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Corrections to "A Global High-Gain Finite-Time Observer"

Tomas Ménard, Emmanuel Moulay and Wilfrid Perruquetti

Abstract—This note fix the proof of Theorem 2 in the article [2].

Equations from the original paper will be denoted with a star (for example (1∗)) whereas equations of the present corrected paper will be denoted without a star (for example (1)).

I. THE ERROR

The function  \( \hat{V}_\alpha \) used in the proof of Theorem 2 in [2], and derived from Theorem 10 in [3], is not \( C^1 \) with respect to \((e, \alpha)\). Indeed, one has

\[
\frac{\partial}{\partial e_k} \left[ \frac{1}{\alpha_k} \alpha_k \right] = \frac{1}{\alpha_k q} \left[ \frac{1}{\alpha_k q} \right]^{-1}.
\]

(1)

Hence, when \( \alpha \to 1 \), \( \frac{1}{\alpha_k q} \to 1 \) and when one of the component of \( e \) goes to zero, the limit \( \lim_{(e, \alpha) \to (1,0)} \frac{1}{\alpha_k q} \left[ \frac{1}{\alpha_k q} \right]^{-1} \) does not exist. Thus the function \( \hat{V}_\alpha \) cannot be used as a candidate Lyapunov function.

II. THE FIX

Let us first recall Theorem 2 from [2].

Theorem 1. Let us consider system (3∗) with a bounded input u. Then there exists \( \theta^* \geq 1 \) such that for all \( \theta > \theta^* \) there exists \( e = 0 \) such that system (3∗) admits the following global finite-time observer:

\[
\begin{align*}
\dot{x}_1 &= \dot{x}_2 + k_1 [e_1 \alpha_1 + \rho e_1] + \sum_{j=1}^m g_{1,j} (\dot{x}_1) u_j \\
\vdots \\
\dot{x}_n &= k_n [e_n \alpha_n + \rho e_n] + \varphi (\dot{x}) + \sum_{j=1}^m g_{n,j} (\dot{x}_1) u_j
\end{align*}
\]

for all \( \alpha \in [1 - \epsilon, 1] \), where \( e_1 = x_1 - \dot{x}_1 \), the powers \( \alpha_i \) are defined by \((5∗)\), the gains \( k_i \) by \((6∗)\) and \( \rho = \left( \frac{n^2 \epsilon^2}{2} \right) \).

where \( S_1 \) is defined by \((8∗)\). In addition, the settling time of the error dynamics is bounded by \( T_1(e_0) + T_2(e_0) \) with \( e_0 = x_0 - \dot{x}_0 \), where \( T_1, T_2 \) are respectively given by \((18∗)\) and \((6∗)\).

The statement of Theorem 2 in [2] remains correct, except for the settling time which has to be corrected.

We can define the function \( V_1(e) = e^T S_{\infty}(1) e, \) for \( e \in \mathbb{R}^n \), where \( S_{\infty}(1) \) is the solution of \((7∗)\) for \( \theta = 1 \). This choice corresponds to the linear case, that is \( \alpha = 1 \). Proceeding as in [4], [5], one can construct a candidate Lyapunov function with properties stated next.

Proposition 1. Let \( \alpha \in C^\infty(\mathbb{R}, \mathbb{R}) \) be such that

\[
\alpha = \begin{cases} 0 & \text{on } (-\infty, 1] \\ 1 & \text{on } [2, +\infty) \end{cases}
\] and \( \alpha' \geq 0 \) on \( \mathbb{R} \).

(2)

There exists \( \epsilon > 0 \) such that for all \( \alpha \in [1 - \epsilon, 1 + \epsilon] \), the function \( V_\alpha \) defined as

\[
V_\alpha(e) = \int_0^{+\infty} \frac{1}{l^3} (a \circ V_1(t r^{(1)} e_1, \ldots, t r^{(n)} e_n)) dt
\]

(3)

if \( e \in \mathbb{R}^n \setminus \{0\} \) and \( V_\alpha(0) = 0 \) is well defined, radially unbounded, of class \( C^1(\mathbb{R}^n, \mathbb{R}) \), and satisfies

a) \( V_\alpha(\delta e) = \lambda^2 V_\alpha(e), \) for all \( e \in \mathbb{R}^n \) and \( \lambda > 0 \),

b) \( (\nabla V_\alpha(e), A e - F(S_{\infty}(1)^T C^T, e)) \leq -\gamma V_\alpha(e) \frac{1}{\alpha}, \) for all \( e \in \mathbb{R}^n \), where \( \gamma > 0 \).

where \( F, C \) and \( \delta \alpha \) are defined in [2].

Proof of proposition 1. Let, \( \alpha \in [1 - \frac{1}{n}, +\infty[ \), proceeding as in [4], one directly shows that \( V_\alpha \) is well defined, radially unbounded, \( C^1 \) on \( \mathbb{R}^n \), and homogeneous of degree 2 with respect to the weights \( r(\alpha) \). Then, only point b) remains to prove.

Following the same lines as in [4], there exists \( L > 0 \) such that for all \( e \in \{ e \in \mathbb{R}^n \mid V_\alpha(e) = 1 \} \), one has

\[
\| \nabla V_\alpha(e), A e - F(S_{\infty}(1)^T C^T, e) \| \leq \int_0^L \frac{1}{l^{n+2}} a' \left( V_1 \left( \delta^{(1)} e \right) \right) \times
\]

\[
\left( V_1 \left( \delta^{(1)} e \right), A \delta^{(1)} e - F \left( S_{\infty}(1)^T C^T, \delta^{(1)} e \right) \right) dt
\]

Consider the function \( g(e, t, \alpha) = \left( V_1 \left( \delta^{(1)} e \right), A \delta^{(1)} e - F \left( S_{\infty}(1)^T C^T, \delta^{(1)} e \right) \right) \), where \( e, t, \alpha \in \{ e \in \mathbb{R}^n, V_\alpha(e) = 1 \} \times \{ t \in [l, L] \} \times [1 - \frac{1}{n}, +\infty[ \).

The function \( g \) is continuous, \( (e, t) \) belongs to a compact set and there exists \( \gamma_1 > 0 \) such that the image of \( g \) is included in \( [e_0, \gamma_1] \) for \( (e, t) \in \{ e \in \mathbb{R}^n, V_\alpha(e) = 1 \} \times \{ t \in [l, L] \} \) and \( \gamma_1 = 1 \) (since it corresponds to the linear case). We can then apply Lemma 26.8 in [1] (tube lemma) which gives the existence of \( \gamma > 0 \) such that for all \( e, t, \alpha \in \{ e \in \mathbb{R}^n, V_\alpha(e) = 1 \} \times \{ t \in [l, L] \} \times [1 - \epsilon, 1 + \epsilon], \)

\[
g(e, t, \alpha) \leq -\gamma_1.
\]

Then we have

\[
\| \nabla V_\alpha(e), A e - F(S_{\infty}(1)^T C^T, e) \| \leq -\gamma_1 \int_0^L \frac{1}{l^{n+2}} a' \left( V_1 \left( \delta^{(1)} e \right) \right) dt \leq -\gamma \left( V_\alpha(e) \right)^{\frac{2+\alpha-1}{2}}
\]

(4)

where \( \gamma > 0 \) is a lower bound of \( \gamma_1 \int_0^L \frac{1}{l^{n+2}} a' \left( V_1 \left( \delta^{(1)} e \right) \right) dt \) for \( (e, \alpha) \in \{ e \in \mathbb{R}^n, V_\alpha(e) = 1 \} \times [1 - \epsilon, 1 + \epsilon]. \) Since \( V_\alpha \) is homogeneous of degree 2 with respect to the weights \( r(\alpha) \), inequality (4) is valid for all \( e \in \mathbb{R}^n \).

Now that a new candidate Lyapunov function has been defined, we explain how it will be used to correct the proof.
of Theorem 2 in [2]. Please note that part 1 of the proof is correct, then it has already been proved that every trajectory starting from \( e_0 \in \mathbb{R}^n \) enter the ball \( B_{\|S_\infty(\theta)\|}^n(1) \) after time \( T_1(e_0) = \log(1/V(e_0))/\kappa(\theta) \) (see equation (18*)).

Denote \( \bar{e} = \Delta \theta e \), where \( \Delta \theta = \text{diag} \left[ 1 \quad \frac{1}{\theta} \quad \ldots \quad \frac{1}{\rho^{\frac{1}{\theta}-1}} \right] \), in the remaining, we will show that for every \( \theta \geq \theta_2 \equiv \frac{2}{\gamma}(M_1 + 2) \), there exists \( \epsilon > 0 \) such that the following inequality

\[
\dot{V}_\alpha(\bar{e}) \leq -\left( 2^\alpha - 1 \right) \left( V_\alpha(\bar{e}) \right)^{\frac{2+\alpha-1}{2}} + M_1 V_\alpha(\bar{e}) \tag{5}
\]

holds for every \( \bar{e} \in B_{\|S_\infty(\theta)\|}^n(1) \), \( \alpha \in ]1-\epsilon, 1[ \) , where \( M_1 > 0 \) is a constant independent of \( \theta \). This inequality replaces inequality (19*). Inequality (5) directly implies that the error system (11*) is finite time stable on \( B_{\|S_\infty(\theta)\|}^n(1) \).

Thus, after time \( T_1(e_0) \), the error enters \( B_{\|S_\infty(\theta)\|}^n(1) \) and after time \( T_1(e_0) + T_2(e_0) \) the error reaches the origin, where the settling time \( T_2(e_0) \) is bounded as follows

\[
T_2(e_0) \leq \frac{\ln \left( 1 - \frac{M_1}{2^{\alpha} - 1} V_\alpha(e_0)^{\frac{2+\alpha-1}{2}} \right)}{M_1 \left( \frac{2+\alpha-1}{2} - 1 \right)} \tag{6}
\]

The remaining of the corrected proof is very similar to the original one. The dynamics of \( \bar{e} \) is given by

\[
\dot{\bar{e}} = \theta \left( A\bar{e} - F(S^{-1}_\infty(1)C^T, \bar{e}) - \rho S^{-1}_\infty(1)C^T C\bar{e} \right) + \Delta \theta D(x, \hat{x}, u).
\]

One has

\[
\dot{V}_\alpha(\bar{e}) \triangleq \theta \bar{W}_1 + \bar{W}_2 \tag{7}
\]

with \( \bar{W}_1 = \langle \nabla V_\alpha(\bar{e}), A\bar{e} - F(S^{-1}_\infty(1)C^T, \bar{e}) - \rho S^{-1}_\infty(1)C^T C\bar{e} \rangle \) and \( \bar{W}_2 = \langle \nabla V_\alpha(\bar{e}), \Delta \theta D(x, \hat{x}, u) \rangle \).

Following the same lines as in [2], one can show that there exists \( \theta_2 \geq 0 \) such that for every \( \theta \geq \theta_2 \), there exists \( \epsilon > 0 \) such that for all \( \alpha \in ]1-\epsilon, 1[ \) inequality (5) holds true.

**REFERENCES**


