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Abstract — Shell-type structures were and are being analysed using the Finite Element Method with 2D shell-type elements. However, for laminates a full 3D solution is required since the complexity of the material behaviour law. The Proper Generalized Decomposition can obtain 3D solutions while keeping the 2D complexity of 2D shell approaches. However, the PGD fails when dealing with complex geometries. We propose a technique, based on the Domain Decomposition technique, that is able to generate complex geometries from easier ones. Numerical examples show the performance of the method.
Keywords — Proper Generalized Decomposition; Domain Decomposition;
Hibridigizable Discontinuous Galerkin; Complex geometry.

1 Introduction

In order to simulate the behaviour of shell-type of structures, specific shell-type elements were introduced to the Finite Element Method (FEM) framework [1]. This approach permits to simplify the 3D problem to a 2D problem thanks to the used of several simplificative hypotheses. However, they are unable to provide the full 3D solution which is required when composite laminates are used and complex behaviours are involved. The Proper Generalized Decomposition (PGD) [3] is a solver for partial differential equations which permits to obtain a full 3D solution for laminates [4, 5, 6], by using a separated representation of the space. For instance, the previous 3D problem can be decomposed in a series of 2D problems (in-plane) and 1D problems (out-of-plane). Additionally, the PGD also permits to introduce different model parameters, such as material properties or forces, among others, as extra coordinates. See [7] for an extensive analysis of this aspect. However, the PGD-based approach, despite of its efficiency when solving this kind of problems, fails when complex geometries are used. One possibility to solve complex geometries consist of subdividing the full geometry in easier ones. A suitable technique for this purpose is the Domain Decomposition (DD) technique. The subdomains generated with the DD technique are coupled via artificial (internal) boundaries. A full description of the Domain Decomposition method can be found in [8].

In this contribution we propose the use of a Domain Decomposition (DD) approach in order to subdivide the complex structure into simple subdomains and "glue" them via a stabilized approach based on the ideas of the Hibridigizable Discontinuous Galerkin (HDG) method [9]. That is, we use a version of the DD method that weakly imposes the continuity in the solution and also weakly equilibrates the normal fluxes along the interfaces. Each subdomain is solved using the PGD solver with parametric Dirichlet boundary conditions over the internal boundaries. Therefore, when solving the problem that couples all subdomains, the solution for each subdomain is available and only a post-processing of the solution will be required. Therefore, the "gluing" algorithm is independent of the nature of the problem of each subdomain. For instance, we can solve some subdomains with an in-plane out-of-plane approach while the others without space separation. This is possible due to the fact that the input data to the "gluing"

algorithm is the solution of each subdomain explicitly depending on the solution at the interfaces.

2 Problem model

The problem model consists of the 2D heat equation. We denote the temperature field as u, the flux field as $\underline{q}$, and the source therm as f. All these fields being defined in the domain $\Omega \subset \mathbb{R}^2$, of boundary denoted by $\partial\Omega$. Prescribed fluxes denoted by $\bar{\underline{q}}$ are imposed on Γ_N, while given temperatures denoted by $\bar{u}$ are prescribed over the complementary part of the boundary Γ_D. The diffusion problem takes the following weak form:

$$\begin{aligned} &\text{Find } u \in (V + \{w\}) : \forall v \in V \\ &\int_\Omega \nabla v \cdot k \nabla u \, \mathrm{d}\Omega = \int_\Omega v \, f \, \mathrm{d}\Omega + \int_{\Gamma_N} v \, \bar{\underline{q}} \cdot \underline{n} \, \mathrm{d}\Gamma \end{aligned} \tag{1}$$

where $V = \{v \ : \ v \in H^1(\Omega), v|_{\Gamma_D} = 0\}$ and w is a particular temperature field satisfying the Dirichlet boundary conditions, k is the diffusion coefficient and $\underline{n}$ is the outwards normal vector. In this case, we suppose that the domain Ω is complex to be solve with the PGD under a space separated representation. Therefore, the domain Ω will be subdivided in a set of non-overlapped subdomains suitable to be solve with the PGD. The process will be described in section 3.

3 Domain Decomposition

The DD technique allows to solve the global problem defined in the domain Ω by solving smaller problems defined in a set of subdomains $\Omega_i \subset \Omega$ such that $\Omega = \cup_{i=1}^{N} \Omega_i$, where N is the number of subdomains. In this particular implementation we consider the case of non-overlapping subdomains, then $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$. In literature [8] we can find a variety of different methods for the DD technique based on the boundary conditions applied over the internal (artificial) boundaries created to "glue" the subdomains. In our situation we will use a version based on the HDG method [9] which permits non-conforming discretizations for each subdomain (u_i) and along the interface ($\hat{u}$).

The PGD allows to construct a parametric solution, depending on the Dirichlet boundary conditions along the internal interfaces, in each subdomain. Let us assume for a while that we have already available the solutions $u_i(x,y,\underline{\tilde{a}}_i)$, where $\underline{\tilde{a}}_i$ is a vector containing the parameters that defines solution at the interface for the subdomain i. Therefore, we have also available the fluxes crossing the internal boundaries $\underline{q}_i(x,y,\underline{\tilde{a}}_i)$ depending on the same set of parameters. We also assume that there exist a projection Π_i which projects the parameters $\underline{\hat{u}}$ controlling the field $\hat{u} = \mathcal{B}\underline{\hat{u}}$ (being $\mathcal{B}$ a hierarchical polynomial basis), which defines the solution along the internal interface, over the parameters that controls the solution, also along the internal interface, viewed from each subdomain i as follows:

$$\underline{\tilde{a}}_i = \Pi_i \underline{\hat{u}} \tag{2}$$

Under these assumptions, we can write that the solution and the flux field at each subdomain depends on $\underline{\hat{u}}$: $u_i(x,y,\underline{\hat{u}})$ and $\underline{q}_i(x,y,\underline{\hat{u}})$. Then, the only unknown is the field $\hat{u}$ defined only along the internal boundaries. In order to evaluate $\hat{u}$, taking the ideas used in the HDG [9], we define the following global problem defined along all internal interfaces.

Find $\hat{u}(x,y) \in \hat{V}$ such that $\forall \mu \in \hat{V}$:

$$\sum_{i=1}^{N} \int_{\Gamma_i^{\text{int}}} \mu \left(\underline{q}_i \cdot \underline{n}_i + \tau(u_i - \hat{u}) \right) \ \mathrm{d}\Gamma = 0 \tag{3}$$

where $\hat{V} = \{\mu \ : \ \mu \in H^0(\Gamma^{\text{int}})\}$, $\Gamma^{\text{int}} = \cup \Gamma_i^{\text{int}}$, $\underline{n}_i$ is the outwards normal vector of the subdomain i and $\tau = 1$ is a user defined parameter. The first term in (3) ($\underline{q}_i \cdot \underline{n}_i$) tries to balance the fluxes crossing the interfaces, while the second term ($\tau(u_i - \hat{u})$) can be seen as a stabilization term. This last term is useful for non-conforming discretizations between the interface and each subdomain. Note that in the particular case in which the discretizations coincide, the stabilization term vanishes since $u_i|_{\Gamma_i} \equiv \hat{u}|_{\Gamma_i}$. Even if the problem is linear (1), since the parametric solution at each subdomain u_i is obtained with the PGD, the problem defined in (3) will lead to a non-linear algebraic system of equations which will be solved using

the Newton-Rahpson method. The non-linearity comes due to the fact that the PGD solution is compose by products of functions depending on $\underline{\tilde{a}}_i$. Note that since the analytical expressions for both u_i and $\underline{q}_i$ are known, the tangent matrix has also an analytical expression, easing the resolution of the non-linear problem.

4 The PGD with parametric Dirichlet boundary conditions

As indicated before, the solution for the problem defined in (1) will be evaluated with the DD technique introduced in section 3. Thus, we need to solve the problem defined in (1) extended only over the domain Ω_i and for any possible Dirichlet boundary condition over the internal boundaries. In order to avoid unnecessary calculations, the basis used to define the Dirichlet boundary conditions will be also $\mathcal{B}$ and the coefficients $\underline{\tilde{a}}_i$ will be evaluated using the projector Π_i that minimizes the difference between $\hat{u}$ and u_i along the interface, in the L_2-norm sense. The basic concepts to introduce parametric Dirichlet boundary conditions as extra coordinates under the PGD framework can be found in [7]. In this case we detail it in a more complex situation in which all sides of the domain can be under parametric Dirichlet boundary conditions. First we perform the following change of variable:

$$u_i^\star = u_i - \sum_{j=1}^{I_i} \sum_{k=2}^{D_i^j} \Phi_{i,k}^j(x,y) a_{i,k}^j - \sum_{k=1}^{E_i} \Psi_{i,k}(x,y) b_{i,k} \tag{4}$$

where $\Phi_{i,k}^j$ and $\Psi_{i,k}$ are functions defined in all the subdomain. Functions $\Psi_{i,k}$ meets, along the internal boundaries, the linear terms of the basis $\mathcal{B}$ which takes unitary values at corners. Functions $\Phi_{i,k}^j$ meets, also along the internal boundaries, the non-linear terms which are zero at the corners, then we can make independent the different sides I_i. D_i^j is the degree of the polynomial over the side j of the subdomain i and E_i is the number of corners in contact with internal boundaries. $a_{i,k}^j$ and $b_{i,k}$ are the corresponding parameters which compose $\underline{\tilde{a}}_i$. The two last terms in (4) represent the parametric Dirichlet boundary conditions. The solution to the problem (1) restricted to the subdomain i, after applying the change of variable defined in (4) enunciates as follows: find $u_i^\star \in V_i : \forall v \in V_i$

$$\begin{aligned}\int_{\Omega_i} \nabla v \cdot k_i \nabla u_i^\star \, \mathrm{d}\Omega = &- \sum_{j=1}^{I_i} \sum_{k=0}^{D_i^j} \int_{\Omega_i} \nabla v \cdot k_i \nabla \Phi_{i,k}^j a_{i,k}^j \, \mathrm{d}\Omega \\ &- \sum_{k=0}^{E_i} \int_{\Omega_i} \nabla v \cdot k_i \nabla \Psi_{i,k} b_{i,k} \, \mathrm{d}\Omega + \int_{\Omega_i} v \, f \, \mathrm{d}\Omega + \int_{\Gamma_N} v \, \bar{\underline{q}} \cdot \underline{n} \, \mathrm{d}\Gamma \end{aligned} \tag{5}$$

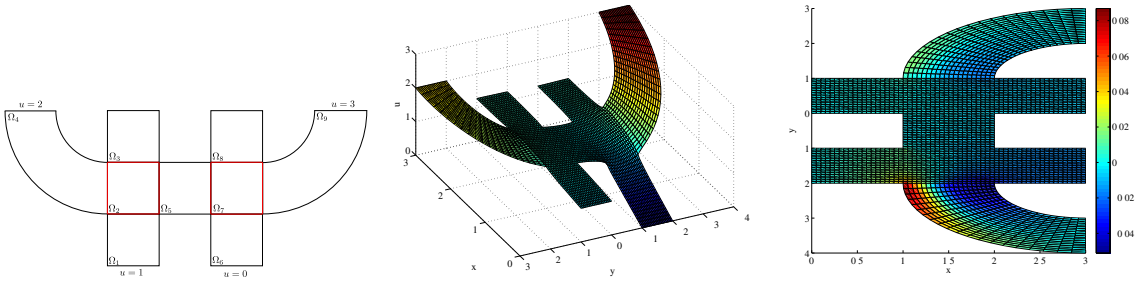
where V_i is the restriction of V to Ω_i. This problem is suitable to be solved with the PGD solver using a separation of the space domain. Note that if we choose a space-separated solution, both functions $\Phi_{i,k}^j$ and $\Psi_{i,k}$ must have a separated representation. The solution of the problem (5) reads as follows:

$$u_i^\star \approx \sum_{l=1}^{M} X_{i,l}(x) Y_{i,l}(y) \prod_{j=1}^{I_i} \prod_{k=2}^{D_i^j} A_{i,k,l}^j(a_{i,k}^j) \prod_{k=0}^{E_i} B_{i,k,l}(b_{i,k}) \tag{6}$$

where M is the number of modes. Then, the solution u_i depends on the parameters that define the Dirichlet boundary condition along the internal boundaries and it can be used to solve the problem (3).

5 Numerical results

The problem here propsed is composed by 9 different subdomains all of them solved using a space separated representation. In the subdomains Ω_4 and Ω_9, a cylindrical coordinates system is used. The rest of subdomains are squares of side 1. Dirichlet boundary conditions are imposed according to the model represented in figure 1a. The rest of the boundaries are isolated. The diffusion coefficient is $k = 1$. The source term is $f = 0$. This problem is solved using a discretization of 21 nodes in x and y directions for each subdomain. The basis for the parametric Dirichlet boundary conditions is of order 6. The



(a) Model of the problem (b) DD+PGD solution (c) Relative error

Figure 1: Model of the problem and solution using the proposed technique (DD-PGD) and the local relative error with respect to the FE solution.

domain defined for the parameters controlling the Dirichlet boundary conditions is $\Omega_s = [-1000, 1000]$ discretized with 1001 equidistributed nodes.

Figure 1b shows the solution for this problem obtained with the proposed technique. Figure 1c shows the error with respect to a FE solution using the same discretization. We observe that the solution obtained with the proposed technique is similar to the FE solution with a relative error around 0.05%. Additionally, the non-linear solver converges to the solution in only 2 iterations, due to the intrinsic linearity of the problem, with an error smaller that 10^{-6}.

6 Conclusions

The PGD technique is of a high interest when simulating the behaviour of laminates due to the capability of obtaining full 3D solutions while keeping a 2D computational complexity. However, when it make use of the space separated representation, the PGD technique is difficult to by applied when non-separable geometries are used, as the one used in the proposed problem. The proposed technique is able to couple a set of different subdomains solved with the PGD under a separated space representation allowing the use of the PGD solver with a space separated representation for addressing complex geometries.

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