Spikes super-resolution with random Fourier sampling

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***Abstract*—We leverage recent results from machine learning to show theoretically and practically that it is possible to stably recover a signal made of few spikes (in the gridless setting) from few random weighted Fourier measurements. Given a free choice of frequencies, a number of measurements lower than with the traditional low-pass filter (uniform sampling of low frequencies) guarantees stable recovery.**

I. Introduction

Recovering sums of Diracs from linear observations with convex continuous variational methods has recently been a subject of interest: it was shown that sums of sufficiently separated Diracs (spikes) can be recovered in a gridless setting from their low-pass filtered version [3], [4], [5]. This super-resolution problem has further been linked with the recovery of probability densities from random empirical generalized moments (sketches) [6], [7]. These results suggest that few spikes can be recovered using random Fourier measurements instead of regular low frequency Fourier measurements.

In this contribution, we show that random weighted Fourier observations guarantee the success of an ideal super-resolution decoder. We then study the effect of different parameters of the problem on the performance of a greedy heuristic used to approach the ideal estimation method. In particular, super-resolution can be performed with fewer measurements than the conventional low-pass filter.

II. Recovery of sums of Diracs

Denote δ_t the Dirac measure at position t in $\mathbb{R}^d$. We aim at recovering elements of the set $\Sigma_{k,\epsilon} = \{\sum_{i=1,k} a_i \delta_{t_i} : \forall k \neq l, \|t_k - t_l\|_2 \geq \epsilon, \|t_i\|_\infty \leq 1,\ a_i \in \mathbb{R}_+, t_i \in \mathbb{R}^d\}$ given some linear observations. The resolution parameter ϵ denotes the minimum separation between spikes. We study the recovery of $x_0 \in \Sigma_{k,\epsilon}$, from noisy linear observations $Ax_0 + e$ with $Ax_0 = (\int_t x_0(t) f_i(t) dt)_{i=1,m}$ where $f_i(t) = e^{j\langle \omega_i, t\rangle}/c_{\omega_i}$, $(\omega_i)_{i=1,m}$ is a set of frequencies in $\mathbb{R}^d$ and c_{ω_i} is a weighting scheme. We consider two such operators:

- A_U**: Uniform Fourier sampling (low-pass filter)**: $(\omega_i)_{i=1,m}$ is a uniform sampling of the set $[-\frac{\pi q}{2}, \frac{\pi q}{2}]^d$ where q is an integer and $m = (2q+1)^d$.
- A_R**: Random weighted Fourier sampling**: Each ω_i is a frequency randomly drawn according to the distribution with probability density proportional to $c_\omega e^{-\sigma^2\|\omega\|_2^2/2}$ (where σ is the parameter tuning the frequency distribution, $c_\omega = \sqrt{2 + \sigma^2\|\omega\|_2^2 + \sigma^4\|\omega\|_2^4}$ is a weighting term).

In 1D, convex recovery from observation with A_U is possible as soon as $m \geq \frac{2}{\epsilon}$ [3] ($m \geq O(1/\epsilon)$ being a necessary condition [10]). With any A, robust and stable recovery is possible [2] with the "ideal decoder"

$$x^* = \underset{x \in \Sigma_{k,\epsilon}}{\mathrm{argmin}} \|Ax - (Ax_0 + e)\|_2 \tag{1}$$

provided A satisfies a restricted isometry property (RIP) on the set $\Sigma_{k,\epsilon} - \Sigma_{k,\epsilon}$ (differences of elements of $\Sigma_{k,\epsilon}$). An operator A obeys a RIP of constant δ on a set $\mathcal{S}$ if for any $x \in \mathcal{S}$,

$$(1-\delta)\|x\|^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|^2. \tag{2}$$

The RIP does not generally hold with typical norms $\|\cdot\|$ in the space $\mathcal{B}$ of Radon measures over $\mathbb{R}^d$ such as the total variation norm. However, when $\|\cdot\| := \|\cdot\|_h = \|h \star \cdot\|_2$ where h is a Gaussian convolution kernel, we establish that the RIP is verified with high probability for the random weighted Fourier measurements A_R for large enough m. The proof relies on concentration of measure arguments [1], [9] and on the (non trivial) fact that the normalized secant set of $\Sigma_{k,\epsilon}$ (the set $(\Sigma_{k,\epsilon} - \Sigma_{k,\epsilon}) \cap S(1)$) has a finite covering number with respect to $\|\cdot\|_h$. Using this we further prove:

Theorem 1. *Let $\sigma_k := \frac{1}{2.4(\ln k + 10)}$, $\epsilon > 0$. Consider $\sigma \leq \sigma_k \epsilon$, $h(t) = e^{-\|t\|_2^2/(2\sigma^2)}$, and assume*

$$m \geq O(k^2 d^3 polylog(k,d) \log(1/\epsilon)). \tag{3}$$

Then the random Fourier sampling operator A_R has the RIP on $\Sigma_{k,\epsilon} - \Sigma_{k,\epsilon}$ with high probability, hence for any $x_0 \in \mathcal{B}$ and $y = Ax_0 + e$, with high probability on the draw of A_R we have

$$\|x^* - x_0\|_h \lesssim \|e\|_2 + d_h(x_0, \Sigma) \tag{4}$$

where x^* is the minimizer in (1) and $d_h(x_0, \Sigma_{k,\epsilon}) = \inf_{x \in \Sigma_{k,\epsilon}} \|x_0 - x\|_h$ is the modeling error.

The important facts are that for fixed k, d, only $m = O(\log(1/\epsilon))$ measures are sufficient for recovery (compared to the low-pass filter version where, in 1D, $m = O(1/\epsilon)$ are necessary). This theorem also gives a suitable frequency distribution for A_R.

III. Experiments and perspectives

CLOMPR is an adaptation of orthogonal matching pursuit with replacement to the off-the-grid setting [6]. It was shown to perform well in a machine learning setting (compressive K-means clustering [7]) with $O(kd)$ measurements being sufficient in practice to recover k spikes (associated to k clusters) measured with A_R. In the experiments below we generated random k-mixtures of ϵ-separated Diracs in dimension $d = 2$, applied A_R (resp. A_U), decoded with CLOMPR, and tested for exact recovery to obtain phase transition curves with respect to various parameters.

In Figure 1, we verify that the frequency distribution parameter σ can be chosen proportional to ϵ. In Figure 2 and Figure 3, a comparison with the low-pass filter A_U is made. For the low-pass filter the number of measurements corresponds to the cut-off frequency and the observations confirm that $m \gtrsim 1/\epsilon$ is needed to recover the spikes. With random Fourier measurements, we observe that recovery is improved beyond this linear limit while not fully reaching the theoretical sufficient behavior of $m \gtrsim O(\log(1/\epsilon))$ when the number k of spikes increases. This may be due to the fact that CLOMPR is only a heuristic not guaranteed to solve (1). Figure 4 shows that recovery is stable to Gaussian noise according to the localization error : $\sum_i \min_k \|t_i^* - t_k\|_2^2$ where t_i^* are the locations of the estimated spikes and t_i are the locations of the spikes of x_0.

These results open an interesting line of research for the design of acquisitions systems for point sources and for theoretical recovery guarantees in a similar set up, e.g. in radio astronomy [8]. In particular, can we improve the theoretical bound for the number of measurements to $O(kd \cdot polylog(k,d))$? Is it possible to match the bound $O(\log(1/\epsilon))$ using a practical reconstruction scheme ?

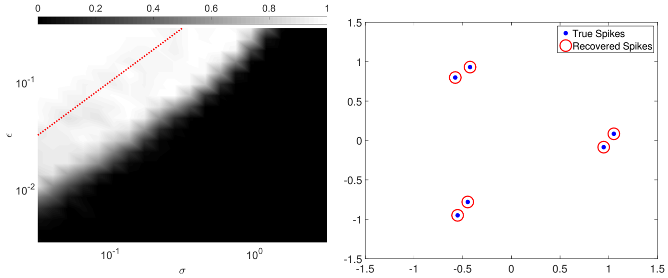

Fig. 1. Left: Probability of successful reconstruction of a mixture of 6 ϵ-separated Diracs, using CLOMPR, depending on σ (sampling scheme) and ϵ. Reconstruction is declared successfull if the localization error is lower than $\epsilon/3$. Red line: $\sigma \propto \epsilon$. Right : An example of perfect reconstruction.

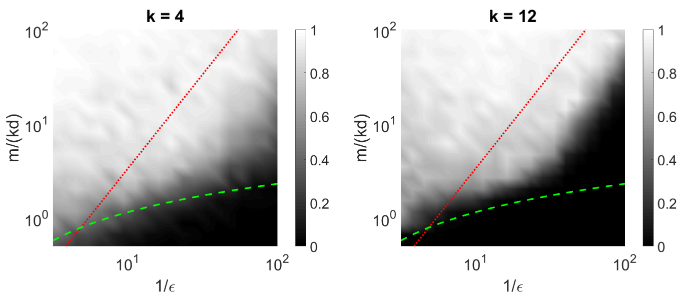

Fig. 2. Probability of successful reconstruction depending on the number of measurements m and separation ϵ for random Fourier observations. Red line: linear relation $m \propto 1/\epsilon$. Green line: theoretical dependency in $\log(1/\epsilon)$. Left: $k = 4$ spikes. Right: $k = 12$ spikes

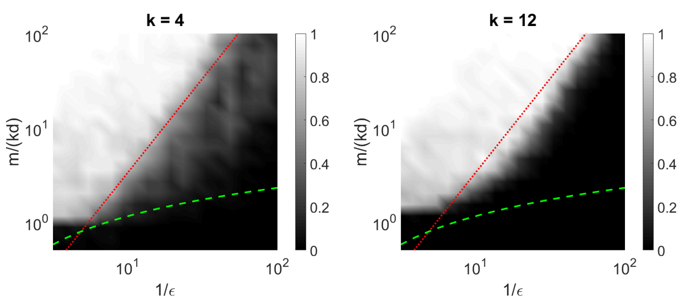

Fig. 3. Probability of successful reconstruction depending on the number of measurements m and separation ϵ for regular Fourier observations (ideal low-pass filter). Red line: linear relation $m \propto 1/\epsilon$. Green line: theoretical dependency in $\log(1/\epsilon)$. Left: $k = 4$ spikes. Right: $k = 12$ spikes.

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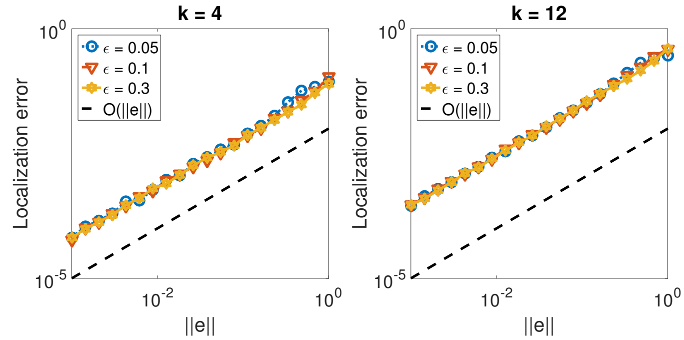

Fig. 4. Behavior of the reconstruction error with respect to noise for different separations. Left : $k = 4$ spikes. Right : $k = 12$ spikes.

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