Corrections to ”A Global High-Gain Finite-Time Observer”

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Abstract—**This note fix the proof of Theorem 2 in the article [2].**

Equations from the original paper will be denoted with a star (for example (1^*)) whereas equations of the present corrected paper will be denoted without a star (for example (1)).

I. The error

The function $\tilde{V}_\alpha$ used in the proof of Theorem 2 in [2], and derived from Theorem 10 in [3], is not C^1 with respect to (e, α). Indeed, one has

$$\frac{\partial\left(\lceil e_k \rfloor^{\frac{1}{\alpha_{k-1} q}}\right)}{\partial e_k} = \frac{1}{\alpha_k q} \lceil e_k \rfloor^{\frac{1}{\alpha_k q}-1}. \tag{1}$$

Hence, when $\alpha \to 1$, $\frac{1}{\alpha_k q} \to 1$ and when one of the component of e goes to zero, the limit $\lim_{(\alpha,e)\to(1,e_0)} \frac{1}{\alpha_k q} \lceil e_k \rfloor^{\frac{1}{\alpha_k q}-1}$ does not exist. Thus the function $\tilde{V}_\alpha$ cannot be used as a candidate Lyapunov function.

II. The fix

Let us first recall Theorem 2 from [2].

Theorem 1. *Let us consider system (3^*) with a bounded input u. Then there exists $\theta^* \geq 1$ such that for all $\theta > \theta^*$ there exists $\epsilon > 0$ such that system (3^*) admits the following global finite-time observer:*

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + k_1(\lceil e_1 \rfloor^{\alpha_1} + \rho e_1) + \sum_{j=1}^m g_{1,j}(\hat{x}_1) u_j \\ \vdots \\ \dot{\hat{x}}_n = k_n(\lceil e_1 \rfloor^{\alpha_n} + \rho e_1) + \varphi(\hat{x}) + \sum_{j=1}^m g_{n,j}(\hat{x}) u_j \end{cases}$$

for all $\alpha \in]1-\epsilon, 1[$, where $e_1 = x_1 - \hat{x}_1$, the powers α_i are defined by (5^*), the gains k_i by (6^*) and $\rho = \left(\frac{n^2\theta^{\frac{2}{3}} S_1 + 1}{2}\right)$ where S_1 is defined by (8^*).
In addition, the settling time of the error dynamics is bounded by $T_1(e_0) + T_2(e_0)$ (with $e_0 = x_0 - \hat{x}_0$), where T_1, T_2 are respectively given by (18^*) and (6).

The statement of Theorem 2 in [2] remains correct, except for the settling time which has to be corrected.

We can define the function $V_1(e) = e^T S_\infty(1) e$, for $e \in \mathbb{R}^n$, where $S_\infty(1)$ is the solution of (7^*) for $\theta = 1$. This choice corresponds to the linear case, that is $\alpha = 1$. Proceeding as in [4], [5], one can construct a candidate Lyapunov function with properties stated next.

Proposition 1. *Let $a \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ be such that*

$$a = \begin{cases} 0 & \text{on } (-\infty, 1] \\ 1 & \text{on } [2, +\infty) \end{cases} \qquad \text{and } a' \geq 0 \text{ on } \mathbb{R}. \tag{2}$$

There exists $\epsilon > 0$ such that for all $\alpha \in]1-\epsilon, 1+\epsilon[$, the function $\bar{V}_\alpha$ defined as

$$\bar{V}_\alpha(e) = \int_0^{+\infty} \frac{1}{t^3} (a \circ V_1)(t^{r_1(\alpha)} e_1, \ldots, t^{r_n(\alpha)} e_n) dt \tag{3}$$

if $e \in \mathbb{R}^n \backslash \{0\}$ and $\bar{V}_\alpha(0) = 0$ is well defined, radially unbounded, of class $\mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, and satisfies

a) $\bar{V}_\alpha(\delta_\lambda^{r(\alpha)} e) = \lambda^2 \bar{V}_\alpha(e)$, for all $e \in \mathbb{R}^n$ and $\lambda > 0$.
b) $\langle \nabla \bar{V}_\alpha(e), Ae - F(S_\infty^{-1}(1) C^T, e) \rangle \leq -\gamma (\bar{V}_\alpha(e))^{\frac{1+\alpha}{2}}$, for all $e \in \mathbb{R}^n$, where $\gamma > 0$.

where F,C and $\delta_\lambda^{r(\alpha)}$ are defined in [2].

Proof of proposition 1. Let, $\alpha \in]1 - \frac{1}{n}, +\infty[$, proceeding as in [4], one directly shows that $\bar{V}_\alpha$ is well defined, radially unbounded, $\mathcal{C}^1$ on $\mathbb{R}^n$, and homogeneous of degree 2 with respect to the weights $r(\alpha)$. Then, only point b) remains to prove.
Following the same lines as in [4], there exists $l, L > 0$ such that for all $e \in \{e \in \mathbb{R}^n \mid \bar{V}_\alpha(e) = 1\}$, one has

$$\langle \nabla \bar{V}_\alpha(e), Ae - F(S_\infty^{-1}(1) C^T, e) \rangle = \int_l^L \frac{1}{t^{\alpha+2}} a'\left(V_1\left(\delta_t^{r(\alpha)} e\right)\right) \times \left\langle \nabla V_1\left(\delta_t^{r(\alpha)} e\right), A\delta_t^{r(\alpha)} e - F\left(S_\infty^{-1}(1) C^T, \delta_t^{r(\alpha)} e\right)\right\rangle dt$$

Consider the function $g(e, t, \alpha) = \left\langle \nabla V_1\left(\delta_t^{r(\alpha)} e\right), A\delta_t^{r(\alpha)} e - F\left(S_\infty^{-1}(1) C^T, \delta_t^{r(\alpha)} e\right)\right\rangle$, where $(e, t, \alpha) \in \{e \in \mathbb{R}^n, \bar{V}_\alpha(e) = 1\} \times \{t \in [l, L]\} \times]1 - \frac{1}{n}, +\infty[$. The function g is continuous, (e, t) belongs to a compact set and there exists $\gamma_1 > 0$ such that the image of g is included in $]-\infty, -\gamma_1[$ for $(e, t) \in \{e \in \mathbb{R}^n, \bar{V}_\alpha(e) = 1\} \times \{t \in [l, L]\}$ and $\alpha = 1$ (since it corresponds to the linear case). We can then apply Lemma 26.8 in [1] (tube lemma) which gives the existence of $\epsilon > 0$ such that for all $(e, t, \alpha) \in \{e \in \mathbb{R}^n, \bar{V}_\alpha(e) = 1\} \times \{t \in [l, L]\} \times]1-\epsilon, 1+\epsilon[$, $g(e, t, \alpha) \leq -\gamma_1$.
Then we have

$$\begin{aligned} &\langle \nabla \bar{V}_\alpha(e), Ae - F(S_\infty^{-1}(1) C^T, e) \rangle \\ &\leq -\gamma_1 \int_l^L \frac{1}{t^{\alpha+2}} a'\left(V_1\left(\delta_t^{r(\alpha)} e\right)\right) dt \leq -\gamma \left(\bar{V}_\alpha(e)\right)^{\frac{2+\alpha-1}{2}} \end{aligned} \tag{4}$$

where $\gamma > 0$ is a lower bound of $\gamma_1 \int_l^L \frac{1}{t^{\alpha+2}} a'\left(V_1\left(\delta_t^{r(\alpha)} e\right)\right) dt$ for $(e, \alpha) \in \{e \in \mathbb{R}^n, \bar{V}_\alpha(e) = 1\} \times]1-\epsilon, 1+\epsilon[$. Since $\bar{V}_\alpha$ is homogeneous of degree 2 with respect to the weights $r(\alpha)$, inequality (4) is valid for all $e \in \mathbb{R}^n$. □

Now that a new candidate Lyapunov function has been defined, we explain how it will be used to correct the proof

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of Theorem 2 in [2]. Please note that part 1 of the proof is correct, then it has already been proved that every trajectory starting from $e_0 \in \mathbb{R}^n$ enter the ball $\mathcal{B}_{\|.\|_{S_\infty(\theta)}}(1)$ after time $T_1(e_0) = \log(1/V(e_0))/\kappa(\theta)$ (see equation (18*)).
Denote $\bar{e} = \Delta_\theta e$, where $\Delta_\theta = \text{diag}\begin{bmatrix} 1 & \frac{1}{\theta} & \dots & \frac{1}{\theta^{n-1}} \end{bmatrix}$, in the remaining, we will show that for every $\theta \geq \theta_2 \stackrel{\triangle}{=} \frac{2}{\gamma}(M_1 + 2)$, there exists $\epsilon > 0$ such that the following inequality

$$\dot{\bar{V}}_\alpha(\bar{e}) \leq -\left(\frac{\gamma}{2}\theta - 1\right)\left(\bar{V}_\alpha(\bar{e})\right)^{\frac{2+\alpha-1}{2}} + M_1 \bar{V}_\alpha(\bar{e}) \qquad (5)$$

holds for every $\bar{e} \in \mathcal{B}_{\|.\|_{S_\infty(1)}}(1)$, $\alpha \in]1-\epsilon, 1[$, where $M_1 > 0$ is a constant independent of θ. This inequality replaces inequality (19*). Inequality (5) directly implies that the error system (11*) is finite time stable on $\mathcal{B}_{\|.\|_{S_\infty(\theta)}}(1)$. Thus, after time $T_1(e_0)$, the error enters $\mathcal{B}_{\|.\|_{S_\infty(\theta)}}(1)$ and after time $T_1(e_0) + T_2(e_0)$ the error reaches the origin, where the settling time $T_2(e_0)$ is bounded as follows

$$T_2(e_0) \leq \frac{ln\left(1 - \frac{M_1}{\frac{\gamma}{2}\theta - 1}\bar{V}_\alpha(e_0)^{1-\frac{2+\alpha-1}{2}}\right)}{M_1(\frac{2+\alpha-1}{2} - 1)}. \qquad (6)$$

The remaining of the corrected proof is very similar to the original one. The dynamics of $\bar{e}$ is given by

$$\begin{aligned} \dot{\bar{e}} &= \theta\left(A\bar{e} - F\left(S_\infty^{-1}(1)C^T, \bar{e}\right) - \rho S_\infty^{-1}(1)C^T C\bar{e}\right) \\ &\quad + \Delta_\theta D(x, \hat{x}, u). \end{aligned}$$

One has

$$\dot{\bar{V}}_\alpha(\bar{e}) \stackrel{\triangle}{=} \theta \bar{W}_1 + \bar{W}_2 \qquad (7)$$

with $\bar{W}_1 = \langle \nabla \bar{V}_\alpha(\bar{e}), A\bar{e} - F(S_\infty^{-1}(1)C^T, \bar{e}) - \rho S_\infty^{-1}(1)C^T C\bar{e}\rangle$ and $\bar{W}_2 = \langle \nabla \bar{V}_\alpha(\bar{e}), \Delta_\theta D(x, \hat{x}, u)\rangle$.
Following the same lines as in [2], one can show that there exists $\theta_2 \geq 0$ such that for every $\theta \geq \theta_2$, there exists $\epsilon > 0$ such that for all $\alpha \in]1-\epsilon, 1[$ inequality (5) holds true.

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