Adaptive Attitude Tracking Control of Rigid Body Systems with Unknown Inertia and Gyro-bias

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Abstract—**This note presents a new adaptive attitude tracking controller for rigid body systems, with unknown inertia and unknown gyro-bias, using inertial vector measurements. The proposed control scheme guarantees almost global asymptotic convergence of the attitude and angular velocity to their desired values. Simulation results are provided to illustrate the effectiveness of the proposed approach.**

I. Introduction

The rigid body attitude tracking problem is still relevant, despite having been extensively studied in the literature for several decades. Several solutions have been proposed in the literature in the full state measurement case (*i.e.,* attitude and angular velocity available for feedback) using different attitude representations, see for instance, [1]–[3]. Since there is no sensor that directly measures the orientation, the explicit use of the attitude in the control law calls for efficient attitude estimation algorithms (observers) that reconstruct the attitude from the measurements provided by some appropriate sensors, such as inertial measurements units (IMUs) typically including a gyroscope, an accelerometer and a magnetometer. The attitude can be determined using either static reconstruction algorithms [4] which are vulnerable to measurement noise, or dynamic attitude estimation algorithms such as Kalman-type filters [5] and nonlinear-complimentary filters [6]. The attitude tracking problem with biased angular velocity measurements has been treated in [7] assuming that the attitude is available for feedback. In [8], the attitude control problem has been addressed in the presence of unknown angular velocity bias, using IMU measurements, assuming that the rigid body inertia is known. In [9], [10], for instance, the attitude stabilization problem has been solved without attitude and angular velocity measurements and without the knowledge of the inertia matrix. The proposed control schemes rely directly on measurements in the body frame of some known inertial vectors. The extension to the case of trajectory tracking remains an open problem. In [11] the adaptive attitude tracking problem, with unknown inertia, has been addressed using the measurement in the body frame of a single (non-constant) inertial vector, assuming perfect angular velocity measurements. This observer-based controller is mainly suitable for non-stationary flights such as in satellite applications.

In [12], the attitude tracking problem using IMU measurements, with unknown angular velocity bias and unknown inertia has been addressed. Two control laws were presented; the first one considers only the case of biased angular velocity measurements, and the second one is an extension to the case of unknown inertia matrix. The second controller which considers unknown inertia and gyro-bias simultaneously, relies on the use of the attitude observer of [6] which provides attitude estimates to be used in the tracking control law. The overall certainty-equivalence-type adaptive control scheme that has been proposed (without proof) seems to rely on a conjectured separation principle.

In the present work, we aim to solve the attitude tracking problem in the case where 1) the rigid body inertia is unknown, 2) the measured angular velocity is biased with an unknown constant bias, and 3) the attitude is not directly available for measurement. To handle the three above mentioned constraints simultaneously, we derive an adaptive control scheme that relies only on biased angular velocity measurements and body-frame measurements of some known inertial vectors. The control design relies on a transformation that allows to linearly parameterize some terms in the system's vector field with respect to the unknown inertia matrix [3], [9], [13], [14]. Our approach is different from the one proposed in [12], and does not rely on the certainty-equivalence principle with separate observer/controller design, but rather relies on a direct injection of the measurements in the control law, and the stability of the interconnection observer-controller is proven as a whole. Moreover, the number of adaptations and the order of the proposed tracking controller are lower than that of [12].

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II. Background

A. Preliminaries

The quaternion set $\mathbb{Q}$ is a four-dimensional vector space over the reals, which forms a group with the quaternion multiplication denoted by "$\odot$", which is distributive, associative but not commutative. The multiplication of two quaternions $P = (p_0, p)$ and $Q = (q_0, q)$ is defined as

$$P \odot Q = (p_0 q_0 - p^T q \ , \ p_0 q + q_0 p + p \times q), \quad (1)$$

and has the quaternion $(1, \mathbf{0})$ as the identity element. Note that, for a given quaternion $Q = (q_0, q)$, one has $Q \odot Q^{-1} = Q^{-1} \odot Q = (1, \mathbf{0})$, where $Q^{-1} = \frac{(q_0, -q)}{\|Q\|^2}$.
Note that in the case where $Q = (q_0, q)$ is a unit-quaternion, the inverse is given by $Q^{-1} = (q_0, -q)$.
The unit quaternion $Q = (q_0, q)$, composed of a scalar component $q_0 \in \mathbb{R}$ and a vector component $q \in \mathbb{R}^3$, represents the orientation of the inertial frame $\mathcal{I}$ with respect to the body-attached frame $\mathcal{B}$, and are subject to the constraint $q_0^2 + q^T q = 1$. The rotation matrix, related to the unit-quaternion Q, that brings the inertial frame into the body-attached frame, can be obtained through the Rodrigues formula $R = \mathcal{R}(Q)$ with the mapping $\mathcal{R} : \mathbb{S}^3 \to SO(3)$ is defined as

$$\begin{aligned} \mathcal{R}(Q) &= I_3 + 2q_0 S(q) + 2S^2(q) \\ &= (q_0^2 - q^T q) I_3 + 2qq^T + 2q_0 S(q) \end{aligned} \quad (2)$$

where I_3 is the 3-by-3 identity matrix and $S(x)$ is the skew-symmetric matrix associated to the vector $x \in \mathbb{R}^3$ such that $S(x)V = x \wedge V$ for any vector $V \in \mathbb{R}^3$, where $\wedge$ denotes the vector cross product of $\mathbb{R}^3$. Note that $\mathcal{R}(Q) = \mathcal{R}(-Q)$ for every $Q \in \mathbb{Q}$ and $\mathcal{R}$ defines a two-sheet covering of $SO(3)$ by $\mathbb{Q}$, *i.e.,* for every $R \in SO(3)$ there exist exactly two distinct quaternions satisfying $\mathcal{R}(Q) = R$. As a consequence every vector field f defined on $\mathbb{Q}$ so that $f(-Q) = -f(Q)$ for every $Q \in \mathbb{Q}$ defines a vector field $\tilde{f}$ on $SO(3)$.

Throughout this paper, we will denote by $(0, X)$ the quaternion associated to the three-dimensional vector X. A vector $x_{\mathcal{I}}$ expressed in the inertial frame $\mathcal{I}$ can be expressed in the body frame $\mathcal{B}$ by $x_{\mathcal{B}} = R^T x_{\mathcal{I}}$ or equivalently in terms of unit-quaternion as $(0, x_{\mathcal{B}}) = Q^{-1} \odot (0, x_{\mathcal{I}}) \odot Q$, where Q is the unit-quaternion associated to R by (2).

Let us define the following mapping $vect : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$, such that for a given matrix $A \in \mathbb{R}^{n\times n}$, we associate the vector $vect(A) = [v_1, \dots, v_n]^T$, where v_i, $i = 1, \dots, n$, are the row vectors of the matrix A.

B. Equations of motion

In this work, we consider a rigid body whose rotational dynamics are governed by

$$\Sigma_R : \begin{cases} \dot{Q} = \frac{1}{2} Q \odot (0, \omega), \\ I_b \dot{\omega} = \tau - S(\omega) I_b \omega, \end{cases} \tag{3}$$

where ω is the angular velocity of the rigid body expressed in the body-attached frame $\mathcal{B}$, τ is the external torque applied to the system expressed in $\mathcal{B}$ and $I_b \in \mathbb{R}^{3\times3}$ is a symmetric positive definite constant inertia matrix (assumed to be unknown) of the rigid body with respect to $\mathcal{B}$ of the form

$$I_b = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix}.$$

III. Main Results

A. Problem statement

Let us define the desired attitude trajectory in terms of the rotation matrix $R_d(t)$ governed by the following dynamics,

$$\dot{R}_d = R_d S(\omega_d),$$

with $\omega_d(t)$ being the desired angular velocity vector.

An equivalent desired unit-quaternion $Q_d(t)$ is defined as $R_d(t) = \mathcal{R}(Q_d(t))$. Its dynamics are governed by

$$\dot{Q}_d = \frac{1}{2} Q_d \odot (0, \omega_d). \tag{4}$$

The following assumptions are used throughout the paper:

Assumption A1. The rigid body is equipped with sensors that provide measurements (in the body-attached frame) of constant and known inertial vectors $r_i \in \mathbb{R}^3$, $i = 1, \dots, n \geq 2$. At least two vectors, among the n inertial vectors, are non-collinear. The vector measurements in the body-attached frame are denoted by $b_i \in \mathbb{R}^3$, $i = 1, \dots, n$. The vectors r_i and b_i are related by $b_i = R^T r_i$.

Assumption A2. The attitude (Q or R) is unknown (*i.e.,* unavailable for feedback).

Assumption A3. The measured angular velocity is assumed to be biased, so that the relation between the actual and measured velocities is given by

$$\omega = \omega_m + \delta,$$

where δ is the unknown constant bias, ω and ω_m are the actual and the measured velocity vectors respectively.

Assumption A4. The inertia matrix I_b is assumed to be unknown.

Assumption A5. The desired angular velocity vector ω_d and its first to sixth derivatives are bounded.

Our objective is to design a control input τ guaranteeing Almost Global Asymptotic Convergence (AGAC) of the body attitude and angular velocity to their desired values, under the above assumptions. This means that there exists an equilibrium point Eq (in the appropriate state space) such that, for almost every initial condition (with respect to the Lebesgue measure in the state space), the corresponding trajectory of the closed loop system converges to Eq.

B. Linearly parameterized model for the control

Let us consider Assumptions A3 and A4 and define the following parameters

$$\begin{aligned} \theta_1 &= \delta \in \mathbb{R}^3, \\ \theta_2 &= S(\delta) I_b \delta \in \mathbb{R}^3, \\ \theta_3 &= (I_{11}, I_{22}, I_{33}, I_{23}, I_{13}, I_{12})^T \in \mathbb{R}^6, \\ \theta_4 &= vect(S(\delta) I_b - S(I_b \delta)) \in \mathbb{R}^9, \\ \Theta^T &= [\ \theta_2^T \quad \theta_3^T \quad \theta_4^T\ ] \in \mathbb{R}^{18}. \end{aligned}$$

Using the second equation of (3), we can write the following

$$I_b(\dot{\omega} - \dot{\omega}_d) = -(S(\omega_m) F_1(\omega_m) + F_1(\dot{\omega}_d))\theta_3 - F_2(\omega_m)\theta_4 - \theta_2 + \tau, \tag{5}$$

where $F_1(\omega)$ is defined as

$$F_1(\omega) = \begin{bmatrix} \omega_1 & 0 & 0 & 0 & \omega_3 & \omega_2 \\ 0 & \omega_2 & 0 & \omega_3 & 0 & \omega_1 \\ 0 & 0 & \omega_3 & \omega_2 & \omega_1 & 0 \end{bmatrix}$$

and $F_2(\omega)$ as

$$F_2(\omega) = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_1 & \omega_2 & \omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_1 & \omega_2 & \omega_3 \end{bmatrix}.$$

The model given by Equation (5) can be written in a linear parameterizations form as

$$I_b(\dot{\omega} - \dot{\omega}_d) = -G(\omega_m, \dot{\omega}_d)\Theta + \tau, \tag{6}$$

with

$$G(\omega_m, \dot{\omega}_d) = [\ I_3 \quad S(\omega_m) F_1(\omega_m) + F_1(\dot{\omega}_d) \quad F_2(\omega_m)\ ] \in \mathbb{R}^{3\times18}$$

We also assume that ω_d verifies the following additional assumption.

Assumption A6. We assume that

$$\limsup_{t\to\infty} |\det J_\delta(\omega_d, \dot{\omega}_d)| > 0, \tag{7}$$

where the 15×15 matrix-valued function $J_\delta(\omega_d, \dot{\omega}_d)$ is given by

$$J_\delta(\omega_d, \dot{\omega}_d) = \begin{bmatrix} \frac{d}{dt}\left[H_\delta(\omega_d, \dot{\omega}_d)\right] \\ \vdots \\ \frac{d^5}{dt^5}\left[H_\delta(\omega_d, \dot{\omega}_d)\right] \end{bmatrix}.$$

where $H_\delta(\omega_d, \dot{\omega}_d)$ is a 3×15 matrix-valued function of the time t given by

$$H_\delta(\omega_d, \dot{\omega}_d) = [\ S(\omega_d - \delta) F_1(\omega_d - \delta) + F_1(\dot{\omega}_d) \quad F_2(\omega_d - \delta)\ ].$$

Assumption A6 is tailored to insure the following convergence result.

Lemma 1. *Let $\delta \in \mathbb{R}^3$ and $\omega_d : [0, \infty) \to \mathbb{R}^3$ satisfying Assumption A6. Assume that there exists a measurable function $\Psi : [0, \infty) \to \mathbb{R}^{15}$ such that $\lim_{t\to\infty} J_\delta(\omega_d, \dot{\omega}_d)\Psi(t) = 0$. Then $\liminf_{t\to\infty} \Psi(t) = 0$.*

Proof: Assumption A6 implies that $\liminf_{t\to\infty} \|J_\delta^{-1}\|$ is finite and since $\Psi = J_\delta^{-1}(J_\delta \Psi)$ one immediately deduces the conclusion. ∎

Remark 1. *Assumption A6 can be seen as a persistence of excitation condition for the biased desired angular velocity $\omega_d - \delta$ together with its first six time derivatives.*

C. Control design

Define n vectors b_i^d and n vectors $\hat{b}_i$ corresponding to the desired and estimated vectors such that $b_i^d = R_d^T r_i$ and $\hat{b}_i = \hat{R}^T r_i$, for $i = 1, \ldots, n$.

According to the model given by (6), we propose the following adaptive control law

$$\tau = G(\omega_m, \dot{\omega}_d - \dot{\hat{\theta}}_1)\hat{\Theta} + z_\gamma - \alpha\bar{\omega}, \tag{8}$$

with $\bar{\omega} = \omega_m + \hat{\theta}_1 - \omega_d$ and

$$z_\gamma = \sum_{i=1}^{n} \gamma_i S(b_i^d) b_i;\; z_\rho = \sum_{i=1}^{n} \rho_i S(\hat{b}_i) b_i,$$

where $\alpha > 0$, $\gamma_i > 0$ and $\rho_i > 0$ are constant scalar gains.

The attitude estimator is given by

$$\dot{\hat{Q}} = \frac{1}{2}\hat{Q} \odot (0, \hat{\omega}), \tag{9}$$

with $\hat{\omega} = \omega_m + \hat{\theta}_1 - z_\rho$.

The adaptation scheme is given by

$$\begin{aligned} \dot{\hat{\theta}}_1 &= \Gamma_1 Proj(-(z_\gamma + z_\rho), \hat{\theta}_1, \theta_m), \\ \dot{\hat{\Theta}} &= \Gamma_2 Proj(-G(\omega_m, \dot{\omega}_d - \dot{\hat{\theta}}_1)^T \bar{\omega}, \hat{\Theta}, \Theta_m), \end{aligned} \tag{10}$$

with $||\hat{\theta}_1(0)|| \leq \theta_m$ and $||\hat{\Theta}(0)|| \leq \Theta_m$. The matrices Γ_1 and Γ_2 are real symmetric positive definite. The positive parameters θ_m and Θ_m are the upper bounds of θ_1 and Θ, *i.e.,* $||\theta_1|| \leq \theta_m$, $||\Theta|| \leq \Theta_m$. The projection operator $Proj$ is defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ as follows:

$$Proj(x, \hat{y}, y_0) = x - \frac{\eta_1 \eta_2}{4(\epsilon^2 + 2\epsilon y_0)^{n+1} y_0^2}\hat{y}, \tag{11}$$

with

$$\begin{aligned} \eta_1 &= \begin{cases} (\hat{y}^T\hat{y} - y_0^2)^{n+1} & \text{if } \hat{y}^T\hat{y} > y_0^2, \\ 0 & \text{otherwise,} \end{cases} \\ \eta_2 &= 0.5\hat{y}^T x + \sqrt{(0.5\hat{y}^T x)^2 + \mu^2}, \end{aligned}$$

where ϵ and μ are arbitrary real positive constants and n is an arbitrary positive integer. Let $\bar{y}$ be a constant vector in $B_{y_0} = \{y \in \mathbb{R}^n \mid ||y|| \leq y_0\}$, $\hat{y}(0) \in B_{y_0}$ and $\tilde{y} = \bar{y} - \hat{y}$. Consider the adaptation algorithm $\dot{\hat{y}} = Proj(x, \hat{y}, y_0)$, then the following properties hold [15] for every $t \geq 0$:

P_1) $||\hat{y}(t)|| \leq y_0 + \epsilon$,
P_2) $-\tilde{y}(t)^T Proj(-x, \hat{y}(t), y_0) \leq x^T \tilde{y}(t)$.
P_3) $Proj(x, \hat{y}, y_0) \in \mathcal{C}^n$.

It is worth pointing out that the choice of this smooth projection algorithm is motivated by some technical reasons in the proof of our theorem that will be provided later. In fact, we will require the parameters estimates to be at least six times differentiable, and hence the integer n involved in the projection mechanism has to satisfy $n \geq 6$.

D. Convergence analysis

Let us define the estimation error $\bar{R} = R\hat{R}^T$ and the tracking error $\tilde{R} = RR_d^T$ of the attitude that correspond to the unit quaternion errors $\bar{Q} = Q \odot \hat{Q}^{-1} \equiv (\bar{q}_0, \bar{q})$ and $\tilde{Q} = Q \odot Q_d^{-1} \equiv (\tilde{q}_0, \tilde{q})$ respectively. The estimation error dynamics are given by

$$\dot{\bar{Q}} = \begin{bmatrix} \dot{\bar{q}}_0 \\ \dot{\bar{q}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\bar{q}^T \hat{R}(\omega - \hat{\omega}) \\ \frac{1}{2}(\bar{q}_0 I + S(\bar{q}))\hat{R}(\omega - \hat{\omega}) \end{bmatrix}, \tag{12}$$

with $\omega - \hat{\omega} = z_\rho + \tilde{\theta}_1$.

The tracking error dynamics are given by

$$\dot{\tilde{Q}} = \begin{bmatrix} \dot{\tilde{q}}_0 \\ \dot{\tilde{q}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\tilde{q}^T R_d(\omega - \omega_d) \\ \frac{1}{2}(\tilde{q}_0 I + S(\tilde{q}))R_d(\omega - \omega_d) \end{bmatrix}, \tag{13}$$

where $\omega - \omega_d = \bar{\omega} + \tilde{\theta}_1$, with $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$.

Before stating our main results, we recall the following useful lemma given in [16] that will be used throughout the paper.

Lemma 2. *Assume that there are n vectors b_i, $i = 1, \ldots, n$ measured in the body attached frame, corresponding to n known inertial vectors r_i, $i = 1, \ldots, n$. Assume that the constant parameters γ_i and ρ_i are strictly positive and at least two vectors among the r_i vectors are non-collinear. Then, the following properties hold*

1) *The vectors z_γ and z_ρ satisfy*

$$z_\gamma \equiv \sum_{i=1}^{n} \gamma_i S(b_i^d) b_i = -2R_d^T(\tilde{q}_0 I - S(\tilde{q}))W_\gamma \tilde{q}, \tag{14}$$

$$z_\rho \equiv \sum_{i=1}^{n} \rho_i S(\hat{b}_i) b_i = -2\hat{R}^T(\bar{q}_0 I - S(\bar{q}))W_\rho \bar{q}, \tag{15}$$

where the matrices $W_\gamma = -\sum_{i=1}^{n} \gamma_i S(r_i)^2$ and $W_\rho = -\sum_{i=1}^{n} \rho_i S(r_i)^2$ are real symmetric and positive definite. If the gains γ_i, ρ_i, $i = 1, \ldots, n$, are such that W_γ and W_ρ have two by two distinct eigenvalues, the following holds true.

2) *$z_\gamma = 0$ is equivalent to $(\tilde{q}_0 = 0, \tilde{q} = v_\gamma)$ or $(\tilde{q}_0 = \pm 1, \tilde{q} = 0)$, where v_γ is a unit eigenvector of W_γ.*
3) *$z_\rho = 0$ is equivalent to $(\bar{q}_0 = 0, \bar{q} = v_\rho)$ or $(\bar{q}_0 = \pm 1, \bar{q} = 0)$, where v_ρ is a unit eigenvector of W_ρ.*

The closed loop attitude error dynamics are given by

$$\dot{\tilde{Q}} = \begin{bmatrix} \dot{\tilde{q}}_0 \\ \dot{\tilde{q}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\tilde{q}^T R_d(\bar{\omega} + \tilde{\theta}_1) \\ \frac{1}{2}(\tilde{q}_0 I + S(\tilde{q}))R_d(\bar{\omega} + \tilde{\theta}_1) \end{bmatrix}, \tag{16}$$

$$\dot{\bar{Q}} = \begin{bmatrix} \dot{\bar{q}}_0 \\ \dot{\bar{q}} \end{bmatrix} = \begin{bmatrix} \bar{q}_0 \bar{q}^T W_\rho \bar{q} - \frac{1}{2}\bar{q}^T \hat{R}\tilde{\theta}_1 \\ -(I - \bar{q}\bar{q}^T)W_\rho \bar{q} + \frac{1}{2}(\bar{q}_0 I + S(\bar{q}))\hat{R}\tilde{\theta}_1 \end{bmatrix}, \tag{17}$$

$$I_b \dot{\bar{\omega}} = -\alpha\bar{\omega} + z_\gamma - G(\omega_m, \dot{\omega}_d - \dot{\hat{\theta}}_1)\tilde{\Theta}, \tag{18}$$

$$\dot{\tilde{\theta}}_1 = -\Gamma_1 Proj(-(z_\gamma + z_\rho), \hat{\theta}_1, \theta_m), \tag{19}$$

$$\dot{\tilde{\Theta}} = -\Gamma_2 Proj(-G(\omega_m, \dot{\omega}_d - \dot{\hat{\theta}}_1)^T \bar{\omega}, \hat{\Theta}, \Theta_m)\ . \tag{20}$$

where $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$ and $\tilde{\Theta} = \Theta - \hat{\Theta}$.

Note that these dynamics are non-autonomous. Define $X = (\tilde{Q}, \bar{Q}, \bar{\omega}, \tilde{\theta}_1, \tilde{\Theta})$ in the state space $\mathcal{X} := \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{18}$. Note that $\mathcal{X}$ has dimension 30. The above dynamics can be written as

$$\dot{X} = f(X, R_d(t), \omega_d(t), \dot{\omega}_d(t)), \tag{21}$$

where f is a time-varying vector field defined on $\mathcal{X}$. Now, one can state our main result in the following theorem:

Theorem 1. *Consider the rigid body dynamics* (3) *with the adaptive control scheme* (8)-(10)*, resulting in the closed loop attitude error dynamics given by* (16)-(20)*. Then under Assumptions A1-A5 and the gains γ_i, ρ_i, $i = 1, \ldots, n$, are chosen such that W_γ and W_ρ have two by two distinct eigenvalues, all the signals of the closed loop-system are bounded and*

i) $\lim_{t\to\infty}(\tilde{\theta}_1(t), \omega(t) - \omega_d(t)) = (0, 0)$, $\lim_{t\to\infty}(\tilde{q}_0(t), \tilde{q}(t)) = ((\pm 1, 0)$ *or* $(0, v_\gamma))$ *and* $\lim_{t\to\infty}(\bar{q}_0(t), \bar{q}(t)) = ((\pm 1, 0)$ *or* $(0, v_\rho))$ *where v_γ and v_ρ are, respectively, the unit eigenvectors of W_γ and W_ρ.*
ii) *The undesired equilibria characterized by $\bar{q}_0 = 0$ and/or $\tilde{q}_0 = 0$ are unstable.*
iii) *If, in addition Assumption A6 is satisfied, then $\lim_{t\to\infty} \tilde{\Theta}(t) = 0$.*
iv) *Moreover, if the gain matrices W_γ and W_ρ satisfy the following condition*

$$4\lambda_{min}(M) - TrM - \alpha Tr I_b^{-1} > 0,\ \textit{for}\ M \in \{W_\gamma, W_\rho\}, \tag{22}$$

then, for almost any initial condition $X_0 \in \mathcal{X}$, the corresponding trajectory of (21) *converges to a point of* $\widetilde{\Omega}_1 = \{((\pm 1, 0), (\pm 1, 0), 0, 0, 0)\}$.

Proof. Let us consider the following Lyapunov function candidate

$$\begin{array}{rcl} V &=& 2\tilde{q}^T W_\gamma \tilde{q} + 2\bar{q}^T W_\rho \bar{q} + \frac{1}{2}\bar{\omega}^T I_b \bar{\omega} \\ &+& \frac{1}{2}\tilde{\theta}_1^T \Gamma_1^{-1}\tilde{\theta}_1 + \frac{1}{2}\tilde{\Theta}^T\Gamma_2^{-1}\tilde{\Theta} \end{array}. \tag{23}$$

The time derivative of (23), in view of (16), (17), (18) and Property P_2) of the projection operator, is given by

$$\dot{V} \leq -\alpha \bar{\omega}^T \bar{\omega} - z_\rho^T z_\rho. \tag{24}$$

According to (24), V is non-increasing along trajectories of the dynamical system, implying that $\bar{\omega}$, $\bar{q}$, $\tilde{q}$, $\tilde{\theta}_1$ and $\tilde{\Theta}$ are bounded and V converges to a non negative limit. One checks easily that $\ddot{V}$ is bounded for every trajectory of the system, implying that $\dot{V}$ is uniformly continuous, hence that $\dot{V} \to 0$. On the other hand, the time derivative of z_ρ and z_γ are given by

$$\begin{aligned} \dot{z}_\rho &= -S(\omega - \tilde{\theta}_1) z_\rho + \left(\sum_{i=1}^n \rho_i S(\hat{b}_i) S(b_i)\right)(z_\rho + \tilde{\theta}_1), \\ \dot{z}_\gamma &= -S(\omega_d) z_\gamma + \left(\sum_{i=1}^n \rho_i S(b_i^d) S(b_i)\right)(\omega - \omega_d), \end{aligned}$$

which are clearly bounded. Since, for any given trajectory of the closed loop system, $\ddot{V}$ is bounded, one deduces that $\bar{\omega} \to 0$ and $z_\rho \to 0$ as time tends to infinity. Using Lemma 2, one sees that $(\bar{q}_0 I - S(\bar{q}))W_\rho \bar{q} \to 0$ as time tends to infinity, which implies that either $(\bar{q}_0, \bar{q}) \to (\pm 1, 0)$ or $(\bar{q}_0, \bar{q}) \to (0, v_\rho)$ where v_ρ is a unit eigenvector of W_ρ. Since $\ddot{\bar{q}}_0$ and $\ddot{\bar{q}}$ are bounded, and in view of the previous results, it is clear that $\dot{\bar{q}}_0 \to 0$ and $\dot{\bar{q}} \to 0$. Hence, in view of (17) one can conclude that $\tilde{\theta}_1 \to 0$ then t tends to infinity. Using (19), and the fact that $z_\rho \to 0$ and $\dot{z}_\gamma$ is bounded, it can be concluded that $\dot{\tilde{\theta}}_1 \to 0$ and therefore $z_\gamma \to 0$. Using Lemma 2, one concludes that either $(\tilde{q}_0, \tilde{q}) \to (\pm 1, 0)$ or $(\tilde{q}_0, \tilde{q}) \to (0, v_\gamma)$ as t tends to infinity, where v_γ is a unit eigenvector of W_γ. We just proved item (i) of the Theorem.

Now, let us prove item (ii). From earlier proven facts, it is clear that $\lim_{t\to\infty} \tilde{\Theta}(t) = \Theta_l$ with $||\Theta_l|| \leq 2\Theta_m + \epsilon$. The undesired equilibria characterized by $\bar{q}_0 = 0$ and/or $\tilde{q}_0 = 0$ are given by $X_1 = (v_\gamma, v_\rho, 0, 0, \Theta_l)$, $X_2 = (v_\gamma, 0, 0, 0, \Theta_l)$ and $X_3 = (0, v_\rho, 0, 0, \Theta_l)$. Let us show that $X_1 = (\tilde{q} = v_\gamma, \bar{q} = v_\rho), \bar{\omega} = 0, \tilde{\theta}_1 = 0, \tilde{\Theta} = \Theta_l)$ is unstable. The proof of instability of X_2 and X_3 follow similar steps and hence omitted here.

The proof consists in showing that there exists $X_1^* = ((\tilde{q}_0^*, \tilde{q}^*), (\bar{q}_0^*, \bar{q}^*), \bar{\omega}^*, \tilde{\theta}_1^*, \tilde{\Theta}^*) \in \mathcal{X}$ arbitrarily close to X_1 such that $V(\tilde{q}^*, \bar{q}^*, \bar{\omega}^*, \tilde{\theta}_1^*, \tilde{\Theta}^*) < V(v_\gamma, v_\rho, 0, 0, \Theta_l)$. This, with the fact that V is non-increasing on $\mathcal{X}$ proves the instability of X_1.

Let us apply small rotations on the unit-quaternion $(0, v_\gamma)$ and $(0, v_\rho)$; that is

$$\begin{pmatrix} \tilde{q}_0^* \\ \tilde{q}^* \end{pmatrix} = \begin{pmatrix} 0 \\ v_\gamma \end{pmatrix} \odot \begin{pmatrix} \tilde{\eta}_0 \\ \tilde{\eta} \end{pmatrix} = \begin{pmatrix} -\tilde{\eta}^T v_\gamma \\ \tilde{\eta}_0 v_\gamma + S(v_\gamma)\tilde{\eta} \end{pmatrix}. \tag{25}$$

$$\begin{pmatrix} \bar{q}_0^* \\ \bar{q}^* \end{pmatrix} = \begin{pmatrix} 0 \\ v_\rho \end{pmatrix} \odot \begin{pmatrix} \bar{\eta}_0 \\ \bar{\eta} \end{pmatrix} = \begin{pmatrix} -\bar{\eta}^T v_\rho \\ \bar{\eta}_0 v_\rho + S(v_\rho)\bar{\eta} \end{pmatrix}. \tag{26}$$

Letting $\Delta V = V(\tilde{q}^*, \bar{q}^*, \bar{\omega}^*, \tilde{\theta}_1^*, \tilde{\Theta}^*) - V(v_\gamma, v_\rho, 0, 0, \Theta_l)$, one gets

$$\begin{array}{rcl} \Delta V &=& +\frac{1}{2}\bar{\omega}^{*T} I_b \bar{\omega}^* + \frac{1}{2}\tilde{\theta}_1^{*T}\Gamma_1^{-1}\tilde{\theta}_1^* + \frac{1}{2}\tilde{\Theta}^{*T}\Gamma_2^{-1}\tilde{\Theta}^* \\ &-& \frac{1}{2}\tilde{\Theta}_l^T \Gamma_2^{-1}\tilde{\Theta}_l + 2(\tilde{\eta}_0^2 - 1)\lambda_\gamma + 2(S(\tilde{\eta})v_\gamma)^T W_\gamma (S(\tilde{\eta})v_\gamma) \\ &+& 2(\bar{\eta}_0^2 - 1)\lambda_\rho + 2(S(\bar{\eta})v_\rho)^T W_\rho (S(\bar{\eta})v_\rho). \end{array} \tag{27}$$

where we used the fact that $W_\gamma v_\gamma = \lambda_\gamma v_\gamma$ and $W_\rho v_\rho = \lambda_\rho v_\rho$ since v_γ and v_ρ are the unit eigenvectors associated, respectively, to λ_γ and λ_ρ.

Let us pick $\tilde{\eta} = \tilde{\epsilon} v_\gamma$ and $\bar{\eta} = \bar{\epsilon} v_\rho$, with $|\tilde{\epsilon}|$ and $|\bar{\epsilon}|$ arbitrarily small. The unit-quaternion $(\tilde{\eta}_0, \tilde{\eta}) = (1 - \tilde{\epsilon}^2, \tilde{\epsilon} v_\gamma)$, corresponds to a rotation about v_γ by an angle $\theta_{\tilde{\epsilon}} = 2 \arcsin \tilde{\epsilon}$. Similarly, The unit-quaternion $(\bar{\eta}_0, \bar{\eta}) = (1 - \bar{\epsilon}^2, \bar{\epsilon} v_\rho)$, corresponds to a rotation about v_ρ by an angle $\theta_{\bar{\epsilon}} = 2 \arcsin \bar{\epsilon}$.

With this choice, (27) leads to

$$\begin{array}{rcl} \Delta V &=& +\frac{1}{2}\bar{\omega}^{*T} I_b \bar{\omega}^* + \frac{1}{2}\tilde{\theta}_1^{*T}\Gamma_1^{-1}\tilde{\theta}_1^* + \frac{1}{2}\tilde{\Theta}^{*T}\Gamma_2^{-1}\tilde{\Theta}^* \\ &-& \frac{1}{2}\tilde{\Theta}_l^T \Gamma_2^{-1}\tilde{\Theta}_l + 2(\tilde{\eta}_0^2 - 1)\lambda_\gamma + 2(\bar{\eta}_0^2 - 1)\lambda_\rho. \end{array} \tag{28}$$

It follows that $\Delta V < 0$ as long as

$$\begin{array}{l} \tilde{\epsilon}^2 = 1 - \tilde{\eta}_0^2 > \\ \frac{1}{4\lambda_\gamma}(\bar{\omega}^{*T} I_b \bar{\omega}^* + \tilde{\theta}_1^{*T}\Gamma_1^{-1}\tilde{\theta}_1^* + \tilde{\Theta}^{*T}\Gamma_2^{-1}\tilde{\Theta}^* - \tilde{\Theta}_l^T\Gamma_2^{-1}\tilde{\Theta}_l), \end{array} \tag{29}$$

or

$$\begin{array}{l} \bar{\epsilon}^2 = 1 - \bar{\eta}_0^2 > \\ \frac{1}{4\lambda_\rho}(\bar{\omega}^{*T} I_b \bar{\omega}^* + \tilde{\theta}_1^{*T}\Gamma_1^{-1}\tilde{\theta}_1^* + \tilde{\Theta}^{*T}\Gamma_2^{-1}\tilde{\Theta}^* - \tilde{\Theta}_l^T\Gamma_2^{-1}\tilde{\Theta}_l), \end{array} \tag{30}$$

Consequently, there exist $\bar{\omega}^*$, $\tilde{\theta}_1^*$, $\tilde{\epsilon}$ and $\bar{\epsilon}$ arbitrarily small in magnitude, and $\tilde{\Theta}^*$ arbitrarily close to Θ_l, such that X_1^* is arbitrarily close to X_1 and $\Delta V < 0$. Since the Lyapunov function V is shown to be non-increasing, it is clear that X_1 is unstable.

Now, under the additional assumption A6, we prove item (iii). Notice that $\tilde{\Theta}^T \Gamma_2^{-1}\tilde{\Theta}$ admits a limit as t tends to infinity and therefore, to prove the lemma it is enough to prove that $\liminf_{t\to\infty} \|\tilde{\Theta}(t)\| = 0$. Notice also that the projection algorithm is smooth and all the parametric estimation errors are bounded as well as their first to sixth time derivatives. Since $\bar{\omega} \to 0$ and $\dddot{\bar{\omega}}$ is bounded, it is clear that $\dot{\bar{\omega}} \to 0$, which in view of (18) and the fact that $z_\gamma \to 0$ implies that $G(\omega_m, \dot{\omega}_d - \dot{\hat{\theta}}_1)\tilde{\Theta}$ tends to zero at t goes to infinity. Defining

$$G_d := G(\omega_d - \theta_1, \dot{\omega}_d) = \left[\begin{array}{cc} I_3 & H_\delta(\omega_d, \dot{\omega}_d) \end{array}\right], \tag{31}$$

one can easily show that

$$G(\omega_m, \dot{\omega}_d - \dot{\hat{\theta}}_1) - G_d \to 0,$$

as t tends to infinity, since $(\omega - \omega_d) \to 0$ and $\dot{\hat{\theta}}_1 \to 0$. Consequently, $G_d\tilde{\Theta} \to 0$ as t goes to infinity. Since $\frac{d^2}{dt^2}(G_d\tilde{\Theta})$ is bounded, one deduces that $\frac{d}{dt}(G_d\tilde{\Theta})$ is uniformly continuous, and hence tends to zero as t tends to infinity. Using Property P3), and the boundedness of all signals involved in the closed-loop system, one can easily show that $\dot{\tilde{\Theta}}$ tends to zero. The latter fact combined with the convergence to zero of $\frac{d}{dt}(G_d\tilde{\Theta})$ yield that $\frac{dG_d}{dt}\tilde{\Theta}$ tends to zero as t tends to infinity. One can also show that $G_d\tilde{\Theta}$ is sufficiently differentiable, thanks to property P3) of the projection algorithm. Therefore, by an easy induction argument, it can be concluded that, for $0 \leq k \leq 5$,

$$\lim_{t\to\infty} \frac{d^{(k)}G_d}{dt^k}\tilde{\Theta} = 0.$$

Setting $\Psi(t) := (\tilde{\theta}_3^T, \tilde{\theta}_4^T)^T$, one has

$$G_d\tilde{\Theta} = \tilde{\theta}_2 + H_\delta(\omega_d, \dot{\omega}_d)\Psi(t). \tag{32}$$

Differentiating (32) and using the fact that $\dot{\tilde{\theta}}_2 \to 0$ and $\dot{\Psi} \to 0$, one concludes that $\frac{d}{dt}H_\delta(\omega_d, \dot{\omega}_d)\Psi \to 0$ as t goes to infinity. Using similar arguments, one can show that $\frac{d^k}{dt^k}H_\delta(\omega_d, \dot{\omega}_d)\Psi \to 0$ as t goes to infinity, for $1 \leq k \leq 5$. Consequently, $\lim_{t\to\infty} J_\delta(\omega_d, \dot{\omega}_d)\Psi(t) = 0$, where $J_\delta(\omega_d, \dot{\omega}_d)$ has been defined in Assumption A6. According to Lemma 1, it follows that $\lim_{t\to\infty}\Psi(t) = 0$, and consequently, in view of (32), $\lim_{t\to\infty}\tilde{\theta}_2(t) = 0$. This proves that $\liminf_{t\to\infty}\|\tilde{\Theta}(t)\| = 0$.

Right now, We have proved that

- The trajectories of (21) converge to the following subsets of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{18}$ given by

$$\begin{aligned}\widetilde{\Omega}_1 &= \{((\pm 1,0),(\pm 1,0),0,0,0)\},\\ \widetilde{\Omega}_2 &= \{((\pm 1,0),(0,v_{j\rho}),0,0,0),\ j=1,2,3\},\\ \widetilde{\Omega}_3 &= \{((0,v_{i\gamma}),(\pm 1,0),0,0,0),\ i=1,2,3\},\\ \widetilde{\Omega}_4 &= \{((0,v_{i\gamma}),(0,v_{j\rho}),0,0,0),\ i=1,2,3;\ j=1,2,3\},\end{aligned}$$

  with $v_{i\gamma}$ and $v_{j\rho}$ are unit eigenvectors of W_γ and W_ρ respectively for $1 \leq i \leq j \leq 3$.
- The equilibria of the subsets $\widetilde{\Omega}_2$, $\widetilde{\Omega}_3$ and $\widetilde{\Omega}_4$ are unstable.

Now, we will show that from almost all initial conditions, the closed loop trajectories will converge to $\widetilde{\Omega}_1$ if condition (22) is satisfied, which proves the last statement of the theorem.

Let X_{eq} be an element in some $\widetilde{\Omega}_i$ and write a trajectory as

$$x(\cdot) = X_{eq} + Z(\cdot)$$

where

$$Z = (Z_{\tilde{q}_0}, Z_{\tilde{q}}, Z_{\bar{q}_0}, Z_{\bar{q}}, Z_{\bar{\omega}}, Z_{\tilde{\theta}_1}, Z_{\tilde{\Theta}})^T.$$

First note that $\bar{\omega} = Z_{\bar{\omega}}$ and we set $z_\gamma = Z_\gamma$ and $z_\rho = Z_\rho$ with

$$Z_\gamma = -2R_d^T\left[\lambda_\gamma Z_{\tilde{q}_0} v_\gamma - S(v_\gamma) W_\gamma Z_{\tilde{q}} + Z_{\tilde{q}_0} W_\gamma Z_{\tilde{q}} - S(Z_{\tilde{q}}) W_\gamma Z_{\tilde{q}}\right],$$

if $\tilde{Q} = (0, v_\gamma)$, and

$$Z_\gamma = -2R_d^T\left[W_\gamma Z_{\tilde{q}} + \left(Z_{\tilde{q}_0} I_3 - S(Z_{\tilde{q}})\right) W_\gamma Z_{\tilde{q}}\right],$$

if $\tilde{Q} = (1,0)$ and

$$Z_\rho = \begin{array}{l} -2\hat{R}^T[\lambda_\rho (Z_{\bar{q}_0} I_3 - S(Z_{\bar{q}})) v_\rho - S(v_\rho) W_\rho Z_{\bar{q}} \\ +(Z_{\bar{q}_0} I_3 - S(Z_{\bar{q}})) W_\rho Z_{\bar{q}}], \end{array}$$

if $\bar{Q} = (0, v_\rho)$, and

$$Z_\rho = -2\hat{R}^T\left(W_\rho Z_{\bar{q}} + \left(Z_{\bar{q}_0} I_3 - S(Z_{\bar{q}})\right) W_\rho Z_{\bar{q}}\right),$$

if $\bar{Q} = (1,0)$.

If $\tilde{Q}_{eq} = (0, v_\gamma)$, the corresponding quaternion constraint yields

$$Z_{\tilde{q}_0}^2 + \| v_\gamma + Z_{\tilde{q}} \|^2 = 1, \tag{33}$$

and then

$$\left(Z_{\tilde{q}_0}^2 + \| Z_{\tilde{q}} \|^2\right) + 2v_\gamma^T Z_{\tilde{q}} = 0. \tag{34}$$

Similarly, if $\bar{Q}_{eq} = (0, v_\rho)$, one deduces from the corresponding quaternion constraint that

$$\left(Z_{\bar{q}_0}^2 + \| Z_{\bar{q}} \|^2\right) + 2v_\rho^T Z_{\bar{q}} = 0. \tag{35}$$

If $\tilde{Q}_{eq} = (1,0)$, the corresponding quaternion constraint yields

$$\left(Z_{\tilde{q}_0}^2 + \| Z_{\tilde{q}} \|^2\right) + 2Z_{\tilde{q}_0} = 0, \tag{36}$$

and similarly, if $\bar{Q}_{eq} = (1,0)$, one deduces from the corresponding quaternion constraint that

$$\left(Z_{\bar{q}_0}^2 + \| Z_{\bar{q}} \|^2\right) + 2Z_{\bar{q}_0} = 0. \tag{37}$$

We will actually prove that the points of the state space converging to the undesired equilibrium points in $\widetilde{\Omega}_2$, $\widetilde{\Omega}_3$ and $\widetilde{\Omega}_4$, form a set of measure zero.

Consider, for instance, a point in $\tilde{\Omega}_2$, let say $X_{eq} = ((1,0),(0,v_\rho),0,0,0)$ where v_ρ is a unit-length eigenvector of W_ρ. If $x \in \mathbb{R}^3$, we use $x^\perp$ to denote the vector in the two-dimensional plane $v_\rho^\perp$ given by $x^\perp = x - (v_\rho^T x) v_\rho$ and $W_\rho^\perp$ the restriction of W_ρ to $v_\rho^\perp$. Recall first that the dimension of the state space $\mathcal{X}$ is equal to 30 and, by using the equations (35) and (36), one deduces that

$$Z_{\tilde{q}_0} = -\frac{\| Z_{\tilde{q}} \|^2}{1 + \sqrt{1 - \| Z_{\tilde{q}} \|^2}}, \quad v_\rho^T Z_{\bar{q}} = -\frac{Z_{\bar{q}_0}^2 + \| Z_{\bar{q}}^\perp \|^2}{1 + \sqrt{1 - (Z_{\bar{q}_0}^2 + \| Z_{\bar{q}}^\perp \|^2)}}.$$

The reduced variable Z_{red} is given by

$$Z_{red} = (Z_{\tilde{q}}, Z_{\bar{q}_0}, Z_{\bar{q}}^\perp, Z_{\bar{\omega}}, Z_{\tilde{\theta}_1}, Z_{\tilde{\Theta}})^T,$$

belongs to a smooth manifold $\mathcal{M}_{eq}$ of dimension 30. Fix a neighborhood $\mathcal{N}$ of the origin for the reduced variable so that the projection operators are equal to the corresponding identity operators in $\mathcal{N}$. Then, as long as the corresponding trajectory lies in $\mathcal{N}$ it obeys to the following dynamics

$$\dot{Z}_{red} = A(t) Z_{red} + F(t, Z_{red}), \tag{38}$$

where $A(t)$ and $F(t, Z_{red})$ are given in the appendix.

We decomposed the error dynamics in (38) into a linear part and a super linear one, *i.e,* there exists a positive constant C_0 such that F verifies an estimate of the type

$$\| F(t, Z_{red}) \| \leq C_0 \| Z_{red} \|^2, \quad |\text{div} F(t, Z_{red})| \leq C_0 \| Z_{red} \|, \tag{39}$$

for every $(t, Z_{red}) \in \mathbb{R}_+ \times \mathcal{M}_{eq}$. Note also that the time-varying matrix $A(\cdot)$ does not depend on Z_{red} and its trace is constant and equal to

$$\text{Tr}A(t) \equiv 4\lambda_\rho - \text{Tr}W_\rho - \alpha \text{Tr} I_b^{-1} := \xi. \tag{40}$$

Since the matrix W_ρ satisfies (22), the right-hand side ξ of (40) is strictly positive. Assume now that the conclusion of the theorem does not hold true and more particularly, that there exist a measurable subset set J of $\mathcal{X}$ with positive measure such that all trajectories of (21) starting in J converge to X_{eq}. Let $J(t)$, the image of J at time t by the flow $\psi(t,0)$ of the reduced dynamics. Since $J(t)$ converges to $\{X_{eq}\}$ as t tends to infinity, one can assume, with no loss of generality, that J is chosen close enough to X_{eq} so that $J(t)$ lies in the neighborhood $\mathcal{N}$ for every $t \geq 0$ and therefore $\psi(t,0)$ is the flow associated with the time-varying equation (38). Moreover, if $m(J(t))$ denotes the measure of $J(t)$, then $m(J(t))$ must tend to zero as t tends to infinity.

On the other hand, one has for $t \geq 0$,

$$m(J(t)) = \int_{J(t)} dZ_t = \int_J |\det(t, Z)| dZ,$$

where $\det(t, Z)$ denotes the determinant of $D\psi(t, 0, Z)$, the differential of $\psi(t, 0, Z)$ with respect to the initial condition $Z \in J$. Recall that, for every $t \geq 0$, one has

$$\frac{\partial d(t,Z)}{\partial t} = \left(\text{Tr}A(t) + \text{div} F(t, \psi(t,0,Z))\right) d(t,Z), \quad d(0,Z) = 1.$$

By taking into account (39) and (40), one deduces that $\det(t,Z) \geq e^{\xi t/2}$ for t large enough, hence $m(J(t)) \geq e^{\xi t/2} m(J)$ which tends to infinity as t tends to infinity. We reached a contradiction.

For the other equilibrium points of $\widetilde{\Omega}_2$, $\widetilde{\Omega}_3$ and $\widetilde{\Omega}_4$, one proceeds similarly to show that there does not exist a measurable subset J of $\mathcal{X}$ with positive measure and such that trajectories of (21) starting in J would converge to that equilibrium point. Therefore, for almost any initial condition $X_0 \in \mathcal{X}$, the corresponding trajectory of (21) converges to a point of $\tilde{\Omega}_1$.

□

The following proposition provides a useful result that will help in the design of the parameters γ_i and ρ_i satisfying (22).

Proposition 1. *Let r_1 and r_2 be two non-collinear 3-dimensional vectors. Let $r_3 = r_1 \times r_2$ and $\theta \in (0,\pi)$ the angle between r_1 and r_2. Set $R_i = ||r_i|| > 0$ for $i = 1,2$ and $R_3 = R_1 R_2 \sin(\theta)$. If $W_\gamma = -\sum_{i=1}^3 \gamma_i S(r_i)^2$ defined in Lemma 2, then the following hold true:*

a) *The eigenvalues of W_γ are equal to $\lambda_1 = R_1^2\gamma_1 + R_2^2\gamma_2$, $\lambda_2 = R_3^2\gamma_3 + \frac{\lambda_1}{2}(1 + \sqrt{1-D})$ and $\lambda_3 = R_3^2\gamma_3 + \frac{\lambda_1}{2}(1 - \sqrt{1-D})$, with $D = \frac{4R_1^2R_2^2\gamma_1\gamma_2\sin^2(\theta)}{\lambda_1^2}$.*

b) *Set $\eta = \frac{R_2^2\gamma_2}{R_1^2\gamma_1}$. Note that $D = \frac{4\eta\sin^2(\theta)}{(1+\eta)^2}$. Then, Condition (22) is satisfied, if the parameters γ_i and α are chosen such that*

$$\eta \neq 1,\ 4R_3^2\gamma_3 = \lambda_1(1 + 3\sqrt{1-D}),\ \lambda_1 > \frac{2\alpha Tr I_b^{-1}}{1 - \sqrt{1-D}}. \tag{41}$$

Proof. First of all, notice that $||r_3|| = R_3 > 0$ and $0 < D < 1$ since r_1 and r_2 are non-collinear and $\eta \neq 1$. Set $e_1 = [1\ 0\ 0]^T$, $e_2 = [\cos(\theta)\ \sin(\theta)\ 0]^T$, $e_3 = [0\ 0\ 1]^T$ and $E = [e_1\ e_2\ e_3]$ and $R_r = [\frac{r_1}{R_1}\ \frac{r_2}{R_2}\ \frac{r_3}{R_3}]$. Then the matrix $U = ER_r^{-1}$ is clearly orthogonal. Finally, set $\mu_i = R_i^2\gamma_i$ for $i = 1,2,3$ and $W = -\sum_{i=1}^3 \mu_i S(e_i)^2$. It is immediate to see that $W = UW_\gamma U^T$, implying that W and W_γ have the same eigenvalues. Moreover, a direct computation shows that

$$W = (\textstyle\sum_{i=1}^3 \mu_i)I - \sum_{i=1}^3 \mu_i e_i e_i^T = \begin{bmatrix} \mu_3 + \mu_2\sin^2(\theta) & -\mu_2\cos(\theta)\sin(\theta) & 0 \\ -\mu_2\cos(\theta)\sin(\theta) & \mu_1 + \mu_3 + \mu_2\cos^2(\theta) & 0 \\ 0 & 0 & \mu_1 + \mu_2 \end{bmatrix}. \quad (42)$$

Straightforward calculations lead to the result of item (a). We next define γ_2 and γ_3 according to the definition of η and the second part of Eq.(41). This choice implies that $\lambda_3 < \lambda_2 < \lambda_1$. Finally, the third part of Eq.(41) imposes a choice on γ_1 and yields that Condition (22) is satisfied.

□

IV. Simulation Results

In this section, we present simulation results showing the effectiveness of the proposed adaptive attitude trajectory tracking controller. We have considered for the simulations the inertia matrix $I_b = diag\,(0.0797, 0.0797, 0.1490)$, the gyro bias $\delta = [0.2, 0.1, -0.1]^T$ and the inertial vectors $r_1 = [0,0,1]^T$ and $r_2 = [1,1,1]^T/\sqrt{3}$. The simulation sampling time is $0.01sec$. The control parameters have been chosen as $\gamma_1 = \gamma_2 = 2$, $\rho_1 = \rho_2 = 10$ and $\alpha = 2$. The adaptation gains are $\Gamma_1 = 1$ and $\Gamma_2 = 1$. The parameters of the projections are given by $n = 6$, $\epsilon = 0.1$, $\mu = 0.1$, $\theta_m = 0.3$ and $\Theta_m = 0.3$. The desired angular velocity vector ω_d has been chosen to satisfy the persistency excitation condition mentioned in assumption A6. We performed two simulation tests Test1 and Test2 to show the performance of the proposed control scheme and confirm the avoidance of the unwinding phenomenon. In the first simulation test, we considered the following initial conditions: $\omega(0) = [0,0,0]^T$, $Q_d(0) = [0.8, 0, 0.6, 0]^T$, $\hat{Q}(0) = [-0.8, 0, 0.6, 0]^T$ and $Q(0) = [1,0,0,0]^T$. In the second simulation test, we considered the same initial conditions except for Q, where we started the scalar part of the unit quaternion from a negative value, *i.e.,* $Q(0) = [-1,0,0,0]^T$. Figure 1 shows the evolution of the four components of the unit-quaternion tracking errors with respect to time for Test1 and Test2, respectively. Figure 2 shows the evolution of the unit-quaternion estimation error with respect to time for Test1 and Test2 respectively. We can clearly see that the unwinding phenomenon is avoided since both equilibria given by $\tilde{\Omega}_1$ are asymptotically stable. Figures 3 and 4 show the norm of the input signals and the angular velocity tracking error respectively. Fig. 5 shows that the parameter error $\tilde{\theta}_1$ converges to zero relatively fast, while the rest of the parameter errors $\tilde{\Theta}$ converge to zero relatively slow. This is due to the fact that only $\tilde{\Theta}$ depends on the richness of the reference signal.

V. Conclusion

A new adaptive attitude tracking control scheme, relying on inertial vector measurements, has been proposed for rigid body systems with unknown inertia and unknown angular velocity bias. Global boundedness of the system state variables and almost global asymptotic convergence of the body attitude and angular velocity to their desired values are proven. The convergence of the adaptive parameters to their true values is guaranteed under some kind of persistency of excitation condition on the reference trajectories. Compared to [12], the proposed control scheme involves fewer parameter adaptations lower order dynamics and avoids the unwinding phenomenon. The performance of the proposed controller is illustrated through some simulation results.

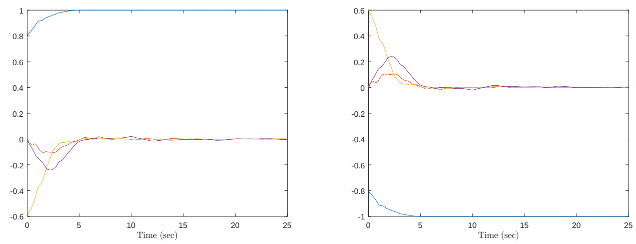

Fig. 1: Unit-quaternion tracking error $\tilde{Q}$ for Test1 and Test2

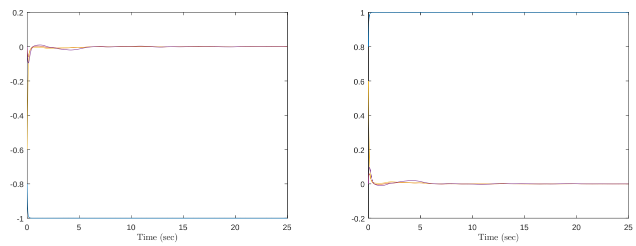

Fig. 2: Unit-quaternion estimation error $\bar{Q}$ for Test1 and Test2

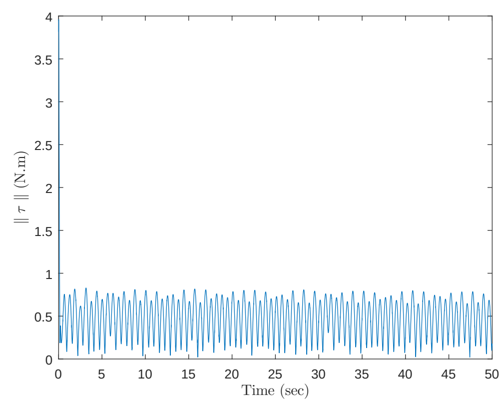

Fig. 3: Input torques $\|\ \tau\ \|$.

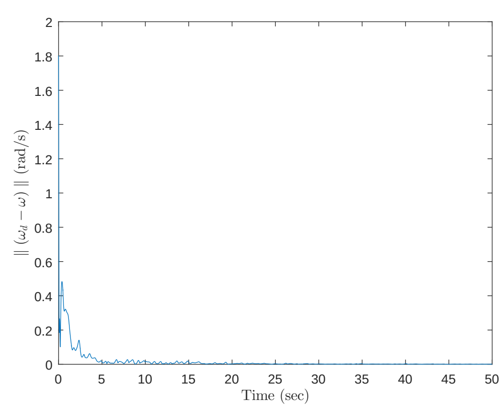

Fig. 4: Angular velocity errors $\|\ \omega - \omega_d\ \|$.

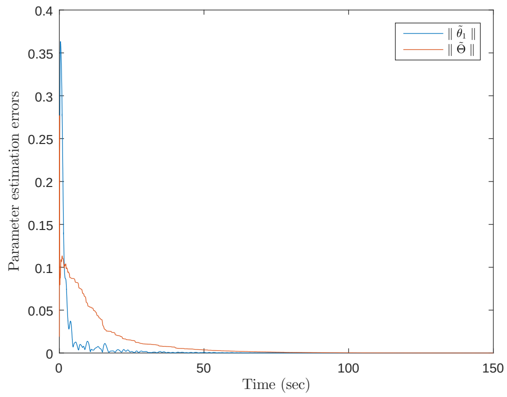

Fig. 5: Parameter estimation errors $\| \tilde{\theta}_1 \|$ in blue and $\| \tilde{\Theta} \|$ in red.

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Appendix

The expressions of $A(t)$ and $F(t, Z_{red})$ are given by

$$A(t) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2}R_d & \frac{1}{2}R_d & 0 \\ 0 & \lambda_\rho & 0 & 0 & -\frac{1}{2}v_\rho^T \hat{R} & 0 \\ 0 & 0 & \lambda_\rho I_2 - W_\rho^\perp & 0 & \frac{1}{2}S(v_\rho)\hat{R} & 0 \\ -2I_b^{-1}R_d^T W_\gamma & 0 & 0 & -\alpha I_b^{-1} & 0 & -I_b^{-1}G_d \\ -2\Gamma_1 R_d^T W_\gamma & -2\lambda_\rho \Gamma_1 \hat{R}^T v_\rho & 2\Gamma_1 \hat{R}^T S(v_\rho)(\lambda_\rho I_2 - W_\rho^\perp) & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_2 G_d^T & 0 & 0 \end{bmatrix}, \quad (43)$$

$$F(t, Z_{red}) = \begin{pmatrix} \frac{1}{2}[(Z_{\tilde{q}_0} Id_3 + S(Z_{\tilde{q}}))R_d(Z_{\bar{\omega}} + Z_{\tilde{\theta}})] \\ Z_{\bar{q}_0}(2\lambda_\rho v_\rho^T Z_{\bar{q}} + Z_{\bar{q}}^T W_\rho Z_{\bar{q}}) - \frac{1}{2}Z_{\bar{q}}^T \hat{R} Z_{\tilde{\theta}} \\ \lambda_\rho v_\rho^T Z_{\bar{q}} Z_{\bar{q}}^\perp + \frac{1}{2}[(Z_{\bar{q}_0} Id_3 + S(Z_{\bar{q}}))\hat{R} Z_{\tilde{\theta}})]^\perp \\ -2I_b^{-1}R_d^T (Z_{\tilde{q}_0} Id_3 - S(Z_{\tilde{q}}))W_\gamma Z_{\tilde{q}} - I_b^{-1}G(Z_{\bar{\omega}} + Z_{\tilde{\theta}}, \Gamma_1(Z_\gamma + Z_\rho))Z_{\tilde{\Theta}} \\ -2\Gamma_1 R_d^T [Z_{\tilde{q}_0} Id_3 - S(Z_{\tilde{q}}))W_\gamma Z_{\tilde{q}}] - 2\Gamma_1 \hat{R}^T [(Z_{\bar{q}_0} I_3 - S(Z_{\bar{q}}))\, W_\rho Z_{\bar{q}}] \\ \Gamma_2 G^T (Z_{\bar{\omega}} + Z_{\tilde{\theta}}, \Gamma_1(Z_\gamma + Z_\rho))Z_{\bar{\omega}} \end{pmatrix}. \quad (44)$$