# BSDEs with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations. Part II: Decoupled mild solutions and Examples.

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# BSDEs with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations. Part II: Decoupled mild solutions and Examples

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**Abstract.** Let $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a family of probability measures, where $E$ is a Polish space, defined on the canonical probability space $\mathrm{D}([0,T],E)$ of $E$-valued càdlàg functions. We suppose that a martingale problem with respect to a time-inhomogeneous generator $a$ is well-posed. We consider also an associated semilinear *Pseudo-PDE* for which we introduce a notion of so called *decoupled mild* solution and study the equivalence with the notion of martingale solution introduced in a companion paper. We also investigate well-posedness for decoupled mild solutions and their relations with a special class of BSDEs without driving martingale. The notion of decoupled mild solution is a good candidate to replace the notion of viscosity solution which is not always suitable when the map $a$ is not a PDE operator.

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## 1 Introduction

The framework of this paper is the canonical space $\Omega = \mathrm{D}([0,T],E)$ of càdlàg functions defined on the interval $[0,T]$ with values in a Polish space $E$. This space will be equipped with a family $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ of probability measures indexed by an initial time $s \in [0,T]$ and a starting point $x \in E$. For each

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$(s,x) \in [0,T] \times E$, $\mathrm{P}^{s,x}$ corresponds to the law of an underlying forward Markov process with time index $[0,T]$, taking values in the Polish state space $E$ which is characterized as the solution of a well-posed martingale problem related to a certain operator $(\mathcal{D}(a), a)$. In the companion paper [6] we have introduced a semilinear equation generated by $(\mathcal{D}(a), a)$, called *Pseudo-PDE* of the type

$$\left\{ \begin{array}{rcll} a(u) + f\left(\cdot, \cdot, u, \Gamma(u)^{\frac{1}{2}}\right) & = & 0 & \text{on } [0,T] \times E \\ u(T, \cdot) & = & g, & \end{array} \right. \tag{1.1}$$

where $\Gamma(u) = a(u^2) - 2ua(u)$ is a potential theory operator called the *carré du champs operator*. A classical solution of (1.1) is defined as an element of $\mathcal{D}(a)$ verifying (1.1). In [6] we have also defined the notion of *martingale solution* of (1.1), see Definition 2.22. A function $u$ is a martingale solution if (1.1) holds replacing the map $a$ (resp. $\Gamma$) with an extended operator $\mathfrak{a}$ (resp. $\mathfrak{G}$) which is introduced in Definition 2.14 (resp. 2.17). The martingale solution extends the (analytical) notion of classical solution, however it is a probabilistic concept. The objectives of the present paper are essentially three.

1. To introduce an alternative notion of (this time analytical) solution, that we call *decoupled mild*, since it makes use of the time-dependent transition kernel associated with $a$. This new type of solution will be shown to be essentially equivalent to the martingale one.

2. To show existence and uniqueness of decoupled mild solutions.

3. To emphasize the link with solutions of forward BSDEs (FBSDEs) without driving martingale introduced in [6].

The aforementioned FBSDEs are of the form

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \sqrt{\frac{d\langle M^{s,x}\rangle_r}{dr}}\right) dr - (M_T^{s,x} - M_t^{s,x}), \tag{1.2}$$

in a stochastic basis $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$ which depends on $(s,x)$. Under suitable conditions, for fixed $(s,x)$, the solution of this FBSDE is a couple $(Y^{s,x}, M^{s,x})$ of càdlàg stochastic processes where $M^{s,x}$ is a martingale. This was introduced and studied in a more general setting in [6], see [31] for a similar formulation.

We refer to the introduction and reference list of previous paper for an extensive description of contributions to non-Brownian type BSDEs. The classical forward BSDE, which is driven by a Brownian motion is of the form

$$\left\{ \begin{array}{rcl} X_t^{s,x} & = & x + \int_s^t \mu(r, X_r^{s,x}) dr + \int_s^t \sigma(r, X_r^{s,x}) dB_r \\ Y_t^{s,x} & = & g(X_T^{s,x}) + \int_t^T f\left(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}\right) dr - \int_t^T Z_r^{s,x} dB_r, \end{array} \right. \tag{1.3}$$

where $B$ is a Brownian motion. Existence and uniqueness for (1.3) was established first supposing mainly Lipschitz conditions on $f$ with respect to the third and fourth variable. $\mu$ and $\sigma$ were also assumed to be Lipschitz (with respect to $x$) and to have linear growth. In the sequel those conditions were considerably relaxed, see [35] and references therein. This is a particular case of a more general (non-Markovian) Brownian BSDE introduced in 1990 by E. Pardoux and S. Peng in [33], after an early work of J.M. Bismut in 1973 in [8].

Equation (1.3) is a probabilistic representation of a semilinear partial differential equation of parabolic type with terminal condition:

$$\left\{\begin{array}{l} \partial_t u + \frac{1}{2}\sum\limits_{i,j\leq d}(\sigma\sigma^\intercal)_{i,j}\partial^2_{x_ix_j}u + \sum\limits_{i\leq d}\mu_i\partial_{x_i}u + f(\cdot,\cdot,u,\sigma\nabla u) = 0 \quad \text{ on } [0,T[\times \mathrm{R}^d \\ u(T,\cdot) = g. \end{array}\right. \tag{1.4}$$

Given, for every $(s,x)$, a solution $(Y^{s,x}, Z^{s,x})$ of the FBSDE (1.3), under some continuity assumptions on the coefficients, see e.g. [34], it was proved that the function $u(s,x) := Y^{s,x}_s$ is a viscosity solution of (1.4), see also [36, 34, 36, 14], for related work.

We prolong this idea in a general case where the FBSDE is (1.2) with solution $(Y^{s,x}, M^{s,x})$. In that case $u(s,x) := Y^{s,x}_s$ will be the decoupled mild solution of (1.1), see Theorem 3.15; in that general context the decoupled mild solution replaces the one of viscosity, for reasons that we will explain below. One celebrated problem in the case of Brownian FBSDEs is the characterization of $Z^{s,x}$ through a deterministic function $v$. This is what we will call the *identification problem*. In general the link between $v$ and $u$ is not always analytically established, excepted when $u$ has some suitable differentiability property, see e.g. [5]: in that case $v$ is closely related to the gradient of $u$. In our case, the notion of decoupled mild solution allows to identify $(u,v)$ as the analytical solution of a deterministic problem. In the literature, the notion of mild solution of PDEs was used in finite dimension in [3], where the authors tackled diffusion operators generating symmetric Dirichlet forms and associated Markov processes thanks to the theory of Fukushima Dirichlet forms, see e.g. [20]. A partial extension to the case of non-symmetric Dirichlet forms is performed in [30]. Infinite dimensional setups were considered for example in [19] where an infinite dimensional BSDE could produce the mild solution of a PDE on a Hilbert space.

Let $B$ be a functional Banach space $(B, \|\cdot\|)$ of real Borel functions defined on $E$ and $A$ be an unbounded operator on $(B, \|\cdot\|)$. In the theory of evolution equations one often considers systems of the type

$$\left\{\begin{array}{rcl} \partial_t u + Au &=& l \text{ on } [0,T]\times\mathrm{R}^d \\ u(T,\cdot) &=& g, \end{array}\right. \tag{1.5}$$

where $l : [0,T]\times \mathrm{R}^d \rightarrow \mathrm{R}$ and $g : \mathrm{R}^d \rightarrow \mathrm{R}$ are such that $l(t,\cdot)$ and $g$ belong to $B$ for every $t \in [0,T]$. The idea of mild solutions consists to consider $A$ (when possible) as the infinitesimal generator of a semigroup of operators $(P_t)_{t\geq 0}$ on $(B, \|\cdot\|)$, in the following sense. There is $\mathcal{D}(A) \subset B$, a dense subset on which $Af = \lim\limits_{t\rightarrow 0^+}\frac{1}{t}(P_tf - f)$. In particular one may think of $(P_t)_{t\geq 0}$ as the heat kernel semi-group and $A$ as $\frac{1}{2}\Delta$ which is the infinitesimal generator of the Brownian motion. The approach of mild solutions is also very popular in the framework of stochastic PDEs see e. g. [9]. When $A$ is a local operator, one solution (in the sense of distributions, or in the sense of evaluation against test functions) to the linear evolution problem with terminal condition (1.5) is the so called *mild solution*

$$u(s,\cdot) = P_{T-s}[g] - \int_s^T P_{r-s}[l(r,\cdot)]dr. \tag{1.6}$$

If $l$ is explicitly a function of $u$ then (1.6) becomes itself an equation and a mild solution would consist in finding a fixed point of (1.6). Let us now suppose the existence of a map $S : \mathcal{D}(S) \subset B \to B$, typically $S$ being the gradient, when $(P_t)$ is the heat kernel semigroup. The natural question is what would be a natural replacement for a *mild solution* for

$$\left\{ \begin{array}{rcl} \partial_t u + Au &=& -f(s,\cdot,u,Su) \text{ on } [0,T]\times \mathrm{R}^d \\ u(T,\cdot) &=& g. \end{array} \right. \tag{1.7}$$

If the domain of $S$ is $B$, then it is not difficult to extend the notion of mild solution to this case. One novelty of our approach consists is considering the case of solutions $u : [0,T] \times \mathrm{R}^d \to \mathrm{R}$ for which $Su(t,\cdot)$ is not defined.

1. Suppose one expects a solution not to be classical, i.e. such that $u(r,\cdot)$ should not belong to the domain of $\mathcal{D}(A)$ but to be in the domain of $S$. In the case of usual PDEs, one thinks of possible solutions which are not $C^{1,2}$ but admitting a gradient, typically viscosity solutions which are differentiable in $x$. In that case the usual idea of mild solutions theory applies to equations of type (1.7). In this setup, inspired by (1.6) a mild solution of the equation is naturally defined as a solution of the integral equation

$$u(s,\cdot) = P_{T-s}[g] + \int_s^T P_{r-s}[f(r,\cdot,u(r,\cdot),Su(r,\cdot))]dr. \tag{1.8}$$

2. However, there may be reasons for which the candidate solution $u$ is such that $u(t,\cdot)$ does not even belong to $\mathcal{D}(S)$. In the case of PDEs it is often the case for viscosity solutions of PDEs which do not admit a gradient. In that case the idea is to replace (1.8) with

$$u(s,\cdot) = P_{T-s}[g] + \int_s^T P_{r-s}[f(r,\cdot,u(r,\cdot),v(r,\cdot))]dr. \tag{1.9}$$

   and to add a second equality which expresses in a *mild* form the equality $v(r,\cdot) = Su(r,\cdot)$.

We will work out previous methodology for the $Pseudo-PDE(f,g)$. In that case $S$ will be given by the mapping $u \longmapsto \Gamma(u)^{\frac{1}{2}}$. If $A = \frac{1}{2}\Delta$ for instance one would have $\Gamma(u)^{\frac{1}{2}} = \|\nabla u\|$. For pedagogical purposes, one can first consider an operator $a$ of type $\partial_t + A$ when $A$ is the generator of a Markovian (time-homogeneous) semigroup. In this case,

$$\begin{array}{rcl} \Gamma(u) & = & \partial_t(u^2) + A(u^2) - 2u\partial_t u - 2uAu \\ & = & A(u^2) - 2uAu. \end{array}$$

Equation

$$\partial_t u + Au + f\left(\cdot, \cdot, u, \Gamma(u)^{\frac{1}{2}}\right) = 0, \tag{1.10}$$

could therefore be decoupled into the system

$$\left\{ \begin{array}{l} \partial_t u + Au + f(\cdot, \cdot, u, v) = 0 \\ v^2 = \partial_t(u^2) + A(u^2) - 2u(\partial_t u + Au), \end{array} \right. \tag{1.11}$$

which furthermore can be expressed as

$$\left\{ \begin{array}{rcl} \partial_t u + Au & = & -f(\cdot, \cdot, u, v) \\ \partial_t(u^2) + A(u^2) & = & v^2 - 2uf(\cdot, \cdot, u, v). \end{array} \right. \tag{1.12}$$

Taking into account the existing notions of mild solution (1.6) (resp. (1.8)), for corresponding equations (1.5) (resp. (1.7)), one is naturally tempted to define a decoupled mild solution of (1.1) as a function $u$ for which there exist $v \geq 0$ such that

$$\left\{ \begin{array}{rcl} u(s, \cdot) & = & P_{T-s}[g] + \int_s^T P_{r-s}[f(r, \cdot, u(r, \cdot), v(r, \cdot))]dr \\ u^2(s, \cdot) & = & P_{T-s}[g^2] - \int_s^T P_{r-s}[v^2(r, \cdot) - 2u(r, \cdot)f(r, \cdot, u(r, \cdot), v(r, \cdot))]dr. \end{array} \right. \tag{1.13}$$

As we mentioned before, our approach is alternative to a possible notion of viscosity solution for the $Pseudo - PDE(f, g)$. That notion will be the object of a subsequent paper, at least in the case when the driver do not depend on the last variable. In the general case the notion of viscosity solution does not fit well because of lack of suitable comparison theorems. On the other hand, even in the recent literature (see [4]) in order to show existence of viscosity solutions specific conditions exist on the driver. In our opinion our approach of decoupled mild solutions for $Pseudo - PDE(f, g)$ constitutes an interesting novelty even in the case of semilinear parabolic PDEs.

The main contributions of the paper are essentially the following. In Section 3.1, Definition 3.4 introduces our notion of decoupled mild solution of (1.1) in the general setup. In section Section 3.2, Proposition 3.7 states that under a square integrability type condition, every martingale solution is a decoupled mild solution of (1.1). Conversely, Proposition 3.8 shows that every decoupled mild solution is a martingale solution. In Theorem 3.9 we prove existence and uniqueness of a decoupled mild solution for (1.1). In Section 3.3, we show how the unique decoupled mild solution of (1.1) can be represented via the FBSDEs (1.2). In Section 4 we develop examples of Markov processes and corresponding operators $a$ falling into our abstract setup. In Section 4.1, we work in the setup of [41], the Markov process is a diffusion with jumps and the corresponding operator is of diffusion type with an additional non-local operator. In Section 4.2

we consider Markov processes associated to pseudo-differential operators (typically the fractional Laplacian) as in [26]. In Section 4.3 we study a semilinear parabolic PDE with distributional drift, and the corresponding process is the solution an SDE with distributional drift as defined in [17]. Finally in Section 4.4 are interested with diffusions on differential manifolds and associated diffusion operators, an example being the Brownian motion in a Riemannian manifold associated to the Laplace-Beltrami operator.

# 2 Preliminaries

In this section we will recall the notations, notions and results of the companion paper [6], which will be used here.

**Notation 2.1.** *In the whole paper, concerning functional spaces we will use the following notations.*

*A topological space $E$ will always be considered as a measurable space with its Borel $\sigma$-field which shall be denoted $\mathcal{B}(E)$. Given two topological spaces, $E, F$, then $\mathcal{C}(E,F)$ (respectively $\mathcal{B}(E,F)$) will denote the set of functions from $E$ to $F$ which are continuous (respectively Borel) and if $F$ is a metric space, $\mathcal{C}_b(E,F)$ (respectively $\mathcal{B}_b(E,F)$) will denote the set of functions from $E$ to $F$ which are bounded continuous (respectively bounded Borel). For any $p \in [1,\infty]$, $d \in \mathrm{N}^*$, $(L^p(\mathrm{R}^d), \|\cdot\|_p)$ will denote the usual Lebesgue space equipped with its usual norm. On a fixed probability space $(\Omega, \mathcal{F}, \mathrm{P})$, for any $p \in \mathrm{N}^*$, $L^p(\mathrm{P})$ will denote the set of random variables (defined up to a.s equality) with finite $p$-th moment. A probability space equipped with a right-continuous filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathrm{T}}, \mathrm{P})$ (where $\mathrm{T}$ is equal to $\mathrm{R}_+$ or to $[0,T]$ for some $T \in \mathrm{R}_+^*$) will be called a* ***stochastic basis*** *and will be said to* ***fulfill the usual conditions*** *if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathrm{P}$-negligible sets. When a stochastic basis is fixed, $\mathcal{P}ro$ denotes the* ***progressive $\sigma$-field*** *on $\mathrm{T}\times\Omega$.*

*On a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathrm{T}}, \mathrm{P})$, we will use the following notations and vocabulary, concerning spaces of stochastic processes, most of them being taken or adapted from [27] or [28]. $\mathcal{M}$ will be the space of càdlàg martingales. For any $p \in [1,\infty]$ $\mathcal{H}^p$ will denote the subset of $\mathcal{M}$ of elements $M$ such that $\underset{t\in\mathrm{T}}{sup}|M_t| \in L^p(\mathrm{P})$ and in this set we identify indistinguishable ele- ments. It is a Banach space for the norm $\|M\|_{\mathcal{H}^p} = \mathrm{E}[|\underset{t\in\mathrm{T}}{sup} M_t|^p]^{\frac{1}{p}}$, and $\mathcal{H}^p_0$ will denote the Banach subspace of $\mathcal{H}^p$ containing the elements starting at zero. If $\mathrm{T} = [0,T]$ for some $T \in \mathrm{R}_+^*$, a stopping time will be considered as a random variable with values in $[0,T] \cup \{+\infty\}$. We define a* ***localizing sequence of stopping times*** *as an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that there exists $N \in \mathrm{N}$ for which $\tau_N = +\infty$. Let $Y$ be a process and $\tau$ a stopping time, we denote $Y^\tau$ the process $t \mapsto Y_{t\wedge\tau}$ which we call* ***stopped process****. If $\mathcal{C}$ is a set of processes, we define its* ***localized class*** *$\mathcal{C}_{loc}$ as the set of processes $Y$ such that there exist a localizing sequence $(\tau_n)_{n\geq 0}$ such that for every $n$, the stopped process $Y^{\tau_n}$ belongs to $\mathcal{C}$. For any $M \in \mathcal{M}_{loc}$, we denote*

*$[M]$ its* ***quadratic variation*** *and if moreover $M \in \mathcal{H}^2_{loc}$, $\langle M \rangle$ will denote its (predictable)* ***angular bracket****. $\mathcal{H}^2_0$ will be equipped with scalar product defined by $(M,N)_{\mathcal{H}^2} = \mathrm{E}[M_T N_T] = \mathrm{E}[\langle M,N\rangle_T]$ which makes it a Hilbert space. Two local martingales $M, N$ will be said to be* ***strongly orthogonal*** *if $MN$ is a local martingale starting in 0 at time 0. In $\mathcal{H}^2_{0,loc}$ this notion is equivalent to $\langle M,N\rangle = 0$.*

As in previous paper [6] we will be interested in a Markov process which is the solution of a martingale problem which we now recall below. For definitions and results concerning Markov processes, the reader may refer to Appendix A. In particular, let $E$ be a Polish space and $T \in \mathrm{R}_+$ be a finite horizon we now consider $\left(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\right)$ the canonical space which was introduced in Notation A.1, and a Markov (canonical) class measurable in time $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$, see Definitions A.5 and A.4. We will also consider the completed stochastic basis $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$, see Definition A.7.

We now recall what the notion of martingale problem associated to an operator introduced in Section 4 of [6].

**Definition 2.2.** *Given a linear algebra $\mathcal{D}(a) \subset \mathcal{B}([0,T]\times E, \mathrm{R})$, and a linear operator a mapping $\mathcal{D}(a)$ into $\mathcal{B}([0,T]\times E, \mathrm{R})$, we say that a set of probability measures $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on $(\Omega, \mathcal{F})$ solves the* **Martingale Problem associated to** *$(\mathcal{D}(a), a)$ if, for any $(s,x) \in [0,T]\times E$, $\mathrm{P}^{s,x}$ verifies*

*(a) $\mathrm{P}^{s,x}(\forall t \in [0,s], X_t = x) = 1$;*

*(b) for every $\phi \in \mathcal{D}(a)$, the process $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dr$, $t \in [s,T]$ is a càdlàg $(\mathrm{P}^{s,x}, (\mathcal{F}_t)_{t\in[s,T]})$-local martingale.*

*We say that the* **Martingale Problem is well-posed** *if for any $(s,x) \in [0,T]\times E$, $\mathrm{P}^{s,x}$ is the only probability measure satisfying the properties (a) and (b).*

As for [6], in the sequel of the paper we will assume the following.

**Hypothesis 2.3.** *The Markov canonical class $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ solves a well-posed Martingale Problem associated to some $(\mathcal{D}(a), a)$ in the sense of Definition 2.2.*

**Notation 2.4.** *For every $(s,x) \in [0,T]\times E$ and $\phi \in \mathcal{D}(a)$, the process $t \mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t,X_t) - \phi(s,x) - \int_s^t a(\phi)(r,X_r)dr\right)$ will be denoted $M[\phi]^{s,x}$.*

$M[\phi]^{s,x}$ is a càdlàg $(\mathrm{P}^{s,x}, (\mathcal{F}_t)_{t\in[0,T]})$-local martingale equal to 0 on $[0,s]$, and by Proposition A.8, it is also a $(\mathrm{P}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]})$-local martingale.

The bilinear operator below was introduced (in the case of time-homogeneous operators) by J.P. Roth in potential analysis (see Chapter III in [37]), and popularized by P.A. Meyer and others in the study of homogeneous Markov processes (see for example Exposé II: L'opérateur carré du champs in [32] or 13.46 in [27]).

**Definition 2.5.** *We introduce the bilinear operator*

$$\Gamma : \begin{array}{rcl} \mathcal{D}(a) \times \mathcal{D}(a) & \to & \mathcal{B}([0,T] \times E) \\ (\phi, \psi) & \mapsto & a(\phi\psi) - \phi a(\psi) - \psi a(\phi). \end{array} \tag{2.1}$$

*The operator $\Gamma$ is called the* ***carré du champs operator****.*

*$\Gamma(\phi, \phi)$ will more simply be denoted $\Gamma(\phi)$, and when this mapping takes positive values, $\Gamma(\phi)^{\frac{1}{2}}$ will denote its point-wise square root.*

The angular bracket of the martingales introduced in Notation 2.4 are expressed via the operator $\Gamma$. Proposition 4.8 of [6], tells the following.

**Proposition 2.6.** *For any $\phi \in \mathcal{D}(a)$ and $(s,x) \in [0,T] \times E$, $M[\phi]^{s,x}$ is in $\mathcal{H}^2_{0,loc}$. Moreover, for any $(\phi, \psi) \in \mathcal{D}(a) \times \mathcal{D}(a)$ and $(s,x) \in [0,T] \times E$ we have in $(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathrm{P}^{s,x})$ and on the interval $[s,T]$*

$$\langle M[\phi]^{s,x}, M[\psi]^{s,x} \rangle = \int_s^{\cdot} \Gamma(\phi, \psi)(r, X_r) dr. \tag{2.2}$$

We introduce the space of square integrable martingales with absolutely continuous angle bracket.

**Notation 2.7.** $\mathcal{H}^{2,abs}_0 := \{M \in \mathcal{H}^2_0 | d\langle M \rangle_t \ll dt\}$. *We will also denote $\mathcal{L}^2(dt \otimes d\mathrm{P})$ the set of (up to indistinguishability) progressively measurable processes $\phi$ such that $\mathrm{E}[\int_0^T \phi_r^2 dr] < \infty$.*

We remark $\mathcal{H}^{2,abs}_0$ corresponds in [6] (Section 3.) to $\mathcal{H}^{2,V}_0$. In this paper we have set $V_t \equiv t$. Proposition 4.11 of [6] says the following.

**Proposition 2.8.** *If Hypothesis 2.3 is verified then under any $\mathrm{P}^{s,x}$,*
*$\mathcal{H}^2_0 = \mathcal{H}^{2,abs}_0$.*

In the sequel, several functional equations will hold up to a **zero potential** set that we recall below.

**Definition 2.9.** *For any $(s,x) \in [0,T] \times E$ we define the* ***potential measure*** *$U(s,x,\cdot)$ on $\mathcal{B}([0,T] \times E)$ by $U(s,x,A) := \mathrm{E}^{s,x}\left[\int_s^T \mathbb{1}_{\{(t,X_t)\in A\}} dt\right]$.*
*A Borel set $A \in \mathcal{B}([0,T] \times E)$ will be said to be* **of zero potential** *if, for any $(s,x) \in [0,T] \times E$ we have $U(s,x,A) = 0$.*

**Notation 2.10.** *Let $p > 0$, we define*

$$\mathcal{L}^p_{s,x} := \left\{ f \in \mathcal{B}([0,T] \times E, \mathrm{R}) : \mathrm{E}^{s,x}\left[\int_s^T |f|^p(r, X_r) dr\right] < \infty \right\},$$

*on which we introduce the usual semi-norm $\|\cdot\|_{p,s,x} : f \mapsto \left(\mathrm{E}^{s,x}\left[\int_s^T |f(r,X_r)|^p dr\right]\right)^{\frac{1}{p}}$.*
*We also denote $\mathcal{L}^0_{s,x} := \left\{ f \in \mathcal{B}([0,T] \times E, \mathrm{R}) : \int_s^T |f|(r,X_r) dr < \infty \, \mathrm{P}^{s,x} \text{ a.s.} \right\}$.*

*For any $p \geq 0$, we then define an intersection of these spaces, i.e.* $\mathcal{L}^p_X := \bigcap_{(s,x)\in[0,T]\times E} \mathcal{L}^p_{s,x}$. *Finally, let $\mathcal{N}$ the linear subspace of $\mathcal{B}([0,T]\times E, \mathrm{R})$ containing all functions which are equal to 0 $U(s,x,\cdot)$ a.e. for every $(s,x)$. For any $p \in \mathrm{N}$, we define the quotient space $L^p_X := \mathcal{L}^p_X/\mathcal{N}$. If $p \geq 1$, $L^p_X$ can be equipped with the topology generated by the family of semi-norms $(\|\cdot\|_{p,s,x})_{(s,x)\in[0,T]\times E}$ which makes it into a separable locally convex topological vector space.*

The statement below was stated in Proposition 4.14 of [6].

**Proposition 2.11.** *Let $f$ and $g$ be in $\mathcal{B}([0,T]\times E, \mathrm{R})$ such that the processes $\int_s^\cdot f(r,X_r)dr$ and $\int_s^\cdot g(r,X_r)dr$ are finite $\mathrm{P}^{s,x}$ a.s. for any $(s,x)\in[0,T]\times E$. Then $f$ and $g$ are equal up a zero potential set if and only if $\int_s^\cdot f(r,X_r)dr$ and $\int_s^\cdot g(r,X_r)dr$ are indistinguishable under $\mathrm{P}^{s,x}$ for any $(s,x)\in[0,T]\times E$.*

We recall that if two functions $f,g$ differ only on a zero potential set then they represent the same element of $L^0_X$. We recall our notion of **extended generator**.

**Definition 2.12.** *We first define the* ***extended domain*** *$\mathcal{D}(\mathfrak{a})$ as the set functions $\phi\in\mathcal{B}([0,T]\times E,\mathrm{R})$ for which there exists $\psi\in\mathcal{B}([0,T]\times E,\mathrm{R})$ such that under any $\mathrm{P}^{s,x}$ the process*

$$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot \psi(r,X_r)dr\right), \tag{2.3}$$

*(which is not necessarily càdl'ag) has a càdlàg modification in $\mathcal{H}^2_0$.*

Proposition 4.16 in [6] states the following.

**Proposition 2.13.** *Let $\phi\in\mathcal{B}([0,T]\times E,\mathrm{R})$. There is at most one (up to zero potential sets) $\psi\in\mathcal{B}([0,T]\times E,\mathrm{R})$ such that under any $\mathrm{P}^{s,x}$, the process defined in (2.3) has a modification which belongs to $\mathcal{M}_{loc}$.*

*If moreover $\phi\in\mathcal{D}(a)$, then $a(\phi)=\psi$ up to zero potential sets. In this case, according to Notation 2.4, for every $(s,x)\in[0,T]\times E$, $M[\phi]^{s,x}$ is the $\mathrm{P}^{s,x}$ càdlàg modification in $\mathcal{H}^2_0$ of $\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot\psi(r,X_r)dr\right)$.*

**Definition 2.14.** *Let $\phi\in\mathcal{D}(\mathfrak{a})$ as in Definition 2.12. We denote again by $M[\phi]^{s,x}$, the unique càdlàg version of the process (2.3) in $\mathcal{H}^2_0$. Taking Proposition 2.11 into account, this will not generate any ambiguity with respect to Notation 2.4. Proposition 2.11, also permits to define without ambiguity the operator*

$$\mathfrak{a}: \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & L^0_X \\ \phi & \longmapsto & \psi. \end{array}$$

*$\mathfrak{a}$ will be called the* ***extended generator****.*

We also extend the carré du champs operator $\Gamma(\cdot,\cdot)$ to $\mathcal{D}(\mathfrak{a})\times\mathcal{D}(\mathfrak{a})$. Proposition 4.18 in [6] states the following.

**Proposition 2.15.** *Let $\phi$ and $\psi$ be in $\mathcal{D}(\mathfrak{a})$, there exists a (unique up to zero-potential sets) function in $\mathcal{B}([0,T]\times E,\mathbb{R})$ which we will denote $\mathfrak{G}(\phi,\psi)$ such that under any $\mathbb{P}^{s,x}$, $\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle = \int_s^{\cdot} \mathfrak{G}(\phi,\psi)(r,X_r)dr$ on $[s,T]$, up to indistinguishability. If moreover $\phi$ and $\psi$ belong to $\mathcal{D}(a)$, then $\Gamma(\phi,\psi)=\mathfrak{G}(\phi,\psi)$ up to zero potential sets.*

**Notation 2.16.** *$\mathfrak{G}(\phi,\phi)$ will be denoted $\mathfrak{G}(\phi)$ and if that function takes positive values, $\mathfrak{G}(\phi)^{\frac{1}{2}}$ will denotes its point-wise square root.*

**Definition 2.17.** *The bilinear operator $\mathfrak{G}:\mathcal{D}(\mathfrak{a})\times\mathcal{D}(\mathfrak{a})\longmapsto L^0_X$ will be called the **extended carré du champs operator**.*

According to Definition 2.12, we do not have necessarily $\mathcal{D}(a)\subset\mathcal{D}(\mathfrak{a})$, however we have the following.

**Corollary 2.18.** *If $\phi\in\mathcal{D}(a)$ and $\Gamma(\phi)\in\mathcal{L}^1_X$, then $\phi\in\mathcal{D}(\mathfrak{a})$ and $(a(\phi),\Gamma(\phi)) = (\mathfrak{a}(\phi),\mathfrak{G}(\phi))$ up to zero potential sets.*

We also recall Lemma 5.12 of [6].

**Lemma 2.19.** *Let $(s,x)\in[0,T]\times E$ be fixed and let $\phi,\psi$ be two measurable processes. If $\phi$ and $\psi$ are $\mathbb{P}^{s,x}$-modifications of each other, then they are equal $dt\otimes d\mathbb{P}^{s,x}$ a.e.*

We now keep in mind the Pseudo-Partial Differential Equation (in short Pseudo-PDE), with final condition, that we have introduced in [6].
Let us consider the following data.

1. A measurable final condition $g\in\mathcal{B}(E,\mathbb{R})$;
2. a measurable nonlinear function $f\in\mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.

The equation is

$$\left\{\begin{array}{rcll} a(u)+f\left(\cdot,\cdot,u,\Gamma(u)^{\frac{1}{2}}\right) &=& 0 & \text{on } [0,T]\times E\\ u(T,\cdot) &=& g. & \end{array}\right. \tag{2.4}$$

**Notation 2.20.** *Equation (2.4) will be denoted $Pseudo-PDE(f,g)$.*

**Definition 2.21.** *We will say that $u$ is a **classical solution** of $Pseudo-PDE(f,g)$ if it belongs to $\mathcal{D}(a)$ and verifies (2.4).*

**Definition 2.22.** *A function $u:[0,T]\times E\to\mathbb{R}$ will be said to be a* **martingale solution** *of $Pseudo-PDE(f,g)$ if $u\in\mathcal{D}(\mathfrak{a})$ and*

$$\left\{\begin{array}{rcl} \mathfrak{a}(u) &=& -f(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}})\\ u(T,\cdot) &=& g. \end{array}\right. \tag{2.5}$$

We now fix couple of functions $f\in\mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $g\in\mathcal{B}(E,\mathbb{R})$. Until the end of these preliminaries, we will assume the following.

**Hypothesis 2.23.**

1. $\forall (s,x) \in [0,T] \times E, \quad g(X_T) \in L^2(\mathrm{P}^{s,x})$;
2. $f(\cdot,\cdot,0,0) \in \mathcal{L}^2_X$;
3. *There exists* $K^Y > 0, K^Z > 0$ *such that for all* $t,x,y,y',z,z'$,
   $|f(t,x,y,z) - f(t,x,y',z')| \leq K^Y|y-y'| + K^Z|z-z'|$.

**Remark 2.24.** *If* $f(\cdot,\cdot,0,0)$ *and* $g$ *are bounded then properties 1. and 2. above are satisfied.*

We conclude these preliminaries by stating the theorem of existence and uniqueness of a martingale solution for $Pseudo-PDE(f,g)$. It was the object of Theorem 5.21 of [6].

**Theorem 2.25.** *Let* $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *be a Markov canonical class associated to a transition function measurable in time (see Definitions A.5 and A.4) which fulfills Hypothesis 2.3, i.e. it is a solution of a well-posed Martingale Problem associated with some* $(\mathcal{D}(a), a)$. *Moreover assume that Hypothesis 2.23 holds.*

*Then* $Pseudo-PDE(f,g)$ *has a unique martingale solution.*

We also had shown (see Proposition 5.19 in [6]) that the unique martingale solution is the only possible classical solution if there is one, as stated below.

**Proposition 2.26.** *Under the conditions of previous Theorem 2.25, a classical solution* $u$ *of* $Pseudo-PDE(f,g)$ *such that* $\Gamma(u) \in \mathcal{L}^1_X$, *is also a martingale solution.*

*Conversely, if* $u$ *is a martingale solution of* $Pseudo-PDE(f,g)$ *belonging to* $\mathcal{D}(a)$, *then* $u$ *is a classical solution of* $Pseudo-PDE(f,g)$ *up to a zero-potential set, meaning that the first equality of* (2.4) *holds up to a set of zero potential.*

# 3 Decoupled mild solutions of Pseudo-PDEs

All along this section we will consider a Markov canonical class $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to a transition function $p$ measurable in time (see Definitions A.5, A.4) verifying Hypothesis 2.3 for a certain $(\mathcal{D}(a), a)$. We are also given a couple of functions $f \in \mathcal{B}([0,T]\times E \times \mathrm{R} \times \mathrm{R}, \mathrm{R})$ and $g \in \mathcal{B}(E, \mathrm{R})$ satisfying Hypothesis 2.23.

## 3.1 Definition

As mentioned in the introduction, in this section we introduce a notion of solution of our $Pseudo-PDE(f,g)$ that we will denominate *decoupled mild*, which is a generalization of the mild solution concept for partial differential equation. We will show that such solution exists and is unique. Indeed, that function will be the one appearing in Theorem 3.13.

In what follows, we will be interested in functions $(f,g)$ which satisfy weaker conditions than Hypothesis 2.23 namely the following ones.

**Hypothesis 3.1.** *There exists $C > 0$ such that the following holds.*

1. $\forall (s,x) \in [0,T] \times E, \quad g(X_T) \in L^2(\mathrm{P}^{s,x})$;
2. $f(\cdot,\cdot,0,0) \in \mathcal{L}^2_X$;
3. $\forall (t,x,y,z): \quad |f(t,x,y,z)| \leq |f(t,x,0,0)| + C(|y| + |z|)$.

**Notation 3.2.** *Let $s,t$ in $[0,T]$ with $s \leq t$, $x \in E$ and $\phi \in \mathcal{B}(E,\mathrm{R})$, if the expectation $\mathrm{E}^{s,x}[|\phi(X_t)|]$ is finite, then $P_{s,t}[\phi](x)$ will denote $\mathrm{E}^{s,x}[\phi(X_t)]$ .*

We recall two important measurability properties.

**Remark 3.3.** *Let $\phi \in \mathcal{B}(E,\mathrm{R})$.*

- *Suppose that for any $(s,x,t)$, $\mathrm{E}^{s,x}[|\phi(X_t)|] < \infty$ then by Proposition A.11, $(s,x,t) \longmapsto P_{s,t}[\phi](x)$ is Borel.*
- *Suppose that for every $(s,x)$, $\mathrm{E}^{s,x}[\int_s^T |\phi(X_r)|dr] < \infty$. Then by Lemma A.10, $(s,x) \longmapsto \int_s^T P_{s,r}[\phi](x)dr$ is Borel.*

In our general setup, considering some operator $a$, the equation

$$a(u) + f\left(\cdot,\cdot,u,\Gamma(u)^{\frac{1}{2}}\right) = 0, \tag{3.1}$$

can be naturally decoupled into

$$\left\{ \begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ \Gamma(u) & = & v^2. \end{array} \right. \tag{3.2}$$

Since $\Gamma(u) = a(u^2) - 2ua(u)$, this system of equation will be rewritten as

$$\left\{ \begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ a(u^2) & = & v^2 - 2uf(\cdot,\cdot,u,v). \end{array} \right. \tag{3.3}$$

On the other hand our Markov process $X$ is time non-homogeneous, which leads us to the definition of a decoupled mild solution.

**Definition 3.4.** *Let $(f,g)$ be a couple verifying Hypothesis 3.1.*
*Let $u,v \in \mathcal{B}([0,T] \times E,\mathrm{R})$ be two Borel functions with $v \geq 0$.*

1. *The couple $(u,v)$ will be called* **solution of the identification problem determined by** *$(f,g)$ or simply* **solution of** *$IP(f,g)$ if $u$ and $v$ belong to $\mathcal{L}^2_X$ and if for every $(s,x) \in [0,T] \times E$,*

$$\left\{ \begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right)\right](x)dr \\ u^2(s,x) & = & P_{s,T}[g^2](x) - \int_s^T P_{s,r}\left[v^2(r,\cdot) - 2uf\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right)\right](x)dr. \end{array} \right. \tag{3.4}$$

2. *The function $u$ will be called* **decoupled mild solution** *of $Pseudo-PDE(f,g)$ if there is a function $v$ such that the couple $(u,v)$ is a solution of $IP(f,g)$.*

**Lemma 3.5.** *Let $u, v \in \mathcal{L}^2_X$, and let $f$ be a Borel function satisfying Hypothesis 3.1, then $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$ and $uf(\cdot,\cdot,u,v)$ to $\mathcal{L}^1_X$.*

*Proof.* Thanks to the growth condition on $f$ in Hypothesis 3.1, there exists a constant $C > 0$ such that for any $(s,x) \in [0,T] \times E$,

$$\begin{array}{rl} & \mathrm{E}^{s,x}\left[\int_t^T f^2(r,X_r,u(r,X_r),v(r,X_r))dr\right] \\ \leq & C\mathrm{E}^{s,x}\left[\int_t^T (f^2(r,X_r,0,0) + u^2(r,X_r) + v^2(r,X_r))dr\right] < \infty, \end{array} \tag{3.5}$$

since we have assumed that $u^2, v^2$ belong to $\mathcal{L}^1_X$, and thanks to Hypothesis 3.1. This means that $f^2(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^1_X$. Since $2|uf(\cdot,\cdot,u,v)| \leq u^2 + f^2(\cdot,\cdot,u,v)$ then $uf(\cdot,\cdot,u,v)$ also belongs to $\mathcal{L}^1_X$. ☐

**Remark 3.6.** *Consequently, under the assumptions of Lemma 3.5 all the terms in* (3.4) *make sense.*

## 3.2 Existence and uniqueness of a solution

**Proposition 3.7.** *Assume that $(f,g)$ verifies Hypothesis 3.1 and let $u \in \mathcal{L}^2_X$ be a martingale solution of $Pseudo-PDE(f,g)$. Then $(u, \mathfrak{G}(u))$ is a solution of $IP(f,g)$ and in particular, $u$ is a decoupled mild solution of $Pseudo-PDE(f,g)$.*

*Proof.* Let $u \in \mathcal{L}^2_X$ be a martingale solution of $Pseudo-PDE(f,g)$. We emphasize that, taking Definition 2.12 and Proposition 2.15 into account, $\mathfrak{G}(u)$ belongs to $\mathcal{L}^1_X$, or equivalently that $\mathfrak{G}(u)^{\frac{1}{2}}$ belongs to $\mathcal{L}^2_X$. By Lemma 3.5, it follows that $f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right) \in \mathcal{L}^2_X$ and $uf\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right) \in \mathcal{L}^1_X$.
We fix some $(s,x) \in [0,T] \times E$ and the corresponding probability $\mathrm{P}^{s,x}$. We are going to show that

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),\mathfrak{G}(u)^{\frac{1}{2}}(r,\cdot)\right)\right](x)dr \\ u^2(s,x) & = & P_{s,T}[g^2](x) - \int_s^T P_{s,r}\left[\mathfrak{G}(u)(r,\cdot) - 2uf\left(r,\cdot,u(r,\cdot),\mathfrak{G}(u)^{\frac{1}{2}}(r,\cdot)\right)\right](x)dr. \end{array}\right. \tag{3.6}$$

Combining Definitions 2.12, 2.14, 2.22, we know that on $[s,T]$, the process $u(\cdot,X_\cdot)$ has a càdlàg modification which we denote $U^{s,x}$ which is a special semimartingale with decomposition

$$U^{s,x} = u(s,x) - \int_s^{\cdot} f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dr + M[u]^{s,x}, \tag{3.7}$$

where $M[u]^{s,x} \in \mathcal{H}^2_0$. Definition 2.22 also states that $u(T,\cdot) = g$, implying that

$$u(s,x) = g(X_T) + \int_s^T f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dr - M[u]^{s,x}_T \text{ a.s.} \tag{3.8}$$

Taking the expectation, by Fubini's theorem we get

$$\begin{array}{rcl} u(s,x) & = & \mathrm{E}^{s,x}\left[g(X_T) + \int_s^T f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dr\right] \\ & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),\mathfrak{G}(u)^{\frac{1}{2}}(r,\cdot)\right)\right](x)dr. \end{array} \tag{3.9}$$

By integration by parts, we obtain

$$d(U^{s,x})^2_t = -2U^{s,x}_t f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(t,X_t)dt + 2U^{s,x}_{t^-}dM[u]^{s,x}_t + d[M[u]^{s,x}]_t, \tag{3.10}$$

so integrating from $s$ to $T$, we get

$$\begin{array}{rl} & u^2(s,x) \\ = & g^2(X_T) + 2\int_s^T U^{s,x}_r f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dr - 2\int_s^T U^{s,x}_{r^-}dM[u]^{s,x}_r - [M[u]^{s,x}]_T \\ = & g^2(X_T) + 2\int_s^T uf\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dr - 2\int_s^T U^{s,x}_{r^-}dM[u]^{s,x}_r - [M[u]^{s,x}]_T, \end{array} \tag{3.11}$$

where the latter line is a consequence of Lemma 2.19. The next step will consist in taking the expectation in equation (3.11), but before, we will check that $\int_s^\cdot U^{s,x}_{r^-}dM[u]^{s,x}_r$ is a martingale. Thanks to (3.7) and Jensen's inequality, there exists a constant $C>0$ such that

$$\sup_{t\in[s,T]}(U^{s,x}_t)^2 \leq C\left(\int_s^T f^2\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dr + \sup_{t\in[s,T]}(M[u]^{s,x}_t)^2\right). \tag{3.12}$$

Since $M[u]^{s,x}\in\mathcal{H}^2_0$ and $f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)\in\mathcal{L}^2_X$, it follows that $\sup\limits_{t\in[s,T]}(U^{s,x}_t)^2\in L^1(\mathrm{P}^{s,x})$ and Lemma 3.15 in [6] states that $\int_s^\cdot U^{s,x}_{r^-}dM[u]^{s,x}_r$ is a $\mathrm{P}^{s,x}$-martingale. Taking the expectation in (3.11), we now obtain

$$\begin{array}{rcl} u^2(s,x) & = & \mathrm{E}^{s,x}\left[g^2(X_T) + \int_s^T 2uf\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dr - [M[u]^{s,x}]_T\right] \\ & = & \mathrm{E}^{s,x}\left[g^2(X_T) + \int_s^T 2uf\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dr - \langle M[u]^{s,x}\rangle_T\right] \\ & = & \mathrm{E}^{s,x}\left[g^2(X_T)\right] - \mathrm{E}^{s,x}\left[\int_s^T\left(\mathfrak{G}(u) - 2uf\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)\right)(r,X_r)dr\right] \\ & = & P_{s,T}[g^2](x) - \int_s^T P_{s,r}\left[\mathfrak{G}(u)(r,\cdot) - 2u(r,\cdot)f\left(r,\cdot,u(r,\cdot),\mathfrak{G}(u)^{\frac{1}{2}}(r,\cdot)\right)\right](x)dr, \end{array} \tag{3.13}$$

where the third equality derives from Proposition 2.15 and the fourth from Fubini's theorem. This concludes the proof. ☐

We now show the converse result of Proposition 3.7.

**Proposition 3.8.** *Assume that $(f,g)$ verifies Hypothesis 3.1. Every decoupled mild solution of $Pseudo-PDE(f,g)$ is a also a martingale solution. Moreover, if $(u,v)$ solves $IP(f,g)$, then $v^2=\mathfrak{G}(u)$ (up to zero potential sets).*

*Proof.* Let $u$ and $v\geq 0$ be a couple of functions in $\mathcal{L}^2_X$ verifying (3.4). We first note that, the first line of (3.4) with $s=T$, gives $u(T,\cdot)=g$. We fix $(s,x)\in[0,T]\times E$ and the associated probability $\mathrm{P}^{s,x}$, and on $[s,T]$, we set $U_t := u(t,X_t)$ and $N_t := u(t,X_t) - u(s,x) + \int_s^t f(r,X_r,u(r,X_r),v(r,X_r))dr$.

Combining the first line of (3.4) applied in $(s,x)=(t,X_t)$ and the Markov property (A.3), and since $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$ (see Lemma 3.5) we get the a.s. equalities

$$
\begin{array}{rcl}
U_t & = & u(t,X_t) \\
& = & P_{t,T}[g](X_t) + \int_t^T P_{t,r}\left[f\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right)\right](X_t)dr \\
& = & \mathrm{E}^{t,X_t}\left[g(X_T) + \int_t^T f(r,X_r,u(r,X_r),v(r,X_r))dr\right] \\
& = & \mathrm{E}^{s,x}\left[g(X_T) + \int_t^T f(r,X_r,u(r,X_r),v(r,X_r))dr|\mathcal{F}_t\right],
\end{array}
\tag{3.14}
$$

from which we deduce that $N_t = \mathrm{E}^{s,x}\left[g(X_T) + \int_s^T f(r,X_r,u(r,X_r),v(r,X_r))dr|\mathcal{F}_t\right] - u(s,x)$ a.s. So $N$ is a $\mathrm{P}^{s,x}$-martingale. We can therefore consider on $[s,T]$ and under $\mathrm{P}^{s,x}$, $N^{s,x}$ the càdlàg version of $N$, and the special semi-martingale $U^{s,x} := u(s,x) - \int_s^{\cdot} f(r,X_r,u(r,X_r),v(r,X_r))dr + N^{s,x}$ which is a càdlàg version of $U$. By Jensen's inequality for both expectation and conditional expectation, we have

$$
\begin{array}{rcl}
\mathrm{E}^{s,x}[(N^{s,x})^2_t] & = & \mathrm{E}^{s,x}\left[\left(\mathrm{E}^{s,x}\left[g(X_T) + \int_s^T f(r,X_r,u(r,X_r),v(r,X_r))dr|\mathcal{F}_t\right] - u(s,x)\right)^2\right] \\
& \leq & 3u^2(s,x) + 3\mathrm{E}^{s,x}[g^2(X_T)] + 3\mathrm{E}^{s,x}\left[\int_s^T f^2(r,X_r,u(r,X_r),v(r,X_r))dr\right] \\
& < & \infty,
\end{array}
\tag{3.15}
$$

where the second term is finite because of Hypothesis 3.1, and the same also holds for the third one because $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$, see Lemma 3.5. So $N^{s,x}$ is square integrable. We have therefore shown that under any $\mathrm{P}^{s,x}$, the process $u(\cdot,X_\cdot) - u(s,x) + \int_s^{\cdot} f(r,X_r,u(r,X_r),v(r,X_r))dr$ has on $[s,T]$ a modification in $\mathcal{H}^2_0$. Definitions 2.12 and 2.14, justify that $u \in \mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(u) = -f(\cdot,\cdot,u,v)$ and that for any $(s,x) \in [0,T] \times E$, $M[u]^{s,x} = N^{s,x}$.

To conclude that $u$ is a martingale solution of $Pseudo-PDE(f,g)$, there is left to show that $\mathfrak{G}(u) = v^2$, up to zero potential sets. By Proposition 2.15, this is equivalent to show that for every $(s,x) \in [0,T] \times E$, $\langle N^{s,x}\rangle = \int_s^{\cdot} v^2(r,X_r)dr$, in the sense of $\mathrm{P}^{s,x}$-indistinguishability.

We fix again $(s,x) \in [0,T] \times E$ and the associated probability, and now set $N'_t := u^2(t,X_t) - u^2(s,x) - \int_s^t (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dr$. Combining the second line of (3.4) applied in $(s,x) = (t,X_t)$ and the Markov property (A.3), and since $v^2, uf(\cdot,\cdot,u,v)$ belong to $\mathcal{L}^1_X$ (see Lemma 3.5) we get the a.s. equalities

$$
\begin{array}{rcl}
u^2(t,X_t) & = & P_{t,T}[g^2](X_t) - \int_t^T P_{t,r}\left[(v^2(r,\cdot) - 2u(r,\cdot)f\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right))\right](X_t)dr \\
& = & \mathrm{E}^{t,X_t}\left[g^2(X_T) - \int_t^T (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dr\right] \\
& = & \mathrm{E}^{s,x}\left[g^2(X_T) - \int_t^T (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dr|\mathcal{F}_t\right],
\end{array}
\tag{3.16}
$$

from which we deduce that for any $t \in [s,T]$,

$$
N'_t = \mathrm{E}^{s,x}\left[g^2(X_T) - \int_s^T (v^2 - uf(\cdot,\cdot,u,v))(r,X_r)dr|\mathcal{F}_t\right] - u^2(s,x) \text{ a.s.}
$$

So $N'$ is a $\mathrm{P}^{s,x}$-martingale. We can therefore consider on $[s,T]$ and under $\mathrm{P}^{s,x}$, $N'^{s,x}$ the càdlàg version of $N'$.

The process $u^2(s,x) + \int_s^\cdot (v^2 - uf(\cdot,\cdot,u,v))(r,X_r)dr + N'^{s,x}$ is therefore a càdlàg special semi-martingale which is a $P^{s,x}$-version of $u^2(\cdot,X)$ on $[s,T]$. But we also had shown that $U^{s,x} = u(s,x) - \int_s^\cdot f(r,X_r,u(r,X_r),v(r,X_r))dr + N^{s,x}$, is a version of $u(\cdot,X)$, which by integration by parts implies that

$$u^2(s,x) - 2\int_s^\cdot U_r^{s,x} f(\cdot,\cdot,u,v)(r,X_r)dr + 2\int_s^\cdot U_{r^-}^{s,x} dN_r^{s,x} + [N^{s,x}],$$

is another càdlàg semi-martingale which is a $P^{s,x}$-version of $u^2(\cdot,X)$ on $[s,T]$. $\int_s^\cdot (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dr + N'^{s,x}$ is therefore indistinguishable from $-2\int_s^\cdot U_r^{s,x} f(\cdot,\cdot,u,v)(r,X_r)dr + 2\int_s^\cdot U_{r^-}^{s,x} dN_r^{s,x} + [N^{s,x}]$, which can be written $\langle N^{s,x}\rangle - 2\int_s^\cdot U_r^{s,x} f(\cdot,\cdot,u,v)(r,X_r)dr + 2\int_s^\cdot U_{r^-}^{s,x} dN_r^{s,x} + ([N^{s,x}] - \langle N^{s,x}\rangle)$ where $\langle N^{s,x}\rangle - 2\int_s^\cdot U_r^{s,x} f(\cdot,\cdot,u,v)(r,X_r)dr$ is predictable with bounded variation and $2\int_s^\cdot U_{r^-}^{s,x} dN_r^{s,x} + ([N^{s,x}] - \langle N^{s,x}\rangle)$ is a local martingale. By uniqueness of the decomposition of a special semi-martingale, we have

$$\int_s^\cdot (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dr = \langle N^{s,x}\rangle - 2\int_s^\cdot U_r^{s,x} f(\cdot,\cdot,u,v)(r,X_r)dr$$

and by Lemma 2.19,

$$\int_s^\cdot (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dr = \langle N^{s,x}\rangle - 2\int_s^\cdot uf(\cdot,\cdot,u,v)(r,X_r)dr,$$

which finally yields $\langle N^{s,x}\rangle = \int_s^\cdot v^2(r,X_r)dr$ as desired. □

We recall that $(P^{s,x})_{(s,x)\in[0,T]\times E}$ is a Markov canonical class associated to a transition function measurable in time (see Definitions A.5 and A.4) which fulfills Hypothesis 2.3, i.e. it is a solution of a well-posed Martingale Problem associated with $(\mathcal{D}(a), a)$.

**Theorem 3.9.** *Let $(f,g)$ be a couple verifying Hypothesis 2.23. Then $Pseudo-PDE(f,g)$ has a unique decoupled mild solution.*

*Proof.* This derives from Theorem 2.25 and Propositions 3.7, 3.8. □

**Corollary 3.10.** *Assume that $(f,g)$ verifies Hypothesis 2.23. A classical solution $u$ of $Pseudo-PDE(f,g)$ such that $\Gamma(u) \in \mathcal{L}^1_X$, is also a decoupled mild solution.*

*Conversely, if $u$ is a decoupled mild solution of $Pseudo-PDE(f,g)$ belonging to $\mathcal{D}(a)$, then $u$ is a classical solution of $Pseudo-PDE(f,g)$ up to a zero-potential set, meaning that the first equality of* (2.4) *holds up to a set of zero potential.*

*Proof.* The statement holds by Proposition 3.8 and Proposition 2.26. □

## 3.3 Representation of the solution via FBSDEs with no driving martingale

In the companion paper [6], the following family of FBSDEs with no driving martingale indexed by $(s,x) \in [0,T] \times E$ was introduced.

**Definition 3.11.** *Let $(s,x) \in [0,T] \times E$ and the associated stochastic basis $\left(\Omega, \mathcal{F}^{s,x}, (\mathcal{F}^{s,x}_t)_{t\in[0,T]}, \mathrm{P}^{s,x}\right)$ be fixed. A couple $(Y^{s,x}, M^{s,x}) \in \mathcal{L}^2(dt \otimes d\mathrm{P}^{s,x}) \times \mathcal{H}^2_0$, will be said to solve $FBSDE^{s,x}(f,g)$ if it verifies on $[0,T]$, in the sense of indistinguishability*

$$Y^{s,x}_{\cdot} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, Y^{s,x}_r, \sqrt{\frac{d\langle M^{s,x}\rangle_r}{dr}}\right) dr - (M^{s,x}_T - M^{s,x}_{\cdot}). \quad (3.17)$$

*If (3.17) is only satisfied on a smaller interval $[t_0, T]$, with $0 < t_0 < T$, we say that $(Y^{s,x}, M^{s,x})$ solves $FBSDE^{s,x}(f,g)$ on $[t_0,T]$.*

The following result follows from Theorem 3.22 in [6].

**Theorem 3.12.** *Assume that $(f,g)$ verifies Hypothesis 2.23. Then for any $(s,x) \in [0,T] \times E$, $FBSDE^{s,x}(f,g)$ has a unique solution.*

In the following theorem, we summarize the links between the $FBSDE^{s,x}(f,g)$ and the notion of martingale solution of $Pseudo-PDE(f,g)$. These are shown in Theorem 5.14, Remark 5.15, Theorem 5.20 and Theorem 5.21 of [6].

**Theorem 3.13.** *Assume that $(f,g)$ verifies 2.23 and let $(Y^{s,x}, M^{s,x})$ denote the (unique) solution of $FBSDE^{s,x}(f,g)$ for fixed $(s,x)$. Let $u$ be the unique martingale solution of $Pseudo-PDE(f,g)$.*

*For every $(s,x) \in [0,T] \times E$, on the interval $[s,T]$, we have the following.*

- *$Y^{s,x}$ and $u(\cdot, X_{\cdot})$ are $\mathrm{P}^{s,x}$-modifications, and equal $dt \otimes d\mathrm{P}^{s,x}$ a.e.;*
- *$M^{s,x}$ and $M[u]^{s,x}$ are $\mathrm{P}^{s,x}$-indistinguishable.*

*Moreover $u$ belongs to $\mathcal{L}^2_X$ and for any $(s,x) \in [0,T] \times E$, we have $\frac{d\langle M^{s,x}\rangle_t}{dt} = \mathfrak{G}(u)(t, X_t)$ $dt \otimes d\mathrm{P}^{s,x}$ a.e.*

**Remark 3.14.** *The martingale solution $u$ of $Pseudo-PDE$ exists and is unique by Theorem 2.25.*

We can therefore represent the unique decoupled mild solution of $Pseudo-PDE(f,g)$ via the stochastic equations $FBSDE^{s,x}(f,g)$ as follows.

**Theorem 3.15.** *Assume that $(f,g)$ verifies see Hypothesis 2.23 and let $(Y^{s,x}, M^{s,x})$ denote the (unique) solution of $FBSDE^{s,x}(f,g)$ for fixed $(s,x)$.*

*Then for any $(s,x) \in [0,T] \times E$, the random variable $Y^{s,x}_s$ is $\mathrm{P}^{s,x}$ a.s. equal to a constant (which we still denote $Y^{s,x}_s$), and the function*

$$u : (s,x) \longmapsto Y^{s,x}_s \quad (3.18)$$

*is the unique decoupled mild solution of $Pseudo-PDE(f,g)$.*

*Proof.* By Theorem 3.13, there exists a Borel function $u$ such that for every $(s,x) \in [0,T] \times E$, $Y^{s,x}_s = u(s, X_s) = u(s,x)$ $\mathrm{P}^{s,x}$ a.s. and $u$ is the unique martingale solution of $Pseudo - PDE(f,g)$. By Proposition 3.7, it is also its unique decoupled mild solution. $\square$

**Remark 3.16.** *The function $v$ such that $(u,v)$ is the unique solution of the identification problem $IP(f,g)$ also has a stochastic representation since it verifies for every $(s,x) \in [0,T] \times E$, on the interval $[s,T]$,* $\frac{d\langle M^{s,x}\rangle_t}{dt} = v^2(t, X_t)\ dt \otimes d\mathrm{P}^{s,x}$ *a.e. where $M^{s,x}$ is the martingale part of the solution of $FBSDE^{s,x}$.*

Conversely, under the weaker condition Hypothesis 3.1 if one knows the solution of $IP(f,g)$, one can (for every $(s,x)$) produce a version of a solution of $FBSDE^{s,x}(f,g)$ as follows. This is only possible with the notion of decoupled mild solution: even in the case of Brownian BSDEs the knowledge of the viscosity solution of the related PDE would (in general) not be sufficient to reconstruct the family of solutions of the BSDEs.

**Proposition 3.17.** *Assume that $(f,g)$ verifies Hypothesis 3.1. Suppose the existence of a solution $(u,v)$ to $IP(f,g)$, and let $(s,x) \in [0,T] \times E$ be fixed. Then*

$$\left( u(\cdot, X), \quad u(\cdot, X) - u(s,x) + \int_s^{\cdot} f(\cdot,\cdot,u,v)(r, X_r)dr \right) \tag{3.19}$$

*admits on $[s,T]$ a $\mathrm{P}^{s,x}$-version $(Y^{s,x}, M^{s,x})$ which solves $FBSDE^{s,x}$ on $[s,T]$.*

*Proof.* By Proposition 3.8, $u$ is a martingale solution of $Pseudo - PDE(f,g)$ and $v^2 = \mathfrak{G}(u)$. We now fix $(s,x) \in [0,T] \times E$. Combining Definitions 2.14, 2.17 and 2.22, we know that $u(T,\cdot) = g$ and that on $[s,T]$, $u(\cdot, X)$ has a $\mathrm{P}^{s,x}$-version $U^{s,x}$ with decomposition $U^{s,x} = u(s,x) - \int_s^{\cdot} f(\cdot,\cdot,u,v)(r,X_r)dr + M[u]^{s,x}$, where $M[u]^{s,x}$ is an element of $\mathcal{H}^2_0$ of angular bracket $\int_s^{\cdot} v^2(r, X_r)dr$ and is a version of $u(\cdot, X) - u(s,x) + \int_s^{\cdot} f(\cdot,\cdot,u,v)(r,X_r)dr$. By Lemma 2.19, taking into account $u(T,\cdot) = g$, the couple $(U^{s,x}, M[u]^{s,x})$ verifies on $[s,T]$, in the sense of indistinguishability

$$U^{s,x} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, U^{s,x}_r, \sqrt{\frac{d\langle M[u]^{s,x}\rangle_r}{dr}}\right) dr - (M[u]^{s,x}_T - M[u]^{s,x}_{\cdot}) \tag{3.20}$$

with $M[u]^{s,x} \in \mathcal{H}^2_0$ verifying $M[u]^{s,x}_s = 0$ (see Definition 2.14) and $U^{s,x}_s$ is deterministic so in particular is a square integrable r.v. Following a slight adaptation of the proof of Lemma 3.25 in [6] (see Remark 3.18 below), this implies that $U^{s,x} \in \mathcal{L}^2(dt \otimes d\mathrm{P}^{s,x})$ and therefore that that $(U^{s,x}, M[u]^{s,x})$ is a solution of $FBSDE^{s,x}(f,g)$ on $[s,T]$. $\square$

**Remark 3.18.** *Indeed Lemma 3.25 in [6], taking into account Notation 5.5 ibidem, can be applied rigorously only under Hypothesis 2.23 for $(f,g)$. However, the same proof easily allows an extension to our framework Hypothesis 3.1.*

# 4 Examples of applications

We now develop some examples. Some of the applications that we are interested in involve operators which only act on the space variable, and we will extend them to time-dependent functions. The reader may consult Appendix B, concerning details about such extensions. In all the items below there will be a Markov canonical class with transition function measurable in time which is solution of a well-posed Martingale Problem associated to some $(\mathcal{D}(a), a)$ as introduced in Definition 2.2. Therefore all the results of this paper will apply to all the examples below, namely Theorem 2.25, Propositions 2.26, 3.7 and 3.8, Theorem 3.9, Corollaries 3.10 and 3.10, Theorems 3.12, 3.13 and 3.15 and Proposition 3.17. In particular, Theorem 3.9 states in all the cases, under suitable Lipschitz type conditions for the driver $f$, that the corresponding Pseudo-PDE admits a unique decoupled mild solution. In all the examples $T \in \mathbb{R}_+^*$ will be fixed.

## 4.1 Markovian jump diffusions

In this subsection, the state space will be $E := \mathbb{R}^d$ for some $d \in \mathbb{N}^*$. We are given $\mu \in \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, $\alpha \in \mathcal{B}([0,T] \times \mathbb{R}^d, S_+^*(\mathbb{R}^d))$ (where $S_+^*(\mathbb{R}^d)$ is the space of symmetric strictly positive definite matrices of size $d$) and $K$ a Lévy kernel: this means that for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $K(t,x,\cdot)$ is a $\sigma$-finite measure on $\mathbb{R}^d \backslash \{0\}$, $\sup\limits_{t,x} \int \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy) < \infty$ and for every Borel set $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $(t,x) \longmapsto \int_A \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy)$ is Borel. We will consider the operator $a$ defined by

$$\partial_t \phi + \frac{1}{2} Tr(\alpha \nabla^2 \phi) + (\mu, \nabla \phi) + \int \left( \phi(\cdot, \cdot + y) - \phi(\cdot, y) - \frac{(y, \nabla \phi)}{1 + \|y\|^2} \right) K(\cdot, \cdot, dy), \tag{4.1}$$

on the domain $\mathcal{D}(a)$ which is here the linear algebra $\mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ of real continuous bounded functions on $[0,T] \times \mathbb{R}^d$ which are continuously differentiable in the first variable with bounded derivative, and twice continuously differentiable in the second variable with bounded derivatives.

Concerning martingale problems associated to parabolic PDE operators, one may consult [41]. Since we want to include integral operators, we will adopt the formalism of D.W. Stroock in [40]. Its Theorem 4.3 and the penultimate sentence of its proof states the following.

**Theorem 4.1.** *Suppose that $\mu$ is bounded, that $\alpha$ is bounded continuous and that for any $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $(t,x) \longmapsto \int_A \frac{y}{1+\|y\|^2} K(t,x,dy)$ is bounded continuous. Then, for every $(s,x)$, there exists a unique probability $\mathbb{P}^{s,x}$ on the canonical space (see Definition A.1) such that $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r) dr$ is a local martingale for any $\phi \in \mathcal{D}(a)$ and $\mathbb{P}^{s,x}(X_s = x) = 1$. Moreover $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times \mathbb{R}^d}$ defines a Markov canonical class and its transition function is measurable in time.*

The Martingale Problem associated to $(\mathcal{D}(a), a)$ in the sense of Definition 2.2 is therefore well-posed and solved by $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$. In this context, $\mathcal{D}(a)$ is an algebra and for $\phi, \psi$ in $\mathcal{D}(a)$, the carré du champs operator is given by

$$\Gamma(\phi,\psi) = \sum_{i,j\leq d} \alpha_{i,j}\partial_{x_i}\phi\partial_{x_j}\psi + \int_{\mathbb{R}^d\backslash\{0\}} (\phi(\cdot,\cdot+y)-\phi)(\psi(\cdot,\cdot+y)-\psi)K(\cdot,\cdot,dy).$$

**Proposition 4.2.** *Under the assumptions of Theorem 4.1, and if $(f, g)$ verify Hypothesis 2.23, $Pseudo-PDE(f, g)$ admits a unique decoupled mild solution in the sense of Definition 3.4.*

*Proof.* $\mathcal{D}(a)$ is an algebra. Moreover $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ is a Markov class which is measurable in time, and it solves the well-posed Martingale Problem associated to $(\mathcal{D}(a), a)$. Therefore our Theorem 3.9 applies. □

We recall that if $f$ is Lipschitz in $(y, z)$ uniformly in $(t, x)$, and $g, f(\cdot,\cdot, 0, 0)$ are bounded then $(f, g)$ satisfies Hypothesis 2.23.

## 4.2 Pseudo-Differential operators and Fractional Laplacian

This section concerns pseudo-differential operators with negative definite symbol, see [25] for an extensive description. A typical example of such operators will be the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in ]0, 2[$, see Chapter 3 in [12] for a detailed study of this operator. We will mainly use the notations and vocabulary of N. Jacob in [24], [25] and [26], some results being attributed to W. Hoh [21]. We fix $d \in \mathbb{N}^*$. $\mathcal{C}_c^\infty(\mathbb{R}^d)$ will denote the space of real functions defined on $\mathbb{R}^d$ which are infinitely continuously differentiable with compact support and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of fast decreasing real smooth functions also defined on $\mathbb{R}^d$. $\mathcal{F}u$ will denote the Fourier transform of a function $u$ whenever it is well-defined. For $u \in L^1(\mathbb{R}^d)$ we use the convention $\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-i(x,\xi)}u(x)dx$.

**Definition 4.3.** *A function $\psi \in \mathcal{C}(\mathbb{R}^d,\mathbb{R})$ will be said* ***negative definite*** *if for any $k \in \mathbb{N}$, $\xi_1, \cdots, \xi_k \in \mathbb{R}^d$, the matrix $(\psi(\xi^j) + \psi(\xi^l) - \psi(\xi^j - \xi^l))_{j,l=1,\cdots,k}$ is symmetric positive definite.*

*A function $q \in \mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ will be called a* ***continuous negative definite symbol*** *if for any $x \in \mathbb{R}^d$, $q(x, \cdot)$ is continuous negative definite*
*In this case we introduce the pseudo-differential operator $q(\cdot, D)$ defined by*

$$q(\cdot, D)(u)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i(x,\xi)}q(x,\xi)\mathcal{F}u(\xi)d\xi. \tag{4.2}$$

**Remark 4.4.** *By Theorem 4.5.7 in [24], $q(\cdot, D)$ maps the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support into itself. In particular $q(\cdot, D)$ will be defined on $\mathcal{C}_c^\infty(\mathbb{R}^d)$. However, the proof of this Theorem 4.5.7 only uses the fact* *that if $\phi \in \mathcal{C}_c^\infty(\mathrm{R}^d)$ then $\mathcal{F}\phi \in \mathcal{S}(\mathrm{R}^d)$ and this still holds for every $\phi \in \mathcal{S}(\mathrm{R}^d)$. Therefore $q(\cdot, D)$ is well-defined on $\mathcal{S}(\mathrm{R}^d)$ and maps it into $\mathcal{C}(\mathrm{R}^d, \mathrm{R})$.*

A typical example of such pseudo-differential operators is the fractional Laplacian defined for some fixed $\alpha \in ]0,2[$ on $\mathcal{S}(\mathrm{R}^d)$ by

$$(-\Delta)^{\frac{\alpha}{2}}(u)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathrm{R}^d} e^{i(x,\xi)} \|\xi\|^\alpha \mathcal{F}u(\xi) d\xi. \tag{4.3}$$

Its symbol has no dependence in $x$ and is the continuous negative definite function $\xi \mapsto \|\xi\|^\alpha$. Combining Theorem 4.5.12 and 4.6.6 in [26], one can state the following.

**Theorem 4.5.** *Let $\psi$ be a continuous negative definite function satisfying for some $r_0, c_0 > 0$: $\psi(\xi) \geq c_0\|\xi\|^{r_0}$ if $\|\xi\| \geq 1$. Let $M$ be the smallest integer strictly superior to $(\frac{d}{r_0} \vee 2) + d$. Let $q$ be a continuous negative symbol verifying, for some $c, c' > 0$ and $\gamma : \mathrm{R}^d \to \mathrm{R}_+^*$, the following items.*

- *$q(\cdot, 0) = 0$ and $\underset{x\in\mathrm{R}^d}{\sup} |q(x,\xi)| \underset{\xi\to 0}{\longrightarrow} 0$;*
- *$q$ is $\mathcal{C}^{2M+1-d}$ in the first variable and for any $\beta \in \mathrm{N}^d$ with $\|\beta\| \leq 2M + 1 - d$, $\|\partial_x^\beta q\| \leq c(1+\psi)$;*
- *$q(x,\xi) \geq \gamma(x)(1+\psi(x))$ if $x \in \mathrm{R}^d$, $\|\xi\| \geq 1$;*
- *$q(x,\xi) \leq c'(1+\|\xi\|^2)$ for every $(x,\xi)$.*

*Then the homogeneous Martingale Problem associated to $(-q(\cdot, D), \mathcal{S}(\mathrm{R}^d))$ is well-posed (see Definition B.3) and its solution $(\mathrm{P}^x)_{x\in\mathrm{R}^d}$ defines a homogeneous Markov class, see Notation B.1.*

We will now introduce the time-inhomogeneous domain which will be used to extend $\mathcal{D}(-q(\cdot, D)) = \mathcal{S}(\mathrm{R}^d)$.

**Definition 4.6.** *We will denote by $\mathcal{C}^1([0,T], \mathcal{S}(\mathrm{R}^d))$ the set of functions $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathrm{R}^d))$ such that there exists a function $\partial_t\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathrm{R}^d))$ verifying the following. For every $t_0 \in [0,T]$ we have $\frac{1}{(t-t_0)}(\phi(t) - \phi(t_0)) \underset{t\to t_0}{\overset{\mathcal{S}(\mathrm{R}^d)}{\longrightarrow}} \partial_t\phi(t_0)$.*

We recall that $\mathcal{S}(\mathrm{R}^d)$ is a topological algebra, meaning that addition, multiplication and multiplication by a scalar are continuous for its topology.

**Lemma 4.7.** *For any $\phi, \psi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathrm{R}^d))$, we have $\partial_t(\phi\psi) = \psi\partial_t\phi + \phi\partial_t\psi$.*

*Proof.* The proof is very close to the one in R. □

**Notation 4.8.** *We set $\mathcal{D}(\partial_t - q(\cdot, D)) := \mathcal{C}^1([0,T], \mathcal{S}(\mathrm{R}^d))$.*

Elements in $\mathcal{C}([0,T], \mathcal{S}(\mathrm{R}^d))$ will also be seen as functions of two variables, and since convergence in $\mathcal{S}(\mathrm{R}^d)$ implies pointwise convergence, the usual notion of partial derivative coincides with the notation $\partial_t$ introduced in Definition 4.6. Any $\phi \in \mathcal{D}(\partial_t - q(\cdot, D))$ clearly verifies

- $\forall t \in [0,T]$, $\phi(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$ and $\forall x \in \mathbb{R}^d$, $\phi(\cdot,x) \in \mathcal{C}^1([0,T],\mathbb{R})$;
- $\forall t \in [0,T]$, $\partial_t \phi(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$.

Our goal now is to show that $\mathcal{D}(\partial_t - q(\cdot,D)$ also verifies the other items needed to be included in $\mathcal{D}^{max}(\partial_t - q(\cdot,D))$ (see Notation B.5) and therefore that Corollary B.8 applies with this domain.

**Notation 4.9.** *Let $\alpha,\beta \in \mathbb{N}^d$ be multi-indices, we introduce the semi-norm*

$$\|\cdot\|_{\alpha,\beta} : \begin{array}{ccc} \mathcal{S}(\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ \phi & \longmapsto & \underset{x\in\mathbb{R}^d}{sup} |x^\alpha \partial_x^\beta \phi(x)|. \end{array} \tag{4.4}$$

$\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space whose topology is determined by the family of seminorms $\|\cdot\|_{\alpha,\beta}$. In particular those seminorms are continuous.
In what follows, $\mathcal{F}_x$ will denote the Fourier transform taken in the space variable.

**Proposition 4.10.** *Let $\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d))$. Then $\mathcal{F}_x\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d))$. Moreover if $\phi \in \mathcal{C}^1([0,T],\mathcal{S}(\mathbb{R}^d))$, then $\mathcal{F}_x\phi \in \mathcal{C}^1([0,T],\mathcal{S}(\mathbb{R}^d))$ and*
*$\partial_t\mathcal{F}_x\phi = \mathcal{F}_x\partial_t\phi$.*

*Proof.* $\mathcal{F}_x : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$ is continuous, so $\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d)$ implies $\mathcal{F}_x\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d))$. If $\phi \in \mathcal{C}^1([0,T],\mathcal{S}(\mathbb{R}^d))$ then $\partial_t\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d)$ so $\mathcal{F}_x\partial_t\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d)$. Then for any $t_0 \in [0,T]$, the convergence
$\frac{1}{t-t_0}(\phi(t,\cdot)-\phi(t_0,\cdot)) \underset{t\to t_0}{\overset{\mathcal{S}(\mathbb{R}^d)}{\longrightarrow}} \partial_t\phi(t_0,\cdot)$ is preserved by the continuous mapping $\mathcal{F}_x$ meaning that (by linearity)
$\frac{1}{t-t_0}(\mathcal{F}_x\phi(t,\cdot) - \mathcal{F}_x\phi(t_0,\cdot)) \underset{t\to t_0}{\overset{\mathcal{S}(\mathbb{R}^d)}{\longrightarrow}} \mathcal{F}_x\partial_t\phi(t_0,\cdot)$. Since $\mathcal{F}_x\partial_t\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d))$, we have shown that $\mathcal{F}_x\phi \in \mathcal{C}^1([0,T],\mathcal{S}(\mathbb{R}^d))$ and $\partial_t\mathcal{F}_x\phi = \mathcal{F}_x\partial_t\phi$. ☐

**Proposition 4.11.** *If $\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d))$, then for any $\alpha,\beta \in \mathbb{N}^d$,*
*$(t,x) \longmapsto x^\alpha\partial_x^\beta\phi(t,x)$ is bounded.*

*Proof.* Let $\alpha,\beta$ be fixed. Since the maps $\|\cdot\|_{\alpha,\beta} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ are continuous, for every $\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d))$, the application $t \mapsto \|\phi(t,\cdot)\|_{\alpha,\beta}$ is continuous on the compact interval $[0,T]$ and therefore bounded, which yields the result.
☐

**Proposition 4.12.** *If $\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathbb{R}^d))$ and $\alpha,\beta \in \mathbb{N}^d$, then there exist non-negative functions $\psi_{\alpha,\beta} \in L^1(\mathbb{R}^d)$ such that for every $(t,x) \in [0,T]\times\mathbb{R}^d$, $|x^\alpha\partial_x^\beta\phi(t,x)| \le \psi_{\alpha,\beta}(x)$.*

*Proof.* We decompose

$$\begin{array}{rcl} |x^\alpha\partial_x^\beta\phi(t,x)| & = & |x^\alpha\partial_x^\beta\phi(t,x)|\mathbb{1}_{[-1,1]^d}(x) + |x^{\alpha+(2,\cdots,2)}\partial_x^\beta\phi(t,x)|\frac{1}{\underset{i\le d}{\Pi} x_i^2}\mathbb{1}_{\mathbb{R}^d\setminus[-1,1]^d}(x) \\ & \le & C(\mathbb{1}_{[-1,1]^d}(x) + \frac{1}{\underset{i\le d}{\Pi} x_i^2}\mathbb{1}_{\mathbb{R}^d\setminus[-1,1]^d}(x)), \end{array} \tag{4.5}$$

where $C$ is some constant which exists thanks to Proposition 4.11. ☐

**Proposition 4.13.** *Let $q$ be a continuous negative definite symbol verifying the assumptions of Theorem 4.5 and let $\phi \in \mathcal{C}^1([0,T],\mathcal{S}(\mathrm{R}^d))$. Then for any $x \in \mathrm{R}^d$, $t \mapsto q(\cdot,D)\phi(t,x) \in \mathcal{C}^1([0,T],\mathrm{R})$ and $\partial_t q(\cdot,D)\phi = q(\cdot,D)\partial_t\phi$.*

*Proof.* We fix $\phi \in \mathcal{C}^1([0,T],\mathcal{S}(\mathrm{R}^d))$, and $x \in \mathrm{R}^d$. We wish to show that for any $\xi \in \mathrm{R}^d$, $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathrm{R}^d} e^{i(x,\xi)}q(x,\xi)\mathcal{F}_x\phi(t,\xi)d\xi$ is $\mathcal{C}^1$ with derivative $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathrm{R}^d} e^{i(x,\xi)}q(x,\xi)\mathcal{F}_x\partial_t\phi(t,\xi)d\xi$. Since $\phi \in \mathcal{C}^1([0,T],\mathcal{S}(\mathrm{R}^d))$, then $\partial_t\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathrm{R}^d))$ and by Proposition 4.10, $\mathcal{F}_x\partial_t\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathrm{R}^d))$. Moreover since $q$ verifies the assumptions of Theorem 4.5, then $|q(x,\xi)|$ is bounded by $c'(1+\|\xi\|^2)$ for some constant $c'$. Therefore by Proposition 4.12, there exists a non-negative $\psi \in L^1(\mathrm{R}^d)$ such that for every $t,\xi$, $|q(x,\xi)\mathcal{F}_x\partial_t\phi(t,\xi)| \leq \psi(\xi)$. Since by Proposition 4.10, $\mathcal{F}_x\partial_t\phi = \partial_t\mathcal{F}_x\phi$, this implies that for any $(t,\xi)$, $|\partial_t e^{i(x,\xi)}q(x,\xi)\mathcal{F}_x\phi(t,\xi)| \leq \psi(\xi)$. So by the theorem about the differentiation of integrals depending on a parameter, for any $\xi \in \mathrm{R}^d$, $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathrm{R}^d} e^{i(x,\xi)}q(x,\xi)\mathcal{F}_x\phi(t,\xi)d\xi$ is of class $\mathcal{C}^1$ with derivative $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathrm{R}^d} e^{i(x,\xi)}q(x,\xi)\mathcal{F}_x\partial_t\phi(t,\xi)d\xi$.

□

**Proposition 4.14.** *Let $q$ be a continuous negative definite symbol verifying the assumptions of Theorem 4.5 and let $\phi \in \mathcal{C}^1([0,T],\mathcal{S}(\mathrm{R}^d))$. Then $\phi$, $\partial_t\phi$, $q(\cdot,D)\phi$ and $q(\cdot,D)\partial_t\phi$ are bounded.*

*Proof.* Proposition 4.11 implies that any element of $\mathcal{C}([0,T],\mathcal{S}(\mathrm{R}^d))$ is bounded, so we immediately deduce that $\phi$ and $\partial_t\phi$ are bounded.
Since $q$ verifies the assumptions of Theorem 4.5, for any fixed $(t,x) \in [0,T]\times\mathrm{R}^d$, we have

$$\begin{array}{rcl} |q(\cdot,D)\phi(t,x)| & = & \left|\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathrm{R}^d} e^{i(x,\xi)}q(x,\xi)\mathcal{F}_x\phi(t,\xi)d\xi\right| \\ & \leq & C\int_{\mathrm{R}^d}(1+\|\xi\|^2)|\mathcal{F}_x\phi(t,\xi)|d\xi, \end{array} \tag{4.6}$$

for some constant $C$. Since $\phi \in \mathcal{C}([0,T],\mathcal{S}(\mathrm{R}^d))$ then, by Proposition 4.10, $\mathcal{F}_x\phi$ also belongs to $\mathcal{C}([0,T],\mathcal{S}(\mathrm{R}^d))$, and by Proposition 4.11, there exists a positive $\psi \in L^1(\mathrm{R}^d)$ such that for any $(t,\xi)$, $(1+\|\xi\|^2)|\mathcal{F}_x\phi(t,\xi)| \leq \psi(\xi)$, so for any $(t,x)$, $|q(\cdot,D)\phi(t,x)| \leq \|\psi\|_1$.

Similar arguments hold replacing $\phi$ with $\partial_t\phi$ since it also belongs to $\mathcal{C}([0,T],\mathcal{S}(\mathrm{R}^d))$.
□

**Remark 4.15.** $\mathcal{C}^1([0,T],\mathcal{S}(\mathrm{R}^d))$ *seems to be a domain which is particularly appropriate for time-dependent Fourier analysis and it fits well for our framework. On the other hand it is not so fundamental to require such regularity for classical solutions for Pseudo-PDEs, so that we could consider a larger domain. For example the Fréchet algebra* $\mathcal{S}(\mathrm{R}^d)$ *could be replaced with the Banach algebra* $W^{d+3,1}(\mathrm{R}^d)\bigcap W^{d+3,\infty}(\mathrm{R}^d)$ *in all the previous proofs.*

**Corollary 4.16.** *Let $q$ be a continuous negative definite symbol verifying the hypotheses of Theorem 4.5. Then the properties below are valid. $\mathcal{D}(\partial_t - q(\cdot, D))$ is a linear algebra included in $\mathcal{D}^{max}(\partial_t - q(\cdot, D))$ as defined in Notation B.5.*

*Proof.* We recall that, according to Notation 4.8 $\mathcal{D}(\partial_t - q(\cdot, D)) = \mathcal{C}^1([0,T], \mathcal{S}(\mathrm{R}^d))$. The proof follows from Lemma 4.7, Propositions 4.13 and 4.14, and the comments under Notation 4.8. □

**Corollary 4.17.** *Let $q$ be a continuous negative definite symbol verifying the hypotheses of Theorem 4.5, let $(\mathrm{P}^x)_{x\in\mathrm{R}^d}$ be the corresponding homogeneous Markov class exhibited in Theorem 4.5, let $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times\mathrm{R}^d}$ be the corresponding Markov class (see Notation B.1), let $(\mathcal{D}(\partial_t - q(\cdot, D)), \partial_t - q(\cdot, D))$ be as in Notation 4.8. Then*

- *$(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times\mathrm{R}^d}$ solves the well-posed Martingale Problem associated to $(\mathcal{D}(\partial_t - q(\cdot, D)), \partial_t - q(\cdot, D))$;*
- *its transition function is measurable in time.*

*Proof.* The first statement directly comes from Theorem 4.5 and Corollaries 4.16 B.8, and the second from Proposition B.2. □

**Remark 4.18.** *The symbol of the fractional Laplacian $q : (x, \xi) \mapsto \|\xi\|^\alpha$ trivially verifies the assumptions of Theorem 4.5. Indeed, it has no dependence in $x$, so it is enough to set $\psi : \xi \mapsto \|\xi\|^\alpha$, $c_0 = c = c' = 1$, $r_0 = \alpha$ and $\gamma = \frac{1}{2}$.*

The Pseudo-PDE that we focus on is the following.

$$\left\{\begin{array}{rl} \partial_t u - q(\cdot, D)u = f(\cdot, \cdot, u, \Gamma(u)^{\frac{1}{2}}) & \text{on } [0,T] \times \mathrm{R}^d \\ & u(T, \cdot) = g, \end{array}\right. \tag{4.7}$$

where $q$ is a continuous negative definite symbol verifying the assumptions of Theorem 4.5 and $\Gamma$ is the associated carré du champs operator, see Definition 2.5.

**Remark 4.19.** *By Proposition 3.3 in [12], for any $\alpha \in ]0, 2[$, there exists a constant $c_\alpha$ such that for any $\phi \in \mathcal{S}(\mathrm{R}^d)$,*

$$(-\Delta)^{\frac{\alpha}{2}}\phi = c_\alpha PV \int_{\mathrm{R}^d} \frac{(\phi(\cdot + y) - \phi)}{\|y\|^{d+\alpha}} dy, \tag{4.8}$$

*where $PV$ is a notation for principal value, see (3.1) in [12]. Therefore in the particular case of the fractional Laplace operator, the carré du champs operator associated to $(-\Delta)^{\frac{\alpha}{2}}$ is given by*

$$\begin{array}{rl} & \Gamma_\alpha(\phi) \\ = & c_\alpha PV \int_{\mathrm{R}^d} \frac{(\phi^2(\cdot, \cdot + y) - \phi^2)}{\|y\|^{d+\alpha}} dy - 2\phi c_\alpha PV \int_{\mathrm{R}^d} \frac{(\phi(\cdot, \cdot + y) - \phi)}{\|y\|^{d+\alpha}} dy \\ = & c_\alpha PV \int_{\mathrm{R}^d} \frac{(\phi(\cdot, \cdot + y) - \phi)^2}{\|y\|^{d+\alpha}} dy. \end{array} \tag{4.9}$$

**Proposition 4.20.** *Let $q$ be a continuous negative symbol verifying the assumptions of Theorem 4.5, let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be the Markov class which by Corollary 4.17 solves the well-posed Martingale Problem associated to $(\mathcal{D}(\partial_t - q(\cdot,D)), \partial_t - q(\cdot,D))$.*

*For any $(f,g)$ verifying 2.23, $Pseudo-PDE(f,g)$ admits a unique decoupled mild solution in the sense of Definition 3.4.*

*Proof.* The assertion comes from Corollary 4.17 and Theorem 3.9. $\square$

### 4.3 Parabolic semi-linear PDEs with distributional drift

In this section we will use the formalism and results obtained in [17] and [18], see also [38], [10] for more recent developments. In particular the latter paper treats interesting applications to polymers. Those papers introduced a suitable framework of Martingale Problem related to a PDE operator containing a distributional drift $b'$ which is the derivative of a continuous function. [16] established a first work in the $n$-dimensional setting.

Let $b,\sigma \in \mathcal{C}^0(\mathbb{R})$ such that $\sigma > 0$. By mollifier, we intend a function $\Phi \in \mathcal{S}(\mathbb{R})$ with $\int \Phi(x)dx = 1$. We denote $\Phi_n(x) = n\Phi(nx)$, $\sigma_n^2 = \sigma^2 * \Phi_n$, $b_n = b*\Phi_n$. We then define $L_n g = \frac{\sigma_n^2}{2}g'' + b_n' g'$. $f \in \mathcal{C}^1(\mathbb{R})$ is said to be a solution to $Lf = \dot{l}$ where $\dot{l} \in \mathcal{C}^0$, if for any mollifier $\Phi$, there are sequences $(f_n)$ in $\mathcal{C}^2$, $(\dot{l}_n)$ in $\mathcal{C}^0$ such that $L_n f_n = (\dot{l}_n)$, $f_n \xrightarrow{\mathcal{C}^1} f$, $\dot{l}_n \xrightarrow{\mathcal{C}^0} \dot{l}$. We will assume that $\Sigma(x) = \lim\limits_{n\to\infty} 2\int_0^x \frac{b_n'}{\sigma_n^2}(y)dy$ exists in $\mathcal{C}^0$ independently from the mollifier.

By Proposition 2.3 in [17] there exists a solution $h \in \mathcal{C}^1$ to $Lh = 0$, $h(0) = 0$, $h'(0) = 1$. Moreover it verifies $h' = e^{-\Sigma}$. Moreover by Remark 2.4 in [17], for any $\dot{l} \in \mathcal{C}^0$, $x_0, x_1 \in \mathbb{R}$, there exists a unique solution of

$$Lf(x) = \dot{l},\ f \in \mathcal{C}^1,\ f(0) = x_0,\ f'(0) = x_1. \tag{4.10}$$

$\mathcal{D}_L$ is defined as the set of $f \in \mathcal{C}^1$ such that there exists some $\dot{l} \in \mathcal{C}^0$ with $Lf = \dot{l}$. And by Lemma 2.9 in [17] it is equal to the set of $f \in \mathcal{C}^1$ such that $\frac{f'}{h'} \in \mathcal{C}^1$. So it is clearly an algebra. $h$ is strictly increasing, $I$ will denote its image. Let $L^0$ be the classical differential operator defined by $L^0\phi = \frac{\sigma_0^2}{2}\phi''$, where

$$\sigma_0(y) = \begin{cases} (\sigma h')(h^{-1}(y)) & : \ y \in I \\ 0 & : \ y \in I^c. \end{cases} \tag{4.11}$$

Let $v$ be the unique solution to $Lv = 1$, $v(0) = v'(0) = 0$, we will assume that

$$v(-\infty) = v(+\infty) = +\infty, \tag{4.12}$$

which represents a non-explosion condition. In this case, Proposition 3.13 in [17] states that the Martingale Problem associated to $(\mathcal{D}_L, L)$ is well-posed. Its solution will be denoted $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$. By Proposition 2.13, $\mathcal{D}_{L^0} = C^2(I)$. and by Proposition 3.2 in [17], the Martingale Problem associated to $(\mathcal{D}_{L^0}, L^0)$

is also well-posed, we will call $(\mathrm{Q}^{s,x})_{(s,x)\in[0,T]\times\mathrm{R}^d}$ its solution. Moreover under any $\mathrm{P}^{s,x}$ the canonical process is a Dirichlet process, and $h^{-1}(X)$ is a semimartingale that we call $Y$ solving the SDE $Y_t = h(x) + \int_s^t \sigma_0(Y_s)dW_s$ in law, where the law of $Y$ is $\mathrm{Q}^{s,x}$. $X_t$ is a $\mathrm{P}^{s,x}$-Dirichlet process whose martingale component is $\int_s^\cdot \sigma(X_r)dW_r$. $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times\mathrm{R}^d}$ and $(\mathrm{Q}^{s,x})_{(s,x)\in[0,T]\times\mathrm{R}^d}$ both define Markov classes.

We introduce now the domain that we will indeed use.

**Definition 4.21.** *We set*

$$\mathcal{D}(a) = \left\{\phi \in \mathcal{C}^{1,1}([0,T]\times\mathrm{R}) : \frac{\partial_x\phi}{h'} \in \mathcal{C}^{1,1}([0,T]\times\mathrm{R})\right\}, \tag{4.13}$$

*which clearly is a linear algebra. On $\mathcal{D}(a)$, we set $L\phi := \frac{\sigma^2 h'}{2}\partial_x(\frac{\partial_x\phi}{h'})$ and $a(\phi) := \partial_t\phi + L\phi$.*

**Proposition 4.22.** *Let $\Gamma$ denote the carré du champ operator associated to $a$, let $\phi,\psi$ be in $\mathcal{D}(a)$, then $\Gamma(\phi,\psi) = \sigma^2\partial_x\phi\partial_x\psi$.*

*Proof.* We fix $\phi,\psi$ in $\mathcal{D}(a)$. We write

$$\begin{array}{rcl} \Gamma(\phi,\psi) & = & (\partial_t + L)(\phi\psi) - \phi(\partial_t + L)(\psi) - \psi(\partial_t + L)(\phi) \\ & = & \frac{\sigma^2 h'}{2}\left(\partial_x\left(\frac{\partial_x\phi\psi}{h'}\right) - \phi\partial_x\left(\frac{\partial_x\psi}{h'}\right) - \psi\partial_x\left(\frac{\partial_x\phi}{h'}\right)\right) \\ & = & \sigma^2\partial_x\phi\partial_x\psi. \end{array} \tag{4.14}$$

□

Emphasizing that $b'$ is a distribution, the equation that we will study in this section is therefore given by

$$\left\{\begin{array}{l} \partial_t u + \frac{1}{2}\sigma^2\partial_x^2 u + b'\partial_x u + f(\cdot,\cdot,u,\sigma|\partial_x u|) = 0 \quad \text{ on } [0,T]\times\mathrm{R} \\ u(T,\cdot) = g. \end{array}\right. \tag{4.15}$$

**Proposition 4.23.** $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times\mathrm{R}^d}$ *solves the Martingale Problem associated to $(a,\mathcal{D}(a))$.*

*Proof.* $(t,y)\mapsto\phi(t,h^{-1}(y))$ is of class $\mathcal{C}^{1,2}$; moreover $\partial_x\left(\phi(r,\cdot)\circ h^{-1}\right) = \frac{\partial_x\phi}{h'}\circ h^{-1}$ and $\partial_x^2\left(\phi(r,\cdot)\circ h^{-1}\right) = \frac{2L\phi}{\sigma^2 h'^2}\circ h^{-1} = \frac{2L\phi}{\sigma_0^2}\circ h^{-1}$. By Itô formula we have

$$\begin{array}{rcl} \phi(t,X_t) & = & \phi(t,h^{-1}(Y_t)) \\ & = & \phi(s,x) + \int_s^t\left(\partial_t\phi(r,h^{-1}(Y_r)) + \frac{1}{2}\sigma_0^2(Y_r)\partial_x^2\left(\phi(r,\cdot)\circ h^{-1}\right)(Y_r)\right)dr \\ & & + \int_s^t\sigma_0(r,h^{-1}(Y_r))\partial_x\left(\phi(r,\cdot)\circ h^{-1}\right)(Y_r)dW_r \\ & = & \phi(s,x) + \int_s^t\left(\partial_t\phi(r,h^{-1}(Y_r)) + L\phi(r,h^{-1}(Y_r))\right)dr \\ & & + \int_s^t\sigma_0(r,h^{-1}(Y_r))\frac{\partial_x\phi(r,h^{-1}(Y_r))}{h'(Y_r)}dW_r \\ & = & \phi(s,x) + \int_s^t\left(\partial_t\phi(r,X_r) + l(r,X_r)\right)dr + \int_s^t\sigma(r,X_r)\partial_x\phi(r,X_r)dW_r. \end{array} \tag{4.16}$$

Therefore $\phi(t,X_t) - \phi(s,x) - \int_s^t a(\phi)(r,X_r)dr = \int_s^t\sigma(r,X_r)\partial_x\phi(r,X_r)dW_r$ is a local martingale. □

In order to consider the $FBSDE^{s,x}(f,g)$ for functions $(f,g)$ having polynomial growth in $x$ we will show the following result. We formulate here the supplementary assumption, called (TA) in [17]. This means the existence of strictly positive constants $c_1, C_1$ such that

$$c_1 \le \frac{e^{\Sigma}}{\sigma} \le C_1. \tag{4.17}$$

**Proposition 4.24.** *We suppose that* (4.17) *is fulfilled and* $\sigma$ *has linear growth. Then, for any* $p > 0$ *and* $(s,x) \in [0,T] \times \mathrm{R}$, $\mathrm{E}^{s,x}[|X_T|^p] < \infty$ *and* $\mathrm{E}^{s,x}[\int_s^T |X_r|^p dr] < \infty$.

*Proof.* We start by proving the proposition in the divergence form case, meaning that $b = \frac{\sigma^2}{2}$. Let $(s,x)$ and $t \in [s,T]$ be fixed. Thanks to the Aronson estimates, see e.g. [1] and also Section 5. of [17], there is a constant $M > 0$ such that

$$\begin{array}{rcl} \mathrm{E}^{s,x}[|X_t|^p] & = & \int_{\mathrm{R}} |y|^p p_{t-s}(x,y) dy \\ & \le & \frac{M}{\sqrt{t-s}} \int_{\mathrm{R}} |y|^p e^{-\frac{|x-y|^2}{M(t-s)}} dz \\ & = & M^{\frac{3}{2}} \int_{\mathrm{R}} |x + z\sqrt{M(t-s)}|^p e^{-z^2} dz \\ & \le & \sum_{k=0}^p M^{\frac{3+k}{2}} \binom{p}{k} |x|^k |t-s|^{\frac{p-k}{2}} \int_{\mathrm{R}} |z|^{p-k} e^{-z^2} dz, \end{array} \tag{4.18}$$

which (for fixed $(s,x)$) is bounded in $t \in [s,T]$ and therefore Lebesgue integrable in $t$ on $[s,T]$. This in particular shows that $\mathrm{E}^{s,x}[|X_T|^p]$ and $\mathrm{E}^{s,x}[\int_s^T |X_r|^p dr] (= \int_s^T \mathrm{E}^{s,x}[|X_r|^p] dr)$ are finite.

Now we will consider the case in which $X$ only verifies (4.17) and we will add the hypothesis that $\sigma$ has linear growth. Then there exists a process $Z$ (see Lemma 5.6 in [17]) solving an SDE with distributional drift of divergence form generator, and a function $k$ of class $\mathcal{C}^1$ such that $X = k^{-1}(Z)$. The (4.17) condition implies that there exist two constants such that $0 < c \le k'\sigma \le C$ implying that for any $x$, $(k^{-1})'(x) = \frac{1}{k' \circ k^{-1}(x)} \le \frac{\sigma \circ k^{-1}(x)}{c} \le C_2(1 + |k^{-1}(x)|)$, for a positive constant $C_2$. So by Gronwall Lemma there exists $C_3 > 0$ such that $k^{-1}(x) \le C_3 e^{C_2|x|}$, $\forall x \in \mathbb{R}$. Now thank to the Aronson estimates on the transition function $p^Z$ of $Z$, for every $p > 0$, we have

$$\begin{array}{rcl} \mathrm{E}^{s,x}[|X|_t^p] & \le & C_3 \int e^{C_2 p|z|} p_{t-s}^Z(k(x), z) dz \\ & \le & \int e^{C_2 p|z|} \frac{M}{\sqrt{t}} e^{-\frac{|k(x)-z|^2}{Mt}} dz \\ & \le & M^{\frac{3}{2}} \int e^{C_2 p(\sqrt{Mt}|y| + k(x))} e^{-y^2} dy \\ & \le & A e^{Bk(x)}, \end{array} \tag{4.19}$$

where $A, B$ are two constants depending on $p$ and $M$. This implies that $\mathrm{E}^{s,x}[|X_T|^p] < \infty$ and $\mathrm{E}^{s,x}[\int_s^T |X_r|^p dr] < \infty$. ☐

We can now state the main result of this section.

**Proposition 4.25.** *Assume that the non-explosion condition* (4.12) *is verified, that* $f$ *is Lipschitz in* $(y,z)$ *uniformly in* $(t,x)$ *and the validity of one of the two following items.*

- *the (TA) condition* (4.17) *is fulfilled, $\sigma$ has linear growth and $g$ has polynomial growth and $f$ has polynomial growth in $x$ uniformly in $t$;*
- $f(\cdot, \cdot, 0, 0)$ *and $g$ are bounded.*

*Then* (4.15) *has a unique decoupled mild solution $u$ in the sense of Definition 3.4.*

*Proof.* The assertion comes from Theorem 3.9 which applies thanks to Propositions 4.23, 4.24 and B.2. □

**Remark 4.26.** *1. A first analysis linking PDEs (in fact second order elliptic differential equations) with distributional drift and BSDEs was performed by [39]. In those BSDEs the final horizon was a stopping time.*

*2. In [23], the authors have considered a class of BSDEs involving particular distributions.*

## 4.4 Diffusion equations on differential manifolds

In this section, we will provide an example of application in a non Euclidean space. We consider a compact connected smooth differential manifold $M$ of dimension $n$. We denote by $\mathcal{C}^{\infty}(M)$ the linear algebra of smooth functions from $M$ to R, and $(U_i, \phi_i)_{i\in I}$ its atlas. The reader may consult [29] for an extensive introduction to the study of differential manifolds, and [22] concerning diffusions on differential manifolds.

**Lemma 4.27.** *$M$ is Polish.*

*Proof.* By Theorem 1.4.1 in [29] $M$ may be equipped with a Riemannian metric, that we denote by $m$ and its topology may be metricized by the associated distance which we denote by $d$. As any compact metric space, $(M, d)$ is separable and complete so that $M$ is a Polish space. □

We denote by $(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}(\mathcal{F})_{t\in[0,T]})$ the canonical space associated to $M$ and $T$, see Definition A.1.

**Definition 4.28.** *An operator $L : \mathcal{C}^{\infty}(M) \longrightarrow \mathcal{C}^{\infty}(M)$ will be called a* ***smooth second order elliptic non degenerate differential operator on*** *$M$ if for any chart $\phi : U \longrightarrow \mathrm{R}^n$ there exist smooth $\mu : \phi(U) \longrightarrow \mathrm{R}^n$ and $\alpha : \phi(U) \longrightarrow S^*_+(\mathrm{R}^n)$ such that on $\phi(U)$ for any $f \in \mathcal{C}^{\infty}(M)$ we have*

$$Lf(\phi^{-1}(x)) = \frac{1}{2}\sum_{i,j=1}^{n}\alpha^{i,j}(x)\partial_{x_i x_j}(f\circ\phi^{-1})(x) + \sum_{i=1}^{n}\mu^i(x)\partial_{x_i}(f\circ\phi^{-1})(x). \quad (4.20)$$

$\alpha$ and $\mu$ depend on the chart $\phi$ but this dependence will be sometimes omitted and we will say that for some given local coordinates,
$Lf = \frac{1}{2}\sum\limits_{i,j=1}^{n}\alpha^{i,j}\partial_{x_i x_j}f + \sum\limits_{i=1}^{n}\mu^i\partial_{x_i}f$.

The following definition comes from [22], see Definition 1.3.1.

**Definition 4.29.** *Let $L$ denote a smooth second order elliptic non degenerate differential operator on $M$. Let $x \in M$. A probability measure $\mathrm{P}^x$ on $(\Omega, \mathcal{F})$ will be called an **$L$-diffusion starting in** $x$ if*

- $\mathrm{P}^x(X_0 = x) = 1$;
- *for every $f \in \mathcal{C}^\infty(M)$, $f(X) - \int_0^\cdot Lf(X_r)dr$ is a $(\mathrm{P}^x, (\mathcal{F})_{t\in[0,T]}))$ local martingale.*

**Remark 4.30.** *No explosion can occur for continuous stochastic processes with values in a compact space, so there is no need to consider paths in the compactification of $M$ as in Definition 1.1.4 in [22].*

*Theorems 1.3.4 and 1.3.5 in [22] state that for any $x \in M$, there exists a unique $L$-diffusion starting in $x$. Theorem 1.3.7 in [22] implies that those probability measures $(\mathrm{P}^x)_{x\in M}$ define a homogeneous Markov class.*

For a given operator $L$, the carré du champs operator $\Gamma$ is given (in local coordinates) by $\Gamma(\phi, \psi) = \sum_{i,j=1}^{n} \alpha^{i,j}\partial_{x_i}\phi\partial_{x_j}\phi$, see equation (1.3.3) in [22]. We wish to emphasize here that the carré du champs operator has recently become a powerful tool in the study of geometrical properties of Riemannian manifolds. The reader may refer e.g. to [2].

**Definition 4.31.** *$(\mathrm{P}^x)_{x\in M}$ will be called the **$L$-diffusion**. If $M$ is equipped with a specific Riemannian metric $m$ and $L$ is chosen to be equal to $\frac{1}{2}\Delta_m$ where $\Delta_m$ the Laplace-Beltrami operator associated to $m$, then $(\mathrm{P}^x)_{x\in M}$ will be called the **Brownian motion associated to** $m$, see [22] Chapter 3 for details.*

We now fix some smooth second order elliptic non degenerate differential operator $L$ and the $L$-diffusion $(\mathrm{P}^x)_{x\in M}$. We introduce the associated Markov class $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times M}$ as described in Notation B.1, which by Proposition B.2 is measurable in time.

**Notation 4.32.** *We define $\mathcal{D}(\partial_t + L)$ the set of functions $u : [0,T] \times M \longrightarrow \mathrm{R}$ such that, for any chart $\phi : U \longrightarrow \mathrm{R}^n$, the mapping*

$$\begin{array}{rcl} [0,T] \times \phi(U) & \longrightarrow & \mathrm{R} \\ (t,x) & \longmapsto & u(t, \phi^{-1}(x)) \end{array} \tag{4.21}$$

*belongs to $\mathcal{C}^\infty([0,T] \times \phi(U), \mathrm{R})$, the set of infinitely continuously differentiable functions in the usual Euclidean setup.*

**Lemma 4.33.** *$\mathcal{D}(\partial_t + L)$ is a linear algebra included in $\mathcal{D}^{max}(\partial_t + L)$ as defined in Notation B.5.*

*Proof.* For some fixed chart $\phi : U \longrightarrow \mathrm{R}^n$, $\mathcal{C}^\infty([0,T] \times \phi(U), \mathrm{R})$ is an algebra, so it is immediate that $\mathcal{D}(\partial_t + L)$ is an algebra. Moreover, if $u \in \mathcal{D}(\partial_t + L)$, it is clear that

- $\forall x \in M$, $u(\cdot, x) \in \mathcal{C}^1([0,T], \mathrm{R})$ and $\forall t \in [0,T]$, $u(t, \cdot) \in \mathcal{C}^\infty(M)$,

- $\forall t \in [0,T]$, $\partial_t u(t,\cdot) \in \mathcal{C}^\infty(M)$ and $\forall x \in M$, $Lu(\cdot,x) \in \mathcal{C}^1([0,T],\mathrm{R})$.

Given a chart $\phi : U \longrightarrow \mathrm{R}^n$, by the Schwarz Theorem allowing the commutation of partial derivatives (in the classical Euclidean setup), we have for $x \in \phi(U)$

$$\begin{array}{rcl} \partial_t \circ L(u)(t,\phi^{-1}(x)) & = & \frac{1}{2}\sum\limits_{i,j=1}^{n} \alpha^{i,j}(x)\partial_t\partial_{x_i x_j}(u(\cdot,\phi^{-1}(\cdot))(t,x) \\ & + & \sum\limits_{i=1}^{n}\mu^i(x)\partial_t\partial_{x_i}(u(\cdot,\phi^{-1}(\cdot))(t,x) \\ & = & \frac{1}{2}\sum\limits_{i,j=1}^{n} \alpha^{i,j}(x)\partial_{x_i x_j}\partial_t(u(\cdot,\phi^{-1}(\cdot))(t,x) \\ & + & \sum\limits_{i=1}^{n}\mu^i(x)\partial_{x_i}\partial_t(u(\cdot,\phi^{-1}(\cdot))(t,x) \\ & = & L \circ \partial_t(u)(t,\phi^{-1}(x)). \end{array} \tag{4.22}$$

So $\partial_t \circ Lu = L \circ \partial_t u$. Finally $\partial_t u$, $Lu$ and $\partial_t \circ Lu$ are continuous (since they are continuous on all the sets $[0,T] \times U$ where $U$ is the domain of a chart) and are therefore bounded as continuous functions on the compact set $[0,T] \times M$. This concludes the proof. ☐

**Corollary 4.34.** $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times M}$ *solves the well-posed Martingale Problem associated to* $(\partial_t + L, \mathcal{D}(\partial_t + L))$ *in the sense of Definition 2.2.*

*Proof.* The corollary derives from Lemma 4.33 and Corollary B.8. ☐

We fix a couple of functions $(f,g)$ with $f$ Lipschitz in $(y,z)$ uniformly in $(t,x)$, and $g, f(\cdot,\cdot,0,0)$ bounded. We consider the PDE

$$\left\{\begin{array}{l} \partial_t u + Lu + f(\cdot,\cdot,u,(\alpha\nabla u\cdot\nabla u)^{\frac{1}{2}}) = 0 \quad \text{on } [0,T]\times M \\ u(T,\cdot) = g. \end{array}\right. \tag{4.23}$$

Since Theorem 3.9 applies, we have the following result.

**Proposition 4.35.** *Equation* (4.23) *admits a unique decoupled mild solution* $u$ *in the sense of Definition 3.4.*

# Appendices

## A Markov classes

In this Appendix we recall some basic definitions and results concerning Markov processes. For a complete study of homogeneous Markov processes, one may consult [11], concerning non-homogeneous Markov classes, our reference was chapter VI of [13]. The first definition refers to the canonical space that one can find in [27], see paragraph 12.63.

**Notation A.1.** *In the whole section $E$ will be a fixed Polish space (a separable completely metrizable topological space). $E$ will be called the **state space**.*

*We consider $T \in \mathrm{R}_+^*$. We denote $\Omega := \mathrm{D}([0,T],E)$ the space of functions from $[0,T]$ to $E$ right-continuous with left limits and continuous at time $T$, e.g. càdl'ag. For any $t \in [0,T]$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$, and we introduce on $\Omega$ the $\sigma$-field $\mathcal{F} := \sigma(X_r | r \in [0,T])$.*

*On the measurable space $(\Omega, \mathcal{F})$, we introduce the measurable **canonical process***

$$X : \begin{array}{rcl} (t,\omega) & \longmapsto & \omega(t) \\ ([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}) & \longrightarrow & (E, \mathcal{B}(E)), \end{array}$$

*and the right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ where $\mathcal{F}_t := \bigcap_{s\in]t,T]} \sigma(X_r | r \leq s)$, if $t < T$, and $\mathcal{F}_T := \sigma(X_r | r \in [0,T]) = \mathcal{F}$. $\left(\Omega, \mathcal{F}, (X_t)_{t\in[0,T]}, (\mathcal{F}_t)_{t\in[0,T]}\right)$ will be called the **canonical space** (associated to $T$ and $E$). For any $t \in [0,T]$ we denote $\mathcal{F}_{t,T} := \sigma(X_r | r \geq t)$, and for any $0 \leq t \leq u < T$ we will denote $\mathcal{F}_{t,u} := \bigcap_{n\geq 0} \sigma(X_r | r \in [t, u + \frac{1}{n}])$.*

Since $E$ is Polish, we recall that $\mathrm{D}([0,T],E)$ can be equipped with a Skorokhod distance which makes it a Polish metric space (see Theorem 5.6 in chapter 3 of [15], and for which the Borel $\sigma$-field is $\mathcal{F}$, see Proposition 7.1 in Chapter 3 of [15]. This in particular implies that $\mathcal{F}$ is separable, as the Borel $\sigma$-field of a separable metric space.

**Remark A.2.** *Previous definitions and all the notions of this Appendix, extend to a time interval equal to $\mathrm{R}_+$ or replacing the Skorokhod space with the Wiener space of continuous functions from $[0,T]$ (or $\mathrm{R}_+$) to $E$.*

**Definition A.3.** *The function*

$$p : \begin{array}{rcl} (s,x,t,A) & \longmapsto & p(s,x,t,A) \\ [0,T]\times E\times[0,T]\times\mathcal{B}(E) & \longrightarrow & [0,1], \end{array}$$

*will be called **transition function** if, for any $s,t$ in $[0,T]$, $x_0 \in E$, $A \in \mathcal{B}(E)$, it verifies*

1. *$x \mapsto p(s,x,t,A)$ is Borel,*
2. *$B \mapsto p(s,x_0,t,B)$ is a probability measure on $(E,\mathcal{B}(E))$,*
3. *if $t \leq s$ then $p(s,x_0,t,A) = \mathbb{1}_A(x_0)$,*
4. *if $s < t$, for any $u > t$, $\int_E p(s,x_0,t,dy)p(t,y,u,A) = p(s,x_0,u,A)$.*

The latter statement is the well-known **Chapman-Kolmogorov equation**.

**Definition A.4.** *A transition function $p$ for which the first item is reinforced supposing that $(s,x) \longmapsto p(s,x,t,A)$ is Borel for any $t, A$, will be said **measurable in time**.*

**Definition A.5.** *A* ***Markov canonical class*** *associated to a transition function $p$ is a set of probability measures $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on the measurable space $(\Omega,\mathcal{F})$ and verifying for any $t\in[0,T]$ and $A\in\mathcal{B}(E)$*

$$\mathrm{P}^{s,x}(X_t\in A)=p(s,x,t,A), \tag{A.1}$$

*and for any $s\le t\le u$*

$$\mathrm{P}^{s,x}(X_u\in A|\mathcal{F}_t)=p(t,X_t,u,A)\quad \mathrm{P}^{s,x}\ a.s. \tag{A.2}$$

**Remark A.6.** *Formula 1.7 in Chapter 6 of [13] states that for any $F\in\mathcal{F}_{t,T}$ yields*

$$\mathrm{P}^{s,x}(F|\mathcal{F}_t)=\mathrm{P}^{t,X_t}(F)=\mathrm{P}^{s,x}(F|X_t)\ \ \mathrm{P}^{s,x}a.s. \tag{A.3}$$

*Property* (A.3) *will be called* ***Markov property****.*

For the rest of this section, we are given a Markov canonical class $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ with transition function $p$.
We will complete the $\sigma$-fields $\mathcal{F}_t$ of the canonical filtration by $\mathrm{P}^{s,x}$ as follows.

**Definition A.7.** *For any $(s,x)\in[0,T]\times E$ we will consider the $(s,x)$-**completion** $\left(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathrm{P}^{s,x}\right)$ of the stochastic basis $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathrm{P}^{s,x}\right)$ by defining $\mathcal{F}^{s,x}$ as the $\mathrm{P}^{s,x}$-completion of $\mathcal{F}$, by extending $\mathrm{P}^{s,x}$ to $\mathcal{F}^{s,x}$ and finally by defining $\mathcal{F}^{s,x}_t$ as the $\mathrm{P}^{s,x}$-closure of $\mathcal{F}_t$ (meaning $\mathcal{F}_t$ augmented with the $\mathrm{P}^{s,x}$-negligible sets) for every $t\in[0,T]$.*

We remark that, for any $(s,x)\in[0,T]\times E$, $\left(\Omega,\mathcal{F}^{s,x},(\mathcal{F}^{s,x}_t)_{t\in[0,T]},\mathrm{P}^{s,x}\right)$ is a stochastic basis fulfilling the usual conditions, see (1.4) in chapter I of [28]). We recall that considering a conditional expectation with respect to a $\sigma$-field augmented with the negligible sets or not, does not change the result. In particular we have the following.

**Proposition A.8.** *Let $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a Markov canonical class. Let $(s,x)\in[0,T]\times E$ be fixed, $Z$ be a random variable and $t\in[s,T]$, then $\mathrm{E}^{s,x}[Z|\mathcal{F}_t]=\mathrm{E}^{s,x}[Z|\mathcal{F}^{s,x}_t]$ $\mathrm{P}^{s,x}$ a.s., provided that the left-hand (or the right-hand) side is well-defined.*

We state the following technical results of measurability without proofs. The interested reader can find complete proofs in [7], which are adapted from the time-homogeneous theory which the interested reader can find in [11] for instance, see Proposition 10.a and Theorem 39 in its chapter XIV.

**Proposition A.9.** *Let $Z$ be a random variable. If the function $(s,x)\longmapsto \mathrm{E}^{s,x}[Z]$ is well-defined (with possible values in $[-\infty,\infty]$), then at fixed $s\in[0,T]$, $x\longmapsto\mathrm{E}^{s,x}[Z]$ is Borel. If moreover the transition function $p$ is measurable in time then, $(s,x)\longmapsto\mathrm{E}^{s,x}[Z]$ is Borel.*

*In particular if $F\in\mathcal{F}$ be fixed, then at fixed $s\in[0,T]$, $x\longmapsto\mathrm{P}^{s,x}(F)$ is Borel. If the transition function $p$ is measurable in time then, $(s,x)\longmapsto\mathrm{P}^{s,x}(F)$ is Borel.*

**Lemma A.10.** *Assume that the transition function of the Markov canonical class is measurable in time.*

*Let $f \in \mathcal{B}([0,T] \times E, \mathrm{R})$ be such that for every $(s,x) \in [0,T] \times E$, $\mathrm{E}^{s,x}[\int_s^T |f(r,X_r)|dr] < \infty$. Then $(s,x) \longmapsto \mathrm{E}^{s,x}[\int_s^T f(r,X_r)dr]$ is Borel.*

**Proposition A.11.** *Let $f \in \mathcal{B}([0,T] \times E, \mathrm{R})$ be such that for any $(s,x,t)$, $\mathrm{E}^{s,x}[|f(t,X_t)|] < \infty$ then at fixed $s \in [0,T]$, $(x,t) \longmapsto \mathrm{E}^{s,x}[f(t,X_t)]$ is Borel. If moreover the transition function $p$ is measurable in time, then $(s,x,t) \longmapsto \mathrm{E}^{s,x}[f(t,X_t)]$ is Borel.*

# B Technicalities concerning homogeneous Markov classes and martingale problems

We start by introducing homogeneous Markov classes. In this section, we are given a Polish space $E$ and some $T \in \mathrm{R}^*$.

**Notation B.1.** *A mapping $\tilde{p} : E \times [0,T] \times \mathcal{B}(E)$ will be called a* ***homogeneous transition function*** *if $p : (s,x,t,A) \longmapsto \tilde{p}(x,t-s,A)\mathbb{1}_{s<t} + \mathbb{1}_A(x)\mathbb{1}_{s\geq t}$ is a transition function in the sense of Definition A.3. This in particular implies $\tilde{p} = p(0,\cdot,\cdot,\cdot)$.*
*A set of probability measures $(\mathrm{P}^x)_{x\in E}$ on the canonical space associated to $T$ and $E$ (see Notation A.1) will be called a* ***homogeneous Markov class*** *associated to a homogeneous transition function $\tilde{p}$ if*

$$\left\{ \begin{array}{ll} \forall t \in [0,T] \quad \forall A \in \mathcal{B}(E) & ,\mathrm{P}^x(X_t \in A) = \tilde{p}(x,t,A) \\ \forall 0 \leq t \leq u \leq T & ,\mathrm{P}^x(X_u \in A|\mathcal{F}_t) = \tilde{p}(X_t, u-t, A) \quad \mathrm{P}^{s,x} a.s. \end{array} \right. \tag{B.1}$$

*Given a homogeneous Markov class $(\mathrm{P}^x)_{x\in E}$ associated to a homogeneous transition function $\tilde{p}$, one can always consider the Markov class $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to the transition function $p : (s,x,t,A) \longmapsto \tilde{p}(x,t-s,A)\mathbb{1}_{s<t} + \mathbb{1}_A(x)\mathbb{1}_{s\geq t}$. In particular, for any $x \in E$, we have $\mathrm{P}^{0,x} = \mathrm{P}^x$.*

We show that a homogeneous transition function necessarily produces a measurable in time non homogeneous transition function.

**Proposition B.2.** *Let $\tilde{p}$ be a homogeneous transition function and let $p$ be the associated non homogeneous transition function as described in Notation B.1. Then $p$ is measurable in time in the sense of Definition A.4.*

*Proof.* Given that $p : (s,x,t,A) \longmapsto \tilde{p}(x,t-s,A)\mathbb{1}_{s<t} + \mathbb{1}_A(x)\mathbb{1}_{s\geq t}$, it is actually enough to show that $\tilde{p}(\cdot,\cdot,A)$ is Borel for any $A \in \mathcal{B}(E)$. We can also write $\tilde{p} = p(0,\cdot,\cdot,\cdot)$, so $p$ is measurable in time if $p(0,\cdot,\cdot,A)$ is Borel for any $A \in \mathcal{B}(E)$, and this holds thanks to Proposition A.11 applied to $f := \mathbb{1}_A$ . ☐

We then introduce below the notion of homogeneous martingale problems.

**Definition B.3.** *Given $A$ an operator mapping a linear algebra $\mathcal{D}(A) \subset \mathcal{B}_b(E, \mathbb{R})$ into $\mathcal{B}_b(E, \mathbb{R})$, we say that a set of probability measures $(\mathbb{P}^x)_{x \in E}$ on the measurable space $(\Omega, \mathcal{F})$ (see Notation A.1) solves the* ***homogeneous Martingale Problem*** *associated to $(\mathcal{D}(A), A)$ if for any $x \in E$, $\mathbb{P}^x$ satisfies*

- *for every $\phi \in \mathcal{D}(A)$, $\phi(X_\cdot) - \int_0^\cdot A\phi(X_r)dr$ is a $(\mathbb{P}^x, (\mathcal{F}_t)_{t \in [0,T]})$-local martingale;*
- $\mathbb{P}^x(X_0 = x) = 1$.

*We say that this* **homogeneous Martingale Problem is well-posed** *if for any $x \in E$, $\mathbb{P}^x$ is the only probability measure on $(\Omega, \mathcal{F})$ verifying those two items.*

**Remark B.4.** *If $(\mathbb{P}^x)_{x \in E}$ is a homogeneous Markov class solving the homogeneous Martingale Problem associated to some $(\mathcal{D}(A), A)$, then the corresponding $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times E}$ (see Notation B.1) solves the Martingale Problem associated to $(\mathcal{D}(A), A)$ in the sense of Definition 2.2. Moreover if the homogeneous Martingale Problem is well-posed, so is the latter one.*

So a homogeneous Markov process solving a homogeneous martingale problem falls into our setup. We will now see how we can pass from an operator $A$ which only acts on time-independent functions to an evolution operator $\partial_t + A$, and see how our Markov class still solves the corresponding martingale problem.

**Notation B.5.** *Let $E$ be a Polish space and let $A$ be an operator mapping a linear algebra $\mathcal{D}(A) \subset \mathcal{B}_b(E, \mathbb{R})$ into $\mathcal{B}_b(E, \mathbb{R})$.*
*If $\phi \in \mathcal{B}([0,T] \times E, \mathbb{R})$ is such that for every $t \in [0,T]$, $\phi(t, \cdot) \in \mathcal{D}(A)$, then $A\phi$ will denote the mapping $(t, x) \longmapsto A(\phi(t, \cdot))(x)$.*

*We now introduce the time-inhomogeneous domain associated to $A$ which we denote $\mathcal{D}^{max}(\partial_t + A)$ and which consists in functions $\phi \in \mathcal{B}_b([0,T] \times E, \mathbb{R})$ verifying the following conditions:*

- *$\forall x \in E$, $\phi(\cdot, x) \in \mathcal{C}^1([0,T], \mathbb{R})$ and $\forall t \in [0,T]$, $\phi(t, \cdot) \in \mathcal{D}(A)$;*
- *$\forall t \in [0,T]$, $\partial_t \phi(t, \cdot) \in \mathcal{D}(A)$ and $\forall x \in E$, $A\phi(\cdot, x) \in \mathcal{C}^1([0,T], \mathbb{R})$;*
- *$\partial_t \circ A\phi = A \circ \partial_t \phi$;*
- *$\partial_t \phi$, $A\phi$ and $\partial_t \circ A\phi$ belong to $\mathcal{B}_b([0,T] \times E, \mathbb{R})$.*

*On $\mathcal{D}^{max}(\partial_t + A)$ we will consider the operator $\partial_t + A$.*

**Remark B.6.** *With these notations, it is clear that $\mathcal{D}^{max}(\partial_t + A)$ is a sub-linear space of $\mathcal{B}_b([0,T] \times E, \mathbb{R})$. It is in general not a linear algebra, but always contains $\mathcal{D}(A)$, and even $\mathcal{C}^1([0,T], \mathbb{R}) \otimes \mathcal{D}(A)$, the linear algebra of functions which can be written $\sum_{k \leq N} \lambda_k \psi_k \phi_k$ where $N \in \mathbb{N}$, and for any $k$, $\lambda_k \in \mathbb{R}$, $\psi_k \in \mathcal{C}^1([0,T], \mathbb{R})$, $\phi_k \in \mathcal{D}(A)$. We also notice that $\partial_t + A$ maps $\mathcal{D}^{max}(\partial_t + A)$ into $\mathcal{B}_b([0,T] \times E, \mathbb{R})$.*

**Lemma B.7.** *Let us consider the same notations and under the same assumptions as in Notation B.5. Let* $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *be a Markov class solving the well-posed Martingale Problem associated to* $(A,\mathcal{D}(A))$ *in the sense of Definition 2.2. Then it also solves the well-posed martingale problem associated to* $(\partial_t + A, \mathcal{A})$ *for any linear algebra* $\mathcal{A}$ *included in* $\mathcal{D}^{max}(\partial_t + A)$.

*Proof.* We start by noticing that since $\mathcal{D}(A) \subset \mathcal{B}_b(E,\mathrm{R})$ and is mapped into $\mathcal{B}_b(E,\mathrm{R})$, then for any $(s,x) \in [0,T]\times E$ and $\phi \in \mathcal{D}(A)$, $M^{s,x}[\phi]$ is bounded and is therefore a martingale.

We fix $(s,x) \in [0,T]\times E$, $\phi \in \mathcal{D}^{max}(\partial_t + A)$ and $s \leq t \leq u \leq T$ and we will show that

$$\mathrm{E}^{s,x}\left[\phi(u,X_u) - \phi(t,X_t) - \int_t^u (\partial_t + A)\phi(r,X_r)dr\middle|\mathcal{F}_t\right] = 0, \tag{B.2}$$

which implies that $\phi(\cdot,X_\cdot) - \int_s^\cdot (\partial_t + A)\phi(r,X_r)dr$, $t\in[s,T]$ is a $\mathrm{P}^{s,x}$-martingale. We have

$$\begin{array}{rl} & \mathrm{E}^{s,x}[\phi(u,X_u) - \phi(t,X_t)|\mathcal{F}_t] \\ = & \mathrm{E}^{s,x}[(\phi(u,X_t) - \phi(t,X_t)) + (\phi(u,X_u) - \phi(u,X_t))|\mathcal{F}_t] \\ = & \mathrm{E}^{s,x}\left[\int_t^u \partial_t\phi(r,X_t)dr + \left(\int_t^u A\phi(u,X_r)dr + (M^{s,x}[\phi(u,\cdot)]_u - M^{s,x}[\phi(u,\cdot)]_t)\right)|\mathcal{F}_t\right] \\ = & \mathrm{E}^{s,x}\left[\int_t^u \partial_t\phi(r,X_t)dr + \int_t^u A\phi(u,X_r)dr|\mathcal{F}_t\right] \\ = & I_0 - I_1 + I_2, \end{array}$$

where

$$\begin{array}{rcl} I_0 & = & \mathrm{E}^{s,x}\left[\int_t^u \partial_t\phi(r,X_r)dr + \int_t^u A\phi(r,X_r)dr|\mathcal{F}_t\right] \\ I_1 & = & \mathrm{E}^{s,x}\left[\int_t^u (\partial_t\phi(r,X_r) - \partial_t\phi(r,X_t))dr|\mathcal{F}_t\right] \\ I_2 & = & \mathrm{E}^{s,x}\left[\int_t^u (A\phi(u,X_r) - A\phi(r,X_r))dr|\mathcal{F}_t\right]. \end{array}$$

(B.2) will be established if one proves that $I_1 = I_2$. We do this below.

At fixed $r$ and $\omega$, $v \longmapsto A\phi(v,X_r(\omega))$ is $\mathcal{C}^1$, therefore $A\phi(u,X_r(\omega)) - A\phi(r,X_r(\omega)) = \int_r^u \partial_t A\phi(v,X_r(\omega))dv$ and $I_2 = \mathrm{E}^{s,x}\left[\int_t^u\int_r^u \partial_t A\phi(v,X_r)dvdr|\mathcal{F}_t\right]$. Then

$$\begin{array}{rcl} I_1 & = & \mathrm{E}^{s,x}\left[\int_t^u\int_t^r A\partial_t\phi(r,X_v)dvdr|\mathcal{F}_t\right] \\ & + & \mathrm{E}^{s,x}\left[\int_t^u (M^{s,x}[\partial_t\phi(r,\cdot)]_r - M^{s,x}[\partial_t\phi(r,\cdot)]_t)dr|\mathcal{F}_t\right]. \end{array}$$

Since $\partial_t\phi$ and $A\partial_t\phi$ are bounded, $M^{s,x}[\partial_t\phi(r,\cdot)]_r(\omega)$ is uniformly bounded in $(r,\omega)$, so by Fubini's theorem for conditional expectations we have

$$\begin{array}{rl} & \mathrm{E}^{s,x}[\int_t^u (M^{s,x}[\partial_t\phi(r,\cdot)]_r - M^{s,x}[\partial_t\phi(r,\cdot)]_t)dr|\mathcal{F}_t] \\ = & \int_t^u \mathrm{E}^{s,x}[M^{s,x}[\partial_t\phi(r,\cdot)]_r - M^{s,x}[\partial_t\phi(r,\cdot)]_t|\mathcal{F}_t]dr \\ = & 0. \end{array} \tag{B.3}$$

Finally since $\partial_t A\phi = A\partial_t\phi$ and again by Fubini's theorem for conditional expectations, we have $\mathrm{E}^{s,x}\left[\int_t^u\int_r^u \partial_t A\phi(v,X_r)dvdr|\mathcal{F}_t\right] = \mathrm{E}^{s,x}\left[\int_t^u\int_t^r A\partial_t\phi(r,X_v)dvdr|\mathcal{F}_t\right]$ so $I_1 = I_2$ which concludes the proof. $\square$

In conclusion we can state the following.

**Corollary B.8.** *Given a homogeneous Markov class* $(\mathrm{P}^x)_{x\in E}$ *solving a well-posed homogeneous Martingale Problem associated to some* $(\mathcal{D}(A), A)$, *there exists a Markov class* $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *which transition function is measurable in time and such that for any algebra* $\mathcal{A}$ *included in* $\mathcal{D}^{max}(\partial_t+A)$, $(\mathrm{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *solves the well-posed Martingale Problem associated to* $(\partial_t + A, \mathcal{A})$ *in the sense of Definition 2.2.*


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