Polynomial Silent Self-Stabilizing p-Star Decomposition
Mohammed Haddad, Colette Johnen, Sven Köhler

To cite this version:
Mohammed Haddad, Colette Johnen, Sven Köhler. Polynomial Silent Self-Stabilizing p-Star Decomposition. 2016. hal-01514323

HAL Id: hal-01514323
https://hal.archives-ouvertes.fr/hal-01514323
Preprint submitted on 26 Apr 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Abstract

We present a silent self-stabilizing distributed algorithm computing a maximal $p$-star decomposition of the underlying communication network. Under the unfair distributed scheduler, the most general scheduler model, the algorithm converges in at most $12\Delta m + O(m + n)$ moves, where $m$ is the number of edges, $n$ is the number of nodes, and $\Delta$ is the maximum node degree. Regarding the move complexity, our algorithm outperforms the previously known best algorithm by a factor of $\Delta$. While the round complexity for the previous algorithm was unknown, we show a $5\left\lceil \frac{n}{p+r} \right\rceil + 5$ bound for our algorithm.

1. Introduction

Fault-tolerance is among the most important requirements for distributed systems. Self-stabilization is a fault-tolerance technique that deals with transient faults. It was first introduced by Dijkstra [1]. Starting in an arbitrary configuration, a self-stabilizing distributed system converges to a legitimate configuration in finite time by itself, i.e., without any external intervention. This makes self-stabilization an elegant approach for non-masking fault-tolerance [2].

An $H$-decomposition of a graph $G$ subdivides a graph into disjoint components which are isomorphic to $H$. A $p$-star is a complete bipartite graph $K_{1,p}$ with one center node and $p$ leaves. One of the famous and well studied graph decompositions in literature is star decomposition [3, 4, 5]. A decomposition of a graph into stars is a way of expressing the graph as the union of disjoint stars [6]. The problem of star decomposition has several applications including scientific computing, scheduling, load balancing and parallel computing [7], important nodes detection in social networks [8]. Decomposing a graph into stars is also used in parallel computing and programming. This decomposition offers similar feature as Master-Slave paradigm, used in grids [9] and P2P infrastructures [10].

---

1This study has been carried out with financial support from the French State, managed by the French National Research Agency (ANR) in the frame of the “Investments for the future” Programme IdEx Bordeaux - CPU (ANR-10-IDEX-03-02) and by the Sustainability Center Freiburg, Germany. The Sustainability Center Freiburg is a cooperation of the Fraunhofer Society and the University of Freiburg and is supported by grants from the Baden-Württemberg Ministry of Economics and the Baden-Württemberg Ministry of Science, Research and the Arts.

Email addresses: mohammed.haddad@univ-lyon1.fr (Mohammed Haddad), johnen@labri.fr (Colette Johnen), koehlers@informatik.uni-freiburg.de (Sven Köhler)
2 Related Work

Self-stabilizing algorithms have been proposed for a large variety of graph theoretical problems such as finding minimal dominating sets [11], maximal matchings [12, 13], independent sets [14], spanning trees [15], etc. The graph decomposition problem is defined on a graph \( G = (V, E) \), where \( V \) is the set of nodes and \( E \) is the set of edges, such that the graph \( G \) is decomposed into smaller components having specific properties. These properties are often defined on the size of the partitions (clusters), on their shape (subgraphs) or both (patterns). Graph partitioning into clusters was considered by [16] and [17]. In [18], authors considered a particular graph partitioning problem that consists of decomposing the graph into partitions of order \( k \). [18] considered a particular graph decomposition problem that consisted in partitioning a graph with \( k^2 \) nodes into \( k \) partitions of order \( k \). The proposed algorithm relies on self-stabilizing spanning tree construction and converges within \( 3(h+1) \) steps where \( h \) is the height of the spanning tree. Furthermore, [16], [17] and [19] focused on decomposing the graph into clusters while [20] considered decomposition of graphs into triangles. Other self-stabilizing algorithms were proposed for graph colorings [21, 22] that can be considered as decompositions into independent sets.

Observe that the maximal node-disjoint \( p \)-star decomposition problem when restricted to \( p = 1 \) is equivalent to maximal matching problem in graphs. Thus the general problem when \( p \geq 1 \) is NP-complete since generalized matching and general graph factor problems were proved to be NP-complete in [23] and [24] respectively. The maximal matching problem has received much interest due the abundant number of applications in fields as diverse as transversal theory, assignment problems, network flows, and scheduling. Many studies have addressed this problem even in the field of self-stabilization [12, 25, 13, 26]. The best known move complexity for maximal matching problem is \( \mathcal{O}(m) \) and was obtained by [27].

The first self-stabilizing algorithm for the \( p \)-star decomposition problem was proposed in [28]. It finds a maximal decomposition into node-disjoint \( p \)-stars. The decomposition is maximal in the sense that the nodes not part of any \( p \)-star cannot form a \( p \)-star. However, the algorithm proposed in [28] always converges to a unique legitimate configuration according to the input graph and does not guarantee a polynomial move complexity. An improvement was proposed in [29] where authors dealt with the uniqueness of legitimate state and proved their algorithm to converge within \( O(\Delta^2 m) \) moves under the unfair distributed scheduler where \( m \) is the number of edges and \( \Delta \) is maximum node degree in the graph. A bound on the round complexity of the algorithm was not given.

Our Results

In this paper, we improve the move complexity of the previous algorithm to \( 12\Delta m + \mathcal{O}(m+n) \) and prove an \( \mathcal{O}(n) \) bound on the round complexity. The algorithm is proven correct and analyzed under the unfair distributed scheduler, the most powerful adversary. The algorithm does not converge to a unique legitimate configuration. In fact, there is a legitimate configuration for any valid maximal \( p \)-star decomposition.

The paper is organized as follows: Computation model is defined in Section 2, then details of the \( p \)-star decomposition algorithm are presented in Section 3. Proofs of correctness and convergence are given in Sections 4 and 5, respectively.

2. Model of Computation

We model the distributed system as a simple undirected graph \( G = (V, E) \), where \( V \) is a set of nodes, and \( E \) is a set of edges representing the communication links. We denote by \( N(v) \) the set of all neighbors of \( v \) in \( G \), i.e., \( N(v) \) contains all nodes that can communicate with \( v \). We call \( N(v) \) the open neighborhood and denote the closed neighborhood by \( N[v] = N(v) \cup \{v\} \).

A distributed algorithm defines a set of shared variables and a set of rules for each node. In this paper, we only discuss uniform algorithms, i.e., each node has the same set of variables and executes the same set of rules. The (ordered) tuple of the values assigned to the variables of a node is called the local state. The (ordered) tuple of all local states constitutes the current configuration of the distributed system. Each rule is given in the form \( guard(v) \rightarrow action \), where \( guard(v) \) is a Boolean predicate over the variables within the closed neighborhood of node \( v \). The action may read all shared variables within the closed neighborhood.
3 THE P-STAR DECOMPOSITION ALGORITHM

but may modify only the variables of \( v \) itself. We say that a rule is enabled, if its guard evaluates to true. We say that a node is enabled, it any of its rules is enabled. For the algorithm given in this paper, at most one rule per node is enabled at a time.

Executions of a distributed algorithm are represented by a sequence of configurations \( e = \langle e_0, e_1, e_2, \ldots \rangle \), where \( e_0 \) is called the initial configuration. Executions are organized in steps. The \( i \)-th step, where \( e_i \) is reached from \( e_{i-1} \), denoted by \( e_{i-1} \rightarrow e_i \), consists of three phases. In the first phase, an adversarial scheduler selects a non-empty set \( S_i \) of nodes that are enabled in \( e_{i-1} \). In the second phase, all selected nodes perform the action of an enabled rule and compute their new local state. The computed local state is then made visible to neighboring nodes in the third phase. This model is called composite atomicity. If a node was selected, then we say that the node has made a move during that step. The sequence \( \langle S_1, S_2, S_3, \ldots \rangle \) is called the schedule of \( e \).

Three scheduler models are commonly discussed in the literature. The central scheduler selects exactly one enabled node per step. The synchronous scheduler selects all enabled nodes in each step. The most general model is the distributed scheduler, where the adversary may chose any non-empty of enabled nodes. A scheduler may satisfy certain fairness properties. We say that a scheduler is fair, if it selects any continuously enabled process after a finite number of steps. An unfair scheduler may indefinitely delay the move of a node as long as there are other enabled nodes. The algorithm given in this paper is proven correct and analyzed under the most general model, i.e., the unfair distributed scheduler.

A set \( CS \) of configurations is closed if any step from a configuration of \( CS \) reaches a configuration of \( CS \). By extension, a predicate \( Pr \) on configurations is said closed if the set of configurations verifying \( Pr \) is closed.

A distributed algorithm is called self-stabilizing, if it satisfies the following requirements:

- Any execution reaches a legitimate configuration after a finite number of steps. (Convergence)
- The set of legitimate configurations is closed. (Closure)

The legitimacy of a configuration typically expresses that the output of the algorithm is correct.

The algorithm given in this paper is silent, i.e., any execution eventually reaches a terminal configuration in which no node is enabled. Note that for any silent self-stabilizing algorithm, terminal configurations are also legitimate. Its runtime complexity is analyzed with respect to two different metrics: the move and the round complexity. The move complexity is defined as \( \sum_i |S_i| \). The round complexity is obtained by partitioning the execution \( e \) into asynchronous rounds as follows: The first round is the minimal prefix \( \langle c_0, c_1, \ldots, c_2 \rangle \) of \( e \) such that \( V \setminus D_0 \subseteq \bigcup_{i=1}^{\infty} D_i \cup S_i \), where \( D_i \) is the set of nodes that are not enabled in \( c_i \). The second round of \( e \) is the first round of the remaining suffix \( \langle c_2, c_3+1, c_3+2, \ldots \rangle \), and so forth. One asynchronous round allows each node that was enabled at the beginning of the round to make a move, unless the node becomes disabled due to a move by a neighbor.

3. The p-Star Decomposition Algorithm

In this section we describe the implementation of the p-star decomposition algorithm. First we give an overview of how the algorithm works. Assume that some nodes are already a member of a star, while others are not. Nodes indicate to their neighbors whether they are a member of a star or not. In addition, each node indicates whether it may be viable center of a new star. That is the case only if the node itself and \( p \) of its neighbors are not a member of a star, yet.

Nodes keep track of the viable centers within the closed neighborhood. Unless they are a member of a star, they invite the viable center having the minimum identifier to form new p-star. The invitation is updated as needed if the set of viable centers within the closed neighborhood changes. Directing the invitation at the viable center with the minimum identifier makes sure that no deadlocks can occur. Eventually, a viable center is invited by itself and at least \( p \) neighbors. Such a viable center \( v \) then picks \( p \) neighbors as the leaves of the star and assigns them to a new p-star centered at \( v \).

In the remainder of this section, we present the shared variables and predicates used by the algorithm. Then, the rules are presented.
3 THE P-STAR DECOMPOSITION ALGORITHM

3.1. Variables and Predicates

Algorithm 1 gives an overview over the variables, their domain, their meaning, and the predicates that the algorithm uses. Node $v$ is the center of a p-star if and only if $v$ verifies $\text{correctCenter}(v)$. The set stored in the shared variable $\text{leaves}(v)$ contains the $p$ neighbors of $v$ belonging to the star centered in $v$. So $\text{star}(v)$, as defined below, is the set of nodes in the p-star centered in $v$.

Definition 1. $\text{star}(v) = \emptyset$ if $\text{leaves}(v) = \emptyset$ and $\text{star}(v) = \{v\} \cup \text{leaves}(v)$ otherwise.

For any p-star centered in $v$, the following statements are proven (Observation 1, Lemma 2):

- The shared variable $\text{center}$ is equal to $v$ for every node in $\text{star}(v)$.
- Every node in $\text{star}(v)$ verifies the predicate $\text{correctLeaf}$ or $\text{correctCenter}$.
- The star is well formed, i.e., the star contains $v$ and $p$ neighbors of $v$.

The values of the shared variables $\text{center}$ and $\text{leaves}$ in its neighborhood are enough to allow a node $v$ to determine the value of the predicates $\text{correctCenter}(v)$ and $\text{correctLeaf}(v)$; so they suffice to compute the value of $\text{isInStar}(v)$.

Algorithm 1 Shared Variables, Predicates, Macros and Guard predicates

Shared variables of each node $v \in V$

- $\text{center}(v)$ — a node identifier or $\perp$
  - The center of the p-star that $v$ belongs to or the viable center that $v$ invites to form new p-star. The value $\perp$ is used if $v$ is not a member of a p-star and is not inviting any node.
- $\text{leaves}(v)$ — a set of up to $p$ node identifiers
  - The set is empty if $v$ is not the center of a p-star.
  - Otherwise it contains the leaves of the p-star.
- $\text{inStar}(v) \in \text{Boolean}$
  - Indicates whether $v$ is a member of a p-star.
- $\text{viableCenter}(v) \in \text{Boolean}$
  - Indicates whether $v$ is a viable center for a new p-star.
- $\text{lockedCenter}(v) \in \text{Boolean}$
  - Indicates whether the value of $\text{center}(v)$ is locked or not.

Predicates

- $\text{isCenter}(v) \equiv |\text{leaves}(v)| = p$
- $\text{incorrectCenter}(v) \equiv (|\text{leaves}(v)| \neq 0) \land ((\text{center}(v) \neq v) \lor (\exists u \in \text{leaves}(v) : \text{center}(u) \neq v) \lor \neg \text{isCenter}(v) \lor (\text{leaves}(v) \not\subseteq N(v)))$
- $\text{correctLeaf}(v) \equiv (\text{center}(v) \in N(v)) \land \text{isCenter}(\text{center}(v)) \land (v \in \text{leaves}(\text{center}(v)))$
- $\text{correctCenter}(v) \equiv \text{isCenter}(v) \land \neg \text{incorrectCenter}(v)$
- $\text{isInStar}(v) \equiv \text{correctLeaf}(v) \lor \text{correctCenter}(v)$
- $\text{isViableCenter}(v) \equiv \neg \text{isInStar}(v) \land (\{u \in N(v) \mid \neg \text{isStar}(u)\} \geq p)$

Macros

- $\text{bestCenter}(v)$ is the element of $\{u \in N[v] \mid \text{viableCenter}(u) \land \text{leaves}(u) = \emptyset\}$ having the smallest identifier or $\perp$ if the set is empty
- $\text{potentialLeaves}(v)$ is the set $\{u \in N(v) \mid \text{center}(u) = v \land \text{lockedCenter}(u)\}$

Guard Predicates

- $\text{starToUpdate}(v) \equiv \neg \text{isInStar}(v) \land (|\text{potentialLeaves}(v)| \geq p) \land (v = \text{center}(v))$
- $\text{centerToUpdate}(v) \equiv \neg \text{isInStar}(v) \land (\text{center}(v) \neq \text{bestCenter}(v) \lor \neg \text{lockedCenter}(v))$
- $\text{variablesToUpdate}(v) \equiv (\text{star}(v) \neq \text{isInStar}(v)) \lor (\text{viableCenter}(v) \neq \text{isViableCenter}(v)) \lor \neg \text{incorrectCenter}(v)$
Observation 1.  
leaves ≠ ∅ ∧ ¬incorrectCenter(v) ⇒ correctCenter(v)  
leaves(v) = ∅ ⇒ ¬incorrectCenter(v)  
variablesToUpdate(v) ⇒ v is enabled  
incorrectCenter(v) ⇒ v is enabled

Algorithm 2: p-star rules on v

Procedures
- updateBooleans(v):  
  inStar(v) := isInStar(v);  
  viableCenter(v) := isViableCenter(v);  
- updateVariables(v): if incorrectCenter(v) then leaves(v) := ∅;  
  updateBooleans(v);

Rules
RA(v): starToUpdate(v) →  
  leaves(v) := subset of potentialLeaves(v) with exactly p elements;  
  updateBooleans(v);  
RI(v): ¬starToUpdate(v) ∧ centerToUpdate(v) ∧ ¬lockedCenter(v) →  
  lockedCenter(v) := true;  
  center(v) := bestCenter(v);  
  updateVariables(v);  
RGI(v): ¬starToUpdate(v) ∧ centerToUpdate(v) ∧ lockedCenter(v) →  
  lockedCenter(v) := false;  
  updateVariables(v);  
RU(v): ¬starToUpdate(v) ∧ ¬centerToUpdate(v) ∧ variablesToUpdate(v) →  
  updateVariables(v);

Lemma 2. Every node v verifying ¬incorrectCenter(v) and star(v) ≠ ∅ satisfies the following assertions:

\[ |\text{star}(v)| = p + 1 \]  
\[ \text{star}(v) \subseteq N[v] \]  
\[ \text{center}(u) = v \quad \forall u \in \text{star}(v) \]  
\[ \text{correctLeaf}(u) = \text{true} \quad \forall u \in \text{leaves}(v) \]  
\[ \text{isInStar}(u) = \text{true} \quad \forall u \in \text{star}(v) \]

Proof. We have leaves(v) ≠ ∅ and ¬incorrectCenter(v). According to Observation 1, v verifies correctCenter(v).

By definition of correctCenter(v), we have leaves(v) ⊆ N(v), |leaves(v)| = p, so assertions (1) and (2) are verified.

Assertion (3) is verified by v because we have ¬incorrectCenter(v). We conclude that assertion (4) is verified. So assertion (5) is also verified.

3.2. p-Star Construction

The shared variable inStar allows nodes to determine whether they are a viable center of a new p-star. If a node v and at least p neighbors of v are not a member of a p-star, then v may become the center of a new p-star (i.e., v verifies the predicate isViableCenter).

The shared variable viableCenter(v) informs neighbors of node v that in the current configuration, v may become the center of an new p-star. The shared variables inStar and viableCenter are updated by any rule during the execution of procedure updateBooleans. Note that if the value of inStar(v) or viableCenter(v) is not accurate, then v is enabled (Observation 1).
4 CORRECTNESS OF TERMINAL CONFIGURATIONS

In the remainder of the algorithm presentation, we will study the behavior of nodes \((v, u)\) that are not in a \(p\)-star (i.e., nodes in \(CS\)).

**Definition 2.**

Let \(S\) be the set of nodes belonging to a \(p\)-star, i.e., \(S = \bigcup_{v \in V} \text{star}(v)\). Then \(CS = V \setminus S\) is the complement of \(S\), i.e., the nodes that do not belong to a \(p\)-star.

The macro \(\text{bestCenter}(v)\) returns the viable center that \(v\) should invite to create a new \(p\)-star. If \(v\) is not connected to a viable center, then \(\bot\) is returned. More precisely, \(\text{bestCenter}(v)\) chooses the node having the smallest identifier among the viable centers at a distance less than 2 from \(v\). In particular, we have \(\text{viableCenter}(u)\) and \(\text{leaves}(u) = \emptyset\) if \(\text{bestCenter}(v) = u\).

To invite \(u\), node \(v\) sets its shared variable \(\text{center}\) to \(u\) via rule \(\text{RI}\). Rule \(\text{RI}\) is the only rule modifying the value of \(\text{center}(v)\). If rule \(\text{RI}\) is enabled for node \(v\), then \(\neg \text{lockedCenter}(v)\) is verified. Node \(v\) is enabled if \(\text{lockedCenter}(v) \neq \text{true}\) or \(\text{center}(v) \neq \text{bestCenter}(v)\) because \(\text{centerToBeUpdate}(v)\) is verified. So \(v\) will invite node \(\text{bestCenter}(v)\) to be the center of its star or it will set \(\text{center}(v)\) to \(\bot\) if it is isolated (rule \(\text{RI}\)).

The set \(\text{potentialLeaves}(v)\) contains the neighbors of \(v\) that can safely be in the \(p\)-star centered at \(v\) if the \(p\)-star is created during the next step. Any node of this set invites \(v\) to be the potential center and they cannot update their variable \(\text{center}\) during the next step because the value of their shared variable \(\text{lockedCenter}\) is true. To change the value of \(\text{center}\), a node with must first execute rule \(\text{RGI}\) before it can change \(\text{center}\) using rule \(\text{RI}\) in a subsequent step.

Let \(v\) be a node of \(CS\). If the size of \(\text{potentialLeaves}(v)\) is at least \(p\) and \(\text{center}(v) = v\) then \(v\) can form a \(p\)-star centered in itself by executing rule \(\text{RA}\).

3.3. \(p\)-Star Stability

The execution of rule \(\text{RA}\) by some node \(v\) builds a \(p\)-star. After this move, \(v\) is the center of a \(p\)-star containing \(p\) members of \(\text{potentialLeaves}(v)\), as proven in Lemma 4. The \(p\)-star stays unchanged (i.e., the set \(\text{star}(v)\) stays unchanged) along any execution, which is proven in Lemma 3.

**Lemma 3.** Let \(c\) be a configuration where \(v\) verifies \(\text{correctCenter}(v)\). Along any execution from \(c\), the predicate \(\text{correctCenter}(v)\) is verified. The set \(\text{leaves}(v)\) stays unchanged along any execution from \(c\).

**Proof.** Only a move of a node in \(\text{star}(v)\) may change the value of the predicates \(\text{isCenter}(v)\) and \(\text{incorrectCenter}(v)\) or the set \(\text{leaves}(v)\).

According to Lemma 2 and the definitions of \(\text{correctCenter}(v)\) and \(\text{star}(v)\), the nodes of \(\text{star}(v)\) verify the predicate \(\text{isInStar}\). For these nodes, only rule \(\text{RU}\) may be enabled.

The action of rule \(\text{RU}\) does not change the value of \(\text{leaves}(v)\) and \(\text{center}(v)\) since \(\text{incorrectCenter}(v) = \text{false}\) in \(c\). For a node \(u\) of \(\text{leaves}(v)\), the action of rule \(\text{RU}\) does not change the value of \(\text{center}(u)\).

So the values of \(\text{leaves}(v)\), \(\text{isCenter}(v)\), and \(\text{incorrectCenter}(v)\) stay unchanged.

**Lemma 4.** If a node \(v\) executes rule \(\text{RA}\) during the step \(c_1 \rightarrow c_2\), then \(\text{correctCenter}(v)\) is verified in \(c_2\).

**Proof.** In \(c_1\), \(\text{potentialLeaves}(v)\) is a set of at least \(p\) nodes of \(N(v)\) and any node \(u\) of \(\text{potentialLeaves}(v)\) verifies \(\text{center}(u) = v\) and \(\text{lockedCenter}(u)\).

Let \(u\) be a neighbor of \(v\) with \(u \in \text{potentialLeaves}(v)\) in \(c_1\). In \(c_1\), \(u\) is disabled with respect to rule \(\text{RI}\). Hence \(\text{center}(u)\) cannot change during the step \(c_1 \rightarrow c_2\). So in \(c_2\), \(\text{leaves}(v)\) is a set of \(p\) distinct nodes of \(N(v)\) and any node of \(\text{leaves}(v)\) verifies \(\text{center}(u) = v\). Also \(\text{center}(v) = v\) in \(c_2\), according to rule \(\text{RA}\). Thus, \(\text{correctCenter}(v)\) is verified in \(c_2\).

4. Correctness of Terminal Configurations

In this section we establish that any terminal configuration represents a maximal \(p\)-star decomposition of the graph. More precisely, we prove that \(S\) contains only well formed disjoint \(p\)-stars and \(S\) is maximal, i.e., no \(p\)-star can be added to \(S\).
Lemma 5. Let $u, v$ be two nodes verifying the predicate $\neg $incorrectCenter. Then the sets $\text{star}(v)$ and $\text{star}(u)$ are disjoint.

Proof. The claim is trivial if either $\text{star}(u)$ or $\text{star}(v)$ is empty. According to Lemma 2, $\text{center}(w) = u$ (resp. $\text{center}(x) = v$) for all $w \in \text{star}(u)$ (resp. for all $x \in \text{star}(v)$). Thus $\text{star}(u)$ and $\text{star}(v)$ cannot intersect. $\square$

Definition 3. $PS$ denotes the set of nodes of $CS$ having at least $p$ neighbors in $CS$ (definition 2), i.e., $PS = \{z \in CS \mid |N(z) \cap CS| \geq p\}$.

Lemma 6. In any terminal configuration, we have

- $\text{leaves}(z) = \emptyset$, for any node $z$ in $CS$
- $\text{isInStar}(u) = \text{false}$ for any node $x$ in $CS$
- $\text{viableCenter}(z) = \text{true}$ for any node $z$ in $PS$
- $\text{viableCenter}(x) = \text{false}$ for any node $x$ not in $PS$

Proof. Let $w$ be a node of $CS$. According to the definition of $S$, $w \notin \bigcup_{v \in V} \text{leaves}(v)$ and $\text{leaves}(w) = \emptyset$. So, we have $\neg \text{isCenter}(w)$ and $\neg \text{correctLeaf}(w)$. According to Observation 1, for any node $w$ of $CS$, $\text{isInStar}(w) = \text{inStar}(w) = \text{false}$ because $w$ is disabled.

Let $z$ be a node of $PS$. $z$ has at least $p$ neighbors having their shared variable $\text{inStar}$ set to false. Thus $\text{isViableCenter}(z)$ is verified. According to Observation 1, $\text{viableCenter}(z) = \text{true}$, because $z$ is disabled.

Let $\text{CPS}$ denote the set of nodes of $CS$ having at most $p - 1$ neighbors in $CS$ (i.e., $\text{CPS} = CS \setminus PS = \{z \in CS \mid |N(z) \cap CS| < p\}$). Let $cz$ be a node of $\text{CPS}$. Node $cz$ does not have $p$ neighbors having their shared variable $\text{inStar}$ set to false. $\neg \text{isViableCenter}(cz)$ is verified. So, $\text{viableCenter}(cz) = \text{false}$. Let $u$ be a node of $S$. According to Lemma 2, $\text{isInStar}(u) = \text{inStar}(u) = \text{true}$. So, $\text{viableCenter}(u) = \text{false}$ because $u$ is disabled.

We conclude that for any node not in $PS$, $\text{viableCenter}(x) = \text{false}$. $\square$

$PS$ is the set of nodes that could be the center of a $\text{star}$ set containing only nodes that are not yet in a $p$-star.

Lemma 7. In any terminal configuration, $PS$ is empty.

Proof. Assume that $PS$ is not empty. Let $cz$ be the node of $PS$ having the smallest identifier. Consider any node $w \in CS \cap N[cz]$. We have $\text{bestCenter}(w) = cz$ and $\text{isInStar}(w) = \text{false}$ are according to Lemma 6. As $w$ is disabled, $\neg \text{centerToUpdate}(w)$ is verified (Observation 1). We have $\text{lockedCenter}(w) = \text{true}$ and $\text{center}(w) = \text{bestCenter}(w) = cz$.

As $|CS \cap N[cz]| \geq p$, $\text{starToUpdate}(cz)$ is verified; so node $cz$ is enabled with respect to rule RA. That is a contradiction to the assumption that the configuration is terminal. Thus $PS$ is empty. $\square$

Theorem 8. In any terminal configuration, $S = \bigcup_{v \in V} \text{star}(v)$ is a valid maximal $p$-star decomposition.

Proof. According to Observation 1, any node $v$ verify the predicate $\neg \text{incorrectCenter}(v)$ in a terminal configuration.

According to Lemma 2, and the definition of $\text{star}$, $\text{star}(v)$ is empty or is a $p$-star centered to $v$ in a terminal configuration. According to Lemma 5, the $\text{star}$ sets are pairwise disjoint.

According to Lemma 7 and the definitions of $PS$, and $S$ is maximal. $\square$

5. Move Complexity

5.1. Number of RA and RU moves per node

Lemma 9. Every node $v$ executes rule RA at most once.
Proof. After \( v \) executes rule RA, it verifies the predicate \( \text{correctCenter}(v) \) (Lemma 4). From a configuration where \( \text{correctCenter}(v) \) is verified this predicate stays verified along any execution (Lemma 3). So along any execution, \( v \) may perform only the rule RU, because \( \text{isInStar}(v) \) is always verified (Lemma 2).

Lemma 10. For every node \( v \), \( \neg \text{incorrectCenter}(v) \) is a closed predicate.

Proof. Assume \( \neg \text{incorrectCenter}(v) \) is verified in configuration \( c \). According to Observation 1, in \( c \), we have \( \text{leaves}(v) = \emptyset \) or \( \text{isCenter}(v) \). Assuming that \( \text{isCenter}(v) \) in \( c \), the claim follows from Lemma 3. Assume that \( \text{leaves}(v) = \emptyset \) in \( c \). Only the action of rule RA sets \( \text{leaves}(v) \) to a non-empty set. The claim follows from Lemmas 3 and 4.

Lemma 11. If a node \( v \) makes a move, then \( \neg \text{incorrectCenter}(v) \) is verified by the configuration reached immediately after this move.

Proof. Consider a step \( c_1 \rightarrow c_2 \) that contains a move of \( v \). If \( v \) executes rule RA, then the claim follows from Lemma 4. If \( \neg \text{incorrectCenter}(v) \) is verified in \( c_1 \) the claim follows from Lemma 10.

Assume that \( v \) executes rules RI, RGI, or RU from a configuration \( c_1 \) where \( \text{incorrectCenter}(v) \) is verified. The procedure \( \text{updateVariables}(v) \) empties the set \( \text{leaves}(v) \) during the step \( c_1 \rightarrow c_2 \). So, in \( c_2 \) we have \( \text{leaves} = \emptyset \). According to Observation 1, \( \neg \text{incorrectCenter}(v) \) is verified in \( c_2 \).

Lemma 12. Consider a step \( c_1 \rightarrow c_2 \) in which \( \text{correctLeaf}(v) \) changes from false to true. Let \( u \) be the node such that \( \text{center}(v) = u \) in \( c_2 \). Then node \( u \) executes rule RA during the step.

Proof. The proof is by contradiction, i.e., assume that node \( u \) has not executed rule RA during the step. The action of rules RI, RGI, or RU, if they change \( \text{leaves}(u) \), assign the empty set to \( \text{leaves}(u) \). As \( v \in \text{leaves}(u) \) in \( c_2 \), we conclude that \( u \) does not update its set \( \text{leaves} \) during this step.

So \( \text{correctLeaf}(v) \) could change from false to true during this step only if \( \text{center}(v) \) changes and \( \text{leaves}(u) \) does not change. That implies that \( v \) executed rule RI during the step. However, rule RI can only set \( \text{center}(v) \) to \( u \) if \( \text{bestCenter}(v) = u \) which implies that \( \text{leaves}(u) = \emptyset \) in \( c_1 \) and in \( c_2 \). That is a contradiction to \( v \in \text{leaves}(u) \) in \( c_2 \).

Lemma 13. Consider a step \( c_1 \rightarrow c_2 \) in which \( \text{correctLeaf}(v) \) changes from false to true. Along any execution starting in \( c_2 \), \( \text{correctLeaf}(v) \) stays verified.

Proof. Let \( u \) be the node such that \( u = \text{center}(v) \) in \( c_2 \). During the step \( c_1 \rightarrow c_2 \), \( u \) performs the rule RA (Lemma 12). So in \( c_2 \), node \( u = \text{center}(v) \) verifies the predicate \( \text{correctCenter}(u) \) (Lemma 4). According to Lemma 3, \( v \) belongs to \( \text{leaves}(u) \) along any execution from \( c_2 \). In \( c_2 \), \( \text{star}(v) \neq \emptyset \) and \( \neg \text{incorrectCenter}(v) \), so \( \text{correctLeaf}(v) \) stays verified along any execution (Lemma 2).

Lemma 14. For each node \( v \), the value of \( \text{isInStar}(v) \) changes at most 2 times.

Proof. Consider a step \( c_1 \rightarrow c_2 \) in which \( \text{isInStar}(v) \) changes from false to true. This is either due to the fact that \( \text{correctLeaf}(v) \) became true, or \( \text{correctCenter}(v) \) became true.

\( \text{correctCenter}(v) \) is closed (Lemma 3). \( \text{correctLeaf}(v) \) stays verified after that step (Lemma 13).

So \( \text{isInStar}(v) \) may change from false to true at most once; then it keeps the value true. Thus \( \{\text{true, false, true}\} \) is the maximum sequence of values that \( \text{isInStar}(v) \) may assume.

Accounting for an initial inconsistencies, we obtain the following result.

Corollary 15. Each node \( v \) changes the variable \( \text{inStar}(v) \) at most 3 times.

Lemma 16. For each node \( v \), the value of \( \text{isViableCenter}(v) \) changes at most \( 3 \deg(v) + 2 \) times.

Proof. \( \text{isViableCenter}(v) \) may change its value each time a neighbors \( u \in N(v) \) change its value of \( \text{inStar}(v) \). By Corollary 15, this may occur at most \( 3 \deg(v) \) times. Furthermore, \( \text{isViableCenter}(v) \) may also change its value if \( \text{isInStar}(v) \) changes. By Lemma 14, this can occur at most 2 times.
Accounting for an initial inconsistencies, we obtain the following result.

**Corollary 17.** Each node $v$ changes the variable $\text{viableCenter}(v)$ at most $3 \deg(v) + 3$ times.

**Lemma 18.** Each node $v$ executes the rule $RU$ at most $3 \deg(v) + 5$ times.

**Proof.** By definition of $\text{variablesToUpdate}(v)$, rule $RU$ is only enabled if variables $\text{inStar}(v)$ or $\text{viableCenter}(v)$ need updating or if $\text{incorrectCenter}(v) = \text{true}$. The first execution of rule $RU$ may be triggered by initial inconsistencies. Note that by Lemmas 11 and 10, $\text{incorrectCenter}(v)$ never becomes true after the first move of $v$.

Thus all further executions of $RU$ are caused by changes of $\text{isInStar}(v)$ and $\text{isViableCenter}(v)$. The latter can occur at most $3 \deg(v) + 2 + 2$ times by Lemmas 14 and 16.

5.2. Number of RGI and RI moves per node

**Lemma 19.** Per node, the variable $\text{leaves}(v)$ changes at most $2$ times.

**Proof.** Rule RA is the only rule assigning a non-empty set to $\text{leaves}(v)$. By Lemma 9, rule RA can only be executed once by $v$ and afterwards, $\text{leaves}(v)$ does not change. All other rules set $\text{leaves}(v)$ to the empty set if they modify $\text{leaves}(v)$. Thus $\text{leaves}(v)$ can only change twice: once to the empty set and once to a non-empty set.

**Lemma 20.** For every node $v$, the value of $\text{bestCenter}(v)$ changes at most $\deg(v)(3\Delta + 5) + 3 \deg(v) + 5$ times.

**Proof.** $\text{bestCenter}(v)$ changes only if the set of nodes $u \in N[v]$ satisfying the condition (i) $\text{viableCenter}(u) \land \text{leaves}(u) = \emptyset$ changes. By Corollary 17 and Lemma 19, the condition (i) changes its value at most $3 \deg(u) + 3 + 2 \leq 3\Delta + 5$ times on any execution. The claim follows as $N[v]$ contains exactly $\deg(v)$ nodes plus $v$.

**Lemma 21.** Rule $RGI$ is executed at most $\deg(v)(3\Delta + 5) + 3 \deg(v) + 5 + 1$ times.

**Proof.** By definition of $\text{centerToUpdate}(v)$, rule $RGI$ is enabled only if $\text{center}(v) \neq \text{bestCenter}(v)$ and $\text{lockedCenter}(v) = \text{true}$. When the algorithm assigns true to $\text{lockedCenter}(v)$ (rule RI), then it also assigns $\text{bestCenter}(v)$ to $\text{center}(v)$. So $\text{center}(v) \neq \text{bestCenter}(v)$ when $\text{lockedCenter}(v) = \text{true}$ is either due to initial inconsistencies or due to the fact that $\text{bestCenter}(v)$ has changed. By Lemma 20, the latter can only happen $\deg(v)(3\Delta + 5) + 3 \deg(v) + 5$ times.

**Lemma 22.** Rule $RI$ is executed at most $\deg(v)(3\Delta + 5) + 3 \deg(v) + 5 + 2$ times.

**Proof.** For rule RI to be enabled, it must holds that $\text{lockedCenter}(v) = \text{false}$. When executed, rule RI changes $\text{lockedCenter}(v)$ to true and only rule $RGI$ set to false this variable. So any execution of rule RI is either due to an initial inconsistency or must be preceded by an execution of rule RGI. The latter can happen at most $\deg(v)(3\Delta + 5) + 3 \deg(v) + 5 + 1$ times by Lemma 21.

**Theorem 23 (Upper bound on the number of moves).** The algorithm terminates after at most $12\Delta m + O(m + n)$ moves.

**Proof.** By Lemma 9, rule RA can be executed at most 1 time. By Lemma 18, rule RU is executed at most $3 \deg(v) + 5$ times. Rule RGI can be executed at most $\deg(v)(3\Delta + 5) + 3 \deg(v) + 6$ times by Lemma 21, and 22 rule RI is executed at most $\deg(v)(3\Delta + 5) + 3 \deg(v) + 7$ times. So each node $v$ makes at most

$$2 \deg(v)(3\Delta + 5) + 9 \deg(v) + 19 = 6 \deg(v)\Delta + 19 \deg(v) + 19 \text{ moves.}$$

The total number of moves over all nodes is then bounded by

$$\sum_{v \in V} (6 \deg(v) + 19 \deg(v) + 19) = 12\Delta m + 38m + 19n$$
6. Round Complexity

The proofs of the round complexity utilize the notion of *attractors* [30]. An attractor \( A \) is a set of configuration that is closed under the execution of the algorithm. That is, for any execution, configurations subsequent to a configuration of \( A \) are also in \( A \). A self-stabilizing always has two attractors: the set of all configurations and the set of all legitimate configurations.

Let \( \mathcal{C} \) denote the set of all configurations. First we will establish that attractor \( A_0 \), as defined below, is reached in a single round.

**Definition 4.** \( A_0 = \{ c \in \mathcal{C} \mid \forall v \in V : \neg \text{incorrectCenter}(v) \} \)

**6.1. Properties of \( A_0 \)**

The following corollary is a direct consequence of the Observation 1 and Lemmas 10 and 11.

**Corollary 24.** After at most one round of execution starting in any configuration, \( A_0 \) has been reached and \( A_0 \) is closed under any execution of the algorithm.

**6.2. Properties of executions starting in \( A_0 \)**

In the following, we establish that in any configuration of \( A_0 \), at most \( nbMaxStars \) nodes verify the predicate \( \text{isCenter} \).

**Definition 5.** \( nbMaxStars = \lfloor \frac{n}{p+1} \rfloor \).

The following corollary follows from the definition of \( A_0 \), and \( \text{star} \) and Lemmas 3 and 2.

**Corollary 25.** In \( A_0 \), \( \text{isCenter} \) is a closed predicate. In \( A_0 \), if \( \text{isCenter}(v) \) is verified, then \( |\text{star}(v)| = p+1 \).

**Corollary 26.** In any execution from \( A_0 \), the rule RA is performed at most \( nbMaxStars \) times.

**Proof.** Let \( c \) be a configuration of \( A_0 \). According to Corollary 25, if \( \text{isCenter}(v) \) is verified in \( c \) then \( |\text{star}(v)| = p+1 \). According to Lemma 5, in \( c \), if \( u \neq v \) then \( \text{star}(u) \) is disjoint of \( \text{star}(v) \). So, in \( c \), the number of nodes satisfying the predicate \( \text{isCenter} \) is bounded by \( nbMaxStars \).

According to Corollary 25, in \( A_0 \) the predicate \( \text{isCenter} \) is closed. Thus, the rule RA is performed at most \( nbMaxStars \) times along any execution in \( A_0 \). \( \square \)

**6.3. Properties of RA-restricted executions starting in \( A_0 \)**

We call a step \( c_1 \rightarrow c_2 \) RA-restricted, if no node executes rule RA during that step. Similarly, we say that a round is RA-restricted if it consists of only RA-restricted steps. In the remainder we study executions starting in configurations of \( A_0 \). We show that any such execution contains at most 4 consecutive RA-restricted rounds, i.e., after 4 consecutive RA-restricted rounds, either a terminal configuration is reached or in the following round at least one node executes rule RA. In any execution starting in \( A_0 \), there are at most \( nbMaxStars \) RA-unrestricted rounds (Corollary 26). So the number total of rounds, in any execution from \( A_0 \) is at most \( 5nbMaxStars + 4 \).

First, we establish that the value of \( \text{leaves}(v) \) and \( \text{isInStar}(v) \) for any node \( v \) in a configuration of \( A_0 \) stays unchanged during any execution till no node performs the RA action.

**Lemma 27.** Consider an RA-restricted step \( c_1 \rightarrow c_2 \) with \( c_1 \in A_0 \). For any node \( v \), the value of \( \text{leaves}(v) \) does not change during this step.

**Proof.** The rules RI, RGI and RU modify \( \text{leaves}(v) \) if \( \text{incorrectCenter}(v) = \text{true} \) in \( c_1 \). We have \( \text{incorrectCenter}(v) = \text{false} \) in \( c_1 \in A_0 \). Thus, \( \text{leaves}(v) \) does not change during this step. \( \square \)

**Lemma 28.** Consider a step \( c_1 \rightarrow c_2 \) with \( c_1 \in A_0 \). If \( \text{correctCenter}(v) \) changes from false to true, then node \( v \) executed rule RA during this step.
**Lemma 29.** isInStar(v) is a closed under any RA-restricted step \( c_1 \rightarrow c_2 \) with \( c_1 \in A_0 \).

**Proof.** Let \( v \) be a node that verifies isInStar(v) in \( c_1 \). Either correctCenter(v) or correctLeaf(v) is verified, by definition of isInStar.

If correctCenter(v) is verified in \( c_1 \), then according to Lemma 3 correctCenter(v) is also verified in \( c_2 \). So isInStar(v) is verified in \( c_2 \) (Lemma 2).

Let \( u = \text{center}(v) \). If correctLeaf(v) is verified, then \( \text{leaves}(u) \neq \emptyset \) in \( c_1 \). So by definition of \( A_0 \) and Observation 1 we have correctCenter(u) in \( c_1 \) and thus in \( c_2 \) (Lemma 3). Since \( v \in \text{leaves}(u) \) in \( c_2 \) by Lemma 27, we have correctLeaf(v) in \( c_2 \) by Lemma 2. □

**Lemma 30.** Consider an RA-restricted step \( c_1 \rightarrow c_2 \) with \( c_1 \in A_0 \). For every node \( v \), the value of isInStar(v) does not change during this step.

**Proof.** Let \( v \) be a node. A change of isInStar(v) from false to true implies that either correctLeaf(v) or correctCenter(v) changes from false to true. According to Lemmas 12 and 28, such a change occurs only if a node performs the rule RA during this step. Since this step is RA-restricted, this cannot occur. According to Lemma 29, a change of the value of isInStar(v) from true to false cannot happen in \( A_0 \). □

**Definition 6.** \( A_1 = \{ c \in A_0 \mid \forall v \in V : \text{isInStar}(v) = \text{isInStar}(c) \} \)

### 6.4. Properties of \( A_1 \)

**Lemma 31.** Let \( c \) be a configuration of \( A_0 \) and \( c_1 \) the configuration reached after one RA-restricted round from \( c \). We have \( c_1 \in A_1 \) and \( A_1 \) is closed under any RA-restricted step.

**Proof.** By Corollary 24, the set \( A_0 \) is closed and thus \( c_1 \in A_0 \). Let \( v \) be a node with \( \text{isInStar}(v) \neq \text{isInStar}(c) \) in \( c \). Node \( v \) is enabled as long as this inequality holds and no node executes rule RA. Thus node \( v \) performs a move during the first RA-restricted round. After the move of \( v \), we have \( \text{isInStar}(v) = \text{isInStar}(c) \). This equality stays verify until a node does the RA action (Lemma 30). □

**Lemma 32.** Consider an RA-restricted step \( c_1 \rightarrow c_2 \) with \( c_1 \in A_1 \). For every node \( v \), the value of isViableCenter(v) does not change during this step.

**Proof.** The value of isViableCenter(v) may only changes when the value of isInStar(u) for some node \( u \in N(v) \) changes. As \( c_1 \in A_1 \) we have isInStar(u) = isInStar(c) for all nodes \( u \in V \). So no node changes the value of its isInStar variable during this step. □

**Definition 7.** \( A_2 = \{ c \in A_1 \mid \forall v \in V : \neg \text{variablesToUpdate}(v) \} \)

### 6.5. Properties of \( A_2 \)

**Lemma 33.** Let \( c \) be a configuration of \( A_0 \) and \( c_2 \) the the configuration reached after two RA-restricted rounds from \( c \). We have \( c_2 \in A_2 \) and \( A_2 \) is closed under any RA-restricted step.

**Proof.** After the first RA-restricted round from \( c \), the configuration \( c_1 \in A_1 \) is reached (Lemma 31). Since \( A_1 \) is closed we have \( c_2 \in A_1 \).

Let \( v \) be a node with isViableCenter(v) \( \neq \) viableCenter(v) in \( c_1 \). As long as this inequality holds and no node performs RA action, the node \( v \) is enabled. Thus node \( v \) performs a move during the second RA-restricted round. After the move of \( v \), we have isViableCenter(v) = viableCenter(v). This equality stays verify until a node does the RA action (Lemma 32). □
Theorem 38 (Upper bound on the number of rounds). The algorithm terminates after at most $5(nb\text{MaxStars} + 1) = 5\left\lceil \frac{\Delta m}{p^2} \right\rceil + 1$ rounds.

Proof. By Corollary 24, any non-restricted execution reaches $A_0$ after at most 1 round. Afterwards, an execution of rule RA occurs at least every 5 rounds by Corollary 37. Also by Corollary 37, any trailing suffix not containing any executions of rule RA can be at most 4 rounds in length.

By Corollary 26, there are at most nbMaxStars rounds having a RA action along any execution starting in $A_0$. This completes the proof.

7. Conclusion

We studied in this paper the problem of decomposing a graph into node-disjoint $p$-stars from a self-stabilization point of view. This problem is a generalization of maximal matching problem in graphs. We proposed a new self-stabilizing algorithm for maximal $p$-star decomposition problem performing better than both previously proposed algorithms. In fact, we improve the move complexity to $O(\Delta m)$ instead of $O(\Delta^2 m)$ in [29] and solve the uniqueness legitimate configuration problem that [28] suffered from, without losing linearity of round complexity. As future work, we aim to generalize the proposed algorithm to the weighted $p$-star decomposition problem.
References