KINETIC DESCRIPTION OF STRATEGIC BINARY GAMES

Francesco Salvarani, Daniela Tonon

KINETIC DESCRIPTION OF STRATEGIC BINARY GAMES

FRANCESCO SALVARANI AND DANIELA TONON

Version: November 19, 2018

Abstract. In this article, we study the behavior of a population composed by an infinite number of rational individuals, which interact through a binary game. After deducing the kinetic description of the system, we study existence and uniqueness of the resulting equation. We then focus our attention on a linear game and provide its asymptotics in the high-frequency and vanishing payoff case. Several numerical simulations show the quantitative behavior of the model.

1. Introduction

Among the different models that study the interactions of a continuum of agents, a particular approach comes from kinetic theory, [13, 18]. Indeed, under some simplistic assumptions, agents can be seen as particles evolving in time through pairwise interactions. This approach has been extensively exploited for the description of social behaviors (see for example Helbing [20, 19, 21]). In particular, it has been applied to study opinion formation (see the review articles [12, 9] and the references therein) and simple market economy ([14, 16, 6]). As in classical kinetic theory, the interaction between two agents is usually described by a given fixed deterministic *a priori* rule. However, this can be seen as a reducing hypothesis when we want to describe (rational or irrational) agents that are far from interacting always following to the same rule. Several strategies have been proposed to give a more accurate analysis of living systems. When dealing with bounded rationality, i.e. individuals that give possibly different responses to the same situation, interactions can be modeled according to stochastic rules, instead of deterministic ones. This is the idea of kinetic theory for "active particles", see ([4, 5]), introduced to model large systems of interacting entities that posses peculiar functions or specific activities. The description of social interactions through stochastic game theory is also well detailed in [29]. In order to model the volatility of human behaviors, agents are seen as interacting through generalized collisions, i.e. games whose payoffs are known only in terms of a certain probability density. According to this description, the collision between two agents give rise to whatever payoff, with a probability density that may also depend on the distribution function of the population (nonlinearly additive interactions). This probability density is in principle pre-determined by some external factor/hypothesis and does not necessarily come from an optimization strategy studied by the agent herself.

Active particles and stochastic games are well suitable to describe possible irrational behaviors. In this article, we are instead more interested in modeling the evolution of an infinite set of agents that rationally choose the strategy of interaction in order to maximize their payoff. Therefore, we introduce and study a kinetic model for a continuum of rational agents playing a deterministic

Work partially funded by the ANR projects *Kimega* ANR-14-ACHN-0030-01 and *MFG* ANR-16-CE40-0015-01.

binary game, i.e. they interact through collisions whose rule is not fixed a priori but depends on the move strategically chosen by the agent. Being the number of players so large, agents won't be able to collect individual information about the others, their optimization process will instead take into account the global behavior of the whole population via its distribution function as well as the value of their own exchange variable.

When modeling large populations of rational interacting agents, it is natural to consider players with negligible influence on each other and whose strategies are implemented only taking into account the distribution of the population. This hypothesis has been introduced and well justified for static games by Aumann [2] and has lead to the study of non-atomic games (where no individual player affects the overall outcome), see Aumann and Shapley [3]. As a particular case of non-atomic games we cite anonymous games, see [7], a class of games where it is assumed that the payoff of each player depends on her own strategy and only on the number of the other players choosing each strategy (not on the identity of these players). As for large populations of agents playing differential games, the same hypothesis is at the basis of the study of Mean Field Games (MFG) theory. Introduced by Lasry and Lions [24, 25, 26, 27] and simultaneously by Huang, Malhamé and Caines [23, 22], MFG theory describes Nash equilibria in differential games with infinitely many players. Players are supposed to act according to a unique optimization criteria, with an infinitesimal influence over the other players and choosing a strategy that takes into account the behavior of the co-players only through the total mass of the system. The pertinence of MFG theory has been observed in many situations, such as – for example – the optimal production of oil, the study of wealth distribution in developed societies, the dynamics of social networks and crowd motion. To conclude the bibliography on the subject, we quote here the articles [15], which proposes and studies a dynamic model for an ensemble of players in the game-theoretical sense inspired by non-cooperative anonymous games with a continuum of players and MFG and [10, 11], which study a Boltzmann-like mean field model for knowledge growth introduced in the literature by Lucas and Moll in [28].

Let us now enter a bit more into the detail of our model. In the analogy with game theory, we suppose to have:

- an infinite set of players,
- a set of possible actions made by the players,
- a set of game results or issues,
- an order relation between the results which expresses the preferences of a player.

The game we consider will be:

- binary: players interact only pairwise,
- zero-sum: the players' choices can neither increase nor decrease the available resources,
- simultaneous: both players move simultaneously,
- non-cooperative: players cannot form alliances, i.e. every player acts individually,
- with imperfect information: players do not know the moves previously made by all other players neither they know the precise features of the competitor.

In particular, in our model, players do not know the exchange variable of their competitor in the game, moreover their payoff depends on it. They have only a (possibly partial) knowledge of the whole population density and they use a mixed strategy, by taking into account their own exchange variable and the known properties of the distribution profile of the population. The main consequence of this lack of knowledge is given by the high nonlinear structure of the model, which implies some peculiar properties of the solutions.

The aim of the article is twofold: from one hand, we prove the fundamental mathematical properties of the model and, from the other hand, we investigate its relaxation limit in the quasi-invariant case, i.e. when the binary games between agents are, in a suitable sense, a perturbation of the identity and the time variable is rescaled with a scale factor having the same order of magnitude of the aforementioned perturbation. This last problem is usually difficult to analyze if one wants to overcome formal arguments: indeed, in the case of nonlinear equations, standard weak compactness tools are not enough for justifying the limiting procedure, and the strategies of proof are very dependent on the structure of the starting equations. That's why we compute the limit only in a "formal way", strong assumptions about convergence are needed even when we look for limit weak solutions.

The structure of the article is the following: in Section 2 we introduce a general framework for describing binary games by using the kinetic approach and we prove the existence and uniqueness of the solution. Then, in Section 3 we introduce a specific game with a linear interacting rule which promotes wealth redistribution. In Section 4, we prove some additional mathematical properties of this model and in 5, we deduce its quasi-invariant limit (see [8] for another example of quasi-invariant limit procedure in the framework of rational populations described by kinetic equations). Finally, in Section 6, we numerically study the solutions of the model and its long-time behavior.

2. A general framework for the kinetic modeling of binary games

In this section, we introduce a general kinetic system to describe a population of rational agents which interact between themselves through simultaneous binary non-cooperative games with imperfect information.

Let $t \in [0, T]$ be the time variable and $x \in \Omega$, open set of $\mathbb{R}^d$, the exchange variable ($d \in \mathbb{N}$). The system of interacting rational individuals is described by a distribution function $f = f(t, x)$, defined on $[0, T] \times \Omega$ having the following precise meaning: let $D \subseteq \Omega$ be a sub-domain, for all t, the integral

$$\int_D f(t, x)\,\mathrm{d}x$$

represents the number of individuals for which the exchange variable belongs to D. It is obvious that a reasonable hypothesis on f is that it is non-negative and such that $f(t, x) \in L^1(\Omega)$ for all $t \in \mathbb{R}$, or, more generally, that f is a non-negative measure.

Let us denote with $(x', x'_*) \in \Omega \times \Omega$ the exchange variables of two agents before an interaction. We suppose that they interact between themselves through a zero-sum binary game (whose choices are represented by 0 and 1 respectively), having the following payoff rule:

	1	0
1	$a_{11}, -a_{11}$	$a_{10}, -a_{10}$
0	$a_{01}, -a_{01}$	$a_{00}, -a_{00}$

where $a_{ij} = a_{ij}(x', x'_*)$, $i, j = 0, 1$, and all $a_{ij} : \Omega \times \Omega \to \mathbb{R}^d$, $a_{ij} \in C(\Omega \times \Omega)$: we hence suppose that the payoffs depend on the exchange variables of the two players before the interaction. This hypothesis implies that the payoffs vary with respect to the pre-interaction exchange variables. Moreover, we suppose that the players do not know the exchange variable of their competitor in the game. This implies that the agents have an incomplete information about the possible payoffs of the game before playing and hence their strategy should be chosen accordingly. By the indistinguishability of players, we must have for all $i, j = 0, 1$,

$$a_{ij}(x', x'_*) = -a_{ji}(x'_*, x'),$$

in particular the strategies $(1, 0)$ and $(0, 1)$ must induce the same payoff rules.

If we denote with $(x_{ij}, x_{ij*}) \in \Omega \times \Omega$ for $i, j = 0, 1$, the values of the exchange variable of two agents after the interaction through the strategy (i, j), the exchange rules are the following: let $\tilde{\mathcal{F}}_{ij} = (\mathcal{F}_{ij}, \mathcal{F}_{ij*}) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ be the interaction function $(x_{ij}, x_{ij*}) = \tilde{\mathcal{F}}_{ij}(x', x'_*)$ defined as

$$\left\{ \begin{array}{rll} x_{ij} = & \mathcal{F}_{ij}(x', x'_*) = & x' + a_{ij}(x', x'_*) \\ \\ x_{ij*} = & \mathcal{F}_{ij*}(x', x'_*) = & x'_* - a_{ij}(x', x'_*). \end{array} \right. \tag{1}$$

Of course, by the indistinguishability of players, we have that, for all $i, j = 0, 1$

$$\mathcal{F}_{ij}(x', x'_*) = x' + a_{ij}(x', x'_*) = x' - a_{ji}(x'_*, x') = \mathcal{F}_{ji*}(x'_*, x'). \tag{2}$$

Note that it is essential to guarantee the consistency of the exchange mechanism with the domain Ω: i.e. the exchange variables after the interaction must belong to the domain Ω itself. We will therefore suppose, from now on, that the domain Ω is closed with respect to the exchange mechanism. Hence, we must have that for all $i, j = 0, 1$, $x_{ij},\ x_{ij*} \in \Omega$, i.e. $\tilde{\mathcal{F}}_{ij}(\Omega \times \Omega) \subseteq \Omega \times \Omega$. This hypothesis translates into the following condition on the Jacobian J_{ij} of (1):

$$J_{ij} := \left| \frac{\partial(x_{ij}, x_{ij*})}{\partial(x', x'_*)} \right| \leq 1$$

for all $i, j = 0, 1$. Here $|M|$, denotes the determinant of the square matrix M.

In order to avoid pathological situations, we suppose in addiction that the Jacobians are always strictly positive: $J_{ij} > 0$ for all $i, j = 0, 1$. It will be then possible to invert $\tilde{\mathcal{F}}_{ij}$. We denote with $\tilde{\mathcal{F}}_{ij}^{-1} = (\mathcal{F}_{ij}^{-1}, \mathcal{F}_{ij*}^{-1}) : \tilde{\mathcal{F}}_{ij}(\Omega \times \Omega) \to \Omega \times \Omega$ the inverse function $(x'_{ij}, x'_{ij*}) = \tilde{\mathcal{F}}_{ij}^{-1}(x, x_*)$. According to this notation, for all $x', x'_* \in \Omega$, we have $(x', x'_*) = \tilde{\mathcal{F}}_{ij}^{-1}(\tilde{\mathcal{F}}_{ij}(x', x'_*))$ and for all $(x_{ij}, x_{ij*}) \in \tilde{\mathcal{F}}_{ij}(\Omega \times \Omega)$, we have $(x_{ij}, x_{ij*}) = \tilde{\mathcal{F}}_{ij}(\tilde{\mathcal{F}}_{ij}^{-1}(x_{ij}, x_{ij*}))$.

The conservation of the exchange variable during the game is a consequence of the payoff table (it is indeed a zero-sum game):

$$x' + x'_* = x_{ij} + x_{ij*} \quad \forall i, j = 0, 1.$$

Since the payoffs a_{ij} depend on the values of the exchange variables, the choice between the strategies 0 and 1 depends also on the values of the exchange variable of each player. As we already said, in this game, we suppose that the pre-interaction value of the exchange variable of one player is unknown to the other player. The only information the player is allowed to have is some feature of the distribution profile f of the population. Therefore, analyzing her own exchange variable and the known features of the distribution profile of the population, the player is able to choose optimally the move 0 or 1 with a certain probability.

Let

$$\alpha : C([0,T]; L^1(\Omega)) \to C([0,T] \times \Omega),$$

be the player's probability of choosing the strategy labelled with 1. Hence α has the following properties $\forall f \in C([0,T]; L^1(\Omega))$

$$\alpha(f)(t,x) \geq 0 \;\forall\; (t,x) \in [0,T] \times \Omega, \qquad \|\alpha(f)\|_{C([0,T]\times\Omega)} \leq 1.$$

The precise structure of the operator α is a consequence of the payoff table and depends on the known features of the distribution profile of the population.

Furthermore, we suppose that all the individuals cannot choose their competitor. This is equivalent to require that the cross section, which governs the probability that two individuals interact, is a constant $\eta \in \mathbb{R}_+$. In this situation, this parameter can be interpreted as a simple rescaling of the time variable.

Finally, the model takes the following form:

$$\left\{ \begin{array}{l} \dfrac{1}{\eta}\dfrac{\partial f}{\partial t}(t,x) = Q^+(f)(t,x) - f(t,x) \displaystyle\int_\Omega f(t,x_*)\,\mathrm{d}x_* \\ f(0,\cdot) = f^{\mathrm{in}}(\cdot) \end{array} \right. , \tag{3}$$

where, the gain collisional operator is defined as

$$\begin{aligned} Q^+(f)(t,x) := &\int_\Omega \frac{1}{J_{11}} f(t,x'_{11}) f(t,x'_{11*}) \alpha(f)(t,x'_{11}) \alpha(f)(t,x'_{11*}) \chi_{\tilde{\mathcal{F}}_{11}(\Omega\times\Omega)}(x,x_*)\,\mathrm{d}x_* \\ &+ \int_\Omega \frac{1}{J_{10}} f(t,x'_{10}) f(t,x'_{10*}) \alpha(f)(t,x'_{10}) [1-\alpha(f)(t,x'_{10*})] \chi_{\tilde{\mathcal{F}}_{10}(\Omega\times\Omega)}(x,x_*)\,\mathrm{d}x_* \\ &+ \int_\Omega \frac{1}{J_{01}} f(t,x'_{01}) f(t,x'_{01*}) [1-\alpha(f)(t,x'_{01})] \alpha(f)(t,x'_{01*}) \chi_{\tilde{\mathcal{F}}_{01}(\Omega\times\Omega)}(x,x_*)\,\mathrm{d}x_* \\ &+ \int_\Omega \frac{1}{J_{00}} f(t,x'_{00}) f(t,x'_{00*}) [1-\alpha(f)(t,x'_{00})]\,[1-\alpha(f)(t,x'_{00*})] \chi_{\tilde{\mathcal{F}}_{00}(\Omega\times\Omega)}(x,x_*)\,\mathrm{d}x_*. \end{aligned}$$

We recall here that, for all $i, j = 0, 1$, $(x'_{ij}, x'_{ij*}) = \tilde{\mathcal{F}}_{ij}^{-1}(x, x_*)$ are the values of the exchange variables of two agents before the interaction due to the strategy (i,j).

The loss part of the collisional operator takes the simpler form $f(t,x)\int_\Omega f(t,x_*)\,\mathrm{d}x_*$ since the probability of an agent to play is equal to one.

Note that, the general structure of a kinetic model, that describes a set of identical players interacting between themselves through simultaneous binary non-cooperative games with imperfect information, shows a gain term with an intricate dependency with respect to the unknown (much more complex than the standard Boltzmann equation, although the equivalent of the cross section is uniformly bounded).

Equation (3) can be rewritten in a distributional form with respect to the exchange variable. To this end, let us define the duality form, for any $F \in L^1(\Omega)$,

$$\langle F\,,\,\varphi\rangle := \int_\Omega F(x)\varphi(x)\,\mathrm{d}x, \qquad \text{for all } \varphi \in C_b(\Omega) := C(\Omega)\cap L^\infty(\Omega). \tag{4}$$

We have that

$$\begin{cases} \dfrac{1}{\eta}\dfrac{\mathrm{d}}{\mathrm{d}t}\langle f\,,\,\varphi\rangle - \langle Q^+(f),\varphi\rangle + \langle f\,,\,\varphi\rangle \displaystyle\int_\Omega f(t,x)\,\mathrm{d}x = 0 \\[2ex] \langle f(0,\,\cdot\,)\,,\,\varphi\rangle = \displaystyle\int_\Omega f^{\mathrm{in}}(x)\varphi(x)\,\mathrm{d}x \end{cases} \tag{5}$$

for all $\varphi \in C_b(\Omega)$, with $f^{\mathrm{in}} \in L^1(\Omega)$, where

$$\begin{aligned} \langle Q^+(f),\varphi\rangle = &\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)\varphi(x_{11})\,\mathrm{d}x\,\mathrm{d}x_* \\ &+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)]\varphi(x_{10})\,\mathrm{d}x\,\mathrm{d}x_* \\ &+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)\varphi(x_{01})\,\mathrm{d}x\,\mathrm{d}x_* \\ &+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*))]\varphi(x_{00})\,\mathrm{d}x\,\mathrm{d}x_*. \end{aligned}$$

In the computations above, by an abuse of notation, with x_{ij} we mean $x_{ij} = \mathcal{F}_{ij}(x,x_*)$ for all $i,j = 0,1$.

Note that, taking $\varphi(x) = 1$ for all $x \in \Omega$, we have

$$\int_\Omega Q^+(f)(t,x)\,\mathrm{d}x = \left(\int_\Omega f(t,x)\,\mathrm{d}x\right)^2 \quad \text{for all } t \in [0,T]. \tag{6}$$

The model guarantees the conservation of the total number of agents. This property, in analogy with physical notations, will also be indicated as the conservation of the total mass.

Proposition 2.1. *Let $f = f(t,x)$ be a solution of* (3)*, with a nonnegative initial condition $f^{\mathrm{in}} \in L^1(\Omega)$. Then we have*

$$\int_\Omega f(t,x)\,\mathrm{d}x = \|f^{\mathrm{in}}\|_{L^1(\Omega)} =: \rho \qquad \text{for all } t \in [0,T].$$

Proof. The proof is a direct consequence of Equation (5) used with the test function $\varphi(x) = 1$ for all $x \in \Omega$. Recall also Equation (6). $\square$

Another peculiar feature of the system, which is a direct consequence of the zero-sum payoff rule, is the total conservation of the exchange variable:

Proposition 2.2. *Let $f = f(t,x)$ be a solution of* (3)*, with a nonnegative initial condition $f^{\text{in}} \in L^1(\Omega)$. For all $k \in \{1,\dots,d\}$, let $\int_\Omega x^k f^{\text{in}}(t,x)\,\mathrm{d}x < +\infty$, where x^k denotes the k component of the vector $x \in \mathbb{R}^d$. Then we have*

$$\int_\Omega x f(t,x)\,\mathrm{d}x = \int_\Omega x f^{\text{in}}(t,x)\,\mathrm{d}x \qquad \textit{for all } t \in [0,T].$$

With the notation above, we mean that the equality is true coordinate by coordinate.

Proof. The result can be formally obtained by means of the equivalent distributional formulation of $Q^+(f)$, where we stress the use of the change of variables (1): for a general $\varphi \in C_b(\Omega) := C(\Omega) \cap L^\infty(\Omega)$

$$\begin{aligned}
\langle Q^+(f),\varphi\rangle &= \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)\varphi(\mathcal{F}_{11}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)]\varphi(\mathcal{F}_{10}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)\varphi(\mathcal{F}_{01}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*)]\varphi(\mathcal{F}_{00}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}$$

Inverting the role of x and x_*, and exploiting the symmetry of the first and the last integral, we have

$$\begin{aligned}
2\langle Q^+(f),\varphi\rangle &= \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)[\varphi(\mathcal{F}_{11}(x,x_*)) + \varphi(\mathcal{F}_{11}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)]\varphi(\mathcal{F}_{10}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x_*)[1-\alpha(f)(t,x)]\varphi(\mathcal{F}_{10}(x_*,x))\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)\varphi(\mathcal{F}_{01}(x,x_*))\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x_*)]\alpha(f)(t,x)\varphi(\mathcal{F}_{01}(x_*,x))\,\mathrm{d}x\,\mathrm{d}x_* \\
&+ \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*)][\varphi(\mathcal{F}_{00}(x,x_*)) + \varphi(\mathcal{F}_{00}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}$$

Hence

$$
\begin{aligned}
2\langle Q^+(f),\varphi\rangle =& \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)[\varphi(\mathcal{F}_{11}(x,x_*))+\varphi(\mathcal{F}_{11}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)][\varphi(\mathcal{F}_{10}(x,x_*))+\varphi(\mathcal{F}_{01}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)[\varphi(\mathcal{F}_{01}(x,x_*))+\varphi(\mathcal{F}_{10}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*)][\varphi(\mathcal{F}_{00}(x,x_*))+\varphi(\mathcal{F}_{00}(x_*,x))]\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}
$$

Then, we use the symmetry of our model given by (2),

$$
\begin{aligned}
2\langle Q^+(f),\varphi\rangle =& \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)[\varphi(\mathcal{F}_{11}(x,x_*))+\varphi(\mathcal{F}_{11*}(x,x_*))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)][\varphi(\mathcal{F}_{10}(x,x_*))+\varphi(\mathcal{F}_{10*}(x,x_*))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)[\varphi(\mathcal{F}_{01}(x,x_*))+\varphi(\mathcal{F}_{01*}(x,x_*))]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*)][\varphi(\mathcal{F}_{00}(x,x_*))+\varphi(\mathcal{F}_{00*}(x,x_*))]\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}
$$

By setting formally $\varphi(x)=x^k$ and using (1), we obtain

$$
\begin{aligned}
2\langle Q^+(f),x^k\rangle =& \int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)\alpha(f)(t,x_*)[x^k+(x_*)^k]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)\alpha(f)(t,x)[1-\alpha(f)(t,x_*)][x^k+(x_*)^k]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\alpha(f)(t,x_*)[x^k+(x_*)^k]\,\mathrm{d}x\,\mathrm{d}x_*\\
&+\int_{\Omega\times\Omega} f(t,x)f(t,x_*)[1-\alpha(f)(t,x)]\,[1-\alpha(f)(t,x_*)][x^k+(x_*)^k]\,\mathrm{d}x\,\mathrm{d}x_*\\
=& \int_{\Omega\times\Omega} f(t,x)f(t,x_*)[x^k+(x_*)^k]\,\mathrm{d}x\,\mathrm{d}x_* = 2\int_{\Omega\times\Omega} f(t,x)f(t,x_*)x^k\,\mathrm{d}x\,\mathrm{d}x_*.
\end{aligned}
$$

Thus

$$
\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}t}\langle f\,,\,x^k\rangle = \langle Q^+(f),x^k\rangle - \langle f,x^k\rangle\int_\Omega f\,\mathrm{d}x = 0,
$$

and

$$
\int_\Omega x^k f(t,x)\;\mathrm{d}x = \int_\Omega x^k f^{\mathrm{in}}(t,x)\;\mathrm{d}x < +\infty \qquad \text{for all } t\in[0,T].
$$

□

We now prove the following existence and uniqueness theorem:

Theorem 2.3. *Let f^{in} a nonnegative function of class $L^1(\Omega)$. Let $T > 0$ be a given constant and suppose that α is such that for all $f_1, f_2 \in C([0,T]; L^1(\Omega))$,*

$$\|\alpha(f_1) - \alpha(f_2)\|_{C([0,T]\times\Omega)} \leq C\|f_1 - f_2\|_{C([0,T];L^1(\Omega))} \tag{7}$$

for $C \in \mathbb{R}_+$.

Then, the Cauchy problem (3) *admits a unique nonnegative solution in $C^1([0,T]; L^1(\Omega))$.*

Proof. First of all, we transform the Cauchy problem (3) in integral form, using the classical Duhamel formula and the fact that $\int_\Omega f(t,x)\,\mathrm{d}x = \rho$ for all $t \in [0,T]$. We have indeed that

$$f(t,x) = f^{\text{in}}(x)e^{-\eta\rho t} + \eta \int_0^t e^{-\eta\rho(t-s)} Q^+(f)(s,x)\,\mathrm{d}s := \mathcal{A}(f)(t,x). \tag{8}$$

We then apply a contraction argument to the previous equation.

Let us define

$$F := \{\psi \in C([0,T]; L^1(\Omega)) \,:\, \|\psi(t,\cdot)\|_{L^1(\Omega)} = \rho \ \forall t \in [0,T],\ \psi \geq 0\}.$$

F is a Banach space, being a closed subset of $C([0,T]; L^1(\Omega))$. We have to prove that $\mathcal{A}$ maps F into F. First of all, let us observe that, due to the integral form (8), the operator $\mathcal{A}$ maps nonnegative functions into nonnegative functions. Moreover, if $f \in F$ then $Q^+(f) \in C([0,T]; L^1(\Omega))$, and $\|Q^+(f)(t,\cdot)\|_{L^1(\Omega)} = \rho^2$ for all $t \in [0,T]$, due to (6).

Then, for all $f \in F$, and all $t \in [0,T]$,

$$\begin{aligned}
\int_\Omega |\mathcal{A}(f)(t,x)|\,\mathrm{d}x &= e^{-\eta\rho t}\int_\Omega f^{\text{in}}(x)\,\mathrm{d}x + \eta\int_0^t e^{-\eta\rho(t-s)}\int_\Omega Q^+(f)(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&= e^{-\eta\rho t}\rho + \eta\rho^2\int_0^t e^{-\eta\rho(t-s)}\,\mathrm{d}s = \rho.
\end{aligned}$$

Hence $\mathcal{A}(f) \in C([0,T]; L^1(\Omega))$, and

$$\|\mathcal{A}(f)(t,\cdot)\|_{L^1(\Omega)} = \rho \ \ \forall t \in [0,T] \quad \text{and} \quad \|\mathcal{A}(f)\|_{C([0,T];L^1(\Omega))} = \max_{t\in[0,T]} \|\mathcal{A}(f)(t,\cdot)\|_{L^1(\Omega)} = \rho.$$

Let us now prove that, under suitable conditions on T, $\mathcal{A}$ is a contraction in F. Indeed, let $f_i \in F$ for $i = 1, 2$. We compute

$$
\begin{aligned}
&\|\mathcal{A}(f_1)-\mathcal{A}(f_2)\|_{C([0,T];L^1(\Omega))} = \\
&= \max_{t\in[0,T]} \eta \int_\Omega \left| \int_0^t e^{-\eta\rho(t-s)} \big(Q^+(f_1)(s,x) - Q^+(f_2)(s,x)\big) \,\mathrm{d}s \right| \mathrm{d}x \\
&\le \max_{t\in[0,T]} \eta \int_\Omega \int_0^t \left| Q^+(f_1)(s,x) - Q^+(f_2)(s,x) \right| \mathrm{d}s\,\mathrm{d}x \\
&\le \eta \int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^2 (-1)^{i+1} f_i(s,x') f_i(s,x'_*) \alpha(f_i)(s,x')\alpha(f_i)(s,x'_*) \right| \mathrm{d}x'_*\,\mathrm{d}x'\,\mathrm{d}s \\
&+ \eta \int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^2 (-1)^{i+1} f_i(s,x') f_i(s,x'_*) \alpha(f_i)(s,x')[1-\alpha(f_i)(s,x'_*)] \right| \mathrm{d}x'_*\,\mathrm{d}x'\,\mathrm{d}s \\
&+ \eta \int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^2 (-1)^{i+1} f_i(s,x') f_i(s,x'_*) [1-\alpha(f_i)(s,x')]\alpha(f_i)(s,x'_*) \right| \mathrm{d}x'_*\,\mathrm{d}x'\,\mathrm{d}s \\
&+ \eta \int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^2 (-1)^{i+1} f_i(s,x') f_i(s,x'_*) [1-\alpha(f_i)(s,x')]\,[1-\alpha(f_i)(s,x'_*)] \right| \mathrm{d}x'_*\,\mathrm{d}x'\,\mathrm{d}s,
\end{aligned}
$$

where in the last inequality we performed the change of variable $(x,x_*) \to (x',x'_*)$. In the following we will omit the prime notation.

Note that, we can bound all the terms in the previous inequality reasoning in the same way. Let us show, for example, how to bound the first one. We have that

$$
\begin{aligned}
&\int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^2 (-1)^{i+1} f_i(s,x) f_i(s,x_*) \alpha(f_i)(s,x)\alpha(f_i)(s,x_*) \right| \mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^T \int_{\Omega\times\Omega} f_1(s,x_*)\alpha(f_1)(s,x)\alpha(f_1)(s,x_*)\,|f_1(s,x)-f_2(s,x)|\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^T \int_{\Omega\times\Omega} f_2(s,x)\alpha(f_1)(s,x)\alpha(f_1)(s,x_*)\,|f_1(s,x_*)-f_2(s,x_*)|\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^T \int_{\Omega\times\Omega} f_2(s,x)f_2(s,x_*)\alpha(f_1)(s,x_*)\,|\alpha(f_1)(s,x)-\alpha(f_2)(s,x)|\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^T \int_{\Omega\times\Omega} f_2(s,x)f_2(s,x_*)\alpha(f_2)(s,x)\,|\alpha(f_1)(s,x_*)-\alpha(f_2)(s,x_*)|\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Hence using hypothesis (7) and bounds for f_1, f_2, we have

$$\int_0^T \int_{\Omega\times\Omega} \left| \sum_{i=1}^{2} (-1)^{i+1} f_i(s,x) f_i(s,x_*) \alpha(f_i)(s,x)\alpha(f_i)(s,x_*) \right| \mathrm{d}x_* \,\mathrm{d}x\,\mathrm{d}s$$

$$\begin{aligned}
&\leq T\left[\|f_1\|_{C([0,T];L^1(\Omega))} + \|f_2\|_{C([0,T];L^1(\Omega))}\right] \|f_1 - f_2\|_{C([0,T];L^1(\Omega))} \\
&+2\|f_2\|_{C([0,T];L^1(\Omega))} \int_0^T \int_\Omega f_2(s,x)\left|\alpha(f_1)(s,x) - \alpha(f_2)(s,x)\right| \mathrm{d}x\,\mathrm{d}s \\
&\leq T\left[\|f_1\|_{C([0,T];L^1(\Omega))} + \|f_2\|_{C([0,T];L^1(\Omega))}\right] \|f_1 - f_2\|_{C([0,T];L^1(\Omega))} \\
&+2\|f_2\|_{C([0,T];L^1(\Omega))} \int_0^T \|f_2(s,\cdot)\|_{L^1(\Omega)} \|\alpha(f_1) - \alpha(f_2)\|_{C([0,T]\times\Omega)} \,\mathrm{d}s \\
&\leq T\left[\|f_1\|_{C([0,T];L^1(\Omega))} + \|f_2\|_{C([0,T];L^1(\Omega))} + 2C\|f_2\|^2_{C([0,T];L^1(\Omega))}\right] \|f_1 - f_2\|_{C([0,T];L^1(\Omega))}
\end{aligned}$$

The other terms can be estimated in a similar way. Consequently, we deduce that,

$$\|\mathcal{A}(f_1) - \mathcal{A}(f_2)\|_{C([0,T];L^1(\Omega))} \leq 4\eta T\left(2\rho + 2C\rho^2\right)\|f_1 - f_2\|_{C([0,T];L^1(\Omega))}$$

where we used the fact that $f_1,\ f_2 \in F$. Therefore, for

$$T < \frac{1}{8\eta\rho\,(1 + C\rho)},$$

Banach's Fixed Point Theorem guarantees the existence of a unique fixed point, and hence a solution of the Cauchy problem (3) in $C([0,T];L^1(\Omega))$.

In addition, it is immediate to see, from the Cauchy problem (3) and formula (6), that

$$\|\frac{\partial}{\partial t} f\|_{C([0,T];L^1(\Omega))} \leq 2\eta\|f\|^2_{C([0,T];L^1(\Omega))}.$$

Hence $\frac{\partial}{\partial t} f \in C([0,T];L^1(\Omega))$. By a standard bootstrap argument we can extend the existence of the solution to all $T > 0$ fixed since the bound on T depends only on the initial condition and time evolution is mass preserving (Proposition 2.1).

□

3. A linear game

Up to now, we have considered a quite general binary game. However, it is clear that many qualitative and quantitative properties of the equation (such as, for example, the concentration effects or the asymptotic behavior) heavily depend on the precise rules of the game. For this reason, we now focus our attention on a specific game with a linear interacting rule which promotes wealth redistribution.

Let $0 \leq \varepsilon < \frac{1}{2}$ and $(x', x'_*) \in \mathbb{R}_+ \times \mathbb{R}_+$ be the wealths of two agents before an interaction. Our agents interact between themselves through a binary game, having the following payoff rule:

	1	0
1	$0, 0$	$0, 0$
0	$0, 0$	$\varepsilon(x'_* - x'), \varepsilon(x' - x'_*)$

Table 1

It means that, if we denote with $(x, x_*) \in \mathbb{R}_+ \times \mathbb{R}_+$ the wealths of two agents after the interaction, the exchange rules are the following: in the case they play $(0, 0)$, we have

$$\begin{cases} x = (1-\varepsilon)x' + \varepsilon x'_* \\ \\ x_* = \varepsilon x' + (1-\varepsilon)x'_*. \end{cases} \tag{9}$$

These relationships can be inverted in the following way:

$$\begin{cases} x' = \dfrac{1-\varepsilon}{1-2\varepsilon} x - \dfrac{\varepsilon}{1-2\varepsilon} x_* \\ \\ x'_* = -\dfrac{\varepsilon}{1-2\varepsilon} x + \dfrac{1-\varepsilon}{1-2\varepsilon} x_*. \end{cases} \tag{10}$$

Note that the existence of a pre-collisional non-negative pair (x', x'_*) generated by a non-negative post-collisional pair (x, x_*) is not guaranteed, unless we suppose that

$$\frac{\varepsilon}{1-\varepsilon} x \leq x_* \leq \frac{1-\varepsilon}{\varepsilon} x.$$

When the agents play $(1, 1)$, $(1, 0)$ or $(0, 1)$, the interaction rule is simply the identity:

$$\begin{cases} x = x' \\ \\ x_* = x'_*. \end{cases} \tag{11}$$

The Jacobian J of (9) is easily computable and it is always strictly positive (remember $0 \leq \varepsilon < \frac{1}{2}$) and contracting:

$$0 < J := \left| \frac{\partial(x, x_*)}{\partial(x', x'_*)} \right| = (1 - 2\varepsilon) \leq 1.$$

As a consequence of the payoff table we have wealth's conservation during the game, i.e. we are considering a zero-sum game:

$$x + x_* = x' + x'_*.$$

We can therefore see this game as a particular case of the model described in Section 2, we just need to define the probability function α. It is clear that the strategy of the two players will depend on their relative wealth: indeed, the strategy 0 is advantageous if and only if the player is poorer than his competitor. On the other hand, the richest player is always forced to adopt the strategy 1. As in the general case, we suppose that the competitor pre-interaction wealth is unknown to each player. Moreover, we suppose that each agent only knows how many players are poorer than him while the distribution profile f is not completely known. Hence, players will naturally choose the strategy 0 or 1 according to the probability of being poorer or richer than the competitor. In other terms, if a given individual has wealth $x \in \mathbb{R}_+$, he knows the cumulative distribution function

$$(t,x) \mapsto \int_0^x f(t,\xi)\,\mathrm{d}\xi.$$

Let

$$m_0 := \int_{\mathbb{R}_+} f^{\mathrm{in}}(x)\,\mathrm{d}x,$$

where f^{in} is a nonnegative function of class $L^1(\mathbb{R}_+)$. We know from Section 2 that $\forall t \in [0,T]$, $T > 0$ fixed,

$$\int_{\mathbb{R}_+} f(t,x)\,\mathrm{d}x = m_0.$$

Then, if a player has wealth $x' \geq 0$, the probability of meeting a poorer player with wealth $0 \leq x'_* \leq x'$ at a time $t \in [0,T]$ can be easily computed to be

$$\mathbb{P}(x' > x'_* \,|(t,x',x'_*)) = \frac{1}{m_0}\int_0^{x'} f(t,\xi)\,\mathrm{d}\xi.$$

Therefore we have

$$\alpha(f)(t,x') := \frac{1}{m_0}\int_0^{x'} f(t,\xi)\,\mathrm{d}\xi.$$

Note that it is easy to verify that the probability operator α defined satisfies

$$\alpha : C([0,T];L^1(\mathbb{R}_+)) \to C([0,T]\times\mathbb{R}_+),$$

verifies $\forall f \in C([0,T];L^1(\mathbb{R}_+))$,

$$\alpha(f)(t,x) \geq 0 \;\forall\; (t,x) \in [0,T]\times\mathbb{R}_+, \quad \|\alpha(f)\|_{C([0,T]\times\mathbb{R}_+)} \leq 1,$$

and the hypothesis (7). Hence, when two individuals with wealth x' and x'_* interact at a time $t \in [0,T]$, the probability to use the strategy $(0,0)$ is given by

$$\begin{aligned}\mathbb{P}((0,0)\,|(t,x',x'_*)) &= \left(1 - \frac{1}{m_0}\int_0^{x'} f(t,\xi)\,\mathrm{d}\xi\right)\left(1 - \frac{1}{m_0}\int_0^{x'_*} f(t,\xi)\,\mathrm{d}\xi\right) \\ &= \frac{1}{m_0^2}\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right).\end{aligned}$$

By considering the whole population, its time evolution is therefore described by a collisional equation of the following type:

$$\frac{\partial f}{\partial t}(t,x) = \eta\, Q^+(f) - \eta f \int_{\mathbb{R}_+} f\,\mathrm{d}x,$$

where

$$Q^+(f)(t,x) := \int_{\mathbb{R}_+} \alpha(f)(t,x)\alpha(f)(t,x_*)f(t,x)f(t,x_*)\,\mathrm{d}x_*$$

$$+ \int_{\mathbb{R}_+} (1-\alpha(f)(t,x))\,\alpha(f)(t,x_*)f(t,x)f(t,x_*)\,\mathrm{d}x_*$$

$$+ \int_{\mathbb{R}_+} \alpha(f)(t,x)\,(1-\alpha(f)(t,x_*))\,f(t,x)f(t,x_*)\,\mathrm{d}x_*$$

$$+ \int_{\frac{\varepsilon}{1-\varepsilon}x}^{\frac{1-\varepsilon}{\varepsilon}x} \big(1-\alpha(f)(t,x')\big)\big(1-\alpha(f)(t,x'_*)\big)\frac{1}{J}f(t,x')f(t,x'_*)\,\mathrm{d}x_*.$$

Note that $0 \le \varepsilon < \frac{1}{2}$ ensures $\frac{\varepsilon}{1-\varepsilon}x \le \frac{1-\varepsilon}{\varepsilon}x$. As in the general case, the cross section $\eta \in \mathbb{R}_+$ is a parameter which governs the probability that two individuals interact. Even here, we suppose that η is a constant.

Using the definition of α, we can rewrite our equation as

$$\frac{\partial f}{\partial t}(t,x) = Q(f)(t,x),$$

where

$$\begin{aligned} Q(f)(t,x) :=\ & \frac{\eta}{m_0^2}\int_{\frac{\varepsilon}{1-\varepsilon}x}^{\frac{1-\varepsilon}{\varepsilon}x}\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\frac{1}{J}f(t,x')f(t,x'_*)\,\mathrm{d}x_* \\ -\ & \frac{\eta}{m_0^2}\int_{\mathbb{R}_+}\left(\int_{x}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x)f(t,x_*)\,\mathrm{d}x_*. \end{aligned} \tag{12}$$

4. Mathematical properties of the model

In this section we briefly prove some mathematical features of the equation introduced in the previous section. Therefore, we study the Cauchy problem

$$\begin{cases} \frac{\partial f}{\partial t} = Q(f) \\ f(0,x) = f^{\mathrm{in}}(x) \in L^1(\mathbb{R}_+) \end{cases}, \tag{13}$$

where $Q(f)$ is defined by (12).

Remark 4.1. *Since the Cauchy problem* (13) *is a particular formulation of the one in* (3) *for the binary game* (9)*, using the weak formulation, we can prove the conservation of mass*

$$m_0 := \int_0^{+\infty} f(t,x)\,\mathrm{d}x = \int_0^{+\infty} f^{\mathrm{in}}(x)\,\mathrm{d}x$$

as in Proposition 2.1 *an the conservation of the momentum*

$$m_1 := \int_0^{+\infty} x f(t,x)\,\mathrm{d}x = \int_0^{+\infty} x f^{\mathrm{in}}(x)\,\mathrm{d}x$$

for all f^{in} such that $\int_0^{+\infty} x f^{\mathrm{in}}(t,x)\,\mathrm{d}x < +\infty$, as in Proposition 2.2. Moreover, Theorem 2.3 holds and the Cauchy problem (13) admits a unique nonnegative solution in $C^1([0,T];L^1(\mathbb{R}_+))$ for all $T>0$.

Remark 4.2. *Since $f \in C^1([0,T];L^1(\mathbb{R}_+))$, we have that*

$$\int_{\mathbb{R}_+}\left(\frac{1}{m_0}\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right) f(t,x)\,\mathrm{d}x = -\frac{1}{2m_0}\int_{\mathbb{R}_+}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\mathrm{d}x = \frac{m_0}{2},$$

and the operator Q can be written also in the equivalent form

(14)

$$Q(f) = \frac{\eta}{m_0^2}\int_{\frac{\varepsilon}{1-\varepsilon}x}^{\frac{1-\varepsilon}{\varepsilon}x}\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\frac{1}{J} f(t,x')f(t,x'_*)\,\mathrm{d}x_* - \frac{\eta}{2}\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right) f(t,x).$$

However, even if this expression may seem simpler than (12), we will keep the latter form because of its symmetric structure, which will make easier many computations.

Remark 4.3. *Let us note that, due to the regularity of the solution f, we have that its spatial primitive, defined as the function $(t,y) \mapsto \int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi$ for all $(t,y) \in [0,T]\times\mathbb{R}_+$ belongs to $C^1([0,T];W^{1,1}(\mathbb{R}_+))$.*

Let us, therefore, compute its time derivative.

Proposition 4.4. *For $T>0$, let $f \in C^1([0,T];L^1(\mathbb{R}_+))$ be the unique solution of the Cauchy problem (13), with $f^{\mathrm{in}}(x) \in L^1(\mathbb{R}_+)$ and $0 \le \varepsilon < \frac{1}{2}$. Then, for all $t \in [0,T]$, the function $(t,y) \mapsto \int_y^{+\infty} f(t,x)\,\mathrm{d}x$ satisfies*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x =& \frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left[\frac{1}{2}\left(\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right.\\ &\left.+\frac{1}{2}\left(\int_{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2 - \frac{1}{2}\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right]\mathrm{d}x.\end{aligned}$$

Proof. The equation satisfied by $\int_y^{+\infty} f(t,x)\,\mathrm{d}x$ can be recovered performing a change of variable as follows:

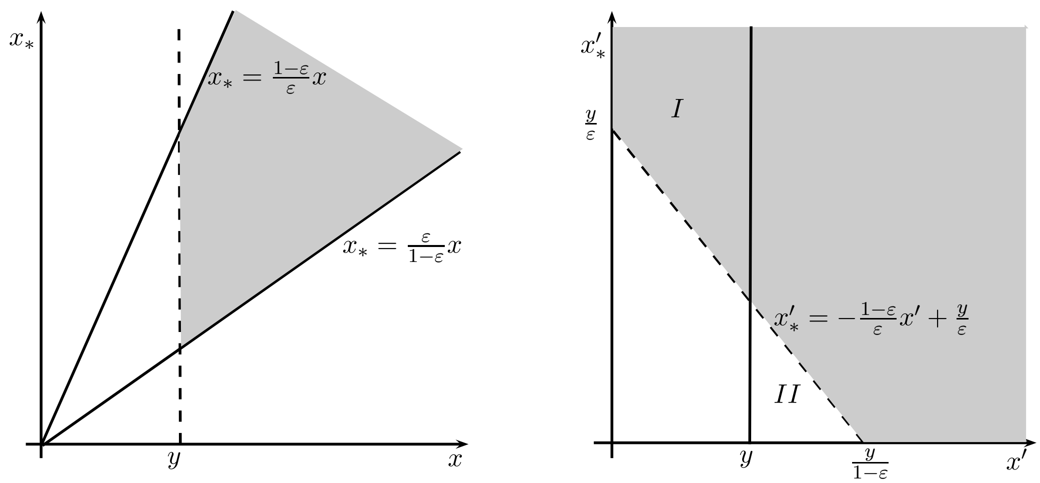

FIGURE 1. Change of variables and integration areas I and II

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x =& \frac{\eta}{m_0^2}\int_y^{+\infty}\int_{\frac{\varepsilon}{1-\varepsilon}x}^{\frac{1-\varepsilon}{\varepsilon}x}\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\frac{1}{J}f(t,x')f(t,x'_*)\,\mathrm{d}x_*\,\mathrm{d}x \\
&-\frac{\eta}{m_0^2}\int_y^{+\infty}\int_{\mathbb{R}_+}\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x)f(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x \\
=& \frac{\eta}{m_0^2}\int_0^{\frac{y}{1-\varepsilon}} f(t,x')\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_{-\frac{1-\varepsilon}{\varepsilon}x'+\frac{y}{\varepsilon}}^{+\infty}\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x'_*)\,\mathrm{d}x'_*\,\mathrm{d}x' \\
&+\frac{\eta}{m_0^2}\int_{\frac{y}{1-\varepsilon}}^{+\infty} f(t,x')\left(\int_{x'}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_0^{+\infty}\left(\int_{x'_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x'_*)\,\mathrm{d}x'_*\,\mathrm{d}x' \\
&-\frac{\eta}{m_0^2}\int_y^{+\infty}\int_{\mathbb{R}_+}\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x)f(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x =& \frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty}\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x \\
&-\frac{\eta}{m_0^2}\int_y^{\frac{y}{1-\varepsilon}} f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_0^{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)f(t,x_*)\,\mathrm{d}x_*\mathrm{d}x.
\end{aligned}
$$

Exchanging the order of integration in the last term, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x =&\frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty}\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right) f(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x\\ &-\frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\int_y^{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}\left(\int_{x_*}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right) f(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x\\ =&\frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left[\frac{1}{2}\left(\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right.\\ &\left.+\frac{1}{2}\left(\int_{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2-\frac{1}{2}\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right]\,\mathrm{d}x,\end{aligned}$$

were we used the fact that $f(t,x)\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi = -\frac{1}{2}\frac{\partial}{\partial x}\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2$ and the convention $\int_{+\infty}^{+\infty} f(t,\xi)\,\mathrm{d}\xi = 0$. ☐

Proposition 4.5. *For $T>0$, let $f\in C^1([0,T];L^1(\mathbb{R}_+))$ be the unique solution of the Cauchy problem* (13)*, with $f^{\mathrm{in}}(x)\in L^1(\mathbb{R}_+)$ and $0\le\varepsilon<\frac{1}{2}$. Suppose f^{in} has compact support $E_0=[x_m,x_M]\subset\mathbb{R}_+$. Then $f(t,\cdot)$ has compact support $E_t\subset[x_m,x_M]$ for all $t\in[0,T]$.*

Proof. We claim that, for all $y\in\mathbb{R}_+$,

$$\int_y^{+\infty} f(t,x)\,\mathrm{d}x \le \int_y^{+\infty} f^{\mathrm{in}}(x)\,\mathrm{d}x\; e^{\frac{\eta m_0 t}{2}} \quad \text{for all } t\in[0,T].$$

Indeed, since for $x\in[0,y]$, $-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}$ varies in $[y,y/\varepsilon]$ and $-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}$ in $[y,y/(1-\varepsilon)]$, we have that

$$\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi \le \int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi \quad\text{and}\quad \int_{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi \le \int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi$$

Moreover we can bound for all $x\in\mathbb{R}_+$ the integral $\int_x^{+\infty} f(t,\xi)$ as

$$\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi \le m_0.$$

Hence, from Proposition 4.4,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x =&\frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left[\frac{1}{2}\left(\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right.\\ &\left.+\frac{1}{2}\left(\int_{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2-\frac{1}{2}\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right]\mathrm{d}x\\ \le&\frac{\eta}{2m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\mathrm{d}x\\ \le&\frac{\eta}{2m_0}\int_0^y f(t,x)\,\mathrm{d}x\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\\ \le&\frac{\eta m_0}{2}\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi.\end{aligned}$$

Using Gronwall's inequality we obtain the claim

$$\int_y^{+\infty} f(t,x)\,\mathrm{d}x \le \int_y^{+\infty} f^{\mathrm{in}}(x)\,\mathrm{d}x\; e^{\frac{\eta m_0 t}{2}}\quad \text{for all } t\in[0,T].$$

Now let $y\in\mathbb{R}_+$ be such that $y\ge x_M$, i.e. $\int_y^{+\infty} f^{\mathrm{in}}(x)\,\mathrm{d}x=0$, then the above inequality implies $\int_y^{+\infty} f(t,x)\,\mathrm{d}x=0$. Hence, $f(t,x)=0$ for a.e. $x\in[y,+\infty[$ and for all $t\in[0,T]$. This shows that $E_t\subset[0,x_M]$ for all $t\in[0,T]$.

We claim now that, for all $y\in\mathbb{R}_+$,

$$\int_0^y f(t,x)\,\mathrm{d}x \le \int_0^y f^{\mathrm{in}}(x)\,\mathrm{d}x\; e^{\frac{\eta m_0}{2}t}\quad \text{for all } t\in[0,T].$$

Indeed, we have

$$\frac{d}{dt}\int_0^y f(t,x)\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{+\infty} f(t,x)\,\mathrm{d}x,$$

Hence, from Proposition 4.4,

$$\begin{aligned}\frac{d}{dt}\int_0^y f(t,x)\,\mathrm{d}x =&-\frac{\eta}{m_0^2}\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\left[\frac{1}{2}\left(\int_{-\frac{1-\varepsilon}{\varepsilon}x+\frac{y}{\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right.\\ &\left.+\frac{1}{2}\left(\int_{-\frac{\varepsilon}{1-\varepsilon}x+\frac{y}{1-\varepsilon}}^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2-\frac{1}{2}\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\right]\mathrm{d}x\\ \le&\frac{\eta}{2m_0^2}\left(\int_y^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)^2\int_0^y f(t,x)\left(\int_x^{+\infty} f(t,\xi)\,\mathrm{d}\xi\right)\mathrm{d}x\\ \le&\frac{\eta m_0}{2}\int_0^y f(t,x)\,\mathrm{d}x.\end{aligned}$$

Using again Gronwall's inequality, we obtain the claim

$$\int_0^y f(t,x)\,\mathrm{d}x \le \int_0^y f^{\mathrm{in}}(x)\,\mathrm{d}x\; e^{\frac{\eta m_0}{2}t} \quad \text{for all } t\in[0,T].$$

Now let $y\in\mathbb{R}_+$ be such that $y\le x_m$, i.e. $\int_0^y f^{\mathrm{in}}(x)\,\mathrm{d}x = 0$, then the above inequality implies $\int_0^y f(t,x)\,\mathrm{d}x = 0$. Hence, $f(t,x)=0$ for a.e. $x\in[0,y]$ and for all $t\in[0,T]$. This shows that $E_t\subset[x_m,+\infty[$ for all $t\in[0,T]$.

Therefore, $E_t\subset[x_m,x_M]$ for all $t\in[0,T]$. □

Remark 4.6. *Note that the previous proposition implies that whenever f^{in} has compact support the momentum is conserved according to Proposition* 2.2.

5. The quasi-invariant limit

In this section we suppose that $\varepsilon \ll 1/2$ and we show how to derive the first-order approximated model starting from (3).

We scale the space and the time variable with a scale parameter of order ε, and we consider the rescaled family of kinetic equations for the unknown $f^\varepsilon = f^\varepsilon(t,x)$, defined for all $0\le\varepsilon<1/2$,

$$\begin{cases} \varepsilon\dfrac{\partial}{\partial t} f^\varepsilon(t,x) = Q(f^\varepsilon)(t,x) \\[2ex] f^\varepsilon(0,x) = f^{\mathrm{in}}(x), \end{cases} \tag{15}$$

where $f^{\mathrm{in}}\in L^1(\mathbb{R}_+)$, is nonnegative, having its support contained in $\overline{E}$, with $E:=(0,x_M)\subset\mathbb{R}_+$. The following result holds.

Theorem 5.1. *Let $\{f^\varepsilon\}_{\varepsilon\in(0,1/2)}$ be a family of solutions to the Cauchy problem* (15)*, supplemented with the same nonnegative initial condition $f^{\mathrm{in}}\in L^1(\mathbb{R}_+)$, with support contained in $\overline{E}$, hence we can consider $f^{\mathrm{in}}\in L^1(E)$. Then, in the limit $\varepsilon\to 0^+$, the family $\{f^\varepsilon\}_{\varepsilon\in(0,1/2)}$ weak* converges, up to a subsequence, to a function $f\in L^\infty([0,T];\mathcal{M}(E))$. Let $F^\varepsilon(t,x) := \int_x^{x_M} f^\varepsilon(t,\xi)d\xi$ then $F^\varepsilon f^\varepsilon$ weak* converges, up to a subsequence, to $G\in L^\infty([0,T];\mathcal{M}(E))$. Let $F(t,x) := \int_x^{x_M} df(t,\xi)$, and suppose that:*

- *$G = Ff$;*
- *$\int_E F^\varepsilon(\cdot,x)f^\varepsilon(\cdot,x)xdx$ converges strongly in $L^1([0,T])$ to $\int_E F(\cdot,x)xdf(\cdot,x)\in L^1([0,T])$;*
- *$\int_E f^{\mathrm{in}}(x)dx = \int_E f^\varepsilon(t,x)dx = \int_E df(t,x)$ for all $t\in[0,T]$.*

Letting f being equal to 0 on $[x_M,+\infty)$, then f is a distributional solution of

$$\begin{cases} \dfrac{\partial}{\partial t} f(t,x) = \dfrac{\eta}{m_0^2}\dfrac{\partial}{\partial x}\left[\displaystyle\int_{\mathbb{R}_+}\left(\int_x^{+\infty}\mathrm{d}f(t,\xi)\right)\left(\int_{x_*}^{+\infty}\mathrm{d}f(t,\xi)\right)(x-x_*)f(t,x)\,\mathrm{d}f(t,x_*)\right] \\[2ex] f(0,x) = f^{\mathrm{in}}(x). \end{cases} \tag{16}$$

Proof. Note that, rescaling in ε does not affect the regularity of f^ε and the conservation properties of the equation. Hence, in particular, for all $0<\varepsilon<1/2$, $f^\varepsilon\in C^1([0,T];L^1(E))$ for a fixed $T>0$ and for all $t\in[0,T]$

$$m_0 := \int_E f^\varepsilon(t,\xi)\,\mathrm{d}\xi = \int_E f^{\mathrm{in}}(\xi)\,\mathrm{d}\xi,$$

where we used the fact that the initial condition f^{in} is the same for all ε. Moreover, due to Proposition 4.5, we have that $f^\varepsilon(t,\cdot)$ has a compact support contained in E for all $t \in [0,T]$.

Now, for all $0<\varepsilon<1/2$, for all $t\in[0,T]$ and $x \in [0,x_M]$, let $F^\varepsilon(t,x) := \int_x^{x_M} f^\varepsilon(t,\xi)\,\mathrm{d}\xi$, then $F^\varepsilon \in C^1([0,T];W^{1,1}(E))$, and $\frac{\partial}{\partial x}F^\varepsilon = -f^\varepsilon$.

In order to show convergence as $\varepsilon \to 0$, we pass to the weak formulation of the kinetic equation. Let $\varphi = \varphi(t,x)$ be a function of class $C_c^\infty([0,T)\times E)$. Using the regularity of f^ε and the definition of $Q(f)$, we have

$$\begin{aligned}
&-\varepsilon\int_0^T\int_E \frac{\partial}{\partial t}\varphi(t,x) f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_E \varphi(0,x) f^{\text{in}}(x)\,\mathrm{d}x = \varepsilon\int_0^T\int_E \varphi(t,x)\frac{\partial}{\partial t} f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
=&\frac{\eta}{m_0^2}\int_0^T\int_{E\times E} F^\varepsilon(t,x)F^\varepsilon(t,x_*)f^\varepsilon(t,x)f^\varepsilon(t,x_*)\varphi(t,x+\varepsilon(x_*-x))\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{\eta}{m_0^2}\int_0^T\int_{E\times E} F^\varepsilon(t,x)F^\varepsilon(t,x_*)f^\varepsilon(t,x)f^\varepsilon(t,x_*)\varphi(t,x)\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We now develop the test function in ε with respect to x

$$\varphi(t,x+\varepsilon(x_*-x)) = \varphi(t,x) + \varepsilon(x_*-x)\frac{\partial}{\partial x}\varphi(t,x) + O(\varepsilon^2).$$

Since

$$\left|\int_0^T\int_{E\times E} F^\varepsilon(t,x)F^\varepsilon(t,x_*)f^\varepsilon(t,x)f^\varepsilon(t,x_*)\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t\right| \le T(m_0)^4$$

we have that

$$\int_0^T\int_{E\times E} F^\varepsilon(t,x)F^\varepsilon(t,x_*)f^\varepsilon(t,x)f^\varepsilon(t,x_*)O(\varepsilon^2)\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t = O(\varepsilon^2)$$

Hence, dividing by ε, we obtain

$$\begin{aligned}
&-\int_0^T\int_E \frac{\partial}{\partial t}\varphi(t,x) f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_E \varphi(0,x) f^{\text{in}}(x)\,\mathrm{d}x =\\
&=\frac{\eta}{m_0^2}\int_0^T\int_{E\times E} F^\varepsilon(t,x)F^\varepsilon(t,x_*)\frac{\partial}{\partial x}\varphi(t,x)f^\varepsilon(t,x)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{\eta}{m_0^2}\int_0^T\int_{E\times E} F^\varepsilon(t,x)F^\varepsilon(t,x_*)\frac{\partial}{\partial x}\varphi(t,x)f^\varepsilon(t,x)f^\varepsilon(t,x_*)x\,\mathrm{d}x_*\,\mathrm{d}x\,\mathrm{d}t + O(\varepsilon)\\
&=\frac{\eta}{m_0^2}\int_0^T\int_E F^\varepsilon(t,x_*)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_*\int_E F^\varepsilon(t,x)f^\varepsilon(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{\eta}{m_0^2}\int_0^T\int_E F^\varepsilon(t,x_*)f^\varepsilon(t,x_*)\,\mathrm{d}x_*\int_E F^\varepsilon(t,x)f^\varepsilon(t,x)x\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t + O(\varepsilon)\\
&= I - II + O(\varepsilon)
\end{aligned}$$

We now analyze the convergence of the above terms.

- Weak* convergence of f^ε in $L^\infty([0,T],\mathcal{M}(E))$: Since $\sup_{t\in[0,T]}\|f^\varepsilon(t,\cdot)\|_{L^1(E)} = m_0$, we have that, up to a subsequence that we will denote with the same index, f^ε weakly* converges respectively to $f \in L^\infty([0,T],\mathcal{M}(E))$, i.e. for all $\phi \in L^1([0,T],C_0(E))$

$$\lim_{\varepsilon\to 0}\int_0^T\int_E \phi(t,x)f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_E \phi(t,x)\,\mathrm{d}f(t,x).$$

- Weak* convergence of $F^\varepsilon f^\varepsilon$ in $L^\infty([0,T],\mathcal{M}(E))$: Let $F(t,x) := \int_x^{x_M} df(t,\xi)$, then $\sup_{t\in[0,T]}\|F^\varepsilon f^\varepsilon\|_{L^1(E)} \le m_0^2$. Hence, $F^\varepsilon f^\varepsilon$ weak* converges, up to a subsequence, to a $G \in L^\infty([0,T];\mathcal{M}(E))$. By hypothesis $G = Ff$. Therefore, for all $\phi \in L^1([0,T],C_0(E))$

$$\lim_{\varepsilon\to 0}\int_0^T\int_E \phi(t,x)F^\varepsilon(t,x)f^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_E \phi(t,x)F(t,x)\,\mathrm{d}f(t,x).$$

Let us analyze I:

$$I = \frac{\eta}{m_0^2}\int_0^T\int_E F^\varepsilon(t,x_*)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_*\int_E F^\varepsilon(t,x)f^\varepsilon(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

For the first term, by hypothesis, we have

$$\lim_{\varepsilon\to 0}\int_0^T\left|\int_E F^\varepsilon(t,x_*)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_* - \int_E F(t,x_*)x_*df(t,x_*)\right|\,\mathrm{d}t = 0.$$

Moreover, we have that $\frac{\partial}{\partial x}\varphi \in C_c^\infty([0,T)\times E) \subset L^1((0,T),C_0(E))$. Hence, defining $H^\varepsilon(t) := \int_E F^\varepsilon(t,x)f^\varepsilon(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x$, we have $H^\varepsilon \in C([0,T])$ and $\sup_{t\in[0,T]}|H^\varepsilon(t)| \le Mm_0^2$ for a positive constant M. Therefore,

$$\begin{aligned}&\int_0^T\left|\int_E F^\varepsilon(t,x_*)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_* - \int_E F(t,x_*)x_*df(t,x_*)\right||H^\varepsilon(t)|\,\mathrm{d}t\\ &\le Mm_0^2\int_0^T\left|\int_E F^\varepsilon(t,x_*)f^\varepsilon(t,x_*)x_*\,\mathrm{d}x_* - \int_E F(t,x_*)x_*df(t,x_*)\,\mathrm{d}x\right|\,\mathrm{d}t \to 0\end{aligned}$$

and, on the other hand,

$$\int_0^T\int_E F(t,x_*)x_*df(t,x_*)\int_E F^\varepsilon(t,x)f^\varepsilon(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

converges to

$$\int_0^T\int_E F(t,x_*)x_*df(t,x_*)\int_E F(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t,$$

being $(t,x) \mapsto \int_E F(t,x_*)x_*df(t,x_*)\frac{\partial}{\partial x}\varphi(t,x) \in L^1((0,T),C_0(E))$. Combining the two, I converges to

$$\frac{\eta}{m_0^2}\int_0^T\int_E F(t,x_*)x_*\,\mathrm{d}f(t,x_*)\int_E F(t,x)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t.$$

Let us analyze II:

$$II = \frac{\eta}{m_0^2}\int_0^T\int_E F^\varepsilon(t,x_*)f^\varepsilon(t,x_*)\,\mathrm{d}x_*\int_E F^\varepsilon(t,x)f^\varepsilon(t,x)x\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Now, by hypothesis, we have

$$\int_E F^\varepsilon(t,x_*)f^\varepsilon(t,x_*)\,\mathrm{d}x_* = \frac{m_0^2}{2} = \int_E F(t,x_*)\,\mathrm{d}f(t,x_*).$$

Moreover, $x\frac{\partial}{\partial x}\varphi \in C_c^\infty([0,T)\times E) \subset L^1([0,T], C_0(E))$. Hence, by weak* convergence, II converges to

$$\int_0^T \int_E F(t,x) x \frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t.$$

Hence, letting $\varepsilon \to 0$ we have

$$\begin{aligned} -\int_0^T \int_E \frac{\partial}{\partial t}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t - \int_E \varphi(0,x) f^{\mathrm{in}}(x)\,\mathrm{d}x = \\ = \frac{\eta}{m_0^2}\int_0^T \int_{E\times E} F(t,x)F(t,x_*)(x_* - x)\,\mathrm{d}f(t,x_*)\frac{\partial}{\partial x}\varphi(t,x)\,\mathrm{d}f(t,x)\,\mathrm{d}t. \end{aligned}$$

Therefore $f \in L^\infty([0,T];\mathcal{M}(E))$ is a distributional solution of

$$\frac{\partial}{\partial t} f(t,x) = \frac{\eta}{m_0^2}\frac{\partial}{\partial x}\left[\int_E F(t,x)F(t,x_*)(x - x_*) f(t,x)\,\mathrm{d}f(t,x_*)\right],$$

with initial condition $f(0,x) = f^{\mathrm{in}}(x)$ for all $x \in E$. Letting f being equal to 0 on $[x_M, +\infty)$ we have the results on $\mathbb{R}_+$. □

6. Numerical simulations

This section collects some numerical results on the model with linear interaction rule, i.e. when the payoff rules of the game are those described in Table 1.

The quantitative results of this section have been obtained by using a particle method. The unknown function f has been discretized by means of a sum of Dirac masses, centered in $x_k(t)$, $1 \le k \le N$, representing a set composed by $N \in \mathbb{N}$ macro-particles that evolve in time.

More precisely, we have approximated the distribution function in this way:

$$f = \sum_{k=1}^N \omega_k\, \delta(x - x_k(t)),$$

where ω_k is the weight of the k-th particle.

Once the number N of numerical particles has been chosen, the problem has been initialized by approximating the initial condition f^{in} with

$$f^{\mathrm{in}}(x,v) = \sum_{k=1}^N \omega_k\, \delta(x - x_k^0),$$

and then the time evolution of the system has been obtained by deducing the time evolution of the macro-particles trough the binary exchange rules (9).

All the weights of the particles are identical and their magnitude has been chosen for reproducing the mass of the initial condition, which has been normalized by convenience:

$$\omega_k = \frac{1}{N}\|f^{\mathrm{in}}\|_{L^1(\mathbb{R}_+)} = \frac{1}{N},$$

for all $1 \le k \le N$.

The main phenomenon which governs the time evolution of the system is the game defined in Table 1, which allows a net gain to the poorer player only. However, the wealth of the competitor being unknown, the agents choose their strategies according to their relative wealth with respect

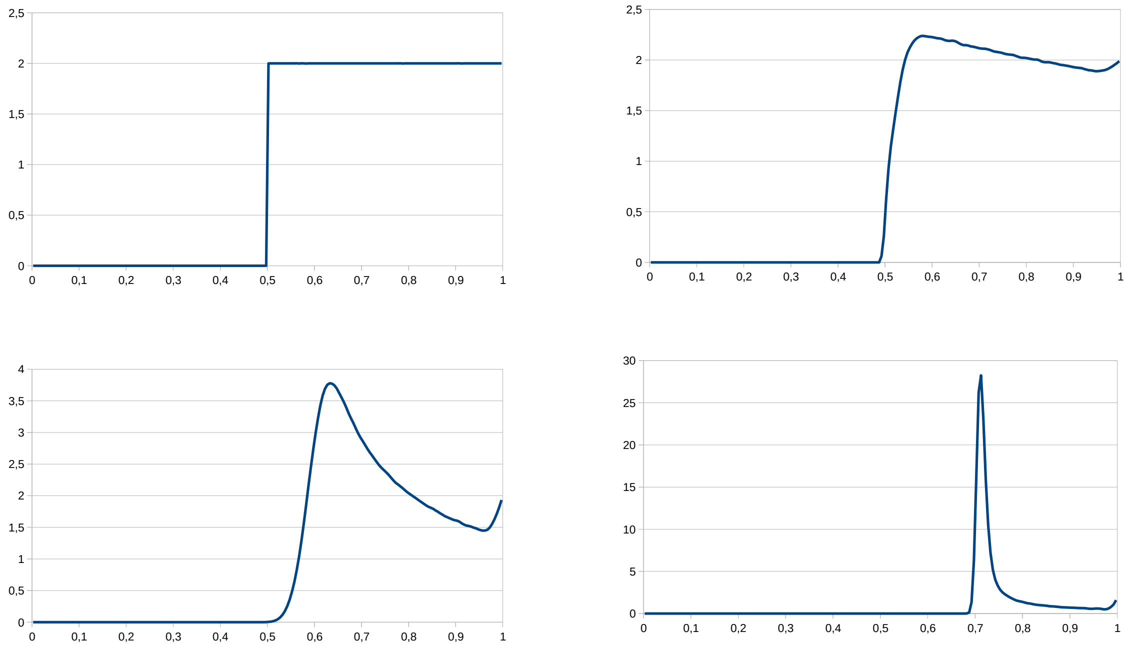

FIGURE 2. Evolution history of the population density with initial condition $f_1^{\rm in}(x) = 2 \times \mathbf{1}_{1/2 \le x \le 1}$, at $t = 0$, $t = 0.1$, $t = 0.65$ and $t = 5$.

to the population. Denote the wealth of the i-th player by x_i. Obviously, the probability that the i-th player is richer than than the j-th player is given by

$$\mathbb{P}(x_i > x_j, (i, j)) = \frac{1}{N} \sum_{k=0}^{N} \mathcal{H}(x_i - x_k), \tag{17}$$

where $\mathcal{H}$ is the Heaviside function. Therefore, a generic player i chooses the strategy 1 with a probability that is given by $\mathbb{P}(x_i > x_j, (i, j))$.

The method is based on the modeling hypotheses which are at the basis of the equations themselves: it is hence robust by its own nature and it can also be easily generalized to more complicated games, possibly with high dimensionality. It is moreover very simple to implement and the evolution law of the game is treated in an exact way, provided that its analytical formulation is known.

Unfortunately, the number of particles that is needed in order to achieve a reasonable accuracy is very large, and it has a great influence on the performances. In our tests we used $N = 2 \times 10^6$ numerical particles, and a time step $\Delta t = 10^{-2}$. The domain of definition of the problem is the interval $\Omega = [0, 1]$, which has been subdivided in 200 sub-intervals of length $\Delta x = 5 \times 10^{-3}$.

The simulations, written in `C`, heavily needed sequences of random numbers. In our tests, we used the standard `rand()` pseudo-random number generator, which has provided a satisfactory approximation of the uniform distribution.

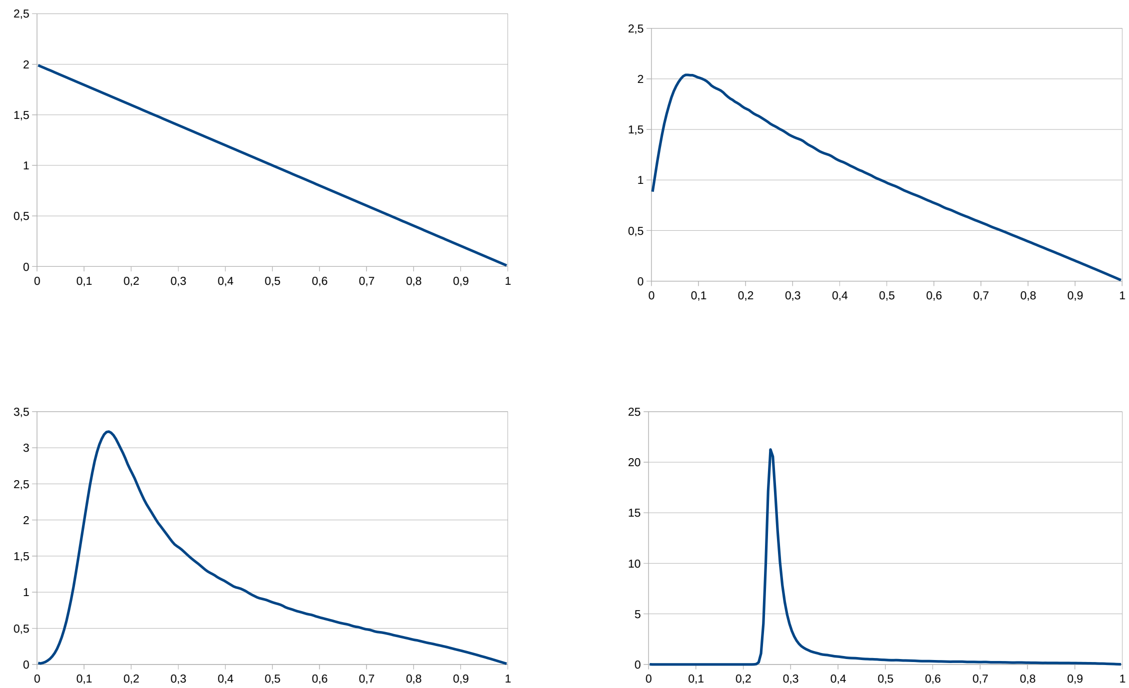

FIGURE 3. Evolution history of the population density with initial condition $f_2^{\text{in}}(x) = 2(1-x)\mathbf{1}_{0\le x\le 1}$, at $t=0$, $t=0.1$, $t=0.65$ and $t=5$.

The numerical reconstruction of the density profile has been obtained by projecting the particles on the mesh by convoluting the Dirac mass which represents the particle and an hat function with mass $1/N$ and base width $4\Delta x$.

For our simulations, we have chosen two initial conditions:

$$f_1^{\text{in}}(x) = 2 \times \mathbf{1}_{1/2\le x\le 1}, \qquad f_2^{\text{in}}(x) = 2(1-x)\mathbf{1}_{0\le x\le 1}.$$

The results of the simulations with initial condition $f_1^{\text{in}}(x) = 2\times \mathbf{1}_{1/2\le x\le 1}$ are plotted in Figure 2. The profiles are ordered from the left to the right and from the top to the bottom and describe the system at various time instants: $t=0$, $t=0.1$, $t=0.65$ and $t=5$.

We note that the support of the distribution function does not vary with respect to time. However, the density numerically tends to concentrate around the only value which is compatible with the conservation of the first moment, i.e. around $x=0.75$ (a longer simulation, up to $t=15$ is in agreement with this expected behavior). We observe moreover that the fraction of the population around the upper bound of the interval, i.e. around $x=1$ practically does not evolve in time. Indeed the richest agents never play the option 0 (since this event is governed by the probability of finding someone richer). As a consequence, a boundary layer appears in $x=1$. It is not caused by any boundary conditions (which are absent in our problem), but it is rather a consequence of the rationality of the agents, which tend to maximize their earnings: the richest agents use a conservative strategy, which suggests them to play 1 exclusively.

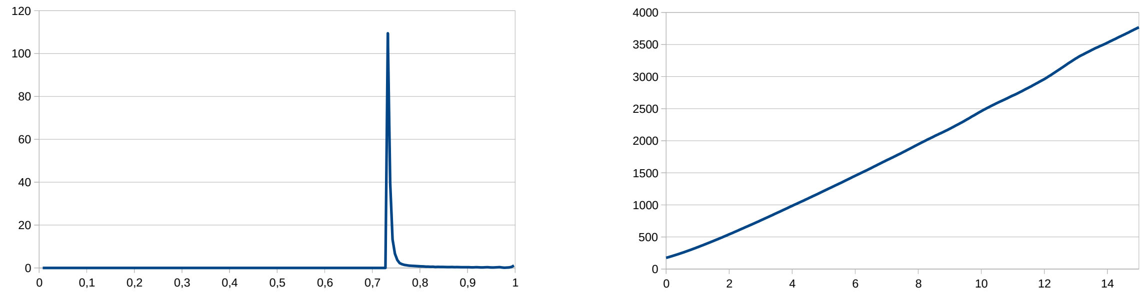

FIGURE 4. Profile of the population density at $t = 15$ (left) and time history of $m_2^{-4/3}$ (right) with initial condition $f_1^{\text{in}}(x) = 2 \times \mathbf{1}_{1/2 \le x \le 1}$.

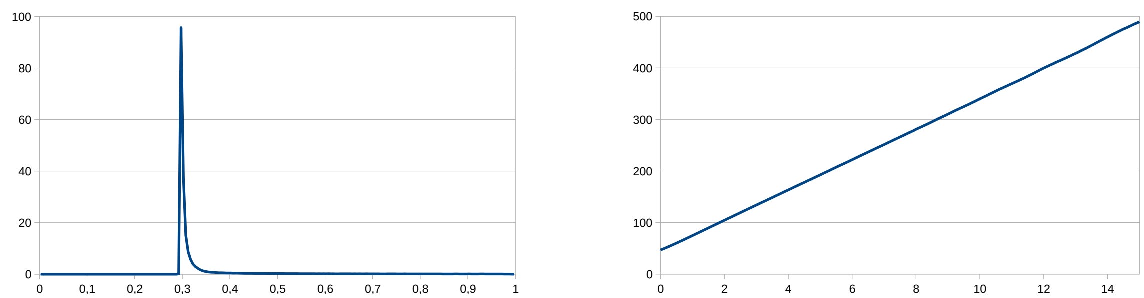

FIGURE 5. Profile of the population density at $t = 15$ (left) and time history of $m_2^{-4/3}$ (right) with initial condition $f_2^{\text{in}}(x) = 2(1 - x)\mathbf{1}_{0 \le x \le 1}$.

In Figure 3 we have collected some results of the simulations with initial condition $f_2^{\text{in}}(x) = 2(1 - x)\mathbf{1}_{0 \le x \le 1}$. Also in this case, the profiles are ordered from the left to the right and from the top to the bottom and describe the system at the same time instants as in Figure 2: $t = 0$, $t = 0.1$, $t = 0.65$ and $t = 5$.

We numerically observed that the support of the distribution function does not evolve in time. In this case, $f_1^{\text{in}}(1) = 0$, and hence no boundary layer exists around $x = 1$. In this case, the time evolution of the system induces a concentration of the density, which is peaked around $x = 0.33$.

In both cases, the convergence speed to the asymptotic profile (which is expected to be a Dirac mass satisfying the conservation of mass and first order moment) is very slow. The numerical simulations in long time seem to suggest a time decay of order $t^{-3/4}$, as shown in Figures 4 and 5, which represent the profile of the solution at $t = 15$ and the time evolution of $m_2^{-4/3}$, with initial condition $f_1^{\text{in}}(x) = 2 \times \mathbf{1}_{1/2 \le x \le 1}$ and $f_2^{\text{in}}(x) = 2(1-x)\mathbf{1}_{0 \le x \le 1}$ respectively, where

$$m_2(t) := \int_{\mathbb{R}_+} \left| x - \frac{m_1}{m_0} \right|^2 f(t,x)\,\mathrm{d}x.$$

We aim to study the long-time behavior of the model and of its quasi-invariant approximation at the analytical level in the near future.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[2] R. J. Aumann. Markets with a continuum of traders. *Econometrica*, 32(1/2):39–50, 1964.

[3] R. J. Aumann and L. S. Shapley. *Values of non-atomic games.* Princeton University Press, 2015.

[4] N. Bellomo. *Modeling complex living systems.* Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Inc., Boston, MA, 2008. A kinetic theory and stochastic game approach.

[5] N. Bellomo, B. Carbonaro, and L. Gramani. On the kinetic and stochastic games theory for active particles: some reasonings on open large living systems. *Math. Comput. Modelling*, 48(7-8):1047–1054, 2008.

[6] M. Bisi, G. Spiga, and G. Toscani. Kinetic models of conservative economies with wealth redistribution. *Commun. Math. Sci.*, 7(4):901–916, 12 2009.

[7] M. Blonski. Anonymous games with binary actions. *Games and Economic Behavior*, 28(2):171 – 180, 1999.

[8] L. Boudin and F. Salvarani. The quasi-invariant limit for a kinetic model of sociological collective behavior. *Kinet. Relat. Models*, 2(3):433–449, 2009.

[9] L. Boudin and F. Salvarani. Modelling opinion formation by means of kinetic equations. In *Mathematical modeling of collective behavior in socio-economic and life sciences*, Model. Simul. Sci. Eng. Technol., pages 245–270. Birkhäuser Boston, Inc., Boston, MA, 2010.

[10] M. Burger, A. Lorz, and M.-T. Wolfram. On a Boltzmann mean field model for knowledge growth. *SIAM J. Appl. Math.*, 76(5):1799–1818, 2016.

[11] M. Burger, A. Lorz, and M.-T. Wolfram. Balanced growth path solutions of a Boltzmann mean field game model for knowledge growth. *Kinet. Relat. Models*, 10(1):117–140, 2017.

[12] C. Castellano, S. Fortunato, and V. Loreto. Statistical physics of social dynamics. *Rev. Mod. Phys.*, 81:591–646, 2009.

[13] C. Cercignani. *The Boltzmann equation and its applications*, volume 67 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1988.

[14] S. Cordier, L. Pareschi, and G. Toscani. On a kinetic model for a simple market economy. *Journal of Statistical Physics*, 120(1):253–277, Jul 2005.

[15] P. Degond, J.-G. Liu, and C. Ringhofer. Large-scale dynamics of mean-field games driven by local Nash equilibria. *J. Nonlinear Sci.*, 24(1):93–115, 2014.

[16] B. Dring and G. Toscani. Hydrodynamics from kinetic models of conservative economies. *Physica A: Statistical Mechanics and its Applications*, 384(2):493–506, 2007.

[17] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[18] R. T. Glassey. *The Cauchy problem in kinetic theory.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.

[19] D. Helbing. Boltzmann-like and Boltzmann-Fokker-Planck equations as a foundation of behavioral models. *Phys. A*, 196:546–573, 1993.

[20] D. Helbing. Stochastic and Boltzmann-like models for behavioral changes, and their relation to game theory. *Phys. A*, 193:241–258, 1993.

[21] D. Helbing. A mathematical model for the behavior of individuals in a social field. *J. Math. Sociol.*, 19(3):189–219, 1994.

[22] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ϵ-Nash equilibria. *IEEE Trans. Automat. Control*, 52(9):1560–1571, 2007.

[23] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, 6(3):221–251, 2006.
[24] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.
[25] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.
[26] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.
[27] P.-L. Lions and J.-M. Lasry. Large investor trading impacts on volatility. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24(2):311–323, 2007.
[28] R. E. Lucas Jr and B. Moll. Knowledge growth and the allocation of time. *Journal of Political Economy*, 122(1):1–51, 2014.
[29] A. Tosin. Kinetic equations and stochastic game theory for social systems. In *Mathematical models and methods for planet Earth*, volume 6 of *Springer INdAM Ser.*, pages 37–57. Springer, Cham, 2014.

F.S.: Université Paris-Dauphine, PSL Research University, Ceremade, UMR CNRS 7534, F-75775 Paris Cedex 16, France & Università degli Studi di Pavia, Dipartimento di Matematica, I-27100 Pavia, Italy

E-mail address: `francesco.salvarani@unipv.it`

D.T.: Université Paris-Dauphine, PSL Research University, Ceremade, UMR CNRS 7534, F-75775 Paris Cedex 16, France

E-mail address: `tonon@ceremade.dauphine.fr`