# Cost allocation protocols for network formation on connection situations

Bruno Escoffier, Laurent Gourvès, Jérôme Monnot, Stefano Moretti

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Bruno Escoffier[†]
Université Paris Dauphine
CNRS UMR UMR 7243
Place du Maréchal de Lattre
de Tassigny, Paris, France

Laurent Gourvès
CNRS UMR UMR 7243
Université Paris Dauphine
Place du Maréchal de Lattre
de Tassigny, Paris, France

Jérôme Monnot
CNRS UMR UMR 7243
Université Paris Dauphine
Place du Maréchal de Lattre
de Tassigny, Paris, France

Stefano Moretti
CNRS UMR UMR 7243
Université Paris Dauphine
Place du Maréchal de Lattre
de Tassigny, Paris, France

## ABSTRACT

The issue of embedding cost-awareness in the design of communication network devices and protocols has been growing at a fast rate in last years. Under certain connection situations, however, network design is not enforced by a central authority. This is the case, for instance, of power control for wireless networks, where the cost of a link is a function of the power needed to send a message to a remote node, which increases with the distance. Here each player wishes to consume as few power as possible to send its request and the main question is how to avoid that players deviate from a socially optimal network.

In this paper, we study strategic games based on connection situations with the objective to coordinate self-interested agents placed on the nodes of a graph to realize a more efficient communication network. We address the problem of the design of cost allocation protocols that may guarantee the convergence of the best response dynamic and we analyze the effects of cost monotonicity and other state-dependent properties on the optimality of a protocol.

## Categories and Subject Descriptors

F.2.2 [**Nonnumerical Algorithms and Problems**]: Computations on discrete structures; G.2.1 [**Combinatorics**]: Combinatorial algorithms

## General Terms

Algorithms , Theory

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## Keywords

Spanning tree, game theory

## 1. INTRODUCTION

Due to economic and environmental concerns, reducing energy consumption in telecommunications is a priority, and the issue of embedding cost-awareness in the design of communication network devices and protocols using game theory has been growing at a fast rate in last years [27, 6]. In particular, game theory applied to connection situations seems to provide a powerful and realistic methodology to analyze the design of cost allocation protocols. A connection situation arises when there is a group of agents (e.g. devices of a communication network) who all want to be connected with a source 0 (e.g. a server), directly or via other agents, and where connections are costly (e.g. due to data traffic costs). Cost sharing problems on connection situations were introduced by Claus and Kleitman [10] and have been studied with the aid of cooperative game theory since the basic paper of Bird [7]. Given a connection situation, Bird [7] introduced an associated coalitional game (known as *minimum cost spanning tree (mcst) game*), where the players are the agents placed on the nodes and the cost incurred by a coalition is the minimal cost of connecting this coalition to the source via links between members of the coalition. Since then, many *cost sharing protocols* have been proposed in the literature of mcst games [16, 17, 28, 20, 21, 24, 28, 9, 21, 29] with various desirable properties including *budget balance* and *cost monotonicity*.

Budget balance consists of satisfying both *cost recovery* (i.e. the cost of the service is recovered from all the players) and *competitiveness* (i.e. no surplus is created because if any surplus is created then a competitor can provide the service at a cheaper cost by reducing the surplus) [11].

Many papers concerning the analysis of cost allocation protocols using coalitional games have focused on cost monotonicity properties, meaning that if some connection costs go down (up), then no agents will pay more (less) (see for instance [12, 28, 8, 5, 2, 8]). In the paper of [12], for instance, a particular cost monotonic protocol was studied, where cost monotonicity means that an agent $i$ does not

pay more if the cost of a link involving $i$ decreases, nothing else changing in the network. The interest for monotonicity properties for protocols in connection situations is explained by the fact that in many real applications, connection costs may increase or decrease with time, and therefore cost allocations which are stable only in the original situation cannot guarantee the cooperation among agents also under the new conditions. This is the case, for instance, for telecommunication networks, where it may happen that at a given moment the cost of connections can increase (*e.g.* as a consequence of an improvement in quality and quantity of services supplied) or decrease (*e.g.* by improving telecommunication technologies). Another reason to analyze cost monotonicity is that it ensures that no customers are motivated to voluntary increase the cost of adjacent links, since according to a cost monotonic allocation protocol no customer will pay less. Note that this kind of considerations arise from interaction situations which are based on cooperative models, where the issue concerning the "strategic" behavior of players is somehow left to the intuition.

In parallel, since the seminal paper [1], another class of related problems has been widely studied in the literature of game theory applied to networks. The basic question of such problems is to describe an endogenous process of the formation of communication links, given some underlying protocol for allocating the benefit of communication, namely the *Myerson value* [23]. The paper by Aumann and Myerson [1] introduced a model of link formation where links are constructed sequentially. Later, Myerson [22] introduced a model of link formation in strategic form where players announce simultaneously the set of players they would like to communicate with, and a link between two nodes is formed if both nodes announce to form it. As in the previous model, the payoffs of players is calculated according to the Myerson value. Later, Slikker and van den Nouweland [26] introduced an extension of such models for link formation that incorporate costs of formation of communication links (see also [18, 19]). An interpretation of these models with costs of link formation is that players at first stage incur the costs of link formation and divide these costs in a fair manner and then, at a second stage, bargain over the division of the benefits (see also [25]).

The objective of this paper is threefold. First, we want to integrate cost allocation protocols for connection situations with the model of network formation with communication costs. In this direction, a game in strategic form is presented where agents are placed on the nodes of a graph and the strategy of each player is to construct a single link which connects himself to another node in the network (that may be another agent or a source). Each link is costly, as in usual connection situations, and the cost of remaining disconnected from the source is larger than any finite cost that should be supported to guarantee the connection with the source. So, agents want to be connected to the source at any cost, but of course they are self-interested to save their own money. Actually, this strategic game has been already introduced in [15], but in that paper the authors focused on a specific cost allocation protocol for trees: the *Bird rule* [7] assigning to each player $i$ the cost of the link from $i$ to the its predecessor on the unique path from the source to $i$ (see also [13, 3, 4] for other strategic models applied to connection situations).

The second objective of this paper is to analyze properties for cost allocation protocols (including but not limited to the Bird one), and in particular the role of cost monotonicity and other state-dependent properties in the convergence of the *best response dynamic* [14]. Contrary to the situation based on a cooperative setting, where cost monotonicity seems to interpret a condition of stability in dynamic situations [20], we show that cost monotonicity of protocols is incompatible with the optimality of associated best response dynamics.

Finally, the third and main goal of this manuscript is to answer an open question concerning the existence of a cost allocation protocol that guarantees the convergence of each best response dynamic to a network of minimum connection cost. This is not the case, for instance, of the cost allocation protocol based on the Bird rule, which allows for the possibility of Nash equilibria which do not correspond to a graph of minimum cost [15].

We start in the next section with a brief introduction of the main notations and the definition of the model of strategic game applied to connection situations. Section 3 focuses on the relation between cost monotonicity and the state-dependent property with the property of optimality for protocols. Section 4 is devoted to two optimal protocols and to the analysis of their properties of convergence. Section 5 concludes.

## 2. PRELIMINARIES AND NOTATIONS

Let $G = (V, E, w)$ be an undirected, connected and weighted graph on $n = |V|$ vertices and $m = |E|$ edges where $V = \{0, 1, \cdots, n-1\}$ and where each edge $e \in E$ has a non negative weight $w(e) \in \mathbb{R}^+$. Node 0 is called the root (or the source) and any other node is an agent who wants to be connected to 0 either directly or via other nodes which are connected to 0. For any set of edges $E'$ we denote by $w(E')$ its total weight: $w(E') = \sum_{e \in E'} w(e)$.

As a notation $V(.)$ and $E(.)$ are two functions which designate the vertex set and edge set of their argument, respectively. The subgraph of $G$ induced by the vertex set $V' \subseteq V$ is denoted by $G[V']$.

We consider a *strategic game form* $(V \setminus \{0\}, \mathcal{N}_G(1) \times \mathcal{N}_G(2) \times \cdots \times \mathcal{N}_G(n-1))$ where the *strategy space* $\mathcal{N}_G(i)$ of every player $i \in \{1, \cdots, n-1\}$ is his neighborhood in the graph. When a player $i$ plays its neighbor $j$ then the edge $(i, j)$ is built. A *state* (or *strategy profile*) $S$ is a vector $(S_1, S_2, \cdots, S_{n-1}) \in \mathcal{N}_G = \mathcal{N}_G(1) \times \mathcal{N}_G(2) \times \cdots \times \mathcal{N}_G(n-1)$. In the following, $S_{-i}$ denotes $S$ from which the strategy of player $i$ was removed and $(S'_i, S_{-i})$ denotes the state $S$ from which $S_i$ was replaced by $S'_i$.

The edges built by the players and associated with $S$ is denoted by $E(S)$ and defined as $\{(i, S_i) : i = 1, \cdots, n-1\}$. Let $S$ be any state. We denote by $\mathrm{con}(S)$ and $\mathrm{dis}(S)$ the players who are connected and disconnected from the source, respectively. Let $CC_S$ be the connected component of $E(S)$ that contains the source and $E(CC_S)$ be the edges of the connected component $CC_S$. $T_S = E(CC_S) \cap E(S)$. Note that $T_S$ is a tree, since $T_S$ is connected by construction and

contains exactly $|T_S| + 1$ vertices. Note also that $\mathtt{con}(S)$ is the vertex set of $CC_S \backslash \{0\}$ and $T_S = \{(i, S_i) : i \in CC_S \backslash \{0\}\}$.

We suppose that every player wants to be connected to the source at the least possible cost. To do so the players interact with a protocol which, given the strategy profile, allocates a cost to the players. More formally, given a graph $G$, a *cost allocation protocol* (or, simply, a *protocol*) is a map $c : (\mathbb{R}^+)^m \times \mathcal{N}_\mathcal{G} \to (\mathbb{R}^+)^{n-1}$, which assigns to every weight vector $w \in (\mathbb{R}^+)^m$ and every state $S \in \mathcal{N}_\mathcal{G}$ a cost vector $(c_1(w, S), \cdots, c_{n-1}(w, S)) \in (\mathbb{R}^+)^{n-1}$ (if the weight function is clear from the context and no confusion arises we simply denote it as $(c_1(S), \cdots, c_{n-1}(S))$).

A cost allocation protocol $c$ such that $\sum_{i \in \mathtt{con}(S)} c_i(S) = w(T_S)$ for every strategy profile $S$ is said *budget balanced*. This property implies that the cost of the edges in the network connected to the source is fully supported by its users. In the remaining of the paper we will focus on budget balanced protocols and the associated *strategic games* $(V \backslash \{0\}, \mathcal{N}_\mathcal{G}, c)$.

Given a protocol $c$, a strategy $x \in \mathcal{N}_G(i)$ is a *better response* of player $i$ with respect to the strategy profile $S$ if $c_i(x, S_{-i}) < c_i(S)$. We say that $x$ is a *best response* when $c_i(x, S_{-i}) = \min_{y \in \mathcal{N}_G(i)} c_i(y, S_{-i})$.

A state $S$ is a Nash equilibrium of the game, if for every player $i$, it holds that $S_i$ is a *better response* of player $i$ to $S_{-i}$. For a state $S$ and a player $i$, let $N_S(i)$ be the sets of strategies of player $i$ resulting from a better response of player $i$, ie., $N_S(i) = \{j \in \{0, \ldots, n-1\} : c_i(j, S_{-i}) < c_i(S)\}$; in particular, $N_S(i) \subseteq \mathcal{N}_G(i)$ and $S$ is a NE iff $N_S(i) = \emptyset$ for every $i \in \{1, \ldots, n-1\}$.

A *Better Response Dynamic* (BRD, also called Nash dynamics) (associated with a protocol $c$) is a sequence of states $S^0, S^1, \ldots$, such that each state $S^k$ (except $S^0$) is resulted by a better response of some player from the state $S^{k-1}$. Note that if a better response dynamic reaches a Nash equilibrium after a finite number of states, then no further changes of strategies are expected (if we assume that a player changes his strategy only if he strictly prefers a different strategy).

## 3. DYNAMICS OF PROTOCOLS

In this section we are interested in analyzing properties of cost allocation protocols in connection situations where agents are continuously prepared to improve their payoff in response to changes made by other agents. How should we design cost allocation protocols to minimize the efficiency loss caused by selfish players that are only willing to perform update leading to an immediate reduction of their individual cost shares?

A natural approach to this problem is the analysis of each BRD associated with a certain protocol. With this objective, the following properties should be considered.

DEFINITION 1 (CONV). *We say that a cost allocation protocol converges to an equilibrium iff every associated BRD reaches a Nash equilibrium.*
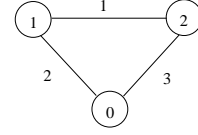


**Figure 1: Illustration of Proposition 1.**

We say that a Nash equilibrium $S$ is *efficient* iff the corresponding graph $E(S)$ is a *minimum cost spanning tree* (*mcst*) (i.e. $w(E(S))$ equals the minimum cost over all networks connecting all nodes in $V$).

DEFINITION 2 (OPT). *We say that a cost allocation protocol is optimal iff every associated BRD reaches an efficient Nash equilibrium.*

Obviously the OPT property implies the CONV one. The protocol based on the Bird rule, which charges each player $i$ in state $S$ with the weight $w(i, S_i)$ is budget balanced, CONV but not OPT [15].

Consider the following property for cost allocation protocols.

DEFINITION 3 (IMON). *We say that a cost allocation protocol is Individually Monotonic iff for every $S \in \mathcal{N}_\mathcal{G}$, $i \in \mathtt{con}(S)$ and $\hat{S}_i \in \mathcal{N}_G(i)$*

$$w(i, \hat{S}_i) \geq w(i, S_i) \Rightarrow c_i(\hat{S}_i, S_{-i}) \geq c_i(S).$$

Looking at the motivations that justify the interest in monotonicity properties in the cooperative setting, one could erroneously argue that individual monotonicity is a good candidate property to guarantee the implementation of a network of minimum cost. However, it is easy to show that there is no cost allocation protocol which satisfies both IMON and OPT properties, implying that a large family of cost monotonic solution from the literature on minimum cost spanning tree games are not optimal in this framework [20].

PROPOSITION 1. *There is no cost allocation protocol which satisfies both IMON and OPT properties.*

PROOF. Suppose it exists a protocol $c$ which is IMON and OPT. Consider the instance of Figure 1 and the suboptimal strategy profile where player 1 plays 2 while player 2 plays 0. If a player changes his strategy then his cost increases. Indeed, if player 1 plays 0 then his cost increases by IMON. Meanwhile, if player 2 plays 1 then he is not connected anymore so his cost is infinite. So this state is a Nash equilibrium but not an efficient one[1]. $\square$

Another property that is particularly valuable in the analysis of the endogenous formation of networks is the state-dependent property, saying that the allocation of the cost of

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[1]For illustrative purposes, this proof is based on a situation with only two players; obviously, the same considerations apply to situations with more than two players.

the network $T_S$ connected to the source in a state $S$ should not depend on the edges not constructed under $S$.

DEFINITION 4 (SDEP). *We say that a cost allocation protocol is State Dependent iff for every state $S$, for every weight functions $w \in (\mathbb{R}^+)^m$ and $w' \in (\mathbb{R}^+)^m$, with $w(e) = w'(e)$ for every $e \in T_S$, then $c_i(w, S) = c_i(w', S)$ for every $i \in con(S)$.*

This property allows for the continuous control of the charge procedure by means of the simple observation of the edges constructed under state $S$, without assuming the knowledge of the weights of the links of the entire network (see for instance the class of *construct and charge* ($CC$)-rules in [21] for a family of cost allocation protocols that meet the requirement of continuous monitoring by the agents involved). Unfortunately, also this property is incompatible with the OPT property, as it is shown by the following proposition.

PROPOSITION 2. *There is no cost allocation protocol which satisfies both SDEP and OPT properties.*

PROOF. By Proposition 1, it is sufficient to prove that if a cost allocation protocol satisfies SDEP and OPT properties than it is satisfies IMON property too.

Suppose it does not. This means there exists $S, i$ and $\hat{S}_i$ such that $w(i, \hat{S}_i) \geq w(i, S_i)$, and $c_i(w, (\hat{S}_i, S_{-i})) < c_i(w, S)$. Take a weight function $w' \in (\mathbb{R}^+)^m$, where $w'(e) = w(e)$ if $e \in T_S \cup \{(i, \hat{S}_i)\}$ or if $e \notin E(CC_S)$, and the cost of all the other edges with vertices in $CC_S$ is $\max_{e \in E}\{w(e)\} + 1$. So the graph $T_S$ is the unique optimal tree with respect to $w'$. By the SDEP property, we have that $c_i(w', (\hat{S}_i, S_{-i})) = c_i(w, (\hat{S}_i, S_{-i})) < c_i(w, S) = c_i(w', S)$, which yields a contradiction with the OPT property. $\square$

Therefore, in order to prosecute in our research for the OPT property, we must renounce to some interesting and well studied properties of cost allocation protocols in the cooperative setting, like *obligation rules* [28], which are cost monotonic, and *CC-rules* [21], which are state-dependent. Nevertheless, next section shows that optimal budget balanced protocols exist.

## 4. TWO OPTIMAL PROTOCOLS

We assume that the edges of $G$ satisfy $w(e_1) \leq w(e_2) \leq \cdots \leq w(e_m)$. In the following, we always compute a mcst with Kruskal's algorithm applied on that edge order.

Let $T_S^*$ be the mcst built using Kruskal's algorithm on the subgraph $G[con(S)]$. We denote by $OPT_S$ the total weight of $T_S^*$, and by $S^*$ a state which corresponds to $T_S^*$.

We say that a player $i \in con(S)$ follows $T_S^*$ in $S$ iff $S_i = S_i^*$. In other words, the strategy of player $i$ is his first neighbor in the unique path from him to the source 0 in the tree $T_S^*$.

Let $\Delta(S) = \sum_{e \in E(CC_S)} w(e) - OPT_S$; it is the difference between the weight of the edges built by the players connected to the source and the minimal weight for connecting these players.

We propose two protocols. Recall that the cost of a non connected player is infinite, so to define a protocole we shall define the cost of connected players (as previously noted, all the players will be connected in any Nash equilibrium).

In the first protocol, all connected players fairly share the cost of an optimal network (namely $\frac{OPT_S}{|con(S)|}$) except one player, denoted by $f(S)$, who is charged $\frac{OPT_S}{|con(S)|}$ plus the extra cost of the current state $\Delta(S)$.

In the second protocol, all connected players who follow the optimal strategy profile $T_S^*$ pay according to the Bird rule while the other connected players (who do not follow $T_S^*$) pay what they should pay in $T_S^*$ with the Bird rule plus an extra cost. We assume that this extra cost is fairly distributed, but actually the result holds for any extra cost.

Our protocols rely on a particular set of players: we define $\hat{V}(S)$ as the players of $con(S)$ such that $S_i \neq S_i^*$ and $con(S) = con(S_i^*, S_{-i})$. In other words, these players do not follow $T_S^*$ and if they unilaterally change their strategy to follow it, then the set of connected players remains unchanged.

LEMMA 1. *The following properties hold for every state $S$: (i) If $\exists i \in con(S)$ such that $S_i \neq S_i^*$ then $\hat{V}(S) \neq \emptyset$. (ii) If $dis(S) \neq \emptyset$, then $\exists i \in dis(S)$ such that $\mathcal{N}_G(i) \cap (con(S) \cup \{0\}) \neq \emptyset$.*

PROOF. Let $S$ be a state.

For $(i)$. Take any player $i \in con(S)$ such that $S_i \neq S_i^*$. Removing the edge $(i, S_i)$ from $E(CC_S)$ provides two connected components $CC_0$ (where the source is) and $CC_i$ (where $i$ is). If $S_i^*$ belongs to $CC_0$ then $i \in \hat{V}(S)$ because $con(S) = con(S_i^*, S_{-i})$. Otherwise $S_i^* \in CC_i$ and consider the path, in $T_S^*$, from $i$ to the source. Denote by $j$ the last node of $CC_i$ visited when walking from $i$ to 0 on the path. It must be $S_j^* \in CC_0$ and $S_j \neq S_j^*$ (otherwise $E(CC_S)$ is not a tree). Thus $j \in \hat{V}(S)$.

For $(ii)$. Assume $dis(S) \neq \emptyset$ and let $j \in dis(S)$. Since $G$ is connected there is a path from the source 0 to $j$. Let $i \in dis(S)$ be the first encountered vertex of this path when we start from 0. Hence, an edge $(i', i)$ belongs to this path with $i' \in (con(S) \cup \{0\})$. $\square$

### 4.1 An egalitarian protocol

Let $f(S) = \min \hat{V}(S)$ be the node of $\hat{V}(S)$ with minimum index if $\hat{V}(S) \neq \emptyset$ and $f(S) = \emptyset$ otherwise (using $(i)$ of Lemma 1). In the protocol, $f(S)$ is charged $\frac{OPT_S}{|con(S)|} + \Delta(S)$ while any other connected node pays $\frac{OPT_S}{|con(S)|}$.

Formally, we get: $c_i(S) = \frac{OPT_S}{|con(S)|}$ for $i \in con(S) \setminus \{f(S)\}$, $c_{f(S)}(S) = \frac{OPT_S}{|con(S)|} + \Delta(S)$ if $f(S) \neq \emptyset$ and $c_i(S) = +\infty$ for $i \in dis(S)$.

One can observe that the total weight of $E(CC_S)$ is always covered by the connected players, i.e., the protocol is budget balanced.
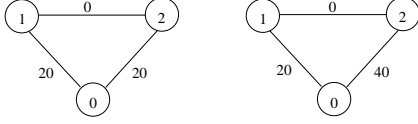
**Figure 2: Two connection situations with two agents, one with symmetric agents and two mcsts (left side) and one asymmetric agents (right side).**

| 1\2 | 0 | 1 |
|---|---|---|
| 0 | 10, 30 | 10, 10 |
| 2 | 10, 10 | $\infty, \infty$ |

| 1\2 | 0 | 1 |
|---|---|---|
| 0 | 10, 50 | 10, 10 |
| 2 | 30, 10 | $\infty, \infty$ |

*Example 1.* Consider the connection situations depicted in Figure 2. Note that on the left side, the agents are symmetric in the graph (two nodes $i$ and $j$ are symmetric in a graph if $w(i, k) = w(j, k)$ for every other node $k$). As a consequence, there are two minimum cost spanning trees of total cost 20: $\{(1, 2), (0, 2)\}$ and $\{(1, 2), (0, 1)\}$. Differently, in the connection situation on the right side, there is a unique mcst of cost 20: $\{(1, 2), (0, 1)\}$.

The strategic games associated with the protocol introduced in this section with respect to the mcst $T_S^* = \{(1, 2), (0, 1)\}$ and the connection situations depicted in Figure 2 are shown in the following tables.

Looking at the left table, corresponding to the network depicted in Figure 2, left side, we observe that even if players 1 and 2 are symmetric, in state $(0, 0)$ player 2 seems to be penalized (he pays more than its direct connection to the source). On the other hand, both Nash equilibria $(2, 0)$ and $(0, 1)$ are efficient.

Instead, in the right table, corresponding to the network in Figure 2, right side, we observe that agent 1 is better off when passing from state $(2, 0)$ to state $(0, 0)$, even if the weight of the link to edge $(1, 2)$ is smaller than the weight of edge $(0, 1)$. As expected, the protocol does not satisfy the IMON property (similarly, we may observe that neither the SDEP property holds: simply increase the cost of the edge $(0, 1)$ from 20 to 22 in the network of Figure 2, right side, all the other costs remaining the same; we have that under the state $(2, 0)$ the protocol would attribute 29 to player 1 and 11 to player 2).

PROPOSITION 3. *Every state corresponding to a mcst is a Nash equilibrium*

PROOF. Let $OPT$ be the weight of any mcst. In a state $S$ corresponding to a mcst, all players pay $OPT_S/(n-1) = OPT/(n-1)$ because $\Delta(S) = 0$. Thus, no player can deviate and decrease his cost. Note that $f(S)$ can be non empty if an optimal solution is not unique. $\square$

PROPOSITION 4. *Any state $S$ which does not correspond to a mcst is not a Nash equilibrium.*

PROOF. If $\Delta(S) = 0$, then $T_S$ is a mcst on $G[\texttt{con}(S)]$ and not on $G$. So, $\texttt{dis}(S) \neq \emptyset$; By $(ii)$ of Lemma 1, there is a player $i \in \texttt{dis}(S)$ which can play $j \in \texttt{con}(S)$. Let $S' = (S_{-i}, j)$ be the state resulting of this modification. We have $c_i(S') < +\infty = c_i(S)$.

If $\Delta(S) > 0$, then the player $f(S)$, who pays $\frac{OPT_S}{|\texttt{con}(S)|} + \Delta(S)$, can play $S_i^*$ and pay $\frac{OPT_S}{|\texttt{con}(S)|}$. $\square$

As indicated in the previous sections, we consider the *better response dynamics* (BRD), a well known process which starts from any given state and, while it is possible, let one player take a better move. We say that BRD converges if it always ends.

Let $\Phi$ be a potential function which maps a state $S$ to the vector

$$(|\texttt{dis}(S)|, |E(S^*) \setminus E(S)|, f(S), |N_S(f(S))|)$$

where we recall that $N_S(f(S))$ is the set if strategies corresponding to a better response for $f(S)$.

A vector $X \in \mathbb{N}^r$ is lexicographically smaller than another vector $Y \in \mathbb{N}^r$, denoted by $X \prec Y$, if one of the following cases occurs:

- $X_1 < Y_1$

- $X_i < Y_i$ for some $i \in \{2, \cdots, r\}$ while $X_j = Y_j$ for all $j < i$.

LEMMA 2. *Let $S$ and $S'$ be two states which only differ on the strategy of one player $i$. If $c_i(S') < c_i(S)$ then $\Phi(S') \prec \Phi(S)$.*

PROOF. Player $i$ has taken a better move. Suppose $i \in \texttt{dis}(S)$. We get that $i \notin \texttt{dis}(S')$ and after $i$'s deviation, the number of disconnected players can only decrease strictly, meaning that $\Phi(S') \prec \Phi(S)$.

Now suppose that $i \in \texttt{con}(S)$. It is clear that $i \in \texttt{con}(S')$ since $i$ takes a better move. It immediately follows that $\texttt{con}(S) = \texttt{con}(S')$ and $OPT_S = OPT_{S'}$.

If $c_i(S) = \frac{OPT_S}{|\texttt{con}(S)|}$ then there is no way for $i$ to change his strategy and decrease his cost so we can assume that $c_i(S) = \frac{OPT_S}{|\texttt{con}(S)|} + \Delta(S)$, implying that $i = f(S)$.

If $i = f(S)$ deviates and plays $S_i' = S_i^*$ instead of $S_i$ then $|E(S^*) \setminus E(S)|$ decreases by one unit while $|\texttt{dis}(S)|$ remains unchanged; thus $\Phi(S') \prec \Phi(S)$.

Now suppose that $i = f(S)$ deviates and plays $S_i' \neq S_i^*$. Then $|E(S^*) \setminus E(S)|$ and $|\texttt{dis}(S)|$ remain unchanged. In addition $i$ belongs to $\hat{V}(S')$, meaning that $f(S) \geq f(S')$. Either $f(S') < f(S)$, implying that $\Phi(S') \prec \Phi(S)$. Otherwise $f(S') = f(S)$. Since $i$ has taken a better move, $c_i(S) = \frac{OPT_S}{|\texttt{con}(S)|} + \Delta(S) > \frac{OPT_{S'}}{|\texttt{con}(S')|} + \Delta(S') = c_i(S')$ from which we

| $1\backslash 2$ | 0 | 1 |
|---|---|---|
| 0 | $20,20$ | $20,0$ |
| 2 | $20,0$ | $\infty,\infty$ |

| $1\backslash 2$ | 0 | 1 |
|---|---|---|
| 0 | $20,40$ | $20,0$ |
| 2 | $40,0$ | $\infty,\infty$ |

deduce that $\Delta(S) > \Delta(S')$ because $OPT_S = OPT_{S'}$ and $\mathtt{con}(S) = \mathtt{con}(S')$. Since the strategies of other players remains unchanged, it follows that $N_{S'}(f(S')) \subset N_S(f(S))$, implying $|N_{S'}(f(S'))| < |N_S(f(S))|$; thus $\Phi(S') \prec \Phi(S)$. $\square$

THEOREM 1. *BRD always converges after at most $\Delta(G)n^3$ rounds where $\Delta(G)$ is the maximum degree of the graph $G$.*

PROOF. Using Lemma 2, each state $S$ is immediately followed by another state $S'$ such that $\Phi(S') \prec \Phi(S)$. Thus BRD can not run into a cycle. Since a finite number of states exists, there is at least one minimal state for $\prec$, meaning that BRD always converge. Now, since for any state $S$, $|\mathtt{dis}(S)| \le n-1$, $|E(S^*) \setminus E(S)| \le n-1$, $f(S) \le n-1$ and $|N_S(f(S))| \le \Delta(G)$, and $\Phi(S)$ always decreases lexicographically when the player who deviates plays a better response, then BRD converges after at most $\Delta(G)(n-1)^3$ rounds. $\square$

COROLLARY 1. *BRD converges after at most $n^3$ rounds if the players play their best response.*

PROOF. Consider the function $\Psi$ which maps any state $S$ to the vector $(|\mathtt{dis}(S)|, |E(S^*) \setminus E(S)|, f(S))$. First, observe that either a disconnected vertex chooses to be connected, or the connected vertex $f(S)$ chooses to follow a strategy leading to a new state $S'$ with $f(S') \ne f(S)$ because $f(S)$ plays a best response. Since $|\mathtt{dis}(S)|$, $|E(S^*) \setminus E(S)|$ and $f(S)$ range from 0 to $n-1$, there are at most $n^3$ values for $\Psi(S)$. $\square$

## 4.2 A Bird's like protocol
In this second protocol, the costs of players are given by:

- If $i \in \mathtt{dis}(S)$, then $c_i(S) = +\infty$,

- If $i \in \mathtt{con}(S) \setminus \hat{V}(S)$, then $c_i(S) = w(i, S_i)$,

- If $i \in \hat{V}(S)$, then $c_i(S) = w(i, S_i^*) + \frac{\Delta(S)}{|\hat{V}(S)|} = w(i, S_i) + \frac{\Delta(S)}{|\hat{V}(S)|}$ where $\Delta(S) = w(T_S) - OPT_{\mathtt{con}(S)}$ (actually here, we can take any cost function $w(i, S_i^*) + g_i(S)$ such that (i) $g_i(S) > 0$ and (ii) $\sum_{i \in \mathtt{con}(S)} g_i(S) = \Delta(S)$.

Note that the protocol is clearly budget balanced.

*Example 2.* Consider again the connection situations depicted in Figure 2.

The strategic games associated with the protocol introduced in this section with respect to the mcst $T_S^* = \{(1,2),(0,1)\}$ and the connection situations depicted in Figure 2 are shown in the following tables.

Looking at the left table, corresponding to the network depicted in Figure 2, left side, we observe that even if players 1 and 2 are symmetric, both Nash equilibria $(2,0)$ and $(0,1)$ are efficient and correspond to the allocation provided by the Bird rule under the network $T_S^*$, which strongly penalizes player 2 in the NE $(2,0)$ where the mcst $\{(1,2),(0,2)\}$ is constructed.

Again, in the right table, corresponding to the network in Figure 2, right side, we observe that the protocol does not satisfy the IMON property, passing from state $(2,0)$ to state $(0,0)$.

PROPOSITION 5. *Every state corresponding to a mcst is a Nash equilibrium*

PROOF. Assume that it is not the case. Thus, there is an optimal state $R$ (with $w(T_R) = OPT$) and another state $S = (R_{-i}, j)$ such that $c_i(S) < c_i(R)$. Note that $\Delta(R) = 0$ so $c_i(R) = w(i, R_i)$. $c_i(S)$ is finite so $i$ is still connected and $\mathtt{con}(S) = V \setminus \{0\}$. The spanning trees $T_R$ and $T_S$ differ only on one edge ($w(i, R_i)$ in $T_R$ and $w(i, S_i)$ in $T_S$) so by optimality of $T_R$ we get that $w(i, R_i) \le w(i, S_i)$. We have $c_i(R) = w(i, R_i) \le w(i, S_i) \le c_i(S)$, contradiction. $\square$

PROPOSITION 6. *Any state $S$ which does not correspond to a mcst is not a Nash equilibrium.*

PROOF. Assume $\mathtt{con}(S) = V$ since otherwise by (ii) of Lemma 1 $S$ is clearly not a Nash equilibrium.

Hence, $\hat{V}(S) \ne \emptyset$ by (i) of Lemma 1 ($w(T_S) \ne OPT$ and $\mathtt{con}(S) = V \setminus \{0\}$). Then, there exists $i \in \hat{V}(S)$ such that $(T_S \cup \{(i, S_i^*)\}) \setminus \{(i, S_i)\}$ is a spanning tree. Consider the state $S' = (S_{-i}, S_i^*)$. Now, we have $i \notin \hat{V}(S')$; so $c_i(S') = w(i, S_i^*)$. On the other hand, $i \in \hat{V}(S)$ implies $c_i(S) = w(i, S_i^*) + \frac{\Delta(S)}{|\hat{V}(S)|} > w(i, S_i^*)$ because $w(T_S) > OPT$. Hence, $c_i(S') < c_i(S)$.

In any case, $S$ is not a NE and the result is proved. $\square$

Let $\Phi(S) = (|\mathtt{dis}(S)|, |\hat{V}(S)|, \sum_{i \in \hat{V}(S)} |E_S(i)|)$ be a potential function, where $E_S(i) = \{j \in \{0, \ldots, n-1\} : w(i,j) < w(i, S_i)\}$.

LEMMA 3. *Let $S$ and $S'$ be two states which only differ on the strategy of a player $i$. If $c_i(S') < c_i(S)$ then $\Phi(S') \prec \Phi(S)$.*

PROOF. Player $i$ has taken a better move. Suppose $i \in \mathtt{dis}(S)$. We get that $i \notin \mathtt{dis}(S')$ and after $i$'s deviation, the number of disconnected players can only decrease strictly, meaning that $\Phi(S') \prec \Phi(S)$.

Now suppose that $i \in \mathtt{con}(S)$. Obviously, $\mathtt{con}(S) = \mathtt{con}(S')$. We prove that $i \in \hat{V}(S)$. Otherwise, $i \in \mathtt{con}(S) \setminus \hat{V}(S)$ and then, $c_i(S) = w(i, S_i) = w(i, S_i^*)$. Now, if $i$ changes its

strategy, then $i \in \hat{V}(S')$, and $c_i(S') = w(i, S_i^*) + \frac{\Delta(S')}{|\hat{V}(S')|} \geq w(i, S_i^*) = c_i(S)$, contradiction.

Hence, $i \in \hat{V}(S)$ and then, $c_i(S) = w(i, S_i^*) + \frac{\Delta(S)}{|\hat{V}(S)|}$. Two possibilities, either $i \in \text{con}(S') \setminus \hat{V}(S')$ or $i \in \hat{V}(S')$.

- If $i \in \text{con}(S') \setminus \hat{V}(S') = \text{con}(S) \setminus \hat{V}(S')$, then $|\hat{V}(S')| = |\hat{V}(S)| - 1$. Hence, $\Phi(S') \prec \Phi(S)$.

- If $i \in \hat{V}(S')$, then $c_i(S') = w(i, S_i^*) + \frac{\Delta(S')}{|\hat{V}(S')|}$. Observe that $\hat{V}(S') = \hat{V}(S)$. Since $c_i(S') < c_i(S)$ by hypothesis, we deduce that $\Delta(S') < \Delta(S)$. Thus, $w(i, S_i') < w(i, S_i)$ which means $|E_{S'}(i)| < |E_S(i)|$. On the other hand, $\forall j \in \hat{V}(S) \setminus \{i\}$, $E_{S'}(j) = E_S(j)$. In conclusion, $\sum_{j \in \hat{V}(S')} |E_{S'}(j)| < \sum_{j \in \hat{V}(S)} |E_S(j)|$ and $\Phi(S') \prec \Phi(S)$.

In any case, $\Phi(S') \prec \Phi(S)$. $\square$

THEOREM 2. *BRD always converges after at most $mn^2$ rounds, where $m$ is the number of edges of $G$.*

PROOF. Using Lemma 3, each state $S$ is immediately followed by another state $S'$ such that $\Phi(S') \prec \Phi(S)$. Thus BRD can not run into a cycle. Since a finite number of states exists, there is at least one minimal state for $\prec$, meaning that BRD always converge. Now, since for any state $S$, $|\text{dis}(S)| \leq n - 1$, $|\hat{V}(S)| \leq n - 1 - |\text{dis}(S)|$ and $\sum_{j \in \hat{V}(S)} |E_S(j)| \leq m_S \leq m$, then BRD converges after at most $m\frac{n^2}{2}$ rounds. $\square$

## 5. CONCLUSIONS

In this paper we have studied cost allocation protocols for connection situations in a strategic setting. We have presented a model that can be easily adapted to the model of network formation on communication graphs, and we have analyzed properties for protocols in relation the the convergence of the best reply dynamics to efficient Nash equilibria. These properties have driven our analysis to the definition of two optimal budget balanced protocols. As a consequence, the question concerning the existence of optimal protocols has been positively answered in this paper.

The method used to define our protocols might lead to the definition of many other optimal protocols, depending on the rule according to which the cost of a mcst is allocated among the players. As illustrated by numerical examples, the inherent limitations of the optimal protocols proposed in this paper is that symmetric players may be treated differently, depending on the choice of an *a priori* selected mcst. The question about the existence of optimal protocols which treat symmetric players in a more equitable manner remains open.

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