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Abstract

The present paper deals with the input-output linearization of nonlinear time-varying delay systems with delays in the input and the output. We introduce an extension of the Lie derivative for time-varying delay systems. We derive sufficient conditions for existence of a causal and bounded nonlinear feedback that linearizes the input-output behavior and stabilizes the system.

Keywords: Nonlinear control systems, Delay systems, Input-output linearization, Lie derivative

1 Introduction

Mathematical models arising in population dynamics or engineering sciences often involve systems with delays. Systems with delays express that at each instant the velocity of the state depends upon the history of its evolution up to that instant. In addition to their natural presence in some complex biological systems, a delay may be introduced in control systems when the control action is not instantaneous. The main difficulty in dealing with delay control systems comes from the infinite dimensional type of their dynamics (see, e.g., [10, 12, 5]).

Input-output linearization is an important tool in nonlinear control theory which aims to apply suitable nonlinear coordinate transformation (depending on the outputs) and an invertible static feedback transformation to a nonlinear control system in order to obtain a linear one in the new coordinates (see [9] and references therein). Various aspects of the input-output linearization problem have been studied in the literature, using different approaches such as the algebraic approach (see, e.g., [3, 19]) or the geometric approach (see, e.g., [2, 16]). These approaches have been extended to encompass nonlinear control systems with multiple (but constant) delays in the state variables as well as in the input and output of the system (see, e.g., [1, 2, 3] for the algebraic approach and [7, 13, 14] for the geometric approach).

In this paper, we consider the input-output linearization problem of nonlinear time-varying delay systems. We propose an extension of the Lie derivative in order to compute the relative degree of the system and design a causal and bounded feedback that linearizes the input-output behavior. If the relative degree is not equal to the system dimension, then a zero-dynamics appears and must be input-to-state stable (ISS) (see, e.g., [6, 17]) in order to avoid the peaking phenomenon (see [18] for more details).

The paper is organized as follows: We first introduce some notations and an extended Lie derivative for time-varying delay systems. In Section 3, we present our main results on complet and partial input-output linearization of single-input single-output (SISO) time-varying delay systems. In Section 4, an illustrative numerical example is given to show the performances of the developed approach. The proofs of the developed results are given in Section 5. Finally, we present some conclusions.

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2 Notations and extended Lie derivative for time-varying delay systems

Throughout, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with norm $\|\cdot\|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, where n and m are two positive integers. Given $\bar{\theta}>0$, we denote by $(\mathcal{W}([-\bar{\theta},0],\mathbb{R}^n),\|\cdot\|_{\mathcal{W}})$ the space of absolutely continuous functions $\varphi:[-\bar{\theta},0]\to\mathbb{R}^n$ with

$$\|\varphi\|_{\mathcal{W}}=\max_{s\in[-\bar{\theta},0]}\|\varphi(s)\|+\left(\int_{-\bar{\theta}}^{0}\|\dot{\varphi}(s)\|^2ds\right)^{1/2}.$$

For $t\geq 0$, $x_t:[-\bar{\theta},0]\to\mathbb{R}^n$ denotes the history function defined as $x_t(\theta)=x(t+\theta)$ for all $\theta\in[-\bar{\theta},0]$.
Next, we introduce the Lie derivative for time-varying delay systems which is a generalization of that introduced in [4] for constant-delay systems. Let us first give some useful notations and definitions.

Definition 1 *Let $\bar{\theta}>0$ be the maximum of the time-delay function $\theta(\cdot)$ which is supposed to be known over $\mathbb{R}$ and takes its values in $(0,\bar{\theta}]$, and let the recursive relation*

$$\tau_{i+1}=\tau_i-\theta\circ\tau_i,\quad \text{for } i\geq 0,$$

where $\tau_0(t)\equiv t$. We denote by δ^i the time delay operator that shifts the time from t to $\tau_i(t)$ and is given by

$$\delta^0\sigma(t)=\sigma(t),\quad \text{and}\quad \delta^i\sigma(t)=\sigma(\tau_i(t)),\quad \text{for } i\geq 1,$$

where σ is a function defined on an interval containing $[t-i\bar{\theta},t]$. The application of δ^i on a composed function $\gamma\circ\sigma(\cdot)$ is given by

$$\delta^i\gamma\circ\sigma(t)=\gamma(\delta^i\sigma(t)),\quad \text{for } i\geq 1.$$

Applied on the product of two functions, this delay operator acts as the following

$$\delta^i\gamma(t)\cdot\sigma(t)=(\delta^i\gamma(t))\cdot(\delta^i\sigma(t)),\quad \text{for } i\geq 1,\tag{1}$$

i.e., the delay spreads to the right. If brackets are present, i.e., we have $(\delta^i\gamma(t))\sigma(t)$, then the delay affects only the first function (here γ).
Now, we can give the Lie derivative definition for time-varying delay systems.

Definition 2 *Let q be a positive integer. Let $f(x,\delta)\equiv f(x,\delta^1x,\cdots,\delta^qx)$ and $g(x,\delta)\equiv g(x,\delta^1x,\cdots,\delta^qx)$ be two vector fields whose components are functions of $\delta^ix(t)$, and $h(x,\delta,t)\equiv h(x,\delta^1x,\cdots,\delta^qx,t)$ a real valued function of t and $\delta^ix(t)$, for $i=0,\cdots,q$. Then, the derivative of h along f at (x,δ,t) is defined as*

$$L_fh(x,\delta,t):=\sum_{i=0}^{q}\frac{\partial h}{\partial\delta^ix}\dot{\tau}_i\delta^if(x,\delta)+\frac{\partial h}{\partial t}(x,\delta,t).\tag{2}$$

Since L_fh is a real-valued function with delays, this operation can be recursively repeated for higher order as

$$L_f^{\rho}h(x,\delta,t):=\sum_{i=0}^{q\rho}\frac{\partial L_f^{\rho-1}h}{\partial\delta^ix}\dot{\tau}_i\delta^if(x,\delta)+\frac{\partial L_f^{\rho-1}h}{\partial t}(x,\delta,t).$$

Notice that (2) is an extension of the Lie Backlünd derivative introduced in [3] in the control theory context. The difference resides in our consideration of time-varying delays which leads to multiplicative coefficients (the time-derivative of the recursive delay function τ_i) affecting the operator δ^i as well as the partial derivative of h with respect to t. We also need the reduced Lie derivative

$$\bar{L}_gh(x,\delta,t):=\sum_{i=0}^{q}\frac{\partial h}{\partial\delta^ix}\dot{\tau}_i\delta^ig(x,\delta).\tag{3}$$

Notice that, contrary to (2), the partial derivative of h with respect to t is not present in (3). If h does not depend explicitly on t, then $\bar{L}_gh$ coincide with L_gh. The reduced Lie derivative will be used when stating and proving our main theorem and they will be always associated to the control vector fields.

3 Main result

Consider the SISO nonlinear time-varying delay system

$$\begin{aligned}
\dot{x}(t) &= f(x(t),\delta) + g_0(x(t),\delta)u(t) + g_1(x(t),\delta)\delta^1 u(t) \\
y(t) &= h(x(t),\delta), \\
x(s) &= \varphi(s), \quad \forall s \in [-q\bar{\theta}, 0], \\
u(s) &= \nu(s), \quad \forall s \in [-\bar{\theta}, 0],
\end{aligned} \tag{4}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system at time t, δ is the time delay operator associated to a sufficiently smooth time-varying delay $\theta : \mathbb{R} \to (0, \bar{\theta}]$ satisfying $\dot{\theta} < 1$, and $\bar{\theta}$ is a positive real number. The condition on the derivative of the delay function θ is important for causality reason. The vector fields $f, g_0, g_1 : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$, and the function $h : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ are sufficiently smooth, such that $f(0,0) = 0$ and $h(0,0) = 0$. The initial condition $\varphi \in \mathcal{C}([-q\bar{\theta}, 0], \mathbb{R}^n)$ and the input $u : [-\bar{\theta}, +\infty) \to \mathbb{R}$ is a Lebesgue measurable function. In the above, $\mathcal{C}([-q\bar{\theta}, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions from $[-q\bar{\theta}, 0]$ into $\mathbb{R}^n$, with the usual norm. Knowing that the Lie derivatives of h may involve the initial state at a certain order of derivation, we suppose that φ satisfies the differential equations of (4) on a sufficiently large interval. The integer q corresponds to the maximal delay order explicitly involved in f, g_0, g_1 and h and it does not mean that f, g_0, g_1 and h have necessarily the same delay orders.

Remark 1 *We abusively note $f(0,0) = 0$ and $h(0,0) = 0$. In fact, here, the second argument of f and h is the delayed state $(\delta^1 x(t), \cdots, \delta^q x(t)) = (0, \cdots, 0)$, see Definition 2.*

Let us recall that a function x is said to be a *solution* of system (4) on $[-q\bar{\theta}, +\infty)$, with initial condition $\varphi \in \mathcal{W}([-q\bar{\theta}, 0], \mathbb{R}^n)$, if x is absolutely continuous on $[0, +\infty)$, $x_t \in \mathcal{W}([-q\bar{\theta}, 0], \mathbb{R}^n)$ for every $t > 0$ and $x(t)$ satisfies (4) for almost every $t > 0$. We suppose that system (4) is forward complete.

3.1 Linearization of the input-output mapping

The problem of input-output linearization, that we study here, consists in introducing suitable new coordinates involving the output h (and thus the state and the delayed state variables) and constructing a causal and bounded feedback transformation (those properties will become clear below) that linearizes (partially or completely) system (4). Similarly to control systems without delays, when we introduce a new coordinate $z = \varphi(x, \delta, t)$, the time-derivative of z can be written (using the Lie derivatives) as follows:

$$\dot{z} = L_f \varphi + \bar{L}_{g_0} \varphi u + \bar{L}_{g_1} \varphi \delta^1 u. \tag{5}$$

Recall that the operator δ spreads to the right, see equation (1), thus $\bar{L}_{g_0} \varphi u$ involves as many delayed controls as the delays present in the function φ. Indeed, suppose that $m \geq 0$ is the highest delay order present in φ, then, according to the definition of the reduced Lie derivative:

$$\bar{L}_{g_0} \varphi u = \sum_{i=0}^{m} \frac{\partial \varphi}{\partial \delta^i x} \dot{\tau}_i (\delta^i g_0(x, \delta)) \cdot (\delta^i u).$$

A similar remark can be made for $\bar{L}_{g_1} \varphi \delta^1 u$, but now the number of delayed controls is no longer m, but $m+1$ (because of the presence of $\delta^1 u$):

$$\bar{L}_{g_1} \varphi \delta^1 u = \sum_{i=0}^{m} \frac{\partial \varphi}{\partial \delta^i x} \dot{\tau}_i (\delta^i g_1(x, \delta)) \cdot (\delta^{i+1} u).$$

Therefore, the reduced Lie derivatives $\bar{L}_{g_0} \varphi$ and $\bar{L}_{g_1} \varphi$ associated to the control vector fields g_0 and g_1, respectively, can be seen as a δ-polynomial and can be developed with respect to the operator δ as follows:

$$\bar{L}_{g_0} \varphi = \sum_{i=0}^{m} a_0^i(x, \delta, t) \delta^i, \quad \bar{L}_{g_1} \varphi = \sum_{i=1}^{m+1} a_1^i(x, \delta, t) \delta^i,$$

where

$$a_0^i(x,\delta,t) = \frac{\partial\varphi}{\partial\delta^i x}\dot{\tau}_i(\delta^i g_0(x,\delta)),\ 0 \le i \le m,$$

$$a_1^i(x,\delta,t) = \frac{\partial\varphi}{\partial\delta^{i-1} x}\dot{\tau}_{i-1}(\delta^{i-1} g_1(x,\delta)),\ 1 \le i \le m+1.$$

Combining these two δ-polynomials allows us to define the δ-polynomial

$$a(\delta] := a^0(x,\delta,t)\delta^0 + \cdots + a^{m+1}(x,\delta,t)\delta^{m+1}, \tag{6}$$

where $a^0 = a_0^0$, $a^i = a_0^i + a_1^i$ and $a^{m+1} = a_1^{m+1}$, for $1 \le i \le m$. Equation (5) can then be rewritten as follows:

$$\dot{z}(t) = L_f\varphi(x,\delta,t) + a(\delta]u(t).$$

The δ-polynomial $a(\delta]$ is an element of the ring of differential time-varying delay operators. See [15] for more details about the algebraic properties of this ring.

Assumption 1 *Throughout, for a δ-polynomial $a(\delta]$, we denote by j the first coefficient of $a(\delta]$ which is non identically zero, i.e., the integer $0 \le j \le m+1$ is such that*

$$a^j(x,\delta,t) \neq 0, \tag{7}$$

$$a^i(x,\delta,t) \equiv 0, \qquad \forall i < j. \tag{8}$$

Furthermore, in this paper, we do not deal with singularities, i.e., condition (7) *will always be interpreted as*

$$a^j(x,\delta,t) \neq 0, \qquad \forall t \ge \tau_j^{-1}(0), \tag{9}$$

where $\tau_j^{-1}(0)$ is, in fact, the time from which the control starts acting on the system. Notice that $\tau_j^{-1}(0)$ is well defined since the time-delay function $\theta(\cdot)$ is completely known over $\mathbb{R}$ (see Definition 1).

Two problems arise when constructing a feedback u of the form

$$a(\delta]u(t) = -L_f\varphi(x,\delta,t) + \delta^j v(t), \quad t \ge \tau_j^{-1}(0), \tag{10}$$

where v is the new control and j is the integer defined by Assumption 1. Since u is described by a recursive equation[1], one problem is its boundedness and the second one is its causality. Some of those problems come from the fact that the drift involves more delays than the control vector fields or vice versa (see [8], where we present examples discussing these two properties).

As stated above, we study the possibility of finding new coordinates z_i, $1 \le i \le \rho$, where the integer ρ is such that $1 \le \rho \le n$, and a causal and bounded feedback u, solution of an equation of the form (10), such that the closed loop subsystem (corresponding to the new variables z_i) is described by a linear ordinary differential equation with finite dimension:

$$\begin{aligned} \dot{z}(t) &= Az(t) + B\delta^j v(t) \\ y(t) &= Cz(t), \end{aligned} \tag{11}$$

where $A \in \mathbb{R}^{\rho\times\rho}$ and $B, C^T \in \mathbb{R}^\rho$ are given by:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad C^T = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

[1]By "recursive" we point out that the construction of u requires a recursive prediction of the values of v (which is chosen as a suitable function of the state and the desired trajectory in order to achieve the desired behavior) over the intervals $[0, \tau_j^{-1}(0)]$ and $[\tau_i^{-1}(0), \tau_{i+1}^{-1}(0)]$, for $i \ge j$, the integer j being defined by Assumption 1.

Using the z-variables, the system's output is z_1, i.e., $z_1 = h(x,\delta)$, so we actually have $z_i = L_f^{i-1}h$, for $1 \le i \le \rho$, and

$$a(\delta]u(t) = -L_f^{\rho}h(x,\delta,t) + \delta^j v(t), \quad t \ge \tau_j^{-1}(0). \tag{12}$$

The integer ρ is in fact the relative degree of the system which can be defined in the same way as the relative degree of finite dimensional control systems (see, e.g., [9]): it corresponds to the number of times that we have to differentiate the output h with respect to time t before the input or the delayed input appears.

Definition 3 *System* (4) *is said fully (resp. partially) linearizable with delay (resp., without delay) if* $\rho = n$ *(resp.* $\rho < n$*) and* $j \ge 1$ *(resp* $j = 0$*).*

As already explained, the feedback u is described by the recursive equation (12). Thus, conditions assuring the boundedness of such a feedback are required:

Lemma 1 *Let a* δ*-polynomial* $a(\delta]$ *of form* (6)*, of degree* $m+1$*, and* j *the integer defined in Assumption 1. Suppose that all* $a^i(\cdot)$*, for* $j \le i \le m+1$*, are bounded,* $v : [-(m+1)\bar\theta, +\infty) \to \mathbb{R}$ *is such that* $-L_f^{\rho}h(x,\delta,t) + \delta^j v(t)$ *is bounded over* $[\tau_j^{-1}(0), +\infty)$ *and* $u : [-(m+1)\bar\theta, +\infty) \to \mathbb{R}$*, defined by* (12)*, is bounded over the interval* $[-(m+1)\bar\theta, \tau_j^{-1}(0)]$*. If there exists a constant* $c > 1$ *such that*

$$\sup_{t \ge \tau_j^{-1}(0)} \left\| \frac{a^i(x,\delta,t)}{a^j(x,\delta,t)} \right\| \le \frac{1}{c(m+1-j)}, \quad \forall i > j, \tag{13}$$

then, for every $t \ge \tau_j^{-1}(0)$*, we have*

$$\|u(t)\| \le \frac{c}{c-1} \sup_{s \ge \tau_j^{-1}(0)} \left\| \frac{-L_f^{\rho}h(x,\delta,s) + \delta^j v(s)}{a^j(x,\delta,s)} \right\| + \varepsilon(t),$$

where $\varepsilon(t)$ *tends to 0 when* t *tends to* $+\infty$*.*

Let u be as in Lemma 1. Thanks to this lemma, we deduce that $u(t)$ stays bounded when $-L_f^{\rho}h(x,\delta,t) + \delta^j v(t)$ is bounded, for $t \ge \tau_j^{-1}(0)$. Our main result is given by the following theorem that gives an explicit coordinate transformation and a causal and bounded feedback that partially or completely linearize system (4).

Theorem 1 *Suppose that system* (4) *satisfies the following conditions:*

1. *There exists a positive integer* $\rho \le n$ *such that*

$$\begin{cases} \bar{L}_{g_0} L_f^i h(x,\delta,t) = \bar{L}_{g_1} L_f^i h(x,\delta,t) = 0, \\ \qquad \forall i = 0, \cdots, \rho-2, \\ \bar{L}_{g_0} L_f^{\rho-1} h(x,\delta,t) \ne 0, \text{ if } g_0 \ne 0, \text{ and} \\ \bar{L}_{g_1} L_f^{\rho-1} h(x,\delta,t) \ne 0, \text{ if } g_0 \equiv 0. \end{cases}$$

2. *The* δ*-polynomial*

$$a(\delta] = \bar{L}_{g_0} L_f^{\rho-1}h + \bar{L}_{g_1} L_f^{\rho-1}h = \textstyle\sum_{i=0}^{\rho q+1} a^i(x,\delta,t)\delta^i$$

satisfies Lemma 1, with $m = \rho q$*.*

3. *If* $\rho < n$*, then*

$$\frac{\partial L_f^{\rho}h}{\partial \delta^i x} \equiv 0 \text{ and } \frac{\partial a^k}{\partial \delta^i x} \equiv 0, \tag{14}$$

for $0 \le i \le j-1$*,* $j \le k \le m+1$*, where* j *is defined in Assumption 1. Moreover, if* $g_0 \ne 0$*, then* $a^j = a_0^j$*.*

4. The state $x(t)$ is known over $[-q(\rho+1)\bar{\theta}, 0]$ and $u(t)$ is known over $[-(\rho q+1)\bar{\theta}, 0]$.

Then a part of system (4) can be transformed to the ρ-th order linear input-output system (11) by applying the following coordinate transformation

$$z(t) = \Phi_1(x,\delta,t) = \begin{pmatrix} h(x,\delta) \\ L_f h(x,\delta,t) \\ \vdots \\ L_f^{\rho-1} h(x,\delta,t) \end{pmatrix} \tag{15}$$

and the causal and bounded feedback

$$a(\delta] u(t) = -L_f^{\rho} h(x,\delta,t) + \delta^j v(t), \quad t \geq \tau_j^{-1}(0). \tag{16}$$

Moreover, if $\rho = n$, then the system is fully linearizable with delay (in the sense of Definition 3) and if, in addition, $j = 0$, then the system is fully linearizable without delay.

Item (1) of the above theorem allows us to construct the new variables z whose corresponding subsystem is linear. The new feedback v is assigned with respect to the desired behavior of $z_\rho = y^{(\rho)}$, the ρ^{th} differentiation of the output. Conditions (2) and (3) deal with the construction of a causal and bounded feedback. Item (2) calls Lemma 1 in order to assures the boundedness of u, while item (3) guarantees its causality. Indeed, from (16), after applying the advance operator δ^{-j} on both sides, we obtain

$$u(t) = -\textstyle\sum_{k=j+1}^{m+1} (\delta^{-j} \frac{a^k}{a^j}(x,\delta,t))\delta^{k-j} u(t) - \delta^{-j} \frac{L_f^{\rho} h}{a^j}(x,\delta,t) + \frac{v(t)}{\delta^{-j} a^j(x,\delta,t)}. \tag{17}$$

We see clearly, that condition (14) avoids the existence of advances in (17) and no causality problem can appear in solving (17). Even that it is less conservative than some related results developed in the literature (see, e.g., [14], for the constant-time delay case), condition (14) is still very restrictive. It says that in order to construct u, we do not need to make a predictor; it could be made less constraining if we ask to be able to predict all states of the original system. This is always the case if the system is fully linearizable, i.e., $\rho = n$, see [11], where this is studied in the context of flatness of a particular delay systems. In that paper, it is also noticed the importance of starting from an asymptotically stable equilibrium point of the system. Our result gives only sufficient conditions. We do not claim that our conditions are also necessary, see [8], where we discussed several examples that do not satisfy the conditions of Theorem 1, but for which we can construct a suitable feedback that (together with the given output) input-output linearizes the system.

3.2 Stability of the closed loop system

By applying feedback (16) and coordinates transformation (15), if $\rho < n$, then system (4) is decomposed into two parts, one is a ρ-th order linear system and the other a remained nonlinear system (*zero-dynamics*) which may contain delayed and advanced states and does not affect the output. Indeed, in this case, we choose $n-\rho$ functions $\phi_{\rho+1}(\cdot), \cdots, \phi_n(\cdot)$, that always exist, completing z_i to a coordinate system:

$$\begin{pmatrix} z(t) \\ \xi(t) \end{pmatrix} = \Phi(x,\delta,t) = \begin{pmatrix} \Phi_1(x,\delta,t) \\ \phi_{\rho+1}(x,\delta) \\ \vdots \\ \phi_n(x,\delta) \end{pmatrix}$$

with $\frac{\partial \Phi}{\partial x}(x,\delta,t)$ of full rank at $(x,\delta,t) = (0,0,t)$. Assume that there exists a mapping Ψ, see [15], that expresses x with respect to (z,ξ,δ,δ^-). By applying coordinates transformation (3.2), system (4) can be rewritten as following:

$$\begin{cases} \dot{z}(t) = & Az(t) + B\delta^j v(t) \\ \dot{\xi}(t) = & G(z,\xi,\delta,\delta^-,v(t)) \\ y(t) = & Cz(t), \end{cases}$$

where G is smooth and affine with respect to the control. Remark that the map Ψ can generate delays and advances (that we denote by δ^-) in the zero-dynamics $\dot{\xi}(t) = G(z, \xi, \delta, \delta^-, v(t))$, and, in particular, may involve advances in the ξ-variables. In the sequel, we suppose that we can choose Ψ in such a way that the zero dynamics does not contain any advances in ξ_i, i.e., we have $\dot{\xi}(t) = G(\xi, \delta, w(t))$, for which $w(t) = (z, \delta, \delta^{-1}, v(t))$ can be seen as an external exogenous input. Let

$$v(t) = \lambda_1 \delta^{-j} z_1(t) + \cdots + \lambda_\rho \delta^{-j} z_\rho(t), \qquad t \geq 0, \tag{18}$$

where λ_i, for $i = 1, \cdots, \rho$, are chosen in such a way that the resulting closed loop system

$$\dot{z}(t) = \tilde{A} z(t), \tag{19}$$

where $\tilde{A} = A + B\Lambda$, with Λ the line vector $\Lambda = (\lambda_1, \cdots, \lambda_\rho)$, is asymptotically stable. Therefore, under this choice of $v(t)$, the state $z(t)$ tends asymptotically to zero when t tends to $+\infty$. Even if the input-output behavior can be stabilized by a feedback, the internal dynamics may be unstable and the global system cannot be stabilized.

In order to characterize the stability of the closed loop system, we have to recall the definition of input-to-state stability. Let us first define the following function classes. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$, $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$ and $\beta(s, \cdot) \to 0$ as $s \to +\infty$. Finally, we define the input-to-state stability (ISS) of time-varying delay systems (see, e.g., [6] and references therein).

Definition 4 *Consider the system*

$$\dot{x}(t) = F(x(t), \delta, w(t)), \tag{20}$$

where n_1, n_2 are two positive integers, $x(t) \in \mathbb{R}^{n_1}$ is the state of the system, $w(t) \in \mathbb{R}^{n_2}$ is an exogenous input, the dynamics F are continuously differentiable, such that $F(0, 0, 0) = 0$. System (20) *is said to be uniformly globally ISS if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that, for any initial time t_0, any initial state $x_{t_0} = \varphi \in \mathcal{W}([-\bar{\theta}, 0], \mathbb{R}^{n_1})$ and any measurable, locally essentially bounded input w, the solution $x(t, t_0, \varphi)$ exists for all $t \geq t_0$ and furthermore, it satisfies*

$$\|x(t, t_0, \varphi)\| \leq \max\left(\beta(\|\varphi\|_{\mathcal{W}}, t - t_0), \gamma(\|w\|_\infty)\right).$$

In our case, we apply the above definition to the zero-dynamics for which the exogenous input is $w(t) = (z, \delta, \delta^-, v(t))$; notice that $w(t)$ is not exogenous for the global system. We will also need the following assumption which guarantees that $w(t)$ is well in the framework of Definition 4.

Assumption 2 *We assume that $L_f^i h(x, \delta, t)$ is essentially bounded over $[-q(\rho + 1)\bar{\theta}, \tau_j^{-1}(0)]$, for all $i \in \{1, \cdots, \rho\}$.*

Proposition 1 *Let $v(\cdot)$ be as in equation* (18)*. Suppose that the matrix $\tilde{A}$ in* (19) *is Hurwitz and that Assumption 2 holds. If the zero-dynamics is input-to-state stable (in the sense of Definition 4) then the global system is locally asymptotically stable around the origin $(z, \xi) = (0, 0)$.*

Proof. The proof derives straightforwardly from Assumption 2 and Definition 4. □

The following corollary gives necessary conditions guaranteeing the global asymptotic stability of the equilibrium point in x-coordinates.

Corollary 1 *Let $u(\cdot)$ be as in equation* (16) *with $v(\cdot)$ given by* (18)*. Suppose that the matrix $\tilde{A}$ in* (19) *is Hurwitz and that Assumption 2 holds. Suppose that the zero-dynamics is input-to-state stable. If the mapping Ψ exists then system* (4) *is locally asymptotically stable around the origin $x = 0$.*

4 Numerical example

In this section, we present an illustrative example similar to that of [14], but with two principal differences: the first one derives from our consideration of time-varying delay (instead of constant-time delay), and the second one comes from our consideration of a delayed input. Consider the following systems:

$$\begin{aligned} \dot{x}(t) &= f(x(t),\delta) + g(x(t),\delta)\delta u(t), \\ x(s) &= \varphi(s), \quad \forall s \in [-2\bar{\theta}, 0], \\ y(t) &= x_3(t), \end{aligned} \tag{21}$$

where

$$f(x,\delta) = \begin{pmatrix} -4x_1 - \delta^1 x_1 - \delta^1 x_3^2 \\ \delta^1 x_1 - \delta^1 x_2 \\ x_2 + x_1\delta^1 x_3 \end{pmatrix}, \; g(x,\delta) = \begin{pmatrix} 0 \\ x_1^2 + 1 \\ 0 \end{pmatrix}.$$

One can easily verify that system (21) satisfies the conditions of Theorem 1. Then, by introducing the following local change of coordinates

$$\begin{pmatrix} z_1 \\ z_2 \\ \xi \end{pmatrix} = \begin{pmatrix} x_3 \\ x_2 + x_1\delta^1 x_3 \\ x_1 \end{pmatrix}$$

and applying the invertible feedback transformation

$$\left(1 + x_1^2\right)\delta^1 u = -L_f^2 h(x,\delta,t) + \delta^1 v(t), \quad t \geq \tau_1^{-1}(0),$$

where $L_f^2 h(x,\delta,t) = (\delta^1 x_1 - \delta^1 x_2) - (4x_1 + \delta^1 x_1 + \delta^1 x_3^2)\delta^1 x_3 + \dot{\tau}_1 x_1(\delta^1 x_2 + \delta^1 x_1 \delta^2 x_3)$, system (21) becomes

$$\begin{cases} \dot{z}(t) = & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta^1 v(t) \\ \dot{\xi}(t) = & -4\xi(t) - \delta^1\xi(t) - \delta^1 z_1^2 \\ y(t) = & z_1(t). \end{cases}$$

According to Definition 3, system (21) is partially linearizable with delay (since the relative degree $\rho = 2 < n = 3$ and $k = 1$). Let $\lambda_1 = -25$ and $\lambda_2 = -10$ and let v be as in equation (18). Under this choice of v, $(z_1(t), z_2(t))$ converges asymptotically to zero when t tends to $+\infty$. Thus, $w(t) := -\delta^1 z_1^2(t)$, seen as an external exogenous input of the zero-dynamics

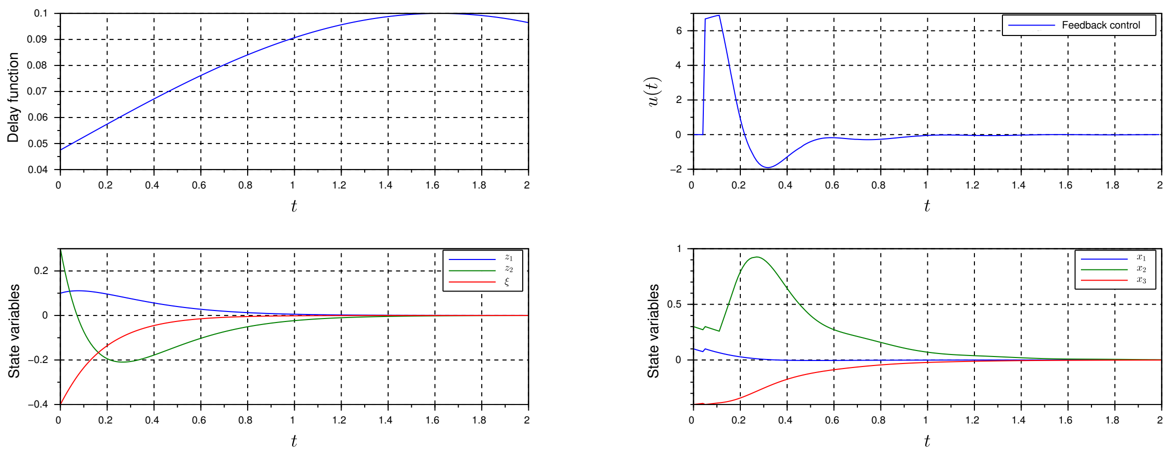
$$\dot{\xi}(t) = -4\xi(t) - \delta^1\xi(t) - w(t) \tag{22}$$

is bounded over $[0, +\infty)$. Then, in order to guarantee the global asymptotic stability of the overall system, it remains to verify that system (22) is uniformly globally ISS. In our case, the ISS property is guaranteed by [6, Theorem 1].

We consider system (21) with initial conditions $x_1(s) = 0.1, x_2(s) = 0.3, x_3(s) = -0.4$, for $s \in [-2\bar{\theta}, 0]$, and variable delay function $\theta(t) = \frac{\bar{\theta}}{2}(1 + \sin(t - \frac{\bar{\theta}}{2}))$, $t \geq 0$, where $\bar{\theta} = 0.1$. Figure 1-(a)[2] shows its behavior in (z,ξ)-coordinates. We clearly see that (z,ξ) tends asymptotically to zero when t becomes large. In Figure 1-(b), we illustrate the behavior of system (21) in x-coordinates. Knowing that our system is partially linearizable with delay, then, as stated in Theorem 1, the feedback control u is calculated by prediction over the intervals $[t, t+\tau_1^{-1}(0)]$ and $[t+\tau_i^{-1}(0), t+\tau_{i+1}^{-1}(0)]$, for $i \geq 1$ and $t \geq 0$. We clearly observe this phenomena

[2]Figures 1 are obtained thanks to the free and open source software Scilab.

(through Figure 1-(right-top)) over the intervals $[0, \tau_1^{-1}(0)] = [0, \bar{\theta}/2]$ and $[\tau_1^{-1}(0), \tau_2^{-1}(0)] \subset [\bar{\theta}/2, 2\bar{\theta}]$ (note that, in this particular case we have $\tau_2^{-1}(0) \simeq 0.103 < 2\bar{\theta}$).



(a)

(b)

Figure 1: The behavior of system (21) in (z, ξ)-coordinates (a). The behavior of system (21) in x-coordinates (b).

5 Proofs

5.1 Proof of Lemma 1

Preliminary remarks. Before proceeding in the proof of Lemma 1, we discuss some preliminary remarks. Throughout, by abuse of notation, we write $a^i(t)$ (resp. $L_f\varphi_\rho(t)$) instead of $a^i(x, \delta, t)$, for $0 \le i \le m+1$, (resp. $L_f\varphi_\rho(x, \delta, t)$). If the integer j (defined in the statement of Lemma 1) is such that $j \ge 1$, we start by introducing a new time-scale $\tilde{t}$ that allows us to shift the time in such a way that $\delta^j u(t)$ becomes $u(\tilde{t})$. In order to do that, remark that $\tau_i(\cdot)$ is a strictly increasing function of t, for $i \in \{1, \cdots, m+1\}$. Indeed, it is easy to verify that

$$\dot{\tau}_{i+1}(t) = \dot{\tau}_i(t)(1 - \dot{\tau}(\tau_i(t))), \quad \text{for } i \in \{1, \cdots, m+1\},$$

from which we can deduce, by recurrence, that $\dot{\tau}_{i+1}(t) > 0$ for every $t \ge 0$. This allows to define a new time-scale $\tilde{t} = \tau_j(t)$, for $t \ge 0$.

By introducing $\tilde{t} = \tau_j(t)$, equation (6) reads

$$\tilde{a}^j(\tilde{t})u(\tilde{t}) + \cdots + \tilde{a}^m(\tilde{t})\delta^{m+1-j}u(\tilde{t}) = -L_f\varphi_\rho(\tilde{t}) + v(\tilde{t}), \tag{23}$$

where $\tilde{a}^i(\tilde{t}) := a^i(\tau_j^{-1}(\tilde{t}))$, for $j \le i \le m+1$, and condition (13) (see the statement of Lemma 1) becomes

$$\sup_{\tilde{t} \ge 0} \left\| \frac{a^i(\tilde{t})}{a^j(\tilde{t})} \right\| \le \frac{1}{c(m+1-j)}, \quad j \le i \le m+1. \tag{24}$$

The idea of the proof of Lemma 1 is to give a bound for $u(\tilde{t})$, for $\tilde{t}$ belonging to a sequence of intervals $[T_k, T_{k+1}]$, where $(T_k)_{k \ge 0}$ is a suitable chosen sequence (see the below paragraph, where we explain how to construct $(T_k)_{k \ge 0}$) and, then, to show that, when k goes to infinity, that bound is finite.
From the assumption made on the delay function $\tau(\cdot)$ (especially that $\tau(t) > 0$ and $\dot{\tau}(t) < 1$, for every $t \ge 0$)

one can easily prove (the proof is left to the reader) the existence of a strictly increasing sequence $(T_k)_{k\geq 0}$ such that

$$T_0 = 0, \qquad T_k = T_{k+1} - \tau(T_{k+1}), \qquad \forall k \geq 0,$$

and $T_k \to +\infty$ with k.

Proof of the lemma. Denote

$$M = \sup_{\tilde{t}\geq 0} \left\| \frac{-L_f\varphi_\rho(\tilde{t}) + v(\tilde{t})}{\tilde{a}^j(\tilde{t})} \right\|.$$

Since τ_i, $1 \leq i \leq m+1$, is an increasing function (as we have just proven), over $[T_0, T_1]$, we have

$$\tau_1(\tilde{t}) \in [\tau_1(T_0), \tau_1(T_1)] = [-\tau(0), 0] \subset [-(m+1)\bar{\theta}, 0],$$

and we can easily verify that

$$\tau_i(\tilde{t}) \in [\tau_i(T_0), \tau_i(T_1)] \subset [-(m+1)\bar{\theta}, 0], \quad \forall 1 < i \leq m+1-j.$$

Then, from relation (23), it follows that, for every $T \in [T_0, T_1]$, we have

$$\begin{aligned} \|u(\tilde{t})\| &\leq \|\frac{\tilde{a}^{j+1}}{\tilde{a}^j} u(\tau_1(\tilde{t}))\| + \cdots + \|\frac{\tilde{a}^{m+1}}{\tilde{a}^j} u(\tau_{m+1-j}(\tilde{t}))\| + M \\ &\leq \left(\|\frac{\tilde{a}^{j+1}}{\tilde{a}^j}\| + \cdots + \|\frac{\tilde{a}^{m+1}}{\tilde{a}^j}\|\right) \sup_{s\leq T_0} \|u(s)\| + M. \end{aligned}$$

Now, recall that according to our assumptions, there exists a constant $c > 1$ such that relation (24) holds for all $\tilde{t} \geq 0$, thus for $\tilde{t} \in [T_0, T_1]$ also. Hence,

$$\|u(\tilde{t})\| \leq \frac{1}{c} \sup_{s\leq T_0} \|u(s)\| + M.$$

The latter inequality allows to estimate $u(\tilde{t})$ over $[T_0, T_1]$. Since, by hypothesis, $u(\tilde{t})$ is bounded over $[-(m+1)\bar{\theta}, 0]$, we can estimate its delays $\delta^i u(\tilde{t})$ over $[-(m+1)\bar{\theta}, T_1]$, for every $i \in \{0, \cdots, m+1-j\}$, and give a bound for $u(\tilde{t})$ over $[T_1, T_2]$. As above, over $[T_1, T_2]$, we have $\tau_1(\tilde{t}) \in [\tau_1(T_1), \tau_1(T_2)] = [-\tau(0), T_1] \subset [-(m+1)\bar{\theta}, T_1]$, and $\tau_i(\tilde{t}) \in [\tau_i(T_1), \tau_i(T_2)] \subset [-(m+1)\bar{\theta}, T_1], \forall 1 < i \leq m+1-j$. Then, for every $T \in [T_1, T_2]$, we obtain

$$\begin{aligned} \|u(\tilde{t})\| &\leq \left(\|\frac{\tilde{a}^{j+1}}{\tilde{a}^j}\| + \cdots + \|\frac{\tilde{a}^{m}}{\tilde{a}^j}\|\right) \sup_{s\leq T_1} \|u(s)\| + M \\ &\leq \left(\frac{1}{c}\right)^2 \sup_{s\leq T_0} \|u(s)\| + \left(1 + \frac{1}{c}\right) M. \end{aligned}$$

By an induction argument, for $k \geq 2$, we deduce that over $[T_k, T_{k+1}]$, we have

$$\|u(\tilde{t})\| \leq \left(\frac{1}{c}\right)^k \| \sup_{s\leq T_0} \|u(s)\| + M \cdot \sum_{i=0}^{i=k} \left(\frac{1}{c}\right)^i.$$

Recall that the constant c is such that $c > 1$, thus, we deduce that for any $\varepsilon > 0$, there exists an integer k_ϵ such that, for $T \geq T_{k_\epsilon}$, we have

$$\|u(\tilde{t})\| \quad \leq \quad \varepsilon + \frac{c}{c-1} M.$$

5.2 Proof of Theorem 1

We assume that system (4) satisfies item (1). Introduce $z_i = L_f^{i-1}h$, for $1 \leq i \leq \rho$, and complete them by the complementary coordinates ξ, that is $\dim z + \dim \xi = n$. The system in the (z, ξ)-coordinates reads:

$$\begin{aligned}
\dot{z}_i &= z_{i+1},\, 1 \leq i \leq \rho - 1, \\
\dot{z}_\rho &= L_f^\rho h + \bar{L}_{g_0} L_f^{\rho-1} h u + \bar{L}_{g_1} L_f^{\rho-1} h \delta^1 u = L_f^\rho h + a(\delta] u(t), \\
\dot{\xi} &= G(z, \xi, \delta, \delta^-, u),
\end{aligned}$$

where G is smooth and affine with respect to the control and the delayed control. By applying the feedback law (16) (recall that according to item (2) of Theorem 1, the δ-polynomial $a(\delta]u(t)$ satisfies Lemma 1, thus the input $u(t)$ stays bounded and, by item (3), it is causal), we obtain

$$\begin{aligned}
\dot{z}_i &= z_{i+1},\, 1 \leq i \leq \rho - 1, \\
\dot{z}_\rho &= \delta^j v(t), \\
\dot{\xi} &= G(z, \xi, \delta, \delta^-, v).
\end{aligned}$$

If $\rho = n$, i.e., the variables ξ are absent, then the system is fully linearizable. If, moreover, $j = 0$, it is fully input-output linearizable without delay (linearizable with delay, otherwise). If $\rho < n$, then the system is partially input-output linearizable.

6 Conclusion

In this paper, we have considered the problem of input-output linearization of nonlinear systems with time-varying delays appearing in the state, in the input and in the output. Since the delays are time-varying, we have introduced a new Lie derivative and have defined a notion of full and partial linearizability with and without delays of SISO systems with such kind of delays. Then, sufficient conditions have been developed in order to guarantee the existence of a causal and bounded linearizing feedback. In a future work, the case of Multi-Inputs Multi-Outputs System with time-varying delays will be studied with respect to linearization and decoupling problem.

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