Optimal stability for a first order coefficient in a non-self-adjoint wave equation from dirichlet-to-neumann map

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OPTIMAL STABILITY FOR A FIRST ORDER COEFFICIENT IN A NON-SELF-ADJOINT WAVE EQUATION FROM DIRICHLET-TO-NEUMANN MAP

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ABSTRACT. This paper is focused on the study of an inverse problem for a non-self-adjoint hyperbolic equation. More precisely, we attempt to stably recover a first order coefficient appearing in a wave equation from the knowledge of Neumann boundary data. We show in dimension n greater than two, a stability estimate of Hölder type for the inverse problem under consideration. The proof involves the reduction to an auxiliary inverse problem for an electro-magnetic wave equation and the use of an appropriate Carleman estimate.
Keywords: Inverse problem, Stability result, Dirichlet-to-Neumann map, Carleman estimate.

1. INTRODUCTION AND MAIN RESULTS

The main purpose of this paper is the study of an inverse problem of determining a coefficient of order one on space appearing in a non-self-adjoint wave equation. Let $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$, be an open bounded set with a sufficiently smooth boundary $\Gamma = \partial\Omega$. For $T > 0$, we denote by $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. We introduce the following initial boundary value problem for the wave equation with a velocity field V,

$$\begin{cases} \mathscr{L}_V u := (\partial_t^2 - \Delta + V\cdot\nabla)u = 0 & \text{in } Q, \\ u_{|t=0} = \partial_t u_{|t=0} = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma, \end{cases} \tag{1.1}$$

where $V \in W^{1,\infty}(\Omega,\mathbb{R}^n)$ is a real vector field and $f \in \mathcal{H}_0^1(\Sigma) := \{f \in H^1(\Sigma),\ f_{|t=0} = 0\}$ is the Dirichlet data that is used to probe the system. We may define the so-called Dirichlet-to-Neumann (DN) map associated with the wave operator $\mathscr{L}_V$ as follows

$$\begin{array}{rcl} \Lambda_V : \mathcal{H}_0^1(\Sigma) & \longrightarrow & L^2(\Sigma) \\ f & \longmapsto & \partial_\nu u, \end{array}$$

where ν denotes the unit outward normal to Γ at x and $\partial_\nu u$ stands for $\nabla u \cdot \nu$.

The inverse problem we address is to determine the velocity field V appearing in (1.1) from the knowlegde of the DN map Λ_V and we aim to derive a stability result for this problem. To our knowledge this paper is the first treating the recovery of a coefficient of order one on space appearing in a wave equation.

The problem of recovering coefficients appearing in hyperbolic equations gained increasing popularity among mathematicians within the last few decades and there are many works related to this topic. But they are mostly concerned with coefficients of order zero on space. In the case where the unknown coefficient is depending only on the spatial variable, Rakesh and Symes [22] proved by means of geometric optics solutions, a uniqueness result in recovering a time-independent potential in a wave equation from global Neumann data. The uniqueness by local Neumann data, was considered by Eskin [14] and Isakov [16]. In [5], Bellassoued, Choulli and Yamamoto proved a log-type stability estimate, in the case where the Neumann data are observed on any arbitrary subset of the boundary. Isakov and Sun [17] proved that the knowledge of local Dirichlet-to-Neumann map yields a stability result of Hölder type in determining a coefficient in a subdomain. As for stability results obtained from global Neumann data, one can see Sun [27], Cipolatti and Lopez [13]. There are also growing publications on the related inverse problems in Riemannian case. We mention e.g the paper of Bellassoued and Dos Santos Ferreira [6], Stefanov and Uhlmann [26] and [20] in which Liu and Oksanen consider the problem of recovering a wave speed c from acoustic boundary measurements modelled by the hyperbolic Dirichlet to Neumann map. Other than the mentioned papers, the recovery of time-dependent coefficients in hyperbolic equations has also been developped recently, we refer e.g to Bellassoued and Ben Aïcha [2, 3] and in the Riemmanian case, we refer to the work of Waters [30], in which a stability of Hölder type was proved, for the identification of the X-ray transform of a time-dependent coefficient in an hyperbolic equation. In [24], R. Salazar considered the stability issue and extended the result of the paper [23] to more general coefficients and he established a stability result for compactly supported coefficients provided T is sufficiently large. For curiosity, the reader can also see [9, 18] and the references therein.

The above papers are concerned only with coefficients of order zero on space. In the case where the unknown coefficient is of order one, we cite for example the paper of Pohjola [21], in which he considered an inverse problem for a steady state convection diffusion equation. He showed by reducing his problem to the case of a stationary magnetic Schrödinger equation that a velocity field can be uniquely determined from the knowledge of Neumann measurements. Cheng, Nakamura and Somersalo [12] treated the same problem and they proved a uniqueness result for more regular coefficients. Salo [25] also studied this problem and proved a uniqness result in the case where the coefficient is Lipschitz continuous. The overall method of proving uniqueness in these papers was based on reducing the inverse problems under investigation to similar ones for self-adjoint operators and applying the maximum principle. We can also refer to the paper [19] in which a uniqueness result for a general non-self-adjoint second-order elliptic operator on a manifold with boundary is addressed.

The stability for problems associated with non self-adjoint operators is never treated before. In this work, we consider this challenging problem and we establish a stability estimate of Hölder type for the recovery of the first order coefficient V appearing in the wave operator $\mathscr{L}_V$ from the knowledge of the DN map Λ_V. The proof of the stability estimate requires the use of an L^2-weighted inequality called a Carleman estimate designed for elliptic operators (see [8, 11]) instead of the maximum principle used in [21].

Before stating our main result, we introduce the admissible set of the coefficients V. Given $M > 0$ and $V_0 \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, we define

$$\mathcal{V}(M, V_0) := \{V \in W^{1,\infty}(\Omega, \mathbb{R}^n),\ \|V\|_{W^{1,\infty}(\Omega)} \leqslant M,\ V = V_0 \text{ on } \Gamma\}.$$

Then our main result can be stated as follows

Theorem 1.1. *Let V_1, $V_2 \in \mathcal{V}$ such that $V_1 - V_2 \in W^{2,\infty}(\Omega)$. Then, there exist positive constants $\kappa \in (0,1)$ and $C > 0$ such that*

$$\|V_1 - V_2\|_{L^2(\Omega)} \leqslant C \|\Lambda_{V_1} - \Lambda_{V_2}\|^{\kappa}.$$

Here the constant C is depending only on Ω and M and $\|\cdot\|$ denotes the norm in $\mathcal{L}(\mathcal{H}_0^1(\Sigma); L^2(\Sigma))$.

The above statement claims stable determination of the velocity field V from the knowledge of the DN map Λ_V, where both the Dirichlet and Neumann data are performed on the whole boundary Σ. By Theorem 1.1, we can readily derive the following

Corollary 1.2. *Let V_1, $V_2 \in \mathcal{V}$. Then, we have that $\Lambda_{V_1} = \Lambda_{V_2}$ implies $V_1 = V_2$ everywhere in Ω.*

We point out that since the hyperbolic operator $\mathscr{L}_V$ is not self-adjoint, then we should first head toward an auxiliary problem for an electro-magnetic wave equation in order to be able to prove our main results.

The remainder of this paper is organized as follows: in Section 2, we reduce the inverse problem associated with the equation (1.1) to a corresponding inverse problem for an electro-magnetic wave equation. By the use of an elliptic Carleman estimate, we give in Section 3 the proof of Theorem 1.1.

2. REDUCTION OF THE PROBLEM

The overall method of proving the stability for the inverse problem under consideration is mainly based on reducing it to an equivalent problem concerning the following electro-magnetic wave equation

$$\begin{cases} \mathscr{H}_{A,q}u := (\partial_t^2 - \Delta_A + q)u = 0 & \text{in } Q, \\ u_{|t=0} = \partial_t u_{|t=0} = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma, \end{cases} \tag{2.2}$$

where $f \in \mathcal{H}_0^1(\Sigma)$ is a non homogeneous Dirichlet data, $A = (a_j)_{1 \leqslant j \leqslant n} \in W^{1,\infty}(\Omega, \mathbb{C}^n)$ is a pure imaginary complex magnetic vector and $q \in L^\infty(\Omega, \mathbb{R})$ is a bounded electric potential. Here Δ_A denotes the magnetic Laplacien and it is given by

$$\Delta_A = \sum_{j=1}^n (\partial_j + i a_j)^2 = \Delta + 2iA \cdot \nabla + i\text{div}\, A - A \cdot A.$$

According to [4, 8, 10], the initial boudary value problem (2.2) is well posed and we have the existence of a unique solution within the following class $u \in \mathcal{C}([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$. Therefore, we may define the DN map $N_{A,q}$ associated with the wave equation (2.2) as follows

$$\begin{array}{rcl} N_{A,q} : \mathcal{H}_0^1(\Sigma) & \longrightarrow & L^2(\Sigma) \\ f & \longmapsto & (\partial_\nu + iA \cdot \nu)u. \end{array}$$

The purpose of this section is to reduce the inverse problem associated with the wave equation (1.1) to an auxiliary problem for (2.2). Note that if A is of real valued then $\mathscr{H}_{A,q}$ is a self-adjoint wave operator. The strategy is mainly inspired by [21, 12, 25]. We specify the choice of the pure imaginary complex vector A and the real function q in such a way $\mathscr{H}_{A,q}$ coïncide with $\mathscr{L}_V$ and the same for the associated DN maps.

We need first to introduce some notations. Let us consider the following set

$$\mathcal{H}_T^1(\Sigma) := \{g \in H^1(\Sigma);\ g_{|t=T} = 0\}.$$

For $g \in H_T^1(\Sigma)$, we define the adjoint operator of Λ_V as follows:

$$\begin{array}{rcl} \Lambda_V^* : \mathcal{H}_T^1(\Sigma) & \longrightarrow & L^2(\Sigma) \\ g & \longmapsto & \partial_\nu v, \end{array}$$

where v here denotes the unique solution of the backward problem

$$\begin{cases} \mathscr{L}_V^* v = \partial_t^2 v - \Delta v - \text{div}(Vv) = 0 & \text{in } Q, \\ v_{|t=T} = \partial_t v_{|t=T} = 0 & \text{in } \Omega, \\ v = g & \text{on } \Sigma. \end{cases}$$

On the other hand, we define the adjoint operator of the DN map $N_{A,q}$ as follows

$$\begin{array}{rcl} N_{A,q}^* : \mathcal{H}_T^1(\Sigma) & \longrightarrow & L^2(\Sigma) \\ g & \longmapsto & (\partial_\nu + iA \cdot \nu)v, \end{array}$$

associated to the backward problem

$$\begin{cases} \mathscr{H}^*_{A,q}v = \mathscr{H}_{\overline{A},q}v = 0 & \text{in } Q, \\ v_{|t=T} = \partial_t v_{|t=T} = 0 & \text{in } \Omega, \\ v = g & \text{on } \Sigma. \end{cases}$$

In the sequel, we shall make use of the following Green formula for the magnetic Laplacian. Let A be a pure imaginary complex vector in $W^{1,\infty}(\Omega,\mathbb{C}^n)$. Then, the following identity holds true

$$\begin{aligned} \int_\Omega \Delta_A u\,\overline{v}\,dx &= -\int_\Omega (\nabla + iA)u\,\overline{(\nabla - iA)v}\,dx + \int_\Gamma (iA\cdot\nu + \partial_\nu)u\overline{v}\,d\sigma \\ &= \int_\Omega \overline{\Delta_{\overline{A}}v}\,u\,dx + \int_\Gamma \Big((\partial_\nu + i\nu\cdot A)u\,\overline{v} - \overline{(\partial_\nu + i\nu\cdot\overline{A})v}\,u\Big)d\sigma, \end{aligned} \tag{2.3}$$

for $u,\, v \in H^1(\Omega)$ such that $\Delta u,\, \Delta\, v \in L^2(\Omega)$. Here $d\sigma$ is the Euclidean surface measure on Γ. Finally, we introduce the admissible sets of the coefficients A and q: for $M > 0$, $A_0 \in W^{1,\infty}(\Omega,\mathbb{C}^n)$ and $q_0 \in L^\infty(\Omega,\mathbb{R})$, we define

$$\mathcal{A}(M,A_0) := \{A \in W^{1,\infty}(\Omega,\mathbb{C}^n),\ \|A\|_{W^{1,\infty}(\Omega)} \leqslant M,\ A = A_0 \text{ on } \Gamma\},$$

and

$$\mathcal{Q}(M,q_0) := \{q \in L^\infty(\Omega,\mathbb{R}),\ \|q\|_{L^\infty(\Omega)} \leqslant M,\ q = q_0 \text{ on } \Gamma\}.$$

We shall now give some properties of the considered operators as well as the associated DN maps. This statement will play a crucial role in proving Theorem 1.1.

Lemma 2.1. *Let $V_1,\, V_2 \in \mathcal{V}$. We define $A_1,\, A_2 \in \mathcal{A}(M,A_0)$ and $q_1,\, q_2 \in \mathcal{Q}(M,q_0)$ by*

$$A_j = \frac{i}{2}V_j, \quad \textit{and} \quad q_j = \frac{1}{4}V_j^2 - \frac{1}{2}\,\mathrm{div}\, V_j, \quad j = 1,\, 2. \tag{2.4}$$

Then, we have

$$\mathscr{H}_{A_j,q_j} = \mathscr{L}_{V_j}, \qquad \mathscr{H}^*_{A_j,q_j} = \mathscr{L}^*_{V_j}, \quad \textit{and} \quad N_{\overline{A_j},q_j} = N_{-A_j,q_j} = N^*_{A_j,q_j}, \quad j = 1,\, 2.$$

Moreover,

$$\|N_{A_1,q_1} - N_{A_2,q_2}\| = \|\Lambda_{V_1} - \Lambda_{V_2}\|, \tag{2.5}$$

where $\|\cdot\|$ stands for the norm in $\mathcal{L}(\mathcal{H}^1_0(\Sigma);L^2(\Sigma))$.

Proof. In light of (2.4), one can easily see that for any $u,\, v \in H^2(Q)$ we have

$$\mathscr{H}_{A_j,q_j}u = (\partial_t^2 - \Delta_{A_j} + q_j(x))u = (\partial_t^2 - \Delta + V_j\cdot\nabla)u = \mathscr{L}_{V_j}u, \tag{2.6}$$

and

$$\mathscr{H}^*_{A_j,q_j}v = (\partial_t^2 - \Delta_{\overline{A_j}} + q_j(x))v = (\partial_t^2 - \Delta_{(-A_j)} + q_j(x))v = \partial_t^2 v - \Delta v - \mathrm{div}(V_j v) = \mathscr{L}^*_{V_j}v. \tag{2.7}$$

A simple application of (2.3) yields $N_{\overline{A},q} = N_{-A,q} = N^*_{A,q}$. We move now to prove (2.5). Let us denote by u_j and v_j, $j = 1,\, 2$, the solutions of

$$\begin{cases} \mathscr{L}_{V_j}u_j = 0 & \text{in } Q, \\ u_{j|t=0} = \partial_t u_{j|t=0} = 0 & \text{in } \Omega, \\ u_j = f & \text{on } \Sigma, \end{cases} \quad ; \quad \begin{cases} \mathscr{L}^*_{V_j}v_j = 0 & \text{in } Q, \\ v_{j|t=T} = \partial_t v_{j|t=T} = 0 & \text{in } \Omega, \\ v_j = g & \text{on } \Sigma, \end{cases} \tag{2.8}$$

where $f \in \mathcal{H}_0^1(\Sigma)$ and $g \in \mathcal{H}_T^1(\Sigma)$. By multiplying the first equation in the left hand side of (2.8) by $\overline{v}_j$ and integrating by parts, we get

$$\int_\Sigma \Lambda_{V_j}(f)\overline{g}\, d\sigma\, dt = \int_Q \Big(-\partial_t u_j \partial_t \overline{v}_j + \nabla u_j \cdot \nabla \overline{v}_j + V_j \cdot \nabla u_j\, \overline{v}_j \Big)\, dx\, dt. \tag{2.9}$$

On the other hand, based on (2.6) and (2.7), u_j and v_j with $j = 1,\, 2$, are also solutions to

$$\left\{ \begin{array}{ll} \mathscr{H}_{A_j,q_j} u_j = 0 & \text{in } Q, \\ u_{j|t=0} = \partial_t u_{j|t=0} = 0 & \text{in } \Omega, \\ u_j = f & \text{on } \Sigma, \end{array} \right. \;;\quad \left\{ \begin{array}{ll} \mathscr{H}^*_{A_j,q_j} v_j = 0 & \text{in } Q, \\ v_{j|t=T} = \partial_t v_{j|t=T} = 0 & \text{in } \Omega, \\ v_j = g & \text{on } \Sigma. \end{array} \right. \tag{2.10}$$

By multiplying the equation in the left hand side of (2.10) by $\overline{v}_j$ and after integrating by parts, we get in light of (2.4) and (2.3),

$$\int_\Sigma N_{A_j,q_j}(f)\overline{g}\, d\sigma\, dt = \int_Q \Big(-\partial_t u_j \partial_t \overline{v}_j + \nabla u_j \cdot \nabla \overline{v}_j + \frac{1}{2} V_j \cdot \nabla u_j \overline{v}_j - \frac{1}{2} V_j \cdot \nabla \overline{v}_j u_j - \frac{1}{2}\mathrm{div} V_j u_j \overline{v}_j \Big)\, dx\, dt.$$

This immediately implies that

$$\int_\Sigma N_{A_j,q_j}(f)\overline{g}\, d\sigma\, dt = \int_Q \Big(-\partial_t u_j \partial_t \overline{v}_j + \nabla u_j \cdot \nabla \overline{v}_j + V_j \cdot \nabla u_j \overline{v}_j \Big)\, dx\, dt - \frac{1}{2}\int_\Sigma V_j \cdot \nu\, u_j \overline{v}_j\, d\sigma\, dt. \tag{2.11}$$

Hence, from (2.9) and (2.11), we find out that

$$\int_\Sigma N_{A_j,q_j}(f)\overline{g}\, d\sigma\, dt = \int_\Sigma \Lambda_{V_j}(f)\overline{g}\, d\sigma\, dt - \frac{1}{2}\int_\Sigma V_j \cdot \nu\, f\, g\, d\sigma\, dt.$$

Owing to the assumption that $V_1 = V_2$ on Γ, we get the desired result. □

Due to Lemma 2.1, the inverse problem under investigation may be equivalently reformulated as to whether the magnetic potential A and the electric potential q in (2.2) can be recovered from the knowledge of $N_{A,q}$. This is the auxiliary inverse problem that we address in the remaining of this section.

As it was noted in Sun [28], the DN map is invariant under a gauge transformation. Namely, given any $\varphi \in \mathcal{C}^2(\overline{\Omega})$ with $\varphi_{|\Gamma} = 0$, one has $N_{A+\nabla\varphi,q} = N_{A,q}$. Hence, the magnetic potential A can not be uniquely determined by $N_{A,q}$. However it is possible to show that the knowledge of the DN map $N_{A,q}$ stably determines the electric potential q and the magnetic field corresponding to the pure imaginary complex potential A which is given by the 2-form $d\alpha_A$ defined as follows

$$d\alpha_A = \sum_{i,j=1}^n \Big(\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_j} \Big) dx_j \wedge dx_i.$$

Actually, this problem is closely related to the one treated in Bellassoued and Ben Joud [4] in the absence of the electric potential, in Bellassoued [1] in the Riemmanian case and in Ben Joud [10]. Compared with the paper of Ben Joud [10], we formulate this auxiliary problem for *less* regular *complex* magnetic potentials.

Theorem 1.1 can then be reduced to the following equivalent statement

Theorem 2.2. *Let $A_1,\, A_2 \in \mathcal{A}(M, A_0)$, and $q_1,\, q_2 \in \mathcal{Q}(M, q_0)$. Assume that $A_1 - A_2 \in W^{2,\infty}(\Omega, \mathbb{C}^n)$ and $q_1 - q_2 \in W^{1,\infty}(\Omega, \mathbb{R})$. Then, there exist $C > 0$ and $\mu \in (0,1)$ such that we have*

$$\|d\alpha_{A_1} - d\alpha_{A_2}\|_{H^{-1}(\Omega)} + \|q_1 - q_2\|_{H^{-1}(\Omega)} \leqslant C \|N_{A_2,q_2} - N_{A_1,q_1}\|^\mu.$$

The above theorem claims stable determination of the magnetic field $d\alpha_A$ and the electric potential q from the global Neumann measurement $N_{A,q}$. Here we improve the result of Ben Joud [10] by considering complex magnetic potentials. The regularity condition imposed on admissible magnetic potentials is also weakened from $W^{3,\infty}(\Omega)$ to $W^{1,\infty}(\Omega)$.

The rest of this section is devoted to proving this auxiliary result.

2.1. **Geometrical optics solutions.** Section 2 mainly aims at the study of the auxiliary inverse problem associated with the electro-magnetic wave equation (2.2), that is the identification of $d\alpha_A$ and q from the DN map $N_{A,q}$. To begin with, we shall first construct geometrical optics solutions for the equation (2.2) associated with a suitable smooth approximation of the magnetic potential (see [7, 21]). For this purpose, we first consider $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and notice that for all $\omega \in \mathbb{S}^{n-1}$ the function

$$\phi(x,t) = \varphi(x + t\omega), \tag{2.12}$$

solves the following transport equation

$$(\partial_t - \omega \cdot \nabla)\phi(x,t) = 0.$$

We will build solutions associated with a suitable smooth approximation of the magnetic potential. This requires to extend the potentials $A_1,\, A_2 \in \mathcal{A}(A_0, M)$ to a larger domain as follows:

Lemma 2.3. *(see[29]) Let Ω be a bounded domain that is compactly contained in $\tilde{\Omega} \subset \mathbb{R}^n$. Let $A_1,\, A_2 \in W^{1,\infty}(\Omega)$ such that $\|A_j\|_{W^{1,\infty}(\Omega)} \leqslant M$, $j = 1, 2$ and $A_1 = A_2$ on Γ. Then, there exist two extensions $\tilde{A}_1,\, \tilde{A}_2 \in W_c^{1,\infty}(\tilde{\Omega})$, such that $\tilde{A}_1 = \tilde{A}_2$ on $\tilde{\Omega}\backslash\Omega$. Moreover, there exists a positive constant $C > 0$ such that*

$$\|\tilde{A}_j\|_{W^{1,\infty}(\tilde{\Omega})} \leqslant C\,M, \quad j = 1,\, 2.$$

Here C is depends only on Ω, $\tilde{\Omega}$ and M.

Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that Supp $\chi \subset B(0,1)$, $0 \leqslant \chi \leqslant 1$, and $\displaystyle\int_{\mathbb{R}^n} \chi(x)\,dx = 1$. For a sufficiently large $\lambda > 0$, we denote $\chi_\lambda(x) = \lambda^{n\alpha}\chi(\lambda^\alpha x)$, with $0 < \alpha \leqslant 1/2$. For $j = 1$, 2, we define the smooth approximations $A_{j,\lambda}^\sharp$ of the extensions $\tilde{A}_j \in W_c^{1,\infty}(\mathbb{R}^n, \mathbb{C})$ as follows:

$$A_{j,\lambda}^\sharp := \chi_\lambda * \tilde{A}_j, \quad j = 1,\, 2. \tag{2.13}$$

This terminology is justified by the fact that $A_{j,\lambda}^\sharp$ gets closer to $\tilde{A}_j$ as λ goes to ∞. This can be seen from the following result:

Lemma 2.4. *Let $\tilde{A} \in W_c^{1,\infty}(\mathbb{R}^n, \mathbb{C}^n)$ be such that $\|\tilde{A}\|_{W^{1,\infty}(\mathbb{R}^n)} \leqslant M$. Then, there exists a positive constant C depending only on M and Ω such that for all $\lambda > 0$ we have*

$$\|\tilde{A} - A_\lambda^\sharp\|_{L^\infty(\mathbb{R}^n)} \leqslant C\lambda^{-\alpha}. \tag{2.14}$$

Moreover, for any multi-index $\gamma \in \mathbb{N}^n$, with $|\gamma| \geqslant 1$, we have

$$\|\partial^\gamma A_\lambda^\sharp\|_{L^\infty(\mathbb{R}^n)} \leqslant C\lambda^{\alpha(|\gamma|-1)}, \tag{2.15}$$

where C is a positive constant depedning only on M and Ω.

Proof. From [15], we have $\tilde{A} \in \mathcal{C}^{0,1}(\mathbb{R}^n)$ and $\|\tilde{A}\|_{\mathcal{C}^{0,1}(\mathbb{R}^n)} \leqslant \|\tilde{A}\|_{W^{1,\infty}(\mathbb{R}^n)}$, thus in light of (2.13), we have

$$\begin{aligned}
|\tilde{A} - A^\sharp_\lambda| &= \Big| \int_{\mathbb{R}^n} \tilde{A}(x-y)\chi_\lambda(y)\,dy - \tilde{A}(x)\Big| \\
&= \Big| \int_{\mathbb{R}^n} \tilde{A}(x-y)\chi(\lambda^\alpha y)\lambda^{n\alpha}\,dy - \tilde{A}(x)\Big| \\
&= \Big| \int_{\mathbb{R}^n} \tilde{A}(x-\lambda^{-\alpha}y)\chi(y) - \tilde{A}(x)\chi(y)\;dy\Big| \\
&\leqslant \|\tilde{A}\|_{\mathcal{C}^{0,1}(\mathbb{R}^n)}\lambda^{-\alpha}\int_{\mathbb{R}^n}|y||\chi(y)|\,dy \\
&\leqslant C\lambda^{-\alpha},
\end{aligned}$$

which completes the proof of the first estimate (2.14). We move now to prove (2.15) . We should first notice that for all multi-index $\gamma \in \mathbb{N}^n$ such that $|\gamma| \geqslant 1$, we have

$$\int_{\mathbb{R}^n} \partial^\gamma \chi(y)\,dy = 0.$$

Thus, from the above observation, we find

$$\begin{aligned}
|\partial^\gamma A^\sharp_\lambda(x)| &= \Big| \int_{\mathbb{R}^n} \tilde{A}(y)\lambda^{n\alpha}\partial^\gamma\Big(\chi(\lambda^\alpha(x-y))\Big)\,dy\Big| \\
&= \Big| \int_{\mathbb{R}^n} \tilde{A}(x-\lambda^{-\alpha}z)\lambda^{\alpha|\gamma|}\,(\partial^\gamma\chi)(z)\,dz\Big| \\
&= \Big| \int_{\mathbb{R}^n} \Big(\tilde{A}(x-\lambda^{-\alpha}z) - A(x)\Big)\,\lambda^{\alpha|\gamma|}\,(\partial^\gamma\chi)(z)\,dz\Big| \\
&\leqslant \|\tilde{A}\|_{\mathcal{C}^{0,1}(\mathbb{R}^n)}\lambda^{\alpha(|\gamma|-1)}\int_{\mathbb{R}^n}|z||(\partial^\alpha\chi)(z)|\,dz \\
&\leqslant C\lambda^{\alpha(|\gamma|-1)}.
\end{aligned}$$

This completes the proof of the Lemma. □

The coming statement claims the existence of particular solutions to the equation (2.2). In the rest of this subsection, we will consider A to be extended as $\tilde{A}$ outside Ω. We denote by A this extension.

Lemma 2.5. *(see* [10]*) Given $\omega \in \mathbb{S}^{n-1}$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Let $A \in W^{1,\infty}(\Omega)$ and $q \in L^\infty(\Omega)$. We consider the function ϕ defined by (2.12). Then, for any $\lambda > 0$ the equation $\mathscr{H}_{A,q}u = 0$ in Q admits a solution*

$$u \in \mathcal{C}([0,T];H^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega)),$$

of the form

$$u(x,t) = \phi(x,t)\,b^\sharp_\lambda(x,t)\,e^{i\lambda(x\cdot\omega+t)} + r(x,t),$$

where

$$b^\sharp_\lambda(x,t) = \exp\Big(i\int_0^t \omega\cdot A^\sharp_\lambda(x+s\omega)\,ds\Big).$$

Here $A^\sharp_\lambda$ is given by (2.13) and the correction term r satisfies

$$r(x,0) = \partial_t r(x,0) = 0,\ \textit{in}\ \Omega,\quad r(x,t) = 0\ \textit{on}\ \Sigma.$$

Moreover, there exists a positive constant $C > 0$ such that

$$\lambda^\alpha\|r\|_{L^2(Q)} + \lambda^{\alpha-1}\|\nabla r\|_{L^2(Q)} \leqslant C\|\varphi\|_{H^3(\mathbb{R}^n)},\qquad 0 < \alpha \leqslant 1/2. \tag{2.16}$$

Proof. In order to prove this lemma, it will be enough to show that if r solves the following equation

$$\begin{cases} \mathscr{H}_{A,q} r = g, & \text{in } Q, \\ r_{|t=0} = \partial_t r_{|t=0} = 0, & \text{in } \Omega, \\ r = 0, & \text{on } \Sigma, \end{cases} \tag{2.17}$$

then the estimate (2.16) is satisfied. Here the function g is given by

$$\begin{aligned} g(x,t) &= -e^{i\lambda(x\cdot\omega+t)}(\partial_t^2 - \Delta_A + q(x))(\phi\, b_\lambda^\sharp)(x,t) \\ &\quad -2i\lambda e^{i\lambda(x\cdot\omega+t)} b_\lambda^\sharp(x,t)(\partial_t - \omega\cdot\nabla)\phi(x,t) \\ &\quad -2i\lambda e^{i\lambda(x\cdot\omega+t)}\phi(x,t)(\partial_t - \omega\cdot\nabla - i(\omega\cdot A))b_\lambda^\sharp(x,t). \end{aligned}$$

This and the fact that ϕ satisfies (2.12) and $b_\lambda^\sharp$ solves the following equation

$$(\partial_t - \omega\cdot\nabla - i(\omega\cdot A_\lambda^\sharp))b_\lambda^\sharp = 0,$$

immediately implies that

$$g(x,t) = -e^{i\lambda(x\cdot\omega+t)}(\partial_t^2 - \Delta_A + q(x))(\phi\, b_\lambda^\sharp)(x,t) + 2\lambda e^{i\lambda(x\cdot\omega+t)}\omega\cdot(A_\lambda^\sharp - A)(x)(b_\lambda^\sharp\phi)(x,t),$$

with $g \in L^1(0,T;L^2(\Omega))$. By setting $w(x,t) = \displaystyle\int_0^t r(x,s)\,ds$, one can see that w solves the hyperbolic problem (2.17) with the right hand side

$$\begin{aligned} F(x,t) = \int_0^t g(x,s)\,ds &= \int_0^t -e^{i\lambda(x\cdot\omega+s)}(\partial_t^2 - \Delta_A + q(x))(\phi\, b_\lambda^\sharp)(x,s)\,ds \\ &+ \int_0^t 2\lambda e^{i\lambda(x\cdot\omega+s)}\omega\cdot(A_\lambda^\sharp - A)(x)(\phi\, b_\lambda^\sharp)(x,s)\,ds = F_1(x,t) + F_2(x,t). \end{aligned} \tag{2.18}$$

Let us put

$$g_1(x,s) = (\partial_t^2 - \Delta_A + q(x))(\phi\, b_\lambda^\sharp)(x,s), \quad \text{and} \quad g_2(x,s) = 2\lambda\omega\cdot(A_\lambda^\sharp - A)(x)(\phi\, b_\lambda^\sharp)(x,s). \tag{2.19}$$

In light of (2.18) and (2.19), we have

$$F_1(x,t) = -\frac{1}{i\lambda}\int_0^t g_1(x,s)\partial_s(e^{i\lambda(x\cdot\omega+s)})\,ds.$$

Thus, by integrating by parts with respect to s and using (2.15), we get

$$\|F_1\|^2_{L^2(Q)} \leqslant \frac{C}{\lambda^2}\Big(\|g_1\|^2_{L^2(Q)} + T\|g_1(\cdot,0)\|^2_{L^2(\Omega)} + T\|\partial_s g_1\|^2_{L^2(Q)}\Big) \leqslant \frac{C}{\lambda^{2-\alpha}}\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \tag{2.20}$$

On the other hand, in view of (2.18) and (2.19) we have

$$F_2(x,t) = \frac{1}{i\lambda}\int_0^t g_2(x,s)\partial_s(e^{i\lambda(x\cdot\omega+s)})\,ds.$$

Again, by integrating by parts with respect to the variable s, we find in view of (2.14)

$$\|F_2\|^2_{L^2(Q)} \leqslant \frac{C}{\lambda^{2\alpha}}\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \tag{2.21}$$

Applying the standard energy estimate for hyperbolic initial boundary value problems to the solution w, we get from (2.20) and (2.21)

$$\|r\|_{L^2(Q)} = \|\partial_t w\|_{L^2(Q)} \leqslant \frac{C}{\lambda^\alpha}\|\varphi\|_{H^3(\mathbb{R}^n)}, \quad 0 < \alpha \leqslant 1/2.$$

By using again the energy estimate applied to the solution r, we get from (2.14)

$$\begin{aligned}\|\nabla r\|_{L^2(Q)} \leqslant C\|g\|_{L^2(Q)} &\leqslant C(\|g_1\|_{L^2(Q)}+\|g_2\|_{L^2(Q)})\\ &\leqslant C\Big(\lambda^\alpha+\frac{1}{\lambda^{\alpha-1}}\Big)\|\varphi\|_{H^3(\mathbb{R}^n)}\\ &\leqslant \frac{C}{\lambda^{\alpha-1}}\|\varphi\|_{H^3(\mathbb{R}^n)},\end{aligned}$$

with $0<\alpha\leqslant 1/2$. This completes the proof of the Lemma. □

By a similar way, we can construct a solution to the backward problem.

Lemma 2.6. *Given $\omega\in\mathbb{S}^{n-1}$ and $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$. We consider the function ϕ defined by (2.12). Then, for any $\lambda>0$ the equation $\mathscr{H}^*_{A,q}v=0$ in Q admits a solution*

$$v\in\mathcal{C}([0,T];H^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega)),$$

of the form

$$v(x,t)=\phi(x,t)\,b^\sharp_\lambda(x,t)\,e^{i\lambda(x\cdot\omega+t)}+r(x,t),$$

where

$$b(x,t)=\exp\Big(i\int_0^t\omega\cdot\overline{A}^\sharp_\lambda(x+s\omega)\,ds\Big),$$

and $r(x,t)$ satisfies

$$r(x,T)=\partial_t r(x,T)=0,\ \text{in }\Omega,\quad r(x,t)=0\ \text{on }\Sigma.$$

Moreover, there exists a positive constant $C>0$ such that

$$\lambda^\alpha\|r\|_{L^2(Q)}+\lambda^{\alpha-1}\|\nabla r\|_{L^2(Q)}\leqslant C\|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{2.22}$$

2.2. **Stability for the magnetic field.** In this section we are going to use the geometrical optics solutions constructed before in order to retrieve a stability result for the determination of the magnetic field $d\alpha_A$ from the DN map $N_{A,q}$. Let us first consider $A_1,A_2\in\mathcal{A}(M,A_0)$ and $q_1,\ q_2\in\mathcal{Q}(M,q_0)$. We define

$$A(x)=(A_1-A_2)(x)\quad\text{and}\quad q(x)=(q_2-q_1)(x).$$

Assume that there exists $\rho>0$ such that $T>\operatorname{Diam}\Omega+4\rho$. We denote

$$\mathcal{D}_\rho=\{x\in\mathbb{R}^n\backslash\Omega,\ \operatorname{dist}(x,\Omega)<\rho\}.$$

Throughout the rest of the paper, we assume that $\operatorname{supp}\varphi\subset\mathcal{D}_\rho$, so that we have

$$\operatorname{supp}\varphi\cap\Omega=\varnothing,\quad\text{and}\quad(\operatorname{supp}\varphi\pm T\omega)\cap\Omega=\varnothing.$$

we recall that A is assumed to be extended as $\tilde{A}$ outside Ω and that we denoted by A this extension. Moreover, we extend q to a $L^\infty(\mathbb{R}^n)$ function by defining it by zero outside Ω. We denote by A and q these extensions.

2.2.1. *Preliminary estimate.* The main purpose of this subsection is to establish the following

Lemma 2.7. *There exists a constant $C>0$ such that for any $\omega\in\mathbb{S}^{n-1}$, the following estimate*

$$\Big|\int_{\mathbb{R}}\omega\cdot A(y-s\omega)\,ds\Big|\leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\beta}\Big)\quad a.e,\ y\in\mathbb{R}^n,$$

holds true for any $\lambda>0$ sufficiently large. Here C depends only on Ω, T and M.

Proof. In view of Lemma 2.5, and using the fact that $\operatorname{supp}\varphi\cap\Omega=\emptyset$, there exists a geometrical optic solution u_2 to the follwoing equation

$$\begin{cases} \mathscr{H}_{A_2,q_2}u_2=0, & \text{in } Q,\\ u_{2|t=0}=\partial_t u_{2|t=0}=0, & \text{in } \Omega, \end{cases}$$

in the following form

$$u_2(x,t)=\varphi_2(x+t\omega)b^\sharp_{2,\lambda}(x,t)e^{i\lambda(x\cdot\omega+t)}+r_2(x,t),$$

with $b^\sharp_{2,\lambda}(x,t)=\exp\Big(i\displaystyle\int_0^t\omega\cdot A^\sharp_{2,\lambda}(x+s\omega)\,ds\Big)$ and r_2 satisfies (2.16). Next, let us denote by $f_\lambda=u_{2|\Sigma}$. Let u_1 be a solution to the follwing system

$$\begin{cases} \mathscr{H}_{A_1,q_1}u_1=0, & \text{in } Q,\\ u_{1|t=0}=\partial_t u_{1|t=0}=0, & \text{in } \Omega,\\ u_1=u_2:=f_\lambda, & \text{on } \Sigma. \end{cases}$$

Putting $u=u_1-u_2$. Then, u is a solution to

$$\begin{cases} \mathscr{H}_{A_1,q_1}u=2iA\cdot\nabla u_2+V_Au_2+qu_2, & \text{in } Q,\\ u_{|t=0}=\partial_t u_{|t=0}=0, & \text{in } \Omega,\\ u=0, & \text{on } \Sigma, \end{cases}\tag{2.23}$$

where $A=A_1-A_2$, $q=q_2-q_1$ and $V_A=i\,\mathrm{div}A+(A_2^2-A_1^2)$. On the other hand, Lemma 2.6 and the fact that $(\operatorname{supp}\varphi\pm T\omega)\cap\Omega=\emptyset$, guarantee the existence of a geometrical optic solution v to

$$\begin{cases} \mathscr{H}^*_{A_1,q_1}v=0, & \text{in } Q,\\ v_{|t=T}=\partial_t v_{|t=T}=0, & \text{in } \Omega, \end{cases}$$

in the following form

$$v(x,t)=\varphi_1(x+t\omega)b^\sharp_{1,\lambda}(x,t)e^{i\lambda(x\cdot\omega+t)}+r_1(x,t),$$

where $b_1(x,t)=\exp\big(i\displaystyle\int_0^t\omega\cdot\overline{A}^\sharp_{1,\lambda}(x+s\omega)\,ds\big)$ and r_1 satisfies (2.22). Multiplying the first equation in (2.23) by v and integrating by parts we get in view of (2.3),

$$\begin{aligned}\int_Q 2iA\cdot\nabla u_2(x,t)\overline{v}(x,t)\,dx\,dt &= \int_\Sigma (N_{A_1,q_1}-N_{A_2,q_2})(f)\overline{v}(x,t)\,d\sigma\,dt\\ &\quad -\int_Q (V_A(x)+q(x))u_2(x,t)\overline{v}(x,t)\,dx\,dt.\end{aligned}\tag{2.24}$$

On the other hand, by replacing u_2 and v by their expressions, we get

$$\begin{aligned}\int_Q 2iA\cdot\nabla u_2(x,t)\overline{v}(x,t)\,dx\,dt &= \int_Q 2iA\cdot\nabla(\phi_2b^\sharp_{2,\lambda})(x,t)(\overline{\phi_1b^\sharp_{1,\lambda}})(x,t)\,dx\,dt\\ &\quad+\int_Q 2iA\cdot\nabla(\phi_2b^\sharp_{2,\lambda})(x,t)\overline{r}_1(x,t)e^{i\lambda(x\cdot\omega+t)}\,dx\,dt\\ &\quad-2\lambda\int_Q\omega\cdot A(x)(\phi_2b^\sharp_{2,\lambda})(x,t)(\overline{\phi_1b^\sharp_{1,\lambda}})(x,t)\,dx\,dt\end{aligned}$$

$$
\begin{aligned}
& -2\lambda \int_Q \omega \cdot A(x)(\phi_2 b^\sharp_{2,\lambda})(x,t)\overline{r}_1(x,t)e^{i\lambda(x\cdot\omega+t)}\,dx\,dt \\
& +2i\int_Q A\cdot\nabla r_2(x,t)\overline{(\phi_1 b^\sharp_{1,\lambda})}(x,t)e^{-i\lambda(x\cdot\omega+t)}\,dx\,dt \\
& +2i\int_Q A\cdot\nabla r_2(x,t)\overline{r}_1(x,t)\,dx\,dt \\
= & -2\lambda\int_Q \omega\cdot A(x)(\phi_2\overline{\phi}_1)(x,t)(b^\sharp_{2,\lambda}\overline{b^\sharp_{1\lambda}})(x,t)\,dx\,dt + I_\lambda.
\end{aligned}
\tag{2.25}
$$

Using the fact that for λ sufficiently large, we have

$$
\|u_2\overline{v}\|_{L^1(Q)} \leqslant C\|\varphi_1\|_{H^3(\mathbb{R}^n)}\|\varphi_2\|_{H^3(\mathbb{R}^n)}, \quad \text{and} \quad |I_\lambda| \leqslant C\lambda^{1-\alpha}\|\varphi_1\|_{H^3(\mathbb{R}^n)}\|\varphi_2\|_{H^3(\mathbb{R}^n)}.
\tag{2.26}
$$

On the other hand from the trace theorem we have

$$
\begin{aligned}
\Big|\int_\Sigma (N_{A_2,q_2}-N_{A_1,q_1})(f_\lambda)\,\overline{v}(x,t)\,d\sigma\,dt\Big| &\leqslant \|N_{A_2,q_2}-N_{A_1,q_1}\|\,\|f_\lambda\|_{H^1(\Sigma)}\|\overline{v}\|_{L^2(\Sigma)} \\
&\leqslant \|N_{A_2,q_2}-N_{A_1,q_1}\|\,\|u_2-r_2\|_{H^2(Q)}\,\|\overline{v}-\overline{r}_1\|_{H^1(Q)} \\
&\leqslant C\lambda^3\|N_{A_2,q_2}-N_{A_1,q_1}\|\,\|\varphi_1\|_{H^3(\mathbb{R}^n)}\|\varphi_2\|_{H^3(\mathbb{R}^n)}.
\end{aligned}
$$

This, (2.25) and (2.26) yield

$$
\begin{aligned}
&\Big|\int_Q \omega\cdot A(x)(\phi_2\overline{\phi}_1)(x,t)(b^\sharp_{2,\lambda}\overline{b^\sharp_{1\lambda}})(x,t)dx\,dt\Big| \\
&\qquad\leqslant C\Big(\lambda^2\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\alpha}\Big)\|\varphi_1\|_{H^3(\mathbb{R}^n)}\|\varphi_2\|_{H^3(\mathbb{R}^n)}.
\end{aligned}
$$

Since $A=0$ outside Ω, then putting $y=x+t\omega$ and $s=t-s$, we get for $\overline{\varphi}_1=\varphi_2=\varphi\in\mathcal{C}_0^\infty(\mathcal{D}_\rho)$,

$$
\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{R}^n}\omega\cdot A(y-t\omega)\varphi^2(y)\exp\Big(-i\int_0^t\omega\cdot A^\sharp_\lambda(y-s\omega)\,ds\Big)\,dy\,dt\Big| \\
&\qquad\leqslant C\Big(\lambda^2\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\alpha}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.
\end{aligned}
$$

Now, using the fact that

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n}\omega\cdot A(y-t\omega)\varphi^2(y)\quad\exp\Big(-i\int_0^t\omega\cdot A^\sharp_\lambda(y-s\omega)\,ds\Big)dy\,dt \\
&\quad=\int_0^T\int_{\mathbb{R}^n}\omega\cdot(A-A^\sharp_\lambda)(y-t\omega)\varphi^2(y)\exp\Big(-i\int_0^t\omega\cdot A^\sharp_\lambda(y-s\omega)ds\Big)dy\,dt \\
&\qquad+\int_0^T\int_{\mathbb{R}^n}\omega\cdot A^\sharp_\lambda(y-t\omega)\varphi^2(y)\exp\Big(-i\int_0^t\omega\cdot A^\sharp_\lambda(y-s\omega)\,ds\Big)dy\,dt.
\end{aligned}
$$

we get from (2.14)

$$
\Big|\int_0^T\int_{\mathbb{R}^n}\omega\cdot A^\sharp_\lambda(y-t\omega)\varphi^2(y)\exp\Big(-i\int_0^t\omega\cdot A^\sharp_\lambda(y-s\omega)\,ds\Big)dy\,dt\Big|\leqslant C\Big(\lambda^2\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\alpha}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.
$$

Therefore, since

$$
\partial_t\Big[\exp\Big(-i\int_0^t\omega\cdot A^\sharp_\lambda(y-s\omega)\,ds\Big)\Big]=-i\omega\cdot A^\sharp_\lambda(y-t\omega)\exp\Big(-i\int_0^t\omega\cdot A^\sharp_\lambda(y-s\omega)\,ds\Big),
$$

we obtain the following estimation

$$\Big|i\int_{\mathbb{R}^n}\varphi^2(y)\Big[\exp\Big(-i\int_0^T\omega\cdot A_\lambda^\sharp(y-s\omega)\Big)-1\Big]\,dy\Big|\leqslant C\Big(\lambda^2\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\alpha}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \tag{2.27}$$

We move now to specify the choice of the function $\varphi\in\mathcal{C}_0^\infty(\mathcal{D}_\rho)$. We set $B(0,r):=\{x\in\mathbb{R}^n;\ |x|<r\}$ for all $r\geqslant 0$. Let $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ be a non-negative function which is supported in the unit ball $B(0,1)$ and such that $\|\psi\|_{L^2(\mathbb{R}^n)}=1$. For $y\in\mathcal{D}_\rho$, we define

$$\varphi_h(x)=h^{-n/2}\psi\Big(\frac{x-y}{h}\Big). \tag{2.28}$$

Then, for $h>0$ sufficiently small such that Supp $\varphi_h\subset\mathcal{D}_\rho$. We can verify that

$$\mathrm{Supp}\,\varphi_h\cap\Omega=\varnothing\quad\text{and}\quad(\mathrm{Supp}\,\varphi_h\pm T\omega)\cap\Omega=\varnothing.$$

Moreover, we have

$$\begin{aligned}\Big|\exp\Big(-i\int_0^T\omega\cdot A_\lambda^\sharp(y-s\omega)\,ds\Big)-1\Big|&=\Big|\int_{\mathbb{R}^n}\varphi_h^2(x)\Big[\exp\Big(-i\int_0^T\omega\cdot A_\lambda^\sharp(y-s\omega)\,ds\Big)-1\Big]\,dx\Big|\\ &\leqslant\ \Big|\int_{\mathbb{R}^n}\varphi_h^2(x)\Big[\exp\Big(-i\int_0^T\omega\cdot A_\lambda^\sharp(y-s\omega)\,ds\Big)-\exp\Big(-i\int_0^T\omega\cdot A_\lambda^\sharp(x-s\omega)\,ds\Big)\Big]\,dx\Big|\\ &\quad+\Big|\int_{\mathbb{R}^n}\varphi_h^2(x)\Big[\exp\Big(-i\int_0^T\omega\cdot A_\lambda^\sharp(x-s\omega)\,ds\Big)-1\Big]dx\Big|.\end{aligned} \tag{2.29}$$

Using the fact that

$$\Big|\int_0^T\Big(i\omega\cdot A_\lambda^\sharp(y-s\omega)-i\omega\cdot A_\lambda^\sharp(x-s\omega)\Big)ds\Big|\leqslant C\,|y-x|,$$

we deduce upon replacing $\varphi=\varphi_h$ in (2.27), the following estimation

$$\Big|\exp\Big(-i\int_0^T\omega\cdot A_\lambda^\sharp(y-s\omega)\,ds\Big)-1\Big|\leqslant C\int_{\mathbb{R}^n}\varphi_h^2(x)\,|y-x|\,dx+C\Big(\lambda^2\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\alpha}\Big)\|\varphi_h\|^2_{H^3(\mathbb{R}^n)}.$$

On the other hand, we have

$$\|\varphi_h\|_{H^3(\mathbb{R}^n)}\leqslant Ch^{-3}\quad\text{and}\quad\int_{\mathbb{R}^n}\varphi_h^2(x)|y-x|\,dx\leqslant Ch.$$

So, we end up getting the following inequality

$$\Big|\exp\Big(-i\int_0^T\omega\cdot A_\lambda^\sharp(y-s\omega)\,ds\Big)-1\Big|\leqslant C\,h+C\Big(\lambda^2\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\alpha}\Big)h^{-6}.$$

Selecting h small such that $h=1/\lambda^\alpha h^6$, that is $h=\lambda^{-\alpha/7}$, we find $\delta>1$ and $0<\beta<\alpha<1$ such that

$$\Big|\exp\Big(-i\int_0^T\omega\cdot A_\lambda^\sharp(y-s\omega)\,ds\Big)-1\Big|\leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\beta}\Big). \tag{2.30}$$

Using the fact that $|X|\leqslant e^M\,|e^X-1|$ for any X real satisfying $|X|\leqslant M$ we found out that

$$\Big|-i\int_0^T\omega\cdot A_\lambda^\sharp(y-s\omega)\,ds\Big|\leqslant e^{MT}\Big|\exp\Big(-i\int_0^T\omega\cdot A(y-s\omega)\,ds\Big)-1\Big|,$$

where $X=\displaystyle\int_0^T -i\omega\cdot A_\lambda^\sharp(y-s\omega)\,ds$. We conclude in light of (2.30) the following the estimate

$$\Big|\int_0^T\omega\cdot A_\lambda^\sharp(y-s\omega)\,ds\Big|\leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\beta}\Big),\quad\text{a.e.}\quad y\in\mathcal{D}_\rho,\quad\omega\in\mathbb{S}^{n-1}. \tag{2.31}$$

By replacing ω by $-\omega$, we get

$$\Big|\int_{-T}^{0}\omega\cdot A^\sharp_\lambda(y-s\omega)\,ds\Big|\leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\beta}\Big),\quad \text{a.e. } y\in\mathcal{D}_\rho,\quad \omega\in\mathbb{S}^{n-1}. \tag{2.32}$$

Bearing in mind that

$$\Big|\int_{-T}^{T}\omega\cdot A(y-s\omega)\,ds\Big|\leqslant\Big|\int_{-T}^{T}\omega\cdot A^\sharp_\lambda(y-s\omega)\,ds\Big|+\Big|\int_{-T}^{T}\omega\cdot (A^\sharp_\lambda-A)(y-s\omega)\,ds\Big|,$$

we can deduce from (2.32) and (2.14) the following estimate

$$\begin{aligned}\Big|\int_{-T}^{T}\omega\cdot A(y-s\omega)\,ds\Big| &\leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\beta}+\frac{1}{\lambda^\alpha}\Big)\\ &\leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\beta}\Big),\end{aligned}$$

for all $y\in\mathcal{D}_\rho$ and $\omega\in\mathbb{S}^{n-1}$. Since $T>\operatorname{Diam}\Omega+4\rho$ and supp $A\subseteq\Omega$ we obtain in view of (2.31)-(2.32),

$$\Big|\int_{\mathbb{R}}\omega\cdot A(y-s\omega)\,ds\Big|\leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\beta}\Big),\quad \text{a.e. } y\in\mathbb{R}^n,\quad \omega\in\mathbb{S}^{n-1}.$$

This completes the proof of the Lemma. □

2.2.2. *An estimate for the magnetic field.* In this section we estimate the magnetic field $d\alpha_{A_1}-d\alpha_{A_2}$ by the use of the lemma proved in the previous section. For this purpose, let us first introduce this notation

$$a_k(x)=(A_1-A_2)(x)\cdot e_k=A(x)\cdot e_k,$$

where $(e_k)_{1\leqslant k\leqslant n}$ is the canonical basis of $\mathbb{R}^n$. On the other hand, we denote by

$$\sigma_{j,k}(x)=\frac{\partial a_k}{\partial x_j}(x)-\frac{\partial a_j}{\partial x_k}(x),\quad j,k=1...n. \tag{2.33}$$

Let $\xi\in\omega^\perp$. By the change of variables $x=z-s\omega\in\omega^\perp\oplus\mathbb{R}\omega=\mathbb{R}^n$, we have the following identity

$$z\cdot\xi=z\cdot\xi-s\omega\cdot\xi=x\cdot\xi,$$

with $dx=d\sigma dt$. Thus, we get

$$\begin{aligned}\int_{\omega^\perp}e^{-iz\cdot\xi}\int_{\mathbb{R}}\omega\cdot A(z-s\omega)\,ds\,d\sigma &= \int_{\omega^\perp}\int_{\mathbb{R}}e^{-iz\cdot\xi+s\xi\cdot\omega}(\omega\cdot A)(z-s\omega)\,ds\,d\sigma\\ &= \int_{\mathbb{R}^n}e^{-ix\cdot\xi}\omega\cdot A(x)\,dx.\end{aligned}$$

Assume that $\Omega\subset B(0,R_1)$, with $R>0$. Using the fact that Supp $A\subset\Omega$, we get from Lemma 2.7

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\omega\cdot A(x)\,dx\Big| &\leqslant \int_{\omega^\perp\cap B(0,R_1)}\Big|e^{-iz\cdot\xi}\int_{\mathbb{R}}\omega\cdot A(z-s\omega)\,ds\Big|\,dz\\ &\leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\beta}\Big).\end{aligned} \tag{2.34}$$

For $\xi\in\mathbb{R}^n$, we define $\omega=\frac{\xi_je_k-\xi_ke_j}{|\xi_je_k-\xi_ke_j|}$. Multiplying (2.34) by $|\xi_je_k-\xi_ke_j|$, we obtain

$$\Big|\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\Big(\xi_ja_k(x)-\xi_ka_j(x)\Big)\,dx\Big|\leqslant C|\xi_je_k-\xi_ke_j|\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\beta}\Big).$$

This together with (2.33) yield

$$|\widehat{\sigma}_{j,k}(\xi)|\leqslant C<\xi>\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\beta}\Big).$$

We are now in position to upper bound the magnetic field induced by the magnetic potential in suitable norms. For this purpose, let $0 < R \leqslant \lambda$. In light of the above reasoning, this can be achieved by decomposing the $H^{-1}(\mathbb{R}^n)$ norm of $\sigma_{j,k}$ as follows

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} = \int_{|\xi|\leqslant R} |\widehat{\sigma}_{j,k}(\xi)|^2 < \xi >^{-2} \, d\xi + \int_{|\xi|>R} |\widehat{\sigma}_{j,k}(\xi)|^2 < \xi >^{-2} \, d\xi.$$

Then, we have

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} \leqslant C\Big(R^n \| < \xi >^{-1} \widehat{\sigma}_{j,k}\|^2_{L^\infty(B(0,R))} + \frac{1}{R^2}\|\sigma_{j,k}\|^2_{L^2(\mathbb{R}^n)}\Big).$$

This entails that

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} \leqslant C\Big(R^n\left(\lambda^{2\delta}\|N_{A_2,q_2} - N_{A_1,q_1}\|^2 + \frac{1}{\lambda^{2\beta}}\right) + \frac{1}{R^2}\Big).$$

Next, we choose $R > 0$ in such away $R^n/\lambda^{2\beta} = 1/R^2$. Thus, we find $\mu_1 > 2$ and $\mu_2 > 0$ such that

$$\begin{aligned} \|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} &\leqslant C\Big(\lambda^{\frac{2\beta}{n+2}+2\delta}\|N_{A_2,q_2} - N_{A_1,q_1}\|^2 + \lambda^{\frac{2\beta n}{n+2}-2\beta}\Big) \\ &\leqslant C\Big(\lambda^{\mu_1}\|N_{A_2,q_2} - N_{A_1,q_1}\|^2 + \lambda^{-\mu_2}\Big). \end{aligned} \tag{2.35}$$

Now we assume that $\|N_{A_2,q_2} - N_{A_1,q_1}\| < c < 1$, and we minimize with respect to λ to end up getting

$$\|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)} \leqslant C\|N_{A_2,q_2} - N_{A_1,q_1}\|^{1/2}.$$

The above estimate remains true in the case where $\|N_{A_2,q_2} - N_{A_1,q_1}\| \geqslant c$, since we have

$$\|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)} \leqslant \frac{2M}{c^{1/2}}c^{1/2} \leqslant \frac{2M}{c^{1/2}}\|N_{A_2,q_2} - N_{A_1,q_1}\|^{1/2}.$$

Therefore, we find out that

$$\|d\alpha_{A_1} - d\alpha_{A_2}\|_{H^{-1}(\Omega)} \leqslant \sum_{j,k}\|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)} \leqslant C\|N_{A_2,q_2} - N_{A_1,q_1}\|^{1/2}. \tag{2.36}$$

2.3. **Stability for the electric potential.** The goal of this section is to prove a stability estimate for the electric potential. The proof involves using the stability estimate we have already obtained for the magnetic field. We will proceed as in [29].

Let $n < p_0 < \infty$. Apply the Hodge decomposition to $A_1 - A_2$ in the space $W^{1,p_0}(\Omega, \mathbb{C}^n)$. We define

$$A_1' = A_1 + \frac{1}{2}\nabla\psi, \quad \text{and} \quad A_2' = A_2 - \frac{1}{2}\nabla\psi, \tag{2.37}$$

with $\psi \in W^{3,p_0}(\Omega) \cap H^1_0(\Omega)$. From Lemma 6.2 given in [29], $A' = A_2' - A_1'$ satisfies

$$\|A'\|_{W^{1,p_0}(\Omega)} \leqslant C\|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^{p_0}(\Omega)} = C\|d\alpha_{A_1'} - d\alpha_{A_2'}\|_{L^{p_0}(\Omega)}. \tag{2.38}$$

Recall that since the DN map is invariant under gauge transformation then we have

$$N_{A_1,q_1} = N_{A_1+\frac{1}{2}\nabla\psi,q_1}, \quad N_{A_2,q_2} = N_{A_2-\frac{1}{2}\nabla\psi,q_2}. \tag{2.39}$$

Throughout the rest of this section, A_j will be replaced by A_j' for $j = 1, 2$.

2.3.1. *Preliminary estimate.*

Lemma 2.8. *There exist a constant $C > 0$ such that for any $\omega \in \mathbb{S}^{n-1}$, the following estimate*

$$\Big|\int_{\mathbb{R}} q(y-t\omega)\,dt\Big| \leqslant C\Big(\lambda^{\delta}\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^{\beta}}\Big),\quad a.\,e.\ y\in\mathbb{R}^n,$$

holds true. Here C depends only on Ω, T and M.

Proof. We start with the identity (2.24) except this time we isolate the electric potential term

$$\begin{aligned}\int_Q q(x)\ u_2(x,t)\overline{v}(x,t)\,dx\,dt &= \int_{\Sigma}(N_{A_1',q_1}-N_{A_2',q_2})(f)\overline{v}(x,t)\,d\sigma\,dt\\ &-\int_Q 2iA'\cdot\nabla u_2(x,t)\overline{v}(x,t)\,dx\,dt-\int_Q V_{A'}(x)u_2(x,t)\overline{v}(x,t)\,dx\,dt,\end{aligned}$$

where $V_{A'}=i\mathrm{div}A'+(A_2'^2-A_1'^2)$. By replacing u_2 and v by their expressions, we get

$$\begin{aligned}\int_Q q(x)u_2(x,t)\overline{v}(x,t)\,dx\,dt \quad &= \quad \int_Q q(x)(\phi_2\overline{\phi}_1)(x,t)(b^{\sharp}_{2,\lambda}\overline{b}^{\sharp}_{1,\lambda})(x,t)\,dx\,dt\\ &+\int_Q q(x)\phi_2(x,t)b^{\sharp}_{2,\lambda}(x,t)e^{i\lambda(x\cdot\omega+t)}\overline{r}_1(x,t)\,dx\,dt\\ &+\int_Q q(x)\overline{\phi}_1(x,t)\overline{b^{\sharp}_{1\lambda}}(x,t)e^{-i\lambda(x\cdot\omega+t)}r_2(x,t)\,dx\,dt\\ &+\int_Q q(x)r_2(x,t)\,\overline{r}_1(x,t)\,dx\,dt.\end{aligned}$$

Therefore, we have the following identity

$$\int_Q q(x)(\phi_2\overline{\phi}_1)(x,t)(b^{\sharp}_{2,\lambda}\overline{b^{\sharp}_{1\lambda}})(x,t)\,dx\,dt=\int_{\Sigma}(N_{A_1',q_1}-N_{A_2',q_2})(f)\overline{v}(x,t)\,d\sigma\,dt+I_{\lambda}, \tag{2.40}$$

where I_λ is given by

$$\begin{aligned}I_{\lambda} \quad = \quad &\int_Q 2iA'\cdot\nabla u_2(x,t)\overline{v}(x,t)\,dx\,dt-\int_Q V_{A'}(x)u_2(x,t)\overline{v}(x,t)\,dx\,dt\\ &-\int_Q q(x)\phi_2(x,t)b^{\sharp}_{2,\lambda}(x,t)e^{i\lambda(x\cdot\omega+t)}\overline{r}_1(x,t)\,dx\,dt-\int_Q q(x)r_2(x,t)\overline{r}_1(x,t)\,dx\,dt\\ &-\int_Q q(x)\overline{\phi}_1(x,t)\overline{b^{\sharp}_{1\lambda}}(x,t)e^{-i\lambda(x\cdot\omega+t)}r_2(x,t)\,dx\,dt.\end{aligned}$$

For λ sufficiently large, we have

$$|I_{\lambda}|\leqslant C\Big(\frac{1}{\lambda^{\alpha}}+\lambda\|A'\|_{L^{\infty}(\Omega)}\Big)\|\varphi_1\|_{H^3(\mathbb{R}^n)}\|\varphi_2\|_{H^3(\mathbb{R}^n)}, \tag{2.41}$$

with $0<\alpha\leqslant 1/2$. On the other hand, by the trace theorem, we have

$$\Big|\int_{\Sigma}(N_{A_1,q_1}-N_{A_2,q_2})(f)\overline{v}(x,t)\,d\sigma\,dt\Big|\leqslant C\lambda^3\|N_{A_1,q_1}-N_{A_2,q_2}\|\|\varphi_1\|_{H^3(\mathbb{R}^n)}\|\varphi_2\|_{H^3(\mathbb{R}^n)}. \tag{2.42}$$

Thus, from (2.40), (2.41) and (2.42) we obtain

$$\begin{aligned}&\Big|\int_Q q(x)(\phi_2\overline{\phi}_1)(x,t)(b^{\sharp}_{2,\lambda}\overline{b}^{\sharp}_{1,\lambda})(x,t)\,dx\,dt\Big|\\ &\qquad\leqslant C\Big(\lambda^3\|N_{A_1,q_1}-N_{A_2,q_2}\|+\lambda\|A'\|_{L^{\infty}(\Omega)}+\frac{1}{\lambda^{\alpha}}\Big)\|\varphi_1\|_{H^3(\mathbb{R}^n)}\|\varphi_2\|_{H^3(\mathbb{R}^n)}.\end{aligned}$$

Since $q=0$ outside Ω, then by the change of variables $y=x+t\omega$, $s=t-s$ we get for $\overline{\varphi}_1=\varphi_2=\varphi$,

$$\begin{aligned}\Big|\int_0^T\int_{\mathbb{R}^n} q(y-t\omega)\varphi^2(y)\ \exp\Big(-i\int_0^t \omega\cdot A'^{\sharp}_\lambda(y-s\omega)\,ds\Big)\,dy\,dt\Big|\\ \leqslant\ C\Big(\lambda^3\|N_{A_1,q_1}-N_{A_2,q_2}\|+\lambda\|A'\|_{L^\infty(\Omega)}+\frac{1}{\lambda^\alpha}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)},\end{aligned}$$

with $A'^{\sharp}_\lambda=\chi_\lambda * A'$. This and the fact that

$$\begin{aligned}\Big|\int_0^T\int_{\mathbb{R}^n} q(y-t\omega)\varphi^2(y)\Big[1-\exp\Big(-i\int_0^t \omega\cdot A'^{\sharp}_\lambda(y-s\omega)\,ds\Big)\Big]\,dy\,dt\Big|\\ \leqslant C\|A'^{\sharp}_\lambda\|_{L^\infty(\mathbb{R}^n)}\|\varphi\|^2_{H^3(\mathbb{R}^n))}\leqslant C\|A'\|_{L^\infty(\Omega)}\|\varphi\|^2_{H^3(\mathbb{R}^n)},\end{aligned}$$

implies that

$$\Big|\int_0^T\int_{\mathbb{R}^n} q(y-t\omega)\varphi^2(y)\,dy\,dt\Big|\leqslant\Big(\lambda^3\|N_{A_1,q_1}-N_{A_2,q_2}\|+\lambda\|A'\|_{L^\infty(\Omega)}+\frac{1}{\lambda^\alpha}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Applying Morrey's inequality given by the following estimate

$$\|A'\|_{\mathcal{C}^{0,1-\frac{n}{p}}(\Omega)}\leqslant C\|A'\|_{W^{1,p}(\Omega)},\quad A'\in W^{1,p}(\Omega),$$

where $n<p\leqslant\infty$ and C a positive constant which depends on p, n and Ω, we get

$$\Big|\int_0^T\int_{\mathbb{R}^n} q(y-t\omega)\varphi^2(y)\,dy\,dt\Big|\leqslant\Big(\lambda^3\|N_{A_1,q_1}-N_{A_2,q_2}\|+\lambda\|A'\|_{W^{1,p_0}(\Omega)}+\frac{1}{\lambda^\alpha}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)},$$

where $n<p_0<\infty$. Hence, in light of (2.38), we find out that

$$\Big|\int_0^T\int_{\mathbb{R}^n} q(y-t\omega)\varphi^2(y)\,dy\,dt\Big|\leqslant\Big(\lambda^3\|N_{A_1,q_1}-N_{A_2,q_2}\|+\lambda\|d\alpha_{A_1}-d\alpha_{A_2}\|_{L^{p_0}(\Omega)}+\frac{1}{\lambda^\alpha}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.\tag{2.43}$$

By interpolating, we have for $s=2/p_0$

$$\begin{aligned}\|d\alpha_{A_1}-d\alpha_{A_2}\|_{L^{p_0}(\Omega)}&\leqslant\ \|d\alpha_{A_1}-d\alpha_{A_2}\|^{1-s}_{L^\infty(\Omega)}\|d\alpha_{A_1}-d\alpha_{A_2}\|^s_{L^2(\Omega)}\\&\leqslant\ C\|d\alpha_{A_1}-d\alpha_{A_2}\|^{s/2}_{H^1(\Omega)}\|d\alpha_{A_1}-d\alpha_{A_2}\|^{s/2}_{H^{-1}(\Omega)}\\&\leqslant\ C\|d\alpha_{A_1}-d\alpha_{A_2}\|^{s/2}_{H^{-1}(\Omega)}.\end{aligned}$$

Therefore, from (2.43) and (2.36), we obtain

$$\begin{aligned}\Big|\int_0^T\int_{\mathbb{R}^n} q(y-t\omega)\varphi^2(y)\,dy\,dt\Big|&\leqslant\ \Big(\lambda^3\|N_{A_1,q_1}-N_{A_2,q_2}\|+\lambda\|N_{A_1,q_1}-N_{A_2,q_2}\|^{s/4}+\frac{1}{\lambda^\alpha}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}\\&\leqslant\ C\Big(\lambda^3\|N_{A_1,q_1}-N_{A_2,q_2}\|+\frac{1}{\lambda^\alpha}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.\end{aligned}$$

Now we just need to proceed as in the determination of the magnetic field. We consider the sequence $(\varphi_h)_h$ defined by (2.28) with $y\in\mathcal{D}_\rho$. Since

$$\begin{aligned}\Big|\int_0^T q(y-t\omega)dt\Big|&=\Big|\int_0^T\int_{\mathbb{R}^n} q(y-t\omega)\,\varphi_h^2(x)\,dx\,dt\Big|\\&\leqslant\Big|\int_0^T\int_{\mathbb{R}^n} q(x-t\omega)\varphi_h^2(x)dx\,dt\Big|+\Big|\int_0^T\int_{\mathbb{R}^n}\Big(q(y-t\omega)-q(x-t\omega)\Big)\varphi_h^2(x)dx\,dt\Big|,\end{aligned}$$

and using the fact that $|q(y-t\omega)-q(x-t\omega)|\leqslant C|y-x|$, we obtain

$$\Big|\int_0^T q(y-t\omega)dt\Big|\leqslant C\Big(\lambda^3\|N_{A_2,q_2}-N_{A_1,q_1}\|+\frac{1}{\lambda^\alpha}\Big)\|\varphi_h\|^2_{H^3(\mathbb{R}^n)}+C\int_{\mathbb{R}^n}|x-y|\varphi_h^2(x)dx.$$

On the other hand, since $\|\varphi_h\|_{H^3(\mathbb{R}^n)} \leqslant Ch^{-3}$ and $\displaystyle\int_{\mathbb{R}^n} |x-y|\varphi_h^2(x)\,dx \leqslant C\,h$, we conclude that

$$\Big|\int_0^T q(y-t\omega)\,dt\Big| \leqslant C\Big(\lambda^3\|N_{A_2,q_2}-N_{A_1,q_1}\| + \frac{1}{\lambda^\alpha}\Big)h^{-6} + C\,h.$$

Selecting h small such that $h = h^{-6}/\lambda^\alpha$. Then, we find two constants $\delta > 1$ and $0 < \beta < \alpha < 1$ such that

$$\Big|\int_0^T q(y-t\omega)\Big| \leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\| + \frac{1}{\lambda^\beta}\Big). \tag{2.44}$$

The estimate (2.44) remains true by replacing ω by $-\omega$. Then we get for all $y \in \mathcal{D}_\rho$,

$$\Big|\int_{-T}^T q(y-t\omega)\,dt\Big| \leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\| + \frac{1}{\lambda^\beta}\Big).$$

Next, using the fact that $q = q_2 - q_1 = 0$ outside Ω and since $T >$Diam $\Omega + 4\rho$, we have

$$\Big|\int_{\mathbb{R}} q(y-t\omega)\,dt\Big| \leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\| + \frac{1}{\lambda^\beta}\Big), \quad \text{a. e. } y \in \mathbb{R}^n,\ \omega \in \mathbb{S}^{n-1}.$$

This completes the proof of the Lemma. □

2.3.2. *Estimate for the electric potential.* This section is devoted to upper bound the electric potential. In light of Lemma 2.8 and arguing as in Section 2.2.2, we get for all $\xi \in \omega^\perp$ the following estimate

$$|\widehat{q}(\xi)| \leqslant C\Big(\lambda^\delta\|N_{A_2,q_2}-N_{A_1,q_1}\| + \frac{1}{\lambda^\beta}\Big). \tag{2.45}$$

By changing $\omega \in \mathbb{S}^{n-1}$ (2.45) holds for all $\xi \in \mathbb{R}^n$. By decomposing the $H^{-1}(\mathbb{R}^n)$ norm of q, we find

$$\begin{aligned}\|q\|^2_{H^{-1}(\mathbb{R}^n)} &= \int_{|\xi|\leqslant R} |\widehat{q}(\xi)|^2 < \xi >^{-2}\ d\xi + \int_{|\xi|>R} |\widehat{q}(\xi)|^2 < \xi >^{-2}\ d\xi \\ &\leqslant C\Big(R^n\|\widehat{q}\|^2_{L^\infty(B(0,R))} + \frac{1}{R^2}\|q\|^2_{L^2(\Omega)}\Big).\end{aligned}$$

Thus, in light of (2.45), we get

$$\|q\|^2_{H^{-1}(\mathbb{R}^n)} \leqslant C\Big(R^n\lambda^{2\delta}\|N_{A_2,q_2}-N_{A_1,q_1}\|^2 + \frac{R^n}{\lambda^{2\beta}} + \frac{1}{R^2}\Big).$$

We choose R such that $R^{n+2} = \lambda^{2\beta}$ and we obtain

$$\|q\|_{H^{-1}(\mathbb{R}^n)} \leqslant C\Big(R^k\|N_{A_2,q_2}-N_{A_1,q_1}\| + \frac{1}{R}\Big),$$

for some positive constant $k > 0$. All the above mentionned statements are valid for λ sufficiently large. Assume that there exists $c > 0$ such that $\|N_{A_2,q_2}-N_{A_1,q_1}\| \leqslant c$. We select

$$R = \|N_{A_2,q_2}-N_{A_1,q_1}\|^{-1/(k+1)}.$$

Thus, λ is sufficietly large and we get

$$\|q\|_{H^{-1}(\mathbb{R}^n)} \leqslant C\|N_{A_2,q_2}-N_{A_1,q_1}\|^{\mu_2}, \quad \mu_2 = 1/(k+1) \in (0,1). \tag{2.46}$$

This completes the proof of Theorem 2.2.

3. PROOF OF THEOREM 1.1

At this stage we are well prepared to deal with the inverse problem under investigation, that is the identification of V appearing in (1.1) from the knwoledge of Λ_V. Based on Lemma 2.1 and Theorem 2.2 we prove the main result of this paper. Let us start by stating the main tool allowing us to prove the stability.

A crucial part of the proof of Theorem 1.1 is an elliptic Carleman estimate designed for the elliptic operator Δ and given in [8, 11] . For formulating our Carleman estimate, we shall first set some notaions: let a subboundary $\Gamma_0 \subset \Gamma$. Assume that there exists a function $\psi \in \mathcal{C}^2(\Omega, \mathbb{R}^n)$ such that

$$\psi(x) > 0, \ \ x \in \Omega, \qquad |\nabla\psi(x)| > 0 \ \ x \in \overline{\Omega}, \qquad \text{and} \quad \partial_\nu \psi(x) \leqslant 0 \ \ x \in \Gamma\backslash\Gamma_0.$$

On the other hand, for any given parameter $\beta > 0$, we define the weight function η as follows

$$\eta(x) = e^{\beta\,\psi(x)} \qquad x \in \Omega.$$

Then the following Carleman estimate holds true:

Proposition 3.1. *(see ([8, 11])) There exist $\gamma_0 > 0$ and $C > 0$ such that for all $\gamma \geqslant \gamma_0$, we have the following estimate:*

$$\int_\Omega (\gamma|\nabla u(x)|^2 + \gamma^3 |u(x)|^2) e^{2\gamma\eta(x)}\,dx \leqslant \int_\Omega |\Delta u(x)|^2 e^{2\gamma\eta(x)}\,dx + \int_{\Gamma_0} \gamma |\partial_\nu u(x)|^2 e^{2\gamma\eta(x)}\,d\sigma,$$

for any $u \in H^2(\Omega)$ such that $u(x) = 0$ on Γ.

Using the above statement, we are now able to stably retrieve the first order coefficient V from the information given by the DN map Λ_V.

3.1. **Stability estimate for the velocity field.** Armed with Proposition 3.1, we turn now to proving the main result of this paper. Let us consider two velocity fields $V_1, V_2 \in \mathcal{V}(V_0, M)$. We define $V = V_1 - V_2$. Our goal is to show that V stably depends on the DN map $\Lambda_{V_1} - \Lambda_{V_2}$. In view of (2.37) and (2.4) we have the existence of a function $\varphi \in W^{3,p_0}(\Omega) \cap H_0^1(\Omega)$ such that

$$V = V_1 - V_2 = -2iA' + \nabla(2i\psi) = V' + \nabla\varphi. \tag{3.47}$$

Then φ is solution to the following equation

$$\begin{cases} \Delta\varphi = \Psi := \mathrm{div}V - \mathrm{div}V' = \mathrm{div}V, & \text{in } \ \Omega, \\ \varphi = 0, & \text{in } \ \Gamma. \end{cases}$$

Thanks to (2.4) and (3.47), we have

$$\Psi = 2(q_2 - q_1) + \frac{1}{2}(V')(V_1 + V_2) + \frac{1}{2}\nabla\varphi\,(V_1 + V_2).$$

By applying Proposition 3.1 to the solution φ and using the fact that $\|V_j\|_{L^\infty(\Omega)} \leqslant M$, $j = 1,\, 2$, we find

$$\begin{aligned} \int_\Omega \gamma|\nabla\varphi(x)|^2 e^{2\gamma\eta(x)}\,dx &\leqslant \int_\Omega |\Delta\varphi(x)|^2 e^{2\gamma\eta(x)}\,dx + \int_{\Gamma_0} \gamma|\partial_\nu\varphi(x)|^2 e^{2\gamma\eta(x)}\,d\sigma \\ &\leqslant C\int_\Omega (|(q_2 - q_1)(x)|^2 + |V'(x)|^2 + |\nabla\varphi(x)|^2) e^{2\gamma\eta(x)} dx + \int_{\Gamma_0} \gamma|\partial_\nu\varphi(x)|^2 e^{2\gamma\eta(x)}\,d\sigma. \end{aligned} \tag{3.48}$$

By taking γ sufficiently large, (3.48) immediately yields

$$\int_\Omega \gamma|\nabla\varphi(x)|^2 e^{2\gamma\eta(x)}\,dx \leqslant C\int_\Omega (|(q_2 - q_1)(x)|^2 + |V'(x)|^2) e^{2\gamma\eta(x)} dx + \int_{\Gamma_0} \gamma|\partial_\nu\varphi(x)|^2 e^{2\gamma\eta(x)}\,d\sigma.$$

This implies that

$$\|\nabla\varphi\|^2_{L^2(\Omega)} \leqslant C\|q_2-q_1\|^2_{L^2(\Omega)} + \|V'\|^2_{L^2(\Omega)} + \|\partial_\nu\varphi\|^2_{L^2(\Gamma_0)}. \tag{3.49}$$

By interpolation and since $\|q_2-q_1\|_{W^{1,\infty}(\Omega)} \leqslant M$, it follows from (2.46) that

$$\|q_2-q_1\|_{L^2(\Omega)} \leqslant \|q_2-q_1\|^{1/2}_{H^1(\Omega)}\|q_2-q_1\|^{1/2}_{H^{-1}(\Omega)} \leqslant C\|N_{A_2,q_2}-N_{A_1,q_1}\|^{\kappa_1}, \tag{3.50}$$

for some $\kappa_1 \in (0,1)$. Moreover, from what has already been shown in Section 2.3, it is readily seen that

$$\|V'\|_{L^2(\Omega)} \leqslant C\|A'\|_{L^2(\Omega)} \leqslant C\|N_{A_1,q_1}-N_{A_2,q_2}\|^{\kappa_2}, \tag{3.51}$$

for some $\kappa_2 > 0$. On the other hand, owing to the assumption that $V = V_1 - V_2 = 0$ on Γ and taking advantage of Trace's Theorem, one gets

$$\|\partial_\nu\varphi\|_{L^2(\Gamma_0)} \leqslant \|V'\|_{L^2(\Gamma)} \leqslant \|V'\|_{H^1(\Omega)} \leqslant \|V'\|^{1/2}_{L^2(\Omega)}\|V'\|^{1/2}_{H^2(\Omega)} \leqslant C\|N_{A_1,q_1}-N_{A_2,q_2}\|^{\kappa_3}, \tag{3.52}$$

for some $\kappa_3 > 0$. In view of (3.49)–(3.52), it is easily understood that

$$\|V_1-V_2\|_{L^2(\Omega)} \leqslant \|V'\|_{L^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)} \leqslant \|N_{A_2,q_2}-N_{A_1,q_1}\|^{\kappa},$$

where $\kappa := \min(\kappa_1,\kappa_2,\kappa_3)$. From (2.5) we deduce the desired result.

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