

# Closed-form expressions of the eigen decomposition of $2 \times 2$ and $3 \times 3$ Hermitian matrices

Charles-Alban Deledalle, Loic Denis, Sonia Tabti, Florence Tupin

► **To cite this version:**

Charles-Alban Deledalle, Loic Denis, Sonia Tabti, Florence Tupin. Closed-form expressions of the eigen decomposition of  $2 \times 2$  and  $3 \times 3$  Hermitian matrices. [Research Report] Université de Lyon. 2017. <hal-01501221>

**HAL Id: hal-01501221**

**<https://hal.archives-ouvertes.fr/hal-01501221>**

Submitted on 4 Apr 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Closed-form expressions of the eigen decomposition of $2 \times 2$ and $3 \times 3$ Hermitian matrices

Charles-Alban Deledalle, Loïc Denis, Sonia Tabti, Florence Tupin

April 3, 2017

## Abstract

The eigen decomposition of covariance matrices is at the core of many data analysis techniques. The study of 2-components or 3-components vector fields typically requires computing numerous eigen decompositions of  $2 \times 2$  or  $3 \times 3$  matrices. This is, for example, the case in the analysis of interferometric or polarimetric SAR images, see MuLoG algorithm (<https://hal.archives-ouvertes.fr/hal-01388858>). The closed-form expression of eigenvalues and eigenvectors then provides a way to derive faster data processing algorithms. This note gives these expressions in the general case (special cases where some coefficients are zero, or the eigenvalues are not separated may not be covered and then require either to introduce a small perturbation of the initial matrix or to derive other expressions).

## 1 Formulæ for $2 \times 2$ Hermitian matrices

We consider the Hermitian matrix  $\mathbf{C}$  defined by:

$$\mathbf{C} = \begin{pmatrix} a & c^* \\ c & b \end{pmatrix} = \begin{pmatrix} v_{1,1} & v_{2,1} \\ v_{1,2} & v_{2,2} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_{1,1}^* & v_{1,2}^* \\ v_{2,1}^* & v_{2,2}^* \end{pmatrix} \quad (1)$$

where  $a$  and  $b$  are real valued,  $c$  is complex valued and  $c^*$  is the complex conjugate of  $c$ .

**Eigenvalues:** The eigenvalues of  $\mathbf{C}$  are given by

$$\begin{cases} \lambda_1 & = & (a + b - \delta)/2 \\ \lambda_2 & = & (a + b + \delta)/2 \end{cases} \quad (2)$$

with  $\delta = \sqrt{4|c|^2 + (a - b)^2}$ .

**Eigenvectors:** The eigenvectors of  $\mathbf{C}$  are given by

$$\begin{cases} v_{1,1} & = & (a - b + \delta)/(2c) & = & (\lambda_2 - b)/c \\ v_{1,2} & = & 1 \\ v_{2,1} & = & (a - b - \delta)/(2c) & = & (\lambda_1 - b)/c \\ v_{2,2} & = & 1 \end{cases} \quad (3)$$

Note that if  $c = 0$ , the matrix  $\mathbf{C}$  is already diagonal so the eigenvalues are  $a$  and  $b$  and the corresponding eigenvectors are  $v_1 = (1, 0)^t$  and  $v_2 = (0, 1)^t$ .

**Changing the eigenvalues:** Let's apply a function  $\mathcal{F} : \mathbb{R} \rightarrow \mathbb{R}$  on the eigenvalues of  $\mathbf{C}$ . We get a matrix  $\tilde{\mathbf{C}}$  defined by:

$$\tilde{\mathbf{C}} \equiv \begin{pmatrix} \tilde{a} & \tilde{c}^* \\ \tilde{c} & \tilde{b} \end{pmatrix} = \begin{pmatrix} v_{1,1} & v_{2,1} \\ v_{1,2} & v_{2,2} \end{pmatrix} \begin{pmatrix} \mathcal{F}(\lambda_1) & 0 \\ 0 & \mathcal{F}(\lambda_2) \end{pmatrix} \begin{pmatrix} v_{1,1}^* & v_{1,2}^* \\ v_{2,1}^* & v_{2,2}^* \end{pmatrix}. \quad (4)$$

For example, when  $\mathcal{F} : x \mapsto \log(x)$ ,  $\tilde{\mathbf{C}}$  is called the matrix logarithm of matrix  $\mathbf{C}$ , or when  $\mathcal{F} : x \mapsto \exp(x)$ ,  $\tilde{\mathbf{C}}$  is called the exponential of matrix  $\mathbf{C}$ .

Based on the previous results, we can derive the expression of  $\tilde{\mathbf{C}}$ :

$$\begin{cases} \tilde{a} &= \left[ (a - b + \delta)\tilde{\lambda}_2 - (a - b - \delta)\tilde{\lambda}_1 \right] / (2\delta), \\ \tilde{b} &= \left[ (b - a + \delta)\tilde{\lambda}_2 - (b - a - \delta)\tilde{\lambda}_1 \right] / (2\delta), \\ \tilde{c} &= c(\tilde{\lambda}_2 - \tilde{\lambda}_1) / \delta, \end{cases} \quad (5)$$

with  $\tilde{\lambda}_1 = \mathcal{F}(\lambda_1)$  and  $\tilde{\lambda}_2 = \mathcal{F}(\lambda_2)$ .

## 2 Formulæ for $3 \times 3$ Hermitian matrices

We now consider the  $3 \times 3$  Hermitian matrix  $\mathbf{C}$  defined by:

$$\mathbf{C} = \begin{pmatrix} a & d^* & f^* \\ d & b & e^* \\ f & e & c \end{pmatrix} = \begin{pmatrix} v_{1,1} & v_{2,1} & v_{3,1} \\ v_{1,2} & v_{2,2} & v_{3,2} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} v_{1,1}^* & v_{1,2}^* & v_{1,3}^* \\ v_{2,1}^* & v_{2,2}^* & v_{2,3}^* \\ v_{3,1}^* & v_{3,2}^* & v_{3,3}^* \end{pmatrix} \quad (6)$$

where  $a, b$  and  $c$  are real-valued,  $d, e$  and  $f$  are complex-valued.

**Eigenvalues:** The eigenvalues of  $\mathbf{C}$  are given by

$$\begin{cases} \lambda_1 = [a + b + c - 2\sqrt{x_1} \cos(\varphi/3)] / 3, \\ \lambda_2 = \{a + b + c + 2\sqrt{x_1} \cos[(\varphi - \pi)/3]\} / 3, \\ \lambda_3 = \{a + b + c + 2\sqrt{x_1} \cos[(\varphi + \pi)/3]\} / 3, \end{cases} \quad (7)$$

with

$$\begin{cases} x_1 &= a^2 + b^2 + c^2 - ab - ac - bc + 3(|d|^2 + |f|^2 + |e|^2) \\ x_2 &= -(2a - b - c)(2b - a - c)(2c - a - b) \\ &\quad + 9[(2c - a - b)|d|^2 + (2b - a - c)|f|^2 + (2a - b - c)|e|^2] - 54 \Re(d^* e^* f) \end{cases} \quad (8)$$

and

$$\varphi = \begin{cases} \operatorname{atan}\left(\frac{\sqrt{4x_1^3 - x_2^2}}{x_2}\right) & \text{if } x_2 > 0 \\ \pi/2 & \text{if } x_2 = 0 \\ \operatorname{atan}\left(\frac{\sqrt{4x_1^3 - x_2^2}}{x_2}\right) + \pi & \text{if } x_2 < 0 \end{cases} \quad (9)$$

**Eigenvectors:** The eigenvectors of  $\mathbf{C}$  are given by:

$$v_1 = \begin{pmatrix} (\lambda_1 - c - e \cdot m_1) / f \\ m_1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} (\lambda_2 - c - e \cdot m_2) / f \\ m_2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} (\lambda_3 - c - e \cdot m_3) / f \\ m_3 \\ 1 \end{pmatrix}, \quad (10)$$

with

$$\begin{cases} m_1 = \frac{d(c-\lambda_1)-e^*f}{f(b-\lambda_1)-de} , \\ m_2 = \frac{d(c-\lambda_2)-e^*f}{f(b-\lambda_2)-de} , \\ m_3 = \frac{d(c-\lambda_3)-e^*f}{f(b-\lambda_3)-de} . \end{cases} \quad (11)$$

(special attention should be paid to cases where  $f = 0$ , or  $f(b - \lambda_1) - de = 0$ ,  $f(b - \lambda_2) - de = 0$  or  $f(b - \lambda_3) - de = 0$ )

**Changing the eigenvalues:** Let's apply a function  $\mathcal{F} : \mathbb{R} \rightarrow \mathbb{R}$  on the eigenvalues of  $\mathbf{C}$ . We get a matrix  $\tilde{\mathbf{C}}$  defined by:

$$\tilde{\mathbf{C}} = \begin{pmatrix} \tilde{a} & \tilde{d}^* & \tilde{f}^* \\ \tilde{d} & \tilde{b} & \tilde{e}^* \\ \tilde{f} & \tilde{e} & \tilde{c} \end{pmatrix} = \begin{pmatrix} v_{1,1} & v_{2,1} & v_{3,1} \\ v_{1,2} & v_{2,2} & v_{3,2} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{pmatrix} \begin{pmatrix} \mathcal{F}(\lambda_1) & 0 & 0 \\ 0 & \mathcal{F}(\lambda_2) & 0 \\ 0 & 0 & \mathcal{F}(\lambda_3) \end{pmatrix} \begin{pmatrix} v_{1,1}^* & v_{1,2}^* & v_{1,3}^* \\ v_{2,1}^* & v_{2,2}^* & v_{2,3}^* \\ v_{3,1}^* & v_{3,2}^* & v_{3,3}^* \end{pmatrix}. \quad (12)$$

Based on the previous results, we can derive the expression of  $\tilde{\mathbf{C}}$ :

$$\begin{cases} \tilde{a} = (\tilde{\lambda}_1 |\lambda_1 - c - e m_1|^2 + \tilde{\lambda}_2 |\lambda_2 - c - e m_2|^2 + \tilde{\lambda}_3 |\lambda_3 - c - e m_3|^2) / |f|^2 , \\ \tilde{b} = \tilde{\lambda}_1 |m_1|^2 + \tilde{\lambda}_2 |m_2|^2 + \tilde{\lambda}_3 |m_3|^2 , \\ \tilde{c} = \tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3 , \\ \tilde{d} = [\tilde{\lambda}_1 m_1 (\lambda_1 - c - e m_1)^* + \tilde{\lambda}_2 m_2 (\lambda_2 - c - e m_2)^* + \tilde{\lambda}_3 m_3 (\lambda_3 - c - e m_3)^*] / f^* , \\ \tilde{e} = \tilde{\lambda}_1 m_1^* + \tilde{\lambda}_2 m_2^* + \tilde{\lambda}_3 m_3^* , \\ \tilde{f} = [\tilde{\lambda}_1 (\lambda_1 - c - e m_1)^* + \tilde{\lambda}_2 (\lambda_2 - c - e m_2)^* + \tilde{\lambda}_3 (\lambda_3 - c - e m_3)^*] / f^* , \end{cases} \quad (13)$$

with  $\tilde{\lambda}_1 = \mathcal{F}(\lambda_1)/n_1$  and  $\tilde{\lambda}_2 = \mathcal{F}(\lambda_2)/n_2$  and  $\tilde{\lambda}_3 = \mathcal{F}(\lambda_3)/n_3$  and

$$\begin{cases} n_1 = 1 + |m_1|^2 + |\lambda_1 - c - e m_1|^2 / |f|^2 , \\ n_2 = 1 + |m_2|^2 + |\lambda_2 - c - e m_2|^2 / |f|^2 , \\ n_3 = 1 + |m_3|^2 + |\lambda_3 - c - e m_3|^2 / |f|^2 . \end{cases} \quad (14)$$