Expansions in Askey-Wilson polynomials via Bailey transform

Zeya Jia, Jiang Zeng

EXPANSIONS IN ASKEY-WILSON POLYNOMIALS VIA BAILEY TRANSFORM

ZEYA JIA AND JIANG ZENG

ABSTRACT. We prove a general expansion formula in Askey-Wilson polynomials using Bailey transform and Bressoud inversion. As applications, we give new proofs and generalizations of some recent results of Ismail-Stanton and Liu. Moreover, we prove a new q-beta integral formula involving Askey-Wilson polynomials, which includes the Nassrallah-Rahman integral as a special case. We also give a bootstrapping proof of Ismail-Stanton's recent generating function of Askey-Wilson polynomials.

1. Introduction

Andrews [2] demonstrates that q-orthogonal polynomials can play an important role in the theory of mock theta functions by applying the following expansion of a terminating, balanced ${}_5\phi_4$ in a series of Askey-Wilson polynomials, [2, (1.3)],

$$
{}_5\phi_4\left[\begin{matrix} q^{-N}, \rho_1, \rho_2, b, c \\ \rho_1\rho_2 q^{-N}/a, e, f, g \end{matrix}; q, q\right] = \frac{(aq/\rho_1, aq/\rho_2; q)_N}{(aq, aq/\rho_1\rho_2; q)_N}
\times \sum_{n=0}^{\infty} \frac{(q^{-N}, \rho_1, \rho_2, a; q)_n (1-aq^{2n})}{(q, aq/\rho_1, aq/\rho_2, aq^{N+1}; q)_n (1-a)} \left(\frac{aq^{N+1}}{\rho_1\rho_2}\right)^n {}_4\phi_3\left[\begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g \end{matrix}; q, q\right], \tag{1.1}
$$

where N is a non-negative integer, and $qabc = efg$.

As a follow-up to [2], Ismail and Stanton [12] show that Andrews' formula (1.1) is one of many similar expansion formulae in the Askey-Wilson polynomials. In particular, they prove the transformation formula:

$$
{}_{p+1}\phi_p\left[\begin{matrix} a_1, \ldots, a_{p-1}, t_4/z, t_4 z \\ t_1t_4, t_2t_4, t_3t_4, b_1, \ldots, b_{p-3} \end{matrix}; q, \delta\right]
= \sum_{k=0}^{\infty} P_k(x; \mathbf{t}|q) \frac{(a_1, \ldots, a_{p-1}; q)_k}{(t_1t_4, t_2t_4, t_3t_4, b_1, \ldots, b_{p-3}; q)_k}
\times \frac{(-t_4\delta)^k q^{\binom{k}{2}}}{(q, t_1t_2t_3t_4q^{k-1}; q)_k} {}_{p-1}\phi_{p-2}\left[\begin{matrix} a_1q^k, \ldots, a_{p-1}q^k \\ b_1q^k, \ldots, b_{p-3}q^k, t_1t_2t_3t_4q^{2k} \end{matrix}; q, \delta\right], \tag{1.2}
$$

Date: March 30, 2017.
Mathematics Subject Classification(2010) 33D45, 33D05, 05A15.
Keywords: Andrews formula, Askey-Wilson polynomials, Bailey transform, Bressoud inversion, Nassrallah-Rahman integral, expansion formulae.

where $x=\cos\theta$ and $z=e^{i\theta}$, the Askey-Wilson polynomials are defined by

$$P_n(x;\mathbf{t}|q)=t_1^{-n}(t_1t_2,t_1t_3,t_1t_4;q)_n{}_4\phi_3\left[\begin{matrix}q^{-n},t_1t_2t_3t_4q^{n-1},t_1e^{i\theta},t_1e^{-i\theta}\\ t_1t_2,t_1t_3,t_1t_4\end{matrix};q,q\right]. \tag{1.3}$$

Note that taking $p=4$, $a_1=q^{-N}$, $a_2=\rho_1$, $a_3=\rho_2$, $b_1=\rho_1\rho_2q^{-N}/a$, $u=1$ and $\delta=z$ in (1.2) the ${}_{p-1}\phi_{p-2}$, namely ${}_3\phi_2$, series at the right-hand side of the transformation can be summed by q-Pfaff-Saalschütz sum (7.4), we obtain the following result of Liu [17, Theorem 10.1], for any non-negative N and $|z|<1$,

$$\begin{aligned}{}_5\phi_4\left[\begin{matrix}q^{-N},\rho_1,\rho_2,b,c\\ \rho_1\rho_2q^{-N}/a,e,f,g\end{matrix};q,z\right]&=\frac{(aq/\rho_1,aq/\rho_2;q)_N}{(aq,aq/\rho_1\rho_2;q)_N}\\ &\times\sum_{n=0}^{N}\frac{(q^{-N},\rho_1,\rho_2,a;q)_n(1-aq^{2n})}{(q,aq/\rho_1,aq/\rho_2,aq^{N+1};q)_n(1-a)}\left(\frac{aq^{N+1}}{\rho_1\rho_2}\right)^n{}_4\phi_3\left[\begin{matrix}q^{-n},aq^n,b,c\\ e,f,g\end{matrix};q,z\right].\end{aligned} \tag{1.4}$$

The above formula is an extension of Watson's transformation (7.9). Moreover, the $z=q$ case corresponds to Andrews's result (1.1) if $aqbc=efg$.

This paper arose from the desire to understand the Ismail-Stanton formula (1.2) through Bailey's machinery. Actually, Ismail-Stanton derived (1.2) from an expansion formula due to Ismail-Rahman [9], see also [11], which was proved using the orthogonality relation of Askey-Wilson polynomials, while Andrews' original proof of (1.1) used Bailey's transform with a special Bailey pair, which is equivalent to an inversion relation [3, (12.2.8)]. Looking at Ismail-Stanton's formula through Bailey's glance and using an inversion formula due to Bressoud [5], we are able to generalize formula (1.2) in several ways, see Proposition 2.3, Proposition 2.5 and Theorem 2.6.

A fundamental result about Askey-Wilson polynomials is the Askey-Wilson q-beta integral,

$$\int_0^\pi\frac{h(\cos 2\theta;1)}{h(\cos\theta;t_1,t_2,t_3,t_4)}d\theta=\frac{2\pi(t_1t_2t_3t_4;q)_\infty}{(q;q)_\infty\prod_{1\le r<s\le 4}(t_rt_s;q)_\infty}, \tag{1.5}$$

where $\max\{|t_1|,|t_2|,|t_3|,|t_4|\}<1$ and

$$h(\cos\theta;t_1,\ldots,t_r)=\prod_{j=1}^{r}(t_je^{i\theta},t_je^{-i\theta};q)_\infty.$$

Nassrallah-Rahman [18] obtained the following important generalization of (1.5)

$$\begin{aligned}\int_0^\pi\frac{h(\cos 2\theta;1)h(\cos\theta;t_6)}{h(\cos\theta;t_1,t_2,t_3,t_4,t_5)}d\theta&=\frac{2\pi(t_6/t_1,t_6t_1,t_1t_3t_4t_5,t_1t_2t_3t_5,t_1t_2t_3t_4,t_1t_2t_4t_5;q)_\infty}{\prod_{1\le r<s\le 5}(t_rt_s;q)_\infty(q,t_1^2t_2t_3t_4t_5;q)_\infty}\\ &\times{}_8W_7\left(t_1^2t_2t_3t_4t_5/q,t_1t_5,t_1t_2,t_1t_3,t_1t_4,t_1t_2t_3t_4t_5/t_6\ ;q,t_6/t_1\right),\end{aligned} \tag{1.6}$$

where $\max\{|t_1|,|t_2|,|t_3|,|t_4|,|t_5|,|t_6|\}<1$.

When $t_6=t_1t_2t_3t_4t_5$, the above ${}_8W_7$ reduces to 1 and (1.6) becomes the following appealing formula [3, 19]:

$$\int_0^\pi\frac{h(\cos 2\theta;1)h(\cos\theta;t_1t_2t_3t_4t_5)}{h(\cos\theta;t_1,t_2,t_3,t_4,t_5)}d\theta=\frac{2\pi(t_1t_2t_3t_4,t_1t_2t_3t_5,t_1t_2t_4t_5,t_1t_3t_4t_5,t_2t_3t_4t_5;q)_\infty}{(q;q)_\infty\prod_{1\le r<s\le 5}(t_rt_s;q)_\infty}. \tag{1.7}$$

By combining Theorem 2.6 and (1.7) we will generalize the Nassrallah-Rahman integral (1.6) in Theorem 2.7, which includes also two integrals of Liu [17, Theorem 1.6] and Zhang-Wang [21, Theorem 4.3].

This paper is organized as follows. In Section 2, we first state and prove a general transformation, Proposition 2.3, and then derive two interesting expansions in Theorems 2.5 and 2.6. Moreover, we give a generalization of Nassrallah-Rahman integral (1.6) in Theorem 2.7. In Section 3, we derive some recent known results in [9, 10, 12] from our main results. In Section 4, we show how to derive some important known q-integrals from (2.10). In Section 5, we give a "bootstrapping proof" of Ismail-Stanton's generating function for Askey-Wilson polynomials. In Section 6, we give two general transformations and show how to recover two transformations of Ismail-Stanton and Verma [12, 20].

Throughout this paper, we assume that q is a complexe number such that $0 < |q| < 1$ and use standard q-notations in [7, 8]. Moreover, in Section 7, for the reader's convenience, we list all summation and transformation formulae used in our proofs.

2. Main results

Our starting point is the Bailey transform, see [3, Chap. 12] for a gentle introduction.

Lemma 2.1 (Bailey transform)**.** *Subject to conditions on the four sequences α_n, β_n, γ_n and δ_n which make all the infinite series absolutely convergent, if*

$$\beta_n = \sum_{r=0}^{n} \alpha_r \upsilon_{n-r} \nu_{n+r}, \tag{2.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r \upsilon_{r-n} \nu_{r+n}, \tag{2.2}$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{2.3}$$

For our purpose we need to choose suitable sequences (υ_n, ν_n) so that (2.1) can be inverted. First, we recall the following matrix inversion due to Bressoud [5].

Lemma 2.2 (Bressoud's inversion)**.** *For $n, k \geq 0$ let*

$$C_{n,k}(a,b) = \frac{(1-aq^{2k})(b;q)_{n+k}(b/a;q)_{n-k}(b/a)^k}{(1-a)(aq;q)_{n+k}(q;q)_{n-k}}. \tag{2.4}$$

The following inversion formula holds true

$$\beta'_n = \sum_{k=0}^{n} C_{n,k}(a,b)\alpha'_k \Longleftrightarrow \alpha'_n = \sum_{k=0}^{n} C_{n,k}(b,a)\beta'_k. \tag{2.5}$$

Proposition 2.3. *We have*

$$\sum_{n=0}^{\infty}\beta_n\delta_n=\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a;q)_n(a/b;q)_n(b/a)^n}{(1-a)(bq;q)_n(q;q)_n}\times\sum_{k=0}^{n}\frac{(1-bq^{2k})(aq^n;q)_k(q^{-n};q)_k}{(1-b)(bq^{n+1};q)_k(bq^{1-n}/a;q)_k}q^k\beta_k\cdot\sum_{r=0}^{\infty}\frac{(b/a;q)_r(b;q)_{r+2n}}{(q;q)_r(aq;q)_{r+2n}}\delta_{r+n}, \tag{2.6}$$

subject to conditions on the two sequences β_n, δ_n which make all the infinite series absolutely convergent.

Proof. Substituting α'_k by $\frac{(1-a)(a/b)^k}{(1-aq^{2k})}\alpha_k$ and β'_n by β_n in (2.5) we obtain

$$\beta_n=\sum_{k=0}^{n}\frac{(b/a;q)_{n-k}(b;q)_{n+k}}{(q;q)_{n-k}(aq;q)_{n+k}}\alpha_k. \tag{2.7}$$

Inverting (2.7) using (2.5) we obtain

$$\begin{aligned}\alpha_n&=\frac{(1-aq^{2n})(b/a)^n}{(1-a)}\sum_{k=0}^{n}\frac{(1-bq^{2k})(a;q)_{n+k}(a/b;q)_{n-k}(a/b)^k}{(1-b)(bq;q)_{n+k}(q;q)_{n-k}}\beta_k\\&=\frac{(1-aq^{2n})(a;q)_n(a/b;q)_n(b/a)^n}{(1-a)(bq;q)_n(q;q)_n}\sum_{k=0}^{n}\frac{(1-bq^{2k})(aq^n;q)_k(q^{-n};q)_k}{(1-b)(bq^{n+1};q)_k(bq^{1-n}/a;q)_k}q^k\beta_k.\end{aligned}$$

In view of (2.7) and (2.1), we choose two sequences (υ_n,ν_n) as

$$\upsilon_n=\frac{(b/a;q)_n}{(q;q)_n}\quad and\quad \nu_n=\frac{(b;q)_n}{(aq;q)_n}.$$

Then, we can compute γ_n by (2.2)

$$\begin{aligned}\gamma_n&=\sum_{r=n}^{\infty}\delta_r\nu_{r-n}\upsilon_{r+n}\\&=\sum_{r=0}^{\infty}\delta_{r+n}\frac{(b/a;q)_r(b;q)_{r+2n}}{(q;q)_r(aq;q)_{r+2n}}.\end{aligned}$$

Plugging the four sequences α_n, β_n, γ_n and δ_n into the Bailey transform (2.3) yields (2.6). □

Remark 2.4. *The pair (α_n,β_n) satisfying (2.7) is called a WP-Bailey pair, see [1]. When $b=0$ a WP-Bailey pair is called a Bailey pair.*

Setting $\delta_n=\frac{(a_1,\ldots,a_{p-1};q)_n}{(b_1,\ldots,b_{p-1};q)_n}\delta^n$ in Proposition 2.3, we obtain the following general transformation.

Theorem 2.5. *Let δ, a_i, b_i be any complex numbers such that $|a_i| < 1$, $|b_i| < 1$ $(1 \le i \le p-1)$ and $|\delta| < 1$. Under suitable convergence conditions, for any complex sequence $\{\beta_n\}$, we have*

$$\sum_{n=0}^{\infty} \frac{(a_1,\dots,a_{p-1};q)_n}{(b_1,\dots,b_{p-1};q)_n}\delta^n\beta_n = \sum_{n=0}^{\infty} \frac{(a,a/b;q)_n(1-aq^{2n})(b;q)_{2n}}{(q,bq;q)_n(1-a)(aq;q)_{2n}}$$
$$\times \frac{(a_1,\dots,a_{p-1};q)_n}{(b_1,\dots,b_{p-1};q)_n}(b\delta/a)^n \sum_{k=0}^{n} \frac{(1-bq^{2k})(aq^n;q)_k(q^{-n};q)_k}{(1-b)(bq^{n+1};q)_k(bq^{1-n}/a;q)_k} q^k\beta_k$$
$$\times\, {}_{p+1}\phi_p\left[\begin{matrix} a_1q^n,\dots,a_{p-1}q^n, b/a, bq^{2n} \\ b_1q^n,\dots,b_{p-1}q^n, aq^{2n+1}\end{matrix} ; q, \delta\right].$$

If we choose

$$b=0, \quad \beta_n = \frac{(g,h;q)_n}{(q,c,d,e;q)_n}u^n, \quad b_{p-1}=b_{p-2}=0,$$

in Theorem 2.5, then we obtain the following generalisation of (1.2), which is our first main result.

Theorem 2.6. *Let δ, u, c, d, e, g, h, b_i, a_i $(i\in\mathbb{N})$ be any complex numbers such that $|\delta|<1$, $|u|<1$ $|a_i|<1$, $|b_i|<1$ $(1\le i\le p-1)$. Then the following identity holds*

$$(2.8)\quad {}_{p+1}\phi_p\left[\begin{matrix} a_1,\dots,a_{p-1},g,h \\ c,d,e,b_1,\dots,b_{p-3}\end{matrix} ; q, \delta u\right] = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(a_1,\dots,a_{p-1};q)_n}{(q,aq^n,b_1,\dots,b_{p-3};q)_n}\delta^n$$
$$\times\, {}_4\phi_3\left[\begin{matrix} q^{-n},aq^n,g,h \\ c,d,e\end{matrix} ; q, qu\right] {}_{p-1}\phi_{p-2}\left[\begin{matrix} a_1q^n,\dots,a_{p-1}q^n \\ b_1q^n,\dots,b_{p-3}q^n,aq^{2n+1}\end{matrix} ; q, \delta\right].$$

We recover Ismail-Stanton's result (1.2) by choosing, in the above transformation,

$$u=1,\ a=t_1t_2t_3t_4/q,\ g=t_4/z,\ h=t_4z,\ c=t_1t_4,\ d=t_2t_4,\ e=t_3t_4$$

(thus $aqgh = cde$) and then applying Sears' transformation (7.5).

By using the expansion formula (2.8) and integral formula (1.7) we can derive a generalization of Nassrallah-Rahman integral (1.6) in Theorem 2.7, which is our second main result. For convenience we shall use the following compact notation

$$(2.9)\qquad A(\mathbf{t}) := \frac{2\pi(t_1t_2t_3t_4,t_1t_2t_3t_5,t_1t_2t_4t_5,t_1t_3t_4t_5;q)_\infty}{(q,\alpha q;q)_\infty \prod_{1\le r<s\le 5}(t_rt_s;q)_\infty}.$$

Theorem 2.7. *Let $\alpha q = t_1^2t_2t_3t_4t_5$. If $|g|\ne|h|$ and $\max\{|t_i|\}<1$ $(1\le i\le 5)$, then*

$$(2.10)\qquad \int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;t_1,t_2,t_3,t_4,t_5)}\,{}_4\phi_3\left[\begin{matrix} g,h,t_1e^{i\theta},t_1e^{-i\theta} \\ c,d,\alpha qgh/cd\end{matrix} ; q, t_2t_3t_4t_5\right] d\theta$$
$$= A(\mathbf{t})\sum_{n=0}^{\infty}\frac{(1-\alpha q^{2n})}{(1-\alpha)}\frac{(\alpha,t_1t_2,t_1t_3,t_1t_4,t_1t_5;q)_n(-1)^nq^{\binom{n}{2}}(t_2t_3t_4t_5)^n}{(q,t_1t_2t_3t_4,t_1t_2t_3t_5,t_1t_2t_4t_5,t_1t_3t_4t_5;q)_n}$$
$$\times {}_4\phi_3\left[\begin{matrix} q^{-n},\alpha q^n,g,h \\ c,d,\alpha qgh/cd\end{matrix} ; q, q\right].$$

Taking $g=s_4/z$, $h=s_4z$, $c=s_1s_4$, $d=s_2s_4$ and $\alpha q = s_1s_2s_3s_4$ in Theorem 2.7, we get the following formula involving Askey-Wilson polynomials.

Corollary 2.8. *If* $\max\{|t_i|,|s_j|\}<1$ $(1\le i\le 5, 1\le j\le 4)$ *and* $\alpha q=t_1^2t_2t_3t_4t_5$*, then*

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;t_1,t_2,t_3,t_4,t_5)}{}_4\phi_3\left[\begin{matrix} s_4z, s_4/z, t_1e^{i\theta}, t_1e^{-i\theta} \\ s_1s_4, s_2s_4, s_3s_4\end{matrix};q, t_2t_3t_4t_5\right]d\theta \tag{2.11}$$
$$= A(\mathbf{t})\sum_{n=0}^\infty \frac{(1-\alpha q^{2n})}{(1-\alpha)}\frac{(\alpha, t_1t_2, t_1t_3, t_1t_4, t_1t_5;q)_n(-1)^nq^{\binom{n}{2}}(t_2t_3t_4t_5s_4)^nP_n(y;\mathbf{s}|q)}{(q, t_1t_2t_3t_4, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5, s_1s_4, s_2s_4, s_3s_4;q)_n},$$

where $z=e^{i\varphi}$, $y=\cos\varphi$ *and* $P_n(y;\mathbf{s}|q)$ *are Askey-Wilson polynomials.*

In Theorem 2.7 choosing $g=s$, $h=adt/q$, $c=st$ and then letting $d\to\infty$, we obtain

Theorem 2.9. *If* $\max\{|t_i|, |at_1^2t_2t_3t_4t_5/q|\}<1$, $(1\le i\le 5)$ *and* $\alpha q=t_1^2t_2t_3t_4t_5$*, then*

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;t_1,t_2,t_3,t_4,t_5)}{}_3\phi_2\left[\begin{matrix} s, t_1e^{i\theta}, t_1e^{-i\theta} \\ st, at_1^2t_2t_3t_4t_5/q\end{matrix};q, at_2t_3t_4t_5/q\right]d\theta$$
$$= A(\mathbf{t})\sum_{n=0}^\infty \frac{(1-\alpha q^{2n})}{(1-\alpha)}\frac{(\alpha, t_1t_2, t_1t_3, t_1t_4, t_1t_5;q)_n(-1)^nq^{\binom{n}{2}}(t_2t_3t_4t_5)^n}{(q, t_1t_2t_3t_4, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5;q)_n} \tag{2.12}$$
$$\times {}_3\phi_2\left[\begin{matrix} q^{-n}, \alpha q^n, s \\ st, \alpha a\end{matrix};q, at\right]$$
$$= A(\mathbf{t})\sum_{n=0}^\infty \frac{(1-\alpha q^{2n})}{(1-\alpha)}\frac{(\alpha, q/a, t_1t_2, t_1t_3, t_1t_4, t_1t_5;q)_n(\alpha at_2t_3t_4t_5)^n}{(q, \alpha a, t_1t_2t_3t_4, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5;q)_n} \tag{2.13}$$
$$\times {}_3\phi_2\left[\begin{matrix} q^{-n}, \alpha q^n, t \\ st, q/a\end{matrix};q, q\right].$$

Note that (2.13) follows from applying the transformation (7.6) to the last ${}_3\phi_2$ in (2.12) with $c=\alpha q^n,\ b=t,\ d=st,\ e=q/a$. A proof and further applications of Theorem 2.7 will be given in section 4.

3. Applications of Theorem 2.6

In this section, we show that Theorem 2.6 encompasses some results of Ismail-Rahman and Ismail-Stanton in [9, 12].

Theorem 3.1. *For any non-negative* N*, we have*

$$\frac{(b_1/b, b_1/c;q)_N}{(b_1/bc, b_1;q)_N} = \sum_{n=0}^N \frac{(q^{-N}, e, f, g;q)_nq^n(-1)^nq^{\binom{n}{2}}}{(q, aq^n, b_1, bcq^{1-N}/b_1;q)_n}{}_4\phi_3\left[\begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g\end{matrix};q, q\right] \tag{3.1}$$
$$\times\ {}_4\phi_3\left[\begin{matrix} q^{-N+n}, eq^n, fq^n, gq^n \\ b_1q^n, bcq^{1-N+n}/b_1, aq^{2n+1}\end{matrix};q, q\right],$$

and, for $|b_1/bc| < 1$,

$$\frac{(b_1/b, b_1/c;q)_\infty}{(b_1/bc, b_1;q)_\infty} = \sum_{n=0}^{\infty} \frac{(e,f,g;q)_n (b_1/bc)^n (-1)^n q^{\binom{n}{2}}}{(q, aq^n, b_1;q)_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g \end{matrix} ;q,q\right] \times {}_3\phi_2\left[\begin{matrix} eq^n, fq^n, gq^n \\ b_1q^n, aq^{2n+1} \end{matrix} ;q, b_1/bc\right]. \tag{3.2}$$

Proof. Taking $p=5$, $a_1 = q^{-N}$, $a_2 = e$, $a_3 = f$ and $a_4 = g$ in Theorem 2.6, we have

$${}_3\phi_2\left[\begin{matrix} q^{-N}, b, c \\ b_1, b_2 \end{matrix} ;q, \delta u\right] = \sum_{n=0}^{N} \frac{(q^{-N}, e, f, g;q)_n \delta^n (-1)^n q^{\binom{n}{2}}}{(q, aq^n, b_1, b_2;q)_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g \end{matrix} ;q, qu\right] \times {}_4\phi_3\left[\begin{matrix} q^{-N+n}, eq^n, fq^n, gq^n \\ b_1q^n, b_2q^n, aq^{2n+1} \end{matrix} ;q, \delta\right].$$

When $b_2 = bcq^{1-N}/b_1$, $\delta = q$ and $u = 1$, the above ${}_3\phi_2$ series can be summed by q-Pfaff-Saalschütz sum (7.4) and we obtain (3.1). Letting $N \to \infty$ in (3.1) yields (3.2), where taking the limit inside the sum is justified by Tannery's theorem, the discrete analogue of the Lebesgue dominated convergence theorem. We omit the details. □

The following connection formula (3.3) was first proved by Ismail-Rahman-Stanton [11] though the limit case (3.4) appeared in a recent paper of Ismail and Stanton [12, Theorem 3.1].

Theorem 3.2 (Ismail-Rahman-Stanton)**.** *For any non-negative n, we have*

$$(be^{i\theta}, be^{-i\theta};q)_n = \sum_{k=0}^{n} f_{n,k}(b,\mathbf{t}) P_k(x,\mathbf{t}|q) \tag{3.3}$$

where

$$f_{n,k}(b,\mathbf{t}) = \frac{(-b)^k q^{\binom{k}{2}} (q;q)_n (b/t_4, bt_4q^k;q)_{n-k}}{(q, t_1t_2t_3t_4q^{k-1};q)_k (q;q)_{n-k}} {}_4\phi_3\left[\begin{matrix} q^{k-n}, t_2t_4q^k, t_1t_4q^k, t_3t_4q^k \\ bt_4q^k, t_4q^{1-n+k}/b, t_1t_2t_3t_4q^{2k} \end{matrix} ;q,q\right],$$

and

$$\frac{(be^{i\theta}, be^{-i\theta};q)_\infty}{(bt_4, b/t_4;q)_\infty} = \sum_{k=0}^{n} P_k(x,\mathbf{t}|q) \frac{(-b)^k q^{\binom{k}{2}}}{(q, bt_4, t_1t_2t_3t_4q^{k-1};q)_k} \times {}_3\phi_2\left[\begin{matrix} t_2t_4q^k, t_1t_4q^k, t_3t_4q^k \\ bt_4q^k, t_1t_2t_3t_4q^{2k} \end{matrix} ;q, \frac{b}{t_4}\right]. \tag{3.4}$$

Proof. Let $a = t_1t_2t_3t_4/q$, $b = t_4z$, $c = t_4/z$, $b_1 = bt_4$, $e = t_2t_4$, $f = t_3t_4$ and $g = t_1t_4$ and $z = e^{i\theta}$ in (3.1). Then, Sears' transformation (7.5) infers that

$${}_4\phi_3\left[\begin{matrix} q^{-n}, aq^n, b, c \\ e, f, g \end{matrix} ;q,q\right] = \frac{(t_1t_3, t_1t_2;q)_n q^{-\binom{n}{2}}}{(t_2t_4, t_3t_4;q)_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1z, t_1/z \\ t_1t_4, t_1t_3, t_1t_2 \end{matrix} ;q,q\right].$$

Now, replacing n by k and N by n in (3.1) we get (3.3) after simplification. Clearly (3.4) is the limit $n \to \infty$ case of (3.3). □

We can also derive a transformation of Ismail-Rahman-Suslov [10, Theorem 5.3] from Theorem 2.6.

Theorem 3.3 (Ismail-Rahman-Suslov)**.** *We have*

$$\frac{(\alpha,\alpha ab/q;q)_\infty}{(\alpha a,\alpha b;q)_\infty}{}_3\phi_2\left[\begin{matrix} q/a,q/b,s \\ st,\alpha c\end{matrix};q,\alpha abct/q^2\right] \tag{3.5}$$
$$=\sum_{n=0}^{\infty}\frac{(1-\alpha q^{2n})(\alpha abc/q^2)^n(\alpha,q/a,q/b,q/c;q)_n}{(q,\alpha a,\alpha b,\alpha c;q)_n}{}_3\phi_2\left[\begin{matrix} q^{-n},\alpha q^n,t \\ st,q/c\end{matrix};q,q\right].$$

Proof. In (2.8), setting $p=3$ and substituting $a\to\alpha$, $a_1\to q/a$, $a_2\to q/b$, $e\to \alpha qgh/dc$, $\delta\to\alpha ab/q$ and $u=1$ we can sum the ${}_2\phi_1$ by the Gauss sum (7.1) and obtain

$$\frac{(\alpha,\alpha ab/q;q)_\infty}{(\alpha a,\alpha b;q)_\infty}{}_4\phi_3\left[\begin{matrix} q/a,q/b,g,h \\ c,d,\alpha qgh/dc\end{matrix};q,\alpha ab/q\right] \tag{3.6}$$
$$=\sum_{n=0}^{\infty}\frac{(1-\alpha q^{2n})(-1)^n q^{\binom{n}{2}}(\alpha ab/q)^n(\alpha,q/a,q/b;q)_n}{(q,\alpha a,\alpha b;q)_n}{}_4\phi_3\left[\begin{matrix} q^{-n},\alpha q^n,g,h \\ d,c,\alpha qgh/dc\end{matrix};q,q\right].$$

By Sears' transformation (7.5)

$$ {}_4\phi_3\left[\begin{matrix} q^{-n},\alpha q^n,g,h \\ d,c,\alpha qgh/dc\end{matrix};q,q\right]=\frac{(cq^{-n}/\alpha,q^{1-n}gh/dc;q)_n}{(c,\alpha qgh/dc;q)_n}(\alpha q^n)^n$$
$$\times{}_4\phi_3\left[\begin{matrix} q^{-n},\alpha q^n,d/g,d/h \\ d,\alpha q/c,dc/gh\end{matrix};q,q\right]. \tag{3.7}$$

Now, plugging (3.7) into (3.6) and substituting $g\to s$, $d\to st$, $c\to\alpha c$ we obtain

$$\frac{(\alpha,\alpha ab/q;q)_\infty}{(\alpha a,\alpha b;q)_\infty}{}_4\phi_3\left[\begin{matrix} q/a,q/b,s,h \\ st,\alpha c,qh/ct\end{matrix};q,\alpha ab/q\right]$$
$$=\sum_{n=0}^{\infty}\frac{(1-\alpha q^{2n})(\alpha abc/q^2)^n(\alpha,q/a,q/b,q/c;q)_n}{(q,\alpha a,\alpha b,\alpha c;q)_n}\frac{(q^{1-n}h/\alpha tc;q)_n}{(qh/tc;q)_n}(\alpha q^n)^n \tag{3.8}$$
$$\times\,{}_4\phi_3\left[\begin{matrix} q^{-n},\alpha q^n,t,st/h \\ st,q/c,\alpha ct/h\end{matrix};q,q\right].$$

Now, replace h by q^{-m}, for a positive integer m, then let $m\to\infty$ and apply Tannery's theorem. The result is (3.5). □

When $st=q/a$, the ${}_3\phi_2$ at the left-hand side of (3.5) reduces to a ${}_2\phi_1$, which can be summed by (7.1) and we get the following summation formula,

$$\sum_{n=0}^{\infty}\frac{(1-\alpha q^{2n})(\alpha abc/q^2)^n(\alpha,q/a,q/b,q/c;q)_n}{(q,\alpha a,\alpha b,\alpha c;q)_n}{}_3\phi_2\left[\begin{matrix} q^{-n},\alpha q^n,t \\ q/a,q/c\end{matrix};q,q\right] \tag{3.9}$$
$$=\frac{(\alpha,\alpha ab/q,\alpha bc/q,\alpha act/q;q)_\infty}{(\alpha a,\alpha b,\alpha c,\alpha abct/q^2;q)_\infty}.$$

Applying the transformation (7.6) to the above ${}_3\phi_2$ we obtain another result of Ismail-Rahman-Suslov [10, Theortem 5.1].

Corollary 3.4 (Ismail-Rahman-Suslov). *We have*

$$\sum_{n=0}^{\infty}\frac{(1-\alpha q^{2n})(\alpha abc/q^2)^n(\alpha,q/a,q/b,q/c;q)_n}{(q,\alpha a,\alpha b,\alpha c;q)_n}{}_3\phi_2\left[\begin{matrix}q^{-n},\alpha q^n,t\\ q/a,q/c\end{matrix};q,q\right] = \frac{(\alpha,\alpha ab/q,\alpha bc/q,\alpha act/q;q)_\infty}{(\alpha a,\alpha b,\alpha c,\alpha abct/q^2;q)_\infty}. \tag{3.10}$$

Ismail-Rahman-Suslov derived the above two results from their main theorem [10, Theortem 1.1], which exresses a double sum as a linear combination of two ${}_5\phi_4$ sums. We notice that if we make the sustitution $(a,b,c,d,e,f)\to(\alpha,q/b,q/c,q/d,q,\alpha)$ in their Theorem 1.1, then $qa/ef=1$, which annihilates the factor in front of the first ${}_5\phi_4$ and reduces the second ${}_5\phi_4$ to 1 in [10, (1.4)], and we obtain immediately the following remarquable extension of (3.10).

Theorem 3.5.

$$\sum_{n=0}^{\infty}\frac{(1-\alpha q^{2n})(\alpha bcd/q^2)^n(\alpha,q/b,q/c,q/d;q)_n}{(q,\alpha b,\alpha c,\alpha d;q)_n}{}_4\phi_3\left[\begin{matrix}q^{-n},\alpha q^n,g,h\\ q/b,q/c,\alpha bcgh/q\end{matrix};q,q\right] = \frac{(\alpha,\alpha bd/q,\alpha cd/q,\alpha bcg/q,\alpha bch/q,\alpha bcdgh/q^2;q)_\infty}{(\alpha b,\alpha c,\alpha d,\alpha bcgh/q,\alpha bcdg/q^2,\alpha bcdh/q^2;q)_\infty}. \tag{3.11}$$

It seems that (3.11) was first published by Liu [16, Theorem 3].

4. Proof of Theorem 2.7 and its applications

4.1. **Proof of Theorem 2.7.** Choosing $p=3$, $u=1$, $\delta=\alpha a_1a_2/q$,

$$a_1=(q/t_1)e^{i\theta},\quad a_2=(q/t_1)e^{-i\theta},\quad e=\alpha qgh/cd$$

in Theorem 2.5, we can sum the ${}_{p-1}\phi_{p-2}$ by the q-Gauss sum (7.1) and rewrite (2.8) as

$$\frac{(\alpha q,\alpha q/t_1^2;q)_\infty}{h(\cos\theta;\alpha q/t_1)}{}_4\phi_3\left[\begin{matrix}t_1e^{i\theta},t_1e^{-i\theta},g,h\\ c,d,\alpha qgh/dc\end{matrix};q,\alpha q/t_1^2\right] = \sum_{n=0}^{\infty}\frac{(1-\alpha q^{2n})(-1)^nq^{\binom{n}{2}}(\alpha q/t_1^2)^n(\alpha,t_1e^{i\theta},t_1e^{-i\theta};q)_n}{(1-\alpha)(q,\alpha qe^{i\theta}/t_1,\alpha qe^{-i\theta}/t_1;q)_n}{}_4\phi_3\left[\begin{matrix}q^{-n},\alpha q^n,g,h\\ c,d,\alpha qgh/dc\end{matrix};q,q\right]. \tag{4.1}$$

It is clear that the series at the left-hand side is convergent if $|\alpha q/t_1^2|<1$. The convergence of the right-hand side can be justified as follows: if $|h|<|g|$, then one can show (see [14, (1.11)]) that the terminate ${}_4\phi_3$ series has the asymptotic formula

$${}_4\phi_3\left[\begin{matrix}q^{-n},aq^n,g,h\\ c,d,aqgh/dc\end{matrix};q,q\right]\sim\frac{(h,d/g,c/g,qah/dc;q)_\infty g^n}{(c,d,h/g,aqgh/dc;q)_\infty},\qquad n\to\infty, \tag{4.2}$$

N.B. This formula is also given in [10, (1.5)] witout the factor $(h/g;q)_\infty$ in the denominator. Hence, in view of the factor $q^{\binom{n}{2}}$, the series on the right-hand side of (4.1) converges if $|g|\neq|h|$. Hence a sufficient condition of convergence of the infinite series on the two sides of (4.1) is

$$|g|\neq|h|,\quad |\alpha q/t_1^2|<1. \tag{4.3}$$

Since $\alpha q/t_1 = t_1t_2t_3t_4t_5$, we have $h(\cos\theta; t_1t_2t_3t_4t_5) = (\alpha q e^{i\theta}/t_1, \alpha q e^{-i\theta}/t_1; q)_\infty$ and

$$\frac{h(\cos\theta; t_1t_2t_3t_4t_5)(t_1e^{i\theta}, t_1e^{-i\theta}; q)_n}{h(\cos\theta; t_1)(\alpha q e^{i\theta}/t_1, \alpha q e^{-i\theta}/t_1; q)_n} = \frac{h(\cos\theta; t_1t_2t_3t_4t_5q^n)}{h(\cos\theta; t_1q^n)}.$$

Multiplying both sides of (4.1) by

$$\frac{h(\cos 2\theta; 1)h(\cos\theta; t_1t_2t_3t_4t_5)}{h(\cos\theta; t_1, t_2, t_3, t_4, t_5)}$$

and integrating over $0 \le \theta \le \pi$, we have

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; t_1, t_2, t_3, t_4, t_5)} {}_4\phi_3\left[\begin{matrix} t_1e^{i\theta}, t_1e^{-i\theta}, g, h \\ c, d, \alpha q g h/dc \end{matrix}; q, \alpha q/t_1^2\right] d\theta \tag{4.4}$$

$$= \frac{1}{(\alpha q, \alpha q/t_1^2; q)_\infty} \sum_{n=0}^\infty \frac{(1-\alpha q^{2n})(-1)^n q^{\binom{n}{2}} (\alpha q/t_1^2)^n (\alpha; q)_n}{(1-\alpha)(q;q)_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, \alpha q^n, g, h \\ c, d, \alpha q g h/dc \end{matrix}; q, q\right].$$

$$\times \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos\theta; t_1t_2t_3t_4t_5q^n)}{h(\cos\theta; t_1q^n, t_2, t_3, t_4, t_5)} d\theta.$$

The last integral can be evaluated by rescaling $t_1 \to t_1q^n$ in (1.7),

$$\int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos\theta; t_1t_2t_3t_4t_5q^n)}{h(\cos\theta; t_1q^n, t_2, t_3, t_4, t_5)} d\theta = \frac{2\pi(t_1t_2t_3t_4, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5, t_2t_3t_4t_5; q)_\infty}{(q;q)_\infty \prod_{1\le r<s\le 5}(t_rt_s; q)_\infty}$$
$$\times \frac{(t_1t_2, t_1t_3, t_1t_4, t_1t_5; q)_n}{(t_1t_2t_3t_4, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5; q)_n}.$$

Substituting this in (4.4), we obtain (2.10). ☐

4.2. **Nassrallah-Rahman integrals.** We show how to get Nassrallah-Rahman integral (1.6) from our Theorem 2.7 . Let $h = c$ and $g \to \infty$ in (2.10), we have

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; t_1, t_2, t_3, t_4, t_5)} {}_2\phi_1\left[\begin{matrix} t_1e^{i\theta}, t_1e^{-i\theta} \\ d \end{matrix}; q, d/t_1^2\right] d\theta \tag{4.5}$$

$$= \frac{2\pi(t_6/t_1, t_6t_1, t_1t_3t_4t_5, t_1t_2t_3t_5, t_1t_2t_3t_4, t_1t_2t_4t_5; q)_\infty}{\prod_{1\le r<s\le 5}(t_rt_s; q)_\infty (q, t_1^2t_2t_3t_4t_5; q)_\infty}$$

$$\times \sum_{n=0}^\infty \frac{(1-\alpha q^{2n})}{(1-\alpha)} \frac{(\alpha, t_1t_2, t_1t_3, t_1t_4, t_1t_5; q)_n}{(q, t_1t_2t_3t_4, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5; q)_n}$$

$$\times (-1)^n q^{\binom{n}{2}} (t_2t_3t_4t_5)^n {}_2\phi_1\left[\begin{matrix} q^{-n}, \alpha q^n \\ d \end{matrix}; q, d/\alpha\right].$$

Now, the above two ${}_2\phi_1$ series can be summed by q-Gauss summation (7.1) and q-Chu-Vandermonde sum (7.3), respectively,

$${}_2\phi_1\left[\begin{matrix} t_1e^{i\theta}, t_1e^{-i\theta} \\ d \end{matrix}; q, d/t_1^2\right] = \frac{(d/t_1e^{i\theta}, d/t_1e^{-i\theta}; q)_\infty}{(d, d/t_1^2; q)_\infty},$$

$${}_2\phi_1\left[\begin{matrix} q^{-n}, \alpha q^n \\ d \end{matrix}; q, d/\alpha\right] = \frac{(\alpha q/d; q)_n}{(d;q)_n}(-d/\alpha q)^n q^{-\binom{n}{2}}.$$

Plugging these into (4.5), and then taking $d \to t_6t_1$, we get the Nassrallah-Rahman integral (1.6).

In the following, we record some other well-known special cases of Theorem 2.7.

- **Askey-Wilson integral** When $t_5 = 0$, Theorem 2.7 immediately reduces to the Askey-Wilson integral (1.5).
- **Rahman Integral** (1.7) When $c = g$ and $h = d$ in (2.10), in the left hand, the ${}_4\phi_3$ series reduces to a ${}_2\phi_1$ series which can be summed by using q-Gauss summation (7.1),

$$ {}_2\phi_1\left[\begin{matrix} t_1e^{i\theta}, t_1e^{-i\theta} \\ \alpha q \end{matrix} ; q, t_2t_3t_4t_5\right] = \frac{(t_1t_2t_3t_4t_5e^{i\theta}, t_1t_2t_3t_4t_5e^{-i\theta}; q)_\infty}{(\alpha q, t_2t_3t_4t_5; q)_\infty}. $$

On the other hand, using q-Chu-Vandermonde sums (7.2), we have

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, \alpha q^n \\ \alpha q \end{matrix} ; q, q\right] = \frac{(q^{1-n}; q)_n}{(\alpha q; q)_n}(\alpha q^n)^n, $$

which is zero for $n \geq 1$. After some simplification, this integral reduces to (1.7).

- **Ismail integral** Ismail [8, p. 442] uses the following integral to derive Nassrallah-Rahman formula via analytic prolongation.

$$ \int_0^\pi \frac{h(\cos 2\theta; 1)(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_n}{h(\cos\theta; , t_1, t_2, t_3, t_4)} d\theta = \frac{2\pi(\alpha/t_4, \alpha t_4; q)_n (t_1t_2t_3t_4; q)_\infty}{(q; q)_\infty \prod_{1\leq r<s\leq 4}(t_rt_s; q)_\infty} \times {}_4\phi_3\left[\begin{matrix} q^{-n}, t_1t_4, t_2t_4, t_3t_4 \\ \alpha t_4, t_1t_2t_3t_4, q^{1-n}t_4/\alpha \end{matrix} ; q, q\right]. \tag{4.6} $$

When $h = c$, $g \to \infty$, $d = t_6t_1$ and $t_1 \leftrightarrow t_5$ in (2.10), in the right hand side, the inner summation can be written as

$$ {}_8W_7\left(t_5^2t_1t_2t_3t_4/q, t_1t_5, t_2t_5, t_5t_3, t_5t_4, t_1t_2t_3t_4t_5/t_6 \ ; q, t_6/t_5\right). \tag{4.7} $$

Replacing $a = t_5^2t_1t_2t_3t_4/q$, $b = t_5t_4$, $c = t_1t_5$, $d = t_2t_5$, $e = t_5t_3$ and $f = t_1t_2t_3t_4t_5/t_6$ into ${}_8W_7$ transformation (7.10), the above factor is equal to

$$ \frac{(t_5^2t_1t_2t_3t_4, t_5t_4, t_1t_6, t_2t_6, t_3t_6, t_1t_2t_3t_4; q)_\infty}{(t_2t_3t_4t_5, t_1t_3t_4t_5, t_1t_2t_4t_5, t_5t_6, t_1t_2t_3t_6, t_6/t_5; q)_\infty} \times {}_8W_7\left(t_1t_2t_3t_6/q, t_2t_3, t_1t_3, t_1t_2, t_6/t_4, t_6/t_5 \ ; q, t_4t_5\right). \tag{4.8} $$

Applying ${}_8W_7$ transformation (7.11) once more (with $a = t_1t_2t_3t_6/q$, $b = t_2t_3$, $c = t_1t_3$, $d = t_1t_2$, $e = t_6/t_4$, $f = t_6/t_5$), the ${}_8W_7$ series of (4.8) is equal to

$$ \frac{(t_1t_2t_3t_6, t_1t_2t_3t_4t_5/t_6, t_4t_6, t_5t_6; q)_\infty}{(t_1t_2t_3t_4, t_1t_2t_3t_5, t_6^2, t_4t_5; q)_\infty} \times {}_8W_7\left(t_6^2/q, t_6/t_1, t_6/t_2, t_6/t_3, t_6/t_4, t_6/t_5; q, t_1t_2t_3t_4t_5/t_6\right). \tag{4.9} $$

Replacing (4.8) and (4.9) into (4.7), we have another form of Nassrallah-Rahman integral (1.6),

$$ \int_0^\pi \frac{h(\cos 2\theta; 1)h(\cos\theta; t_6)}{h(\cos\theta; t_1, t_2, t_3, t_4, t_5)} d\theta = \frac{2\pi(\prod_{j=1}^5 (t_6t_j; q)_\infty (t_1t_2t_3t_4t_5/t_6; q)_\infty}{(q, t_6^2; q)_\infty \prod_{1\leq r<s\leq 5}(t_rt_s; q)_\infty} \times {}_8W_7\left(t_6^2/q, t_6/t_1, t_6/t_2, t_6/t_3, t_6/t_4, t_6/t_5 \ ; q, t_1t_2t_3t_4t_5/t_6\right), $$

Taking $t_1=t_6q^n$ in the above result, we have

$$\int_0^\pi \frac{h(\cos 2\theta;1)(t_6e^{i\theta},t_6e^{-i\theta};q)_n}{h(\cos\theta;t_2,t_3,t_4,t_5)}d\theta = \frac{2\pi(t_2t_3t_4t_5q^n;q)_\infty(t_6^2q^n;q)_\infty\prod_{j=2}^5(t_jt_6;q)_\infty}{(q,t_6^2;q)_\infty\prod_{j=2}^5(t_jt_6q^n;q)_\infty\prod_{2\le r<s\le 5}(t_rt_s;q)_\infty} \times {}_8W_7\left(t_6^2/q,q^{-n},t_6/t_2,t_6/t_3,t_6/t_4,t_6/t_5\ ;q,t_2t_3t_4t_5q^n\right). \tag{4.10}$$

By using Watson's transformation($a=t_6^2/q$, $b=t_6/t_2$, $c=t_6/t_3$, $d=t_6/t_4$ and $e=t_6/t_5$ in (7.9)), the ${}_8W_7$ series can be reduced to

$$\begin{aligned}&{}_8W_7\left(t_6^2/q,q^{-n},t_6/t_2,t_6/t_3,t_6/t_4,t_6/t_5\ ;q,t_2t_3t_4t_5q^n\right)\\&=\frac{(t_6^2,t_4t_5;q)_n}{(t_4t_6,t_5t_6;q)_n}{}_4\phi_3\left[\begin{matrix}q^{-n},t_6/t_4,t_6/t_5,t_2t_3\\ t_2t_6,t_3t_6,q^{1-n}/t_4t_5\end{matrix};q,q\right]\\&=\frac{(t_6^2,t_4t_5,t_6/t_2,t_2t_3t_4t_5;q)_n}{(t_4t_6,t_5t_6,t_3t_6,t_4t_5;q)_n}{}_4\phi_3\left[\begin{matrix}q^{-n},t_2t_3,t_2t_4,t_2t_5\\ t_2t_6,t_2t_3t_4t_5,q^{1-n}t_2/t_6\end{matrix};q,q\right].\end{aligned}$$

The second step is obtained by Sears' transformation (7.5)($a=t_2t_3$, $b=t_6/t_4$, $c=t_6/t_5$, $d=t_2t_6$, $e=t_3t_6$, $f=q^{1-n}/t_4t_5$). Replacing the above formula into (4.10), we get (4.6) after taking $t_2\leftrightarrow t_4$, $t_1\leftrightarrow t_5$ and $t_6\to\alpha$.

- **Ismail-Stanton-Viennot integral** It is proved in [13] that

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;t_1,t_2,t_3,t_4,t_5)}d\theta = \frac{2\pi(t_1t_2t_3t_4,t_2t_3t_4t_5,t_1t_5;q)_\infty}{(q;q)_\infty\prod_{1\le r<s\le 5}(t_rt_s;q)_\infty} \times {}_3\phi_2\left[\begin{matrix}t_2t_3,t_2t_4,t_3t_4\\ t_1t_2t_3t_4,t_2t_3t_4t_5\end{matrix};q,t_1t_5\right], \tag{4.11}$$

where $\max\{|t_1|,|t_2|,|t_3|,|t_4|,|t_5|\}<1$.

When $g=1$, the integral in (2.10) becomes

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;t_1,t_2,t_3,t_4,t_5)}d\theta = \frac{2\pi(t_1t_2t_3t_4,t_1t_2t_3t_5,t_1t_2t_4t_5,t_1t_3t_4t_5;q)_\infty}{(q,\alpha q;q)_\infty\prod_{1\le r<s\le 5}(t_rt_s;q)_\infty} \times\sum_{n=0}^\infty\frac{(1-\alpha q^{2n})}{(1-\alpha)}\frac{(\alpha,t_1t_2,t_1t_3,t_1t_4,t_1t_5;q)_n(-1)^nq^{\binom{n}{2}}(t_2t_3t_4t_5)^n}{(q,t_1t_2t_3t_4,t_1t_2t_3t_5,t_1t_2t_4t_5,t_1t_3t_4t_5;q)_n}. \tag{4.12}$$

In the right-hand side of (4.12), the summation becomes

$$\sum_{n=0}^\infty\frac{(\alpha,\alpha^{1/2},\alpha^{-1/2},t_1t_2,t_1t_3,t_1t_4,t_1t_5;q)_n}{(q,\alpha^{1/2},\alpha^{-1/2},t_1t_2t_3t_4,t_1t_2t_3t_5,t_1t_2t_4t_5,t_1t_3t_4t_5;q)_n}(-1)^nq^{\binom{n}{2}}(t_2t_3t_4t_5)^n.$$

Setting $a=\alpha$, $b=t_1t_2$, $c=t_1t_3$, $d=t_1t_4$, $e=t_1t_5$ and $N\to 0$ in Watson's transformation (7.9), the above factor is equal to

$$\frac{(\alpha q,t_2t_3;q)_\infty}{(t_1t_2t_3t_5,t_1t_2t_3t_4;q)_\infty}{}_3\phi_2\left[\begin{matrix}t_1t_4,t_1t_5,t_4t_5\\ t_1t_3t_4t_5,t_1t_2t_4t_5\end{matrix};q,t_2t_3\right]. \tag{4.13}$$

Using $_3\phi_2$ transformation (7.8)($a = t_1t_4$, $b = t_1t_5$, $c = t_4t_5$, $d = t_1t_3t_4t_5$, $e = t_1t_2t_4t_5$), the above $_3\phi_2$ series is equal to

$$\frac{(t_1t_5, t_2t_3t_4t_5, t_1t_2t_3t_4; q)_\infty}{(t_1t_3t_4t_5, t_1t_2t_4t_5, t_2t_3; q)_\infty} {}_3\phi_2\left[\begin{matrix} t_2t_4, t_3t_4, t_2t_3 \\ t_2t_3t_4t_5, t_1t_2t_3t_4 \end{matrix}; q, t_1t_5\right]. \tag{4.14}$$

Substituting (4.13) and (4.14) into the integral (4.12), we get (4.11).

Remark 4.1. *In the next section, we will give another proof of* (4.11) *as an application of* (5.1).

4.3. **Two integrals of Liu and Zhang-Wang.** When $h = d$, $c = \alpha u$ and $g = \alpha uv/q$, the $_3\phi_2$ series at the right-hand side of (2.10) can be summed by q-Pfaff-Saalschütz sum (7.4). Thus we recover Liu's result [17, Theorem 1.6].

Theorem 4.2 (Liu)**.**

$$\begin{aligned}\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; t_1, t_2, t_3, t_4, t_5)} {}_3\phi_2\left[\begin{matrix} \alpha uv/q, t_1e^{i\theta}, t_1e^{-i\theta} \\ \alpha u, \alpha v, \end{matrix}; q; t_2t_3t_4t_5\right] d\theta \\ = \frac{2\pi(t_1t_2t_3t_4, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5; q)_\infty}{(q, \alpha q; q)_\infty \prod_{1\le r<s\le 5}(t_rt_s; q)_\infty} \\ \times \sum_{n=0}^\infty \frac{(1-\alpha q^{2n})}{(1-\alpha)} \frac{(\alpha, q/u, q/v, t_1t_2, t_1t_3, t_1t_4, t_1t_5; q)_n (-1)^n q^{\binom{n}{2}} (\alpha^2 uv/t_1^2)^n}{(q, t_1t_2t_3t_4, \alpha u, \alpha v, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5; q)_n},\end{aligned} \tag{4.15}$$

where $\alpha q = t_1^2t_2t_3t_4t_5$ *and* $\max\{|t_1|, |t_2|, |t_3|, |t_4|, |t_5|\} < 1$.

Taking $u = t_1g/\alpha$ and $v = t_1t_5/\alpha$, then (4.15) reduces to

$$\begin{aligned}\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; t_1, t_2, t_3, t_4, t_5)} {}_3\phi_2\left[\begin{matrix} g/t_2t_3t_4, t_1e^{i\theta}, t_1e^{-i\theta} \\ gt_1, t_1t_5 \end{matrix}; q, t_2t_3t_4t_5\right] d\theta \\ = \frac{2\pi(t_1t_2t_3t_4, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5; q)_\infty}{(q, \alpha q; q)_\infty \prod_{1\le r<s\le 5}(t_rt_s; q)_\infty} \\ \times \sum_{n=0}^\infty \frac{(1-\alpha q^{2n})}{(1-\alpha)} \frac{(\alpha, t_1t_2, t_1t_3, t_1t_4, \alpha q/gt_1; q)_n}{(q, gt_1, t_1t_2t_3t_5, t_1t_2t_4t_5, t_1t_3t_4t_5; q)_n} (-1)^n q^{\binom{n}{2}} (gt_5)^n.\end{aligned} \tag{4.16}$$

Using the limit $N \to \infty$ case of Watson's transformation (7.9) with $a = \alpha$, $b = t_1t_2$, $c = t_1t_3$, $d = t_1t_4$, $e = \alpha q/gt_1$, the summation at the right-hand side of (4.16) is transformed to

$$\frac{(t_1^2t_2t_3t_4t_5, g/t_4; q)_\infty}{(t_1t_2t_3t_5, gt_1; q)_\infty} {}_3\phi_2\left[\begin{matrix} t_4t_5, t_1t_4, \alpha q/gt_1 \\ t_1t_2t_4t_5, t_1t_3t_4t_5 \end{matrix}; q, g/t_4\right].$$

Hence (4.16) is equivalent to

$$\begin{aligned}\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; t_1, t_2, t_3, t_4, t_5)} {}_3\phi_2\left[\begin{matrix} g/t_2t_3t_4, t_1e^{i\theta}, t_1e^{-i\theta} \\ gt_1, t_1t_5 \end{matrix}; q, t_2t_3t_4t_5\right] d\theta \\ = \frac{2\pi(t_1t_2t_3t_4, t_1t_2t_4t_5, t_1t_3t_4t_5, g/t_4; q)_\infty}{(q, gt_1; q)_\infty \prod_{1\le r<s\le 5}(t_rt_s; q)_\infty} {}_3\phi_2\left[\begin{matrix} t_4t_5, t_1t_4, \alpha q/gt_1 \\ t_1t_2t_4t_5, t_1t_3t_4t_5 \end{matrix}; q, g/t_4\right].\end{aligned} \tag{4.17}$$

Applying the ${}_3\phi_2$ transformations (7.7) and (7.8) to the above two ${}_3\phi_2$ in (4.17), repectively, we obtain

$$ {}_3\phi_2\left[\begin{matrix} g/t_2t_3t_4, t_1e^{i\theta}, t_1e^{-i\theta} \\ gt_1, t_1t_5 \end{matrix}; q, t_2t_3t_4t_5\right] = \frac{(t_1t_2t_3t_4t_5/g, gt_5; q)_\infty}{(t_1t_5, t_2t_3t_4t_5; q)_\infty} {}_3\phi_2\left[\begin{matrix} g/t_2t_3t_4, ge^{i\theta}, ge^{-i\theta} \\ gt_1, gt_5 \end{matrix}; q, t_1t_2t_3t_4t_5/g\right], $$

and

$$ {}_3\phi_2\left[\begin{matrix} t_4t_5, t_1t_4, \alpha q/gt_1 \\ t_1t_3t_4t_5, t_1t_2t_4t_5 \end{matrix}; q, g/t_4\right] = \frac{(\alpha q/gt_1, gt_1, gt_5; q)_\infty}{(t_1t_3t_4t_5, t_1t_2t_4t_5, g/t_4; q)_\infty} {}_3\phi_2\left[\begin{matrix} g/t_2, g/t_3, g/t_4 \\ gt_1, gt_5 \end{matrix}; q, \alpha q/gt_1\right]. $$

Plugging these into (4.17) and taking $t_1 = a$, $t_2 = b$, $t_3 = c$, $t_4 = d$, $t_5 = f$, we get the following integral formula of Zhang and Wang [21, Theorem 4.3].

Theorem 4.3 (Zhang and Wang)**.**

$$ \int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; a, b, c, d, f)} {}_3\phi_2\left[\begin{matrix} g/bcd, ge^{i\theta}, ge^{-i\theta} \\ ag, fg \end{matrix}; q, \frac{abcdf}{g}\right] d\theta = \frac{2\pi(abcd, bcdf; q)_\infty}{(q, ab, ac, ad, bc, bd, cd, bf, cf, df; q)_\infty} {}_3\phi_2\left[\begin{matrix} g/b, g/c, g/d \\ af, fg, \end{matrix}; q, \frac{abcdf}{g}\right], \tag{4.18} $$

provided $|abcdf/g| < 1$.

5. Ismail-Stanton's generating function of Askey-Wilson polynomials

Ismail and Stanton [12] use the orthogonality relation of Askey-Wilson polynomials and (4.11) to prove the following generating function of Askey-Wilson polynomials.

Theorem 5.1 (Ismail-Stanton)**.** *The Askey-Wilson polynomials have the generating function*

$$ \sum_{n=0}^{\infty} P_n(x, \mathbf{t}|q) c_n(\mathbf{t}, b) = \frac{1}{(be^{i\theta}, be^{-i\theta})_\infty}, \tag{5.1} $$

where

$$ c_n(\mathbf{t}, b) = \frac{b^n(t_2t_3t_4bq^n; q)_\infty}{(q, t_1t_2t_3t_4q^{n-1}; q)_n \Pi_{j=2}^4 (t_jb; q)_\infty} {}_3\phi_2\left[\begin{matrix} t_2t_3q^n, t_2t_4q^n, t_3t_4q^n \\ t_1t_2t_3t_4q^{2n}, t_2t_3t_4bq^n \end{matrix}; q, t_1b\right]. \tag{5.2} $$

Prior to Ismail-Stanton's work, Kim and Stanton [15] proved the following special case of (5.1).

Proposition 5.2 (Kim-Stanton)**.** *We have the following generating function of continuous dual q-Hahn polynomials* $P_n(x; 0, t_2, t_3, t_4|q)$,

$$ \sum_{k=0}^{\infty} \frac{P_k(x; 0, t_2, t_3, t_4|q)}{(q, bt_2t_3t_4; q)_k} b^k = \frac{(bt_2, bt_3, bt_4; q)_\infty}{(bt_2t_3t_4, be^{i\theta}, be^{-i\theta}; q)_\infty}. \tag{5.3} $$

Kim and Stanton's idea is to use the "bootstrapping method": they derive (5.3) from the generating function of Al-Salam-Chihara polynomials $P_n(x;0,0,t_3,t_4|q)$ by using the connection formula, [4,8],

$$\frac{P_n(x;A,t_2,t_3,t_4|q)}{(q,t_2t_3,t_2t_4,t_3t_4;q)_n}=\sum_{k=0}^{n}\frac{P_k(x,\mathbf{t}|q)(At_2t_3t_4q^{n-1};q)_k}{(q,t_2t_3,t_2t_4,t_3t_4,t_1t_2t_3t_4q^{k-1};q)_k}\times\frac{t_1^{n-k}(A/t_1;q)_{n-k}}{(q,t_1t_2t_3t_4q^{2k};q)_{n-k}}. \tag{5.4}$$

We show that the same idea works for Ismail-Stanton's formula (5.1), that is, one can derive (5.1) from (5.3) by using (5.4).

Proof of Theorem 5.1 Letting $A=0$ and summing the two sides of (5.4), multiplied with $\frac{(t_2t_3,t_2t_4,t_3t_4;q)_nb^n}{(bt_2t_3t_4;q)_n}$, over $n\geq 0$, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{P_n(x;0,t_2,t_3,t_4|q)}{(q,bt_2t_3t_4;q)_n}b^n&=\sum_{n=0}^{\infty}\frac{b^n}{(q,bt_2t_3t_4;q)_n}\sum_{k=0}^{n}P_k(x,\mathbf{t}|q)\frac{(q,t_2t_3,t_2t_4,t_3t_4;q)_n}{(q,t_2t_3,t_2t_4,t_3t_4;q)_k}\\&\quad\times\frac{t_1^{n-k}}{(t_1t_2t_3t_4q^{k-1};q)_k(q,t_1t_2t_3t_4q^{2k};q)_{n-k}}\\&=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{b^{n+k}P_k(x,\mathbf{t}|q)}{(bt_2t_3t_4;q)_{n+k}}\frac{(t_2t_3,t_2t_4,t_3t_4;q)_{n+k}}{(q,t_2t_3,t_2t_4,t_3t_4;q)_k}\\&\quad\times\frac{t_1^{n}}{(t_1t_2t_3t_4q^{k-1};q)_k(q,t_1t_2t_3t_4q^{2k};q)_{n}}\\&=\sum_{k=0}^{\infty}\frac{b^kP_k(x,\mathbf{t}|q)}{(q,bt_2t_3t_4,t_1t_2t_3t_4q^{k-1};q)_k}\sum_{n=0}^{\infty}\frac{(t_2t_3q^k,t_2t_4q^k,t_3t_4q^k;q)_n(bt_1)^n}{(q,t_1t_2t_3t_4q^{2k},bt_2t_3t_4q^k;q)_n}.\end{aligned}$$

In view of (5.3), we can rewrite the above equation as:

$$\frac{(bt_2,bt_3,bt_4;q)_\infty}{(bt_2t_3t_4,be^{i\theta},be^{-i\theta};q)_\infty}=\sum_{k=0}^{\infty}\frac{b^kP_k(x,\mathbf{t}|q)}{(q,bt_2t_3t_4,t_1t_2t_3t_4q^{k-1};q)_k}\times{}_3\phi_2\left[\begin{matrix}t_2t_3q^k,t_2t_4q^n,t_3t_4q^k\\t_1t_2t_3t_4q^{2k},bt_2t_3t_4q^k\end{matrix};q,bt_1\right].$$

The result follows then after some simplification. □

As an application of (5.1), we give another proof of Ismail-Stanton-Viennot integral (4.11).

Another Proof of (4.11) In view of (5.1) with $b\to t_5$ the left-hand side of (4.11) is

$$\sum_{n=0}^{\infty}c_n(\mathbf{t},t_5)\frac{(t_1t_4,t_1t_3,t_1t_4;q)_n}{t_1^n}\sum_{k=0}^{n}\frac{(q^{-n},t_1t_2t_3t_4q^{n-1};q)_k}{(q,t_1t_2,t_1t_3,t_1t_4;q)_k}q^k\int_0^\pi\frac{h(\cos 2\theta;1)(t_1e^{i\theta},t_1e^{-i\theta};q)_k}{h(\cos\theta;t_1,t_2,t_3,t_4)}d\theta.$$

The inner integral can be evaluated by replacing $t_1\to t_1q^k$ in the Askey-Wilson integral (1.5),

$$\frac{2\pi(t_1t_2t_3t_4;q)_\infty}{(q;q)_\infty\prod_{1\leq r<s\leq 4}(t_rt_s;q)_\infty}\sum_{n=0}^{\infty}c_n(\mathbf{t},t_5)\frac{(t_1t_4,t_1t_3,t_1t_4;q)_n}{t_1^n}{}_2\phi_1\left[\begin{matrix}q^{-n},t_1t_2t_3t_4q^{n-1}\\t_1t_2t_3t_4\end{matrix};q,q\right]. \tag{5.5}$$

The inner ${}_2\phi_1$ series can be summed by q-Chu-Vandemonde (7.2)

$${}_2\phi_1\left[\begin{matrix}q^{-n},t_1t_2t_3t_4q^{n-1}\\t_1t_2t_3t_4\end{matrix};q,q\right]=\frac{(q^{1-n};q)_n(t_1t_2t_3t_4q^{n-1})^n}{(t_1t_2t_3t_4;q)_n}.$$

Since $(q^{1-n};q)_n=0$ for $n\geq 1$, (5.5) reduces to

$$\frac{2\pi(t_1t_2t_3t_4;q)_\infty}{(q;q)_\infty\prod_{1\leq r<s\leq 4}(t_rt_s;q)_\infty}c_0(\mathbf{t},t_5),$$

which is clearly equal to the right-hand side of (4.11) in view of (5.2). □

6. More transformation formulae

Ismail-Stanton [12, §5-6] proved the following expansion formula:

$$\sum_{n=0}^{\infty}\frac{(az,a/z;q)_n}{(q;q)_n}A_nB_n\delta^n=\sum_{n=0}^{\infty}\frac{(-\delta)^nq^{\binom{n}{2}}}{(q,Cq^{k-1};q)_k}\times\sum_{k=0}^{n}\frac{(q^{-n},Cq^{n-1},az,a/z;q)_kq^kA_k}{(q;q)_k}\sum_{r=0}^{\infty}\frac{\delta^rB_{r+n}}{(q,Cq^{2n};q)_r}. \tag{6.1}$$

They also derive several interesting results from the above identity. We note that (6.1) follows from Proposition 2.3. Indeed, substituting $\beta_n\to\frac{A_nu^n}{(q,\alpha,\beta;q)_n}$, $\delta_n\to(\alpha,\beta;q)_nB_n\delta^n$ and $b\to\gamma$ in (2.6) yields the following result.

Theorem 6.1.

$$\sum_{n=0}^{\infty}A_nB_n\frac{(\delta u)^n}{(q;q)_n}=\sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a;q)_n(a/\gamma;q)_n(\gamma/a)^n}{(1-a)(\gamma q;q)_n(q;q)_n}\times\sum_{k=0}^{n}\frac{(1-\gamma q^{2k})(aq^n;q)_k(q^{-n};q)_k}{(1-\gamma)(\gamma q^{n+1};q)_k(\gamma q^{1-n}/a;q)_k}\frac{(uq)^kA_k}{(q,\alpha,\beta;q)_k}\times\sum_{r=0}^{\infty}\frac{(\gamma/a;q)_r(\gamma;q)_{r+2n}(\alpha,\beta;q)_{r+n}\delta^{r+n}B_{r+n}}{(q;q)_r(aq;q)_{r+2n}}. \tag{6.2}$$

Letting $\gamma=0$ and $A_n\to A_n(b,c;q)_n$, the above formula reduces to

$$\sum_{n=0}^{\infty}\frac{(b,c;q)_n}{(q;q)_n}A_nB_n(\delta u)^n=\sum_{n=0}^{\infty}\frac{(a;q)_n(1-aq^{2n})(-1)^nq^{\binom{n}{2}}}{(q;q)_n(1-a)(aq;q)_{2n}}\times\sum_{k=0}^{n}\frac{(q^{-n},aq^n,b,c;q)_k}{(q,\alpha,\beta;q)_k}(uq)^kA_k\sum_{r=0}^{\infty}\frac{\delta^{r+n}B_{r+n}(\alpha,\beta;q)_{n+r}}{(q,aq^{2n+1};q)_r}. \tag{6.3}$$

Obviously (6.3) reduces to (6.1) when $\alpha=\beta=0$, $b\to az$, $c\to a/z$, $u\to 1$ and $a\to C/q$.

Verma [20] (see also [7, p. 84]) proved the following important expansion formula

$$\sum_{n=0}^{\infty}A_nB_n\frac{(xw)^n}{(q;q)_n}=\sum_{n=0}^{\infty}\frac{(-x)^n}{(q,\gamma q^n;q)_n}q^{\binom{n}{2}}\sum_{j=0}^{n}\frac{(q^{-n},\gamma q^n;q)_j}{(q,\alpha,\beta;q)_j}(wq)^rA_j\sum_{k=0}^{\infty}\frac{(\alpha,\beta;q)_{n+k}}{(q,\gamma q^{2n+1};q)_k}x^kB_{k+n}. \tag{6.4}$$

We note that the above formula corresponds to (6.3) with $b=c=0$, $a=\gamma$, $u=w$ and $\delta=x$.

Besides, the special $\beta_n=\frac{(g,h,y;q)_n}{(q,e,f,t;q)_n}u^n$ case of Proposition 2.5 gives the following result.

Theorem 6.2. *Let δ, u, y, h, t, e, f, g, b_i, a_i ($i \in \mathbb{N}$) be any complex numbers. Then the following formal power series in ζ and u holds*

$$
{}_{p+1}\phi_p\left[\begin{matrix} a_1,\ldots,a_{p-1},y,h,g \\ t,e,f,b_1,\ldots,b_{p-1}\end{matrix};q,u\delta\right] = \sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a,a/b,a_1,\ldots,a_{p-1};q)_n(b\delta a^{-1})^n(b;q)_{2n}}{(1-a)(bq,q,b_1,\ldots,b_{p-1};q)_n(aq;q)_{2n}}
$$
$$
\times {}_8\phi_7\left[\begin{matrix} b,b^{1/2}q,-b^{1/2}q,y,h,g,aq^n,q^{-n} \\ b^{1/2},-b^{1/2},bq^{n+1},bq^{1-n}/a,t,e,f\end{matrix};q,qu\right]{}_{p+1}\phi_p\left[\begin{matrix} a_1q^n,\ldots,a_{p-1}q^n,b/a,bq^{2n} \\ b_1q^n,\ldots,b_{p-1}q^n,aq^{2n+1}\end{matrix};q,\delta\right].
$$

Finally, we record two special cases of Theorem 6.2 when the above ${}_8\phi_7$ is summable in closed form.

- Taking $u=1$, $t=bq/y$, $e=bq/h$, $f=bq/g$ and $b^2q=ayhg$ in Theorem 6.2, the ${}_8\phi_7$ series can be summed by Jackson's summation (7.13)

$$
(6.5)\quad {}_{p+1}\phi_p\left[\begin{matrix} a_1,\ldots,a_{p-1},y,h,g \\ bq/y,bq/h,bq/g,b_1,\ldots,b_{p-1}\end{matrix};q,\delta\right]
$$
$$
= \sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a,a/b;q)_n(a_1,\ldots,a_{p-1};q)_n(b;q)_{2n}}{(1-a)(q;q)_n(b_1,\ldots,b_{p-1})_n(aq;q)_{2n}}
$$
$$
\times\frac{(bq/gh,bq/yg,bq/gy;q)_n}{(bq/y,bq/h,bq/g,by/ygh;q)_n}{}_{p+1}\phi_p\left[\begin{matrix} a_1q^n,\ldots,a_{p-1}q^n,b/a,bq^{2n} \\ b_1q^n,\ldots,b_{p-1}q^n,aq^{2n+1}\end{matrix};q,\delta\right].
$$

- Taking $y=t$, $h=e$, $u=b/ag$ and $f=bq/g$ in Theorem 6.2, the ${}_6\phi_5$ series can be summed by (7.12)

$$
(6.6)\quad {}_{p+1}\phi_p\left[\begin{matrix} a_1,\ldots,a_{p-1},g \\ bq/g,b_1,\ldots,b_{p-1}\end{matrix};q,b\delta/ag\right]
$$
$$
= \sum_{n=0}^{\infty}\frac{(1-aq^{2n})(a,a_1,\ldots,a_{p-1},ag/b;q)_n(bg\delta/a)^n(b;q)_{2n}}{(1-a)(q,b_1,\ldots,b_{p-1},bq/g;q)_n(aq;q)_{2n}}
$$
$$
\times\,{}_{p+1}\phi_p\left[\begin{matrix} a_1q^n,\ldots,a_{p-1}q^n,b/a,bq^{2n} \\ b_1q^n,\ldots,b_{p-1}q^n,aq^{2n+1}\end{matrix};q,\delta\right].
$$

Acknowledgement. This work was supported by the LABEX MILYON (ANR-10-LABX-0070) of Universit de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR). The first author would like to thank Institut Camille Jordan and Université Claude Bernard Lyon 1 for hosting her for the academic year 2014–2015 so that this reseach could take place. This visit was also supported by a grant of CSC Ph.D. student visiting program.

7. Appendix

The following formulae are taken from [7, Appendices II and III].
The q-Gauss sum,

$$ {}_2\phi_1\left[\begin{matrix} a,b \\ c \end{matrix};q,c/ab\right]=\frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty},\qquad (|c/ab|<1). \tag{7.1}$$

The q-Chu-Vandermonde sums,

$$ {}_2\phi_1\left[\begin{matrix} a,q^{-n} \\ c \end{matrix};q,q\right]=\frac{(c/a;q)_n a^n}{(c;q)_n}, \tag{7.2}$$

and, reversing the order of summation,

$$ {}_2\phi_1\left[\begin{matrix} a,q^{-n} \\ c \end{matrix};q,cq^n/a\right]=\frac{(c/a;q)_n}{(c;q)_n}. \tag{7.3}$$

The q-Pfaff-Saalschütz sum,

$$ {}_3\phi_2\left[\begin{matrix} a,b,q^{-n} \\ c,abq^{1-n}/c \end{matrix};q,q\right]=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}. \tag{7.4}$$

Sears' transformation,

$$ {}_4\phi_3\left[\begin{matrix} a,b,c,q^{-n} \\ d,e,f \end{matrix};q,q\right]=\frac{(e/a,f/a;q)_n a^n}{(e,f;q)_n}{}_4\phi_3\left[\begin{matrix} a,d/b,d/c,q^{-n} \\ d,aq^{1-n}/e,aq^{1-n}/f \end{matrix};q,q\right], \tag{7.5}$$

where $def=abcq^{1-n}$.

Transformations of finite ${}_3\phi_2$ series (by sending $c,f\to 0$ in (7.5)),

$$ {}_3\phi_2\left[\begin{matrix} q^{-n},a,b \\ d,e \end{matrix};q,q\right]=\frac{(e/a;q)_n}{(e;q)_n}a^n\,{}_3\phi_2\left[\begin{matrix} q^{-n},a,d/b \\ d,aq^{1-n}/e \end{matrix};q,bq/e\right]. \tag{7.6}$$

Transformations of ${}_3\phi_2$ series,

$$ {}_3\phi_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix};q,de/abc\right]=\frac{(e/a,de/bc;q)_\infty}{(e,de/abc;q)_\infty}{}_3\phi_2\left[\begin{matrix} a,d/b,d/c \\ d,de/bc \end{matrix};q,e/a\right] \tag{7.7}$$

$$ =\frac{(b,de/ab,de/bc;q)_\infty}{(d,e,de/abc;q)_\infty}{}_3\phi_2\left[\begin{matrix} e/b,d/b,de/abc \\ de/ab,de/bc \end{matrix};q,b\right], \tag{7.8}$$

where $\max\{|de/abc|,|e/a|,|b|\}<1$.
Watson's transformation,

$$ {}_8\phi_7\left[\begin{matrix} a,a^{1/2}q,-a^{1/2}q,b,c,d,e,q^{-n} \\ a^{1/2},-a^{1/2},aq/b,aq/c,aq/d,aq/e,aq^{n+1} \end{matrix};q,\frac{a^2q^{n+2}}{bcde}\right] \tag{7.9}$$
$$ =\frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}{}_4\phi_3\left[\begin{matrix} aq/bc,d,e,q^{-n} \\ aq/b,aq/c,deq^{-n}/a \end{matrix};q,q\right]. $$

Transformations of very-well-poised ${}_8\phi_7$ series,

$$
{}_8\phi_7\left[\begin{matrix} a, a^{1/2}q, -a^{1/2}q, b, c, d, e, f \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right] \tag{7.10}
$$

$$
= \frac{(aq, b, bc\mu/a, bd\mu/a, be\mu/a, bf\mu/a; q)_\infty}{(aq/c, aq/d, aq/e, aq/f, \mu q, b\mu/a; q)_\infty}
$$

$$
\times {}_8\phi_7\left[\begin{matrix} \mu, \mu^{1/2}q, -\mu^{1/2}q, aq/bc, aq/bd, aq/be, aq/bf, b\mu/a \\ \mu^{1/2}, -\mu^{1/2}, bc\mu/a, bd\mu/a, be\mu/a, bf\mu/a, aq/b \end{matrix}; q, b\right] \tag{7.11}
$$

$$
= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)_\infty}{(aq/e, aq/f, \lambda q, \lambda q/ef; q)_\infty}
$$

$$
\times {}_8\phi_7\left[\begin{matrix} \lambda, \lambda^{1/2}q, -\lambda^{1/2}q, \lambda b/a, \lambda c/a, \lambda d/a, e, f \\ \lambda^{1/2}, -\lambda^{1/2}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f \end{matrix}; q, \frac{aq}{ef}\right],
$$

where $\lambda = a^2q/bcd$, $\mu = a^3q^3/b^2cdef$ and $\max\{|a^2q^2/bcdef|, |aq/ef|, |b|\} < 1$.
Rogers' ${}_6\phi_5$ summation,

$$
{}_6\phi_5\left[\begin{matrix} a, a^{1/2}q, -a^{1/2}q, b, c, q^{-n} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq^{n+1} \end{matrix}; q, \frac{aq^{n+1}}{bc}\right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}. \tag{7.12}
$$

Jackson's ${}_8\phi_7$ summation,

$$
{}_8\phi_7\left[\begin{matrix} a, a^{1/2}q, -a^{1/2}q, b, c, d, e, q^{-n} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix}; q, q\right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}, \tag{7.13}
$$

where $a^2q = bcdeq^{-n}$.

References

[1] G. E. Andrews, Bailey's transform, lemma, chains and tree, in: Special Functions 2000: Current Perspective and Future Directions, Tempe, AZ, in: NATO Sci. Ser. II Math. Phys. Chem., vol. 30, Kluwer Acad. Publ., Dordrecht, 2001. 1–22.

[2] G. E. Andrews, q-orthogonal polynomials, Rogers-Ramanujan identities, and mock theta functions. Proc. Steklov Inst. Math. 276 (2012), no. 1, 21–32.

[3] G. E. Andrews, R. Askey, R. Roy, Special functions, Cambridge Univ. Press, Cambridge, 1999.

[4] R. Askey, J. A. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. Amer. Math. Soc., 1985.

[5] D. M. Bressoud, A matrix inverse, Proc. Amer. Math. Soc. 88 (1983), 446–448.

[6] N. J. Fine, Basic Hypergeometric Series and Applications, Math. Surveys Monogr. 27, Amer. Math. Soc., Providence, RI, 1988.

[7] G. Gasper, M. Rahman, Basic Hypergeometric Series, second ed., Cambridge Univ. Press, Cambridge, 2004.

[8] M. E. H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Cambridge Univ. Press, Cambridge, 2005.

[9] M. E. H. Ismail, M. Rahman, Connection relations and expansions, Pacific J. Math. 252 (2011), 427–446.

[10] Ismail, Mourad E. H.; Rahman, Mizan; Suslov, Sergei K. Some summation theorems and transformations for q-series. Canad. J. Math. 49 (1997), no. 3, 543–567.

[11] M. E. H. Ismail, M. Rahman, D. Stanton, Quadratic q-exponentials and connection coefficient problems. Proc. Amer. Math. Soc. 127 (1999), no. 10, 2931–2941.

[12] M. E. H. Ismail, D. Stanton, Expansions in the Askey-Wilson polynomials. J. Math. Anal. Appl. 424 (2015), no. 1, 664–674.

[13] M. E. H. Ismail, D. Stanton, G. Viennot, The combinatorics of q-Hermite polynomials and the Askey-Wilson integral. European J. Combin. 8, 1987, 379–392.

[14] Ismail, Mourad E. H.; Wilson, James A. Asymptotic and generating relations for the q-Jacobi and ${}_4\phi_3$ polynomials. J. Approx. Theory 36 (1982), no. 1, 43–54.

[15] J. S. Kim, D. Stanton, Bootstrapping and Askey–Wilson polynomials, J. Math. Anal. Appl. 421 (1) (2015), 501–520.

[16] Z.-G. Liu, An extension of the non-terminating ${}_6\phi_5$ summation and the Askey-Wilson polynomials, J. Difference Equ. Appl. 17(2011), 1401-1411.

[17] Z.-G. Liu, A q-summation formula, the continuous q-Hahn polynomials and the big q-Jacobi polynomials. J. Math. Anal. Appl. 419 (2014), no. 2, 1045–1064.

[18] B. Nassrallah, M. Rahman, Projection formulas, a reproducing kernel and a generating function for q-Wilson polynomials, SIAM J. Math. Anal. 16 (1) (1985), 186–197.

[19] M. Rahman, S. K. Suslov, The Pearson equation and the beta integrals, SIAM. J. Math. Anal., 25 (1994), 646–693.

[20] A. Verma, S. Accad, Some transformations of series with arbitrary terms, Ist. Lombardo Accad. Sci. Lett. Rend. A, 106(1972), 342–353.

[21] Z. Z. Zhang, J. Wang, Two operator identities and their applications to terminating basic hypergeometric series and q-integrals, J. Math. Anal. Appl. 312 (2) (2005), 653–665.

East China Normal University, Department of Mathematics, 500 Dongchuan Road, Shanghai 200241, P. R. China

E-mail address: jiawei163jzy@163.com;

Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne Cedex, France

E-mail address: *zeng@math.univ-lyon1.fr*