal metric coefficient s of isothermic surfaces. It should be the same for the whole associated family. On the level of the Lax representation, it is the coefficient w in this article. The next step then would be to find a discrete analogue of (10).

Some progress in this direction has been achieved in [15] where the associated family in $\mathbb{R}^3$ was described as edge-constraint net with non-planar faces (see Section 4.2). The curvatures were defined there but not the conformal metric. Here we are dealing with discrete isothermic surfaces in non-isothermic parametrization. Let us mention that such triangulated surfaces were recently introduced in [16].

Acknowledgments

The authors wish to thank Udo Hertrich-Jeromin, Wayne Rossman and Tim Hoffmann for the fruitful discussions on this topic, as well as the referees for their helpful remarks.

Funding

DFG Collaborative Research Center TRR 109 'Discretization in Geometry and Dynamics'.

References

1. Lawson, H. B. (1970) Complete minimal surfaces in S^3. *Ann. Math. (2)*, **92**, 335–374.
2. Große-Brauckmann, K., Kusner, R. & Sullivan, J. M. (2001) Triunduloids: embedded constant mean curvature surfaces with three ends and genus zero. *J. Reine Angew. Math.,* **564**, 35–61.
3. Große-Brauckmann, & Polthier, K. (1997) Compact constant mean curvature surfaces with low genus. *Exp. Math.* **6**, 13–32.
4. Oberknapp, B. & Polthier, K. (1997) An algorithm for discrete constant mean curvature surfaces. *Visualization and Mathematics (Berlin-Dahlem, 1995)* (H. C. Hege & K. Polthier eds). Berlin: Springer, pp. 141–161.
5. Bobenko, A. I. & Pinkall, U. (1999) Discretization of surfaces and integrable systems. *Discrete Integrable Geometry and Physics* (A. I. Bobenko & R. Seiler eds). New York: Oxford University Press, pp. 3–58.
6. Bobenko, A. I., Hertrich-Jeromin, U. & Lukyanenko, I. (2014) Discrete constant mean curvature nets in space forms: Steiner's formula and Christoffel duality. *Discrete Comput. Geom.* **52**, 612–629.
7. Burstall, F., Hertrich-Jeromin, U., Rossman, W. & Santos, S. (2015) Discrete special isothermic surfaces. *Geom. Dedicata* **174**, 1–11.
8. Bobenko, A. I. & Suris, Y. B. (2008) *Discrete Differential Geometry: Integrable Structure*. Graduate Studies in Mathematics, vol. 98. Providence, RI: American Mathematical Society, pp. xxiv + 404.
9. Burstall, F., Hertrich-Jeromin, U. & Rossman, W. (2014) Discrete linear Weingarten surfaces. ArXiv 1406.1293.
10. Bobenko, A. I. (1991) Constant mean curvature surfaces and integrable equations. *Russian Math. Surveys*, **46**, 1–45.

11. Bobenko, A. I., Pottmann, H. & Wallner, J. (2010) A curvature theory for discrete surfaces based on mesh parallelism. *Math. Ann.* **348**, 1–24.

12. Bobenko, A. I. & Pinkall, U. (1996) Discrete isothermic surfaces. *J. reine Angew. Math.* **475**, 187–208.

13. Bobenko, A. I. & Suris, Y. B. (2009) Discrete Koenigs nets and discrete isothermic surfaces. *Int. Math. Res. Not.* 1976–2012.

14. Bücking, U. & Matthes, D. (2016) Constructing solutions to the Björling problem for isothermic surfaces by structure preserving discretization. *Advances in Discrete Differential Geometry* (A. I. Bobenko ed.). Berlin: Springer, pp. 309–345.

15. Hoffmann, T., Sageman-Furnas, A. O. & Wardetzky, M. (2016) A discrete parametrized surface theory in $\mathbb{R}^3$. *Int. Math. Res. Not.,* 1–42.

16. Lam W. Y. & Pinkall, U. (2016) Isothermic triangulated surfaces, *Math. Ann.,* doi:10.1007/s00208-016-1424-z.