Numerical estimations of the cost of boundary controls for the equation $y\_t - \varepsilon y\_xx + My\_x = 0$ with respect to ε

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Numerical estimations of the cost of boundary controls for the equation $y_t - \varepsilon y_{xx} + M y_x = 0$ with respect to ε

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Abstract

We numerically examine the cost of the null boundary control for the transport diffusion equation $y_t - \varepsilon y_{xx} + M y_x = 0$, $x \in (0, L)$, $t \in (0, T)$ with respect to the positive parameter ε. It is known that this cost is uniformly bounded with respect to ε if $T \geq T_M$ with $T_M \in [1, 2\sqrt{3}]L/M$ if $M > 0$ and if $T_M \in [2\sqrt{2}, 2(1+\sqrt{3})]L/|M|$ if $M < 0$. We propose a method to approximate the underlying observability constant and then conjecture, through numerical computations, the minimal time of controllability T_M leading to a uniformly bounded cost. Several experiments for $M \in \{-1, 1\}$ are performed and discussed.

Key words: Singular controllability, Lagrangian variational formulation, Numerical approximation.

1 Introduction - Problem statement

Let $L > 0$, $T > 0$ and $Q_T := (0, L) \times (0, T)$. This work is concerned with the null controllability problem for the parabolic equation

$$\begin{cases} y_t - \varepsilon y_{xx} + M y_x = 0 & \text{in} \quad Q_T, \\ y(0, \cdot) = v, \ y(L, \cdot) = 0 & \text{on} \quad (0, T), \\ y(\cdot, 0) = y_0 & \text{in} \quad (0, L). \end{cases} \tag{1}$$

Here we assume that $y_0 \in H^{-1}(0, L)$. $\varepsilon > 0$ is the diffusion coefficient while $M \in \mathbb{R}$ is the transport coefficient; $v = v(t)$ is the control (a function in $L^2(0, T)$) and $y = y(x, t)$ is the associated state. In the sequel, we shall use the following notations :

$$L_\varepsilon y := y_t - \varepsilon y_{xx} + M y_x, \qquad L_\varepsilon^\star \varphi := -\varphi_t - \varepsilon \varphi_{xx} - M \varphi_x.$$

For any $y_0 \in H^{-1}(0, L)$ and $v \in L^2(0, T)$, there exists exactly one solution y to (1), with the regularity $y \in L^2(Q_T) \cap \mathcal{C}([0, T]; H^{-1}(0, L))$ (see for instance [11, Prop. 2.2]). Accordingly, for any final time $T > 0$, the associated null controllability problem at time $T > 0$ is the following: for each $y_0 \in H^{-1}(0, L)$, find $v \in L^2(0, T)$ such that the corresponding solution to (1) satisfies

$$y(\cdot, T) = 0 \text{ in } H^{-1}(0, L). \tag{2}$$

For any $T > 0$, $M \in \mathbb{R}$ and $\varepsilon > 0$, the null controllability for the parabolic type equation (1) holds true. We refer to [13] and [16] using Carleman type estimates. We therefore introduce the non-empty set of null controls

$$\mathcal{C}(y_0, T, \varepsilon, M) := \{(y, v) : v \in L^2(0, T); y \text{ solves (1) and satisfies (2)}\}.$$

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For $\varepsilon = 0$, the system (1) degenerates into a transport equation and is uniformly controllable as soon as T is large enough, according to the speed $|M|$ of transport, precisely as soon as $T \geq L/|M|$. On the other hand, for $\varepsilon > 0$, the asymptotic behavior of the null controls as $\varepsilon \to 0^+$ is less clear, depends on the sign of M, and has been the subject of several works in the last decade.

For any $\varepsilon > 0$, we define the cost of control by the following quantity :

$$K(\varepsilon, T, M) := \sup_{\|y_0\|_{L^2(0,L)}=1} \left\{ \min_{u \in \mathcal{C}(y_0, T, \varepsilon, M)} \|u\|_{L^2(0,T)} \right\}, \tag{3}$$

and denote by T_M the minimal time for which the cost $K(\varepsilon, T, M)$ is uniformly bounded with respect to the parameter ε. In other words, (1) is uniformly controllable with respect to ε if and only if $T \geq T_M$. In [9], J-M. Coron and S. Guerrero proved, using spectral arguments coupled with Carleman type estimates, that

$$T_M \in \begin{cases} [1, 4.3]\dfrac{L}{M} & \text{if} \quad M > 0, \\ [2, 57.2]\dfrac{L}{|M|} & \text{if} \quad M < 0. \end{cases}$$

The lower bounds are obtained using the initial condition $y_0(x) = \sin(\pi x/L)e^{\frac{Mx}{2\varepsilon}}$. The upper bounds are deduced from Carleman type inequalities for the adjoint solution. Then, using complex analysis arguments, O. Glass improved in [14] the previous estimations: precisely, he obtained that

$$T_M \in \begin{cases} [1, 4.2]\dfrac{L}{M} & \text{if} \quad M > 0, \\ [2, 6.1]\dfrac{L}{|M|} & \text{if} \quad M < 0. \end{cases}$$

These authors exhibit an exponential behavior of the L^2-norm of the controls with respect to ε. More recently, P. Lissy in [18, 19] yielded to the following conclusions:

$$T_M \in \begin{cases} [1, 2\sqrt{3}]\dfrac{L}{M} & \text{if} \quad M > 0, \\ [2\sqrt{2}, 2(1+\sqrt{3})]\dfrac{L}{|M|} & \text{if} \quad M < 0. \end{cases} \tag{4}$$

Remark that $2(1+\sqrt{3}) \approx 5.46$. The second lower bound $2\sqrt{2}$ is obtained by considering again the initial data $y_0(x) = \sin(\pi x/L)e^{\frac{Mx}{2\varepsilon}}$.

The main goal of the present work is to approximate numerically the value of T_M, for both $M > 0$ and $M < 0$. This can be done by approximating the cost K for various values of ε and $T > 0$, the ratio L/M being fixed.

In Section 2, we reformulate the cost of control K as the solution of a generalized eigenvalue problem, involving the control operator. In Section 3, we adapt [22], present a robust method to approximate numerically the control of minimal L^2-norm and discuss some experiments, for a given initial data y_0. In Section 4, we solve at the finite dimensional level the related eigenvalue problem using the power iterate method: each iteration requires the resolution of a null controllability problem for (1). We then discuss some experiments with respect to ε and T for $L/M = 1$ and $L/M = -1$ respectively.

2 Reformulation of the controllability cost $K(\varepsilon, T, M)$

We reformulate the cost of control K as the solution of a generalized eigenvalues problem involving the control operator (named as the HUM operator by J.-L. Lions for wave type equations). From

(3), we can write

$$K^2(\varepsilon,T,M) = \sup_{y_0\in L^2(0,L)} \frac{(v,v)_{L^2(0,T)}}{(y_0,y_0)_{L^2(0,L)}}$$

where $v=v(y_0)$ is the null control of minimal $L^2(0,T)$-norm for (1) with initial data y_0 in $L^2(0,L)$. Let us recall that any null control for (1) satisfies the following characterization

$$(v,\varepsilon\varphi_x(0,\cdot))_{L^2(0,T)} + (y_0,\varphi(\cdot,0))_{L^2(0,L)} = 0, \tag{5}$$

for any φ solution of the adjoint problem

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xx} - M\varphi_x = 0 & \text{in} \quad Q_T, \\ \varphi(0,\cdot) = \varphi(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \varphi(\cdot,T) = \varphi_T & \text{in} \quad (0,L), \end{cases} \tag{6}$$

where $\varphi_T\in H^1_0(0,L)$. In particular, the control of minimal L^2-norm is given by $v=\varepsilon\hat\varphi_x(0,\cdot)$ in $(0,T)$ where $\hat\varphi$ solves (6) associated to the initial $\hat\varphi_T$, solution of the extremal

$$\sup_{\varphi_T\in H^1_0(0,L)} J^\star(\varphi_T) := \frac{1}{2}\int_0^T (\varepsilon\varphi_x(0,\cdot))^2 dt + (y_0,\varphi(\cdot,0))_{L^2(0,L)}. \tag{7}$$

Taking $\varphi=\hat\varphi$ associated to $\hat\varphi_T$ in (5), we therefore have

$$(v,v)_{L^2(0,T)} = (v,\varepsilon\hat\varphi_x(0,t))_{L^2(0,T)} = -(y_0,\hat\varphi(\cdot,0))_{L^2(0,T)}. \tag{8}$$

Consequently, if we denote by $\mathcal{A}_\varepsilon : L^2(0,L)\to L^2(0,L)$ the control operator defined by $\mathcal{A}_\varepsilon y_0 := -\hat\varphi(\cdot,0)$, we finally obtain

$$K^2(\varepsilon,T,M) = \sup_{y_0\in L^2(0,L)} \frac{(\mathcal{A}_\varepsilon y_0,y_0)_{L^2(0,L)}}{(y_0,y_0)_{L^2(0,L)}} \tag{9}$$

and conclude that $K^2(\varepsilon,T,M)$ is solution of the following generalized eigenvalue problem :

$$\sup\Big\{\lambda\in\mathbb{R} : \exists\ y_0\in L^2(0,L), y_0\neq 0, \text{ s.t. } \mathcal{A}_\varepsilon y_0 = \lambda y_0 \quad \text{in} \quad L^2(0,L)\Big\}. \tag{10}$$

Remark 1 *The controllability cost is related to the observability constant $C_{obs}(\varepsilon,T,M)$ which appears in the observability inequality for (6)*

$$\|\varphi(\cdot,0)\|^2_{L^2(0,L)} \le C_{obs}(\varepsilon,T,M)\|\varepsilon\varphi_x(0,\cdot)\|^2_{L^2(0,T)}, \quad \forall\varphi_T\in H^1_0(0,L)\cap H^2(0,L)$$

defined by

$$C_{obs}(\varepsilon,T,M) = \sup_{\varphi_T\in H^1_0(0,L)} \frac{\|\varphi(\cdot,0)\|^2_{L^2(0,L)}}{\|\varepsilon\varphi_x(0,\cdot)\|^2_{L^2(0,T)}}. \tag{11}$$

Precisely, we get that $K(\varepsilon,T,M)=\sqrt{C_{obs}(\varepsilon,T,M)}$ (see [8], Remark 2.98).

Remark 2 *We may reformulate as well the previous extremal problem over $H^1_0(0,L)$ (seen as the dual space of $H^{-1}(0,L)\ni y(\cdot,T)$) in term of a generalized eigenvalue problem; we proceed as follows.*

We introduce the operators A_ε and B_ε given by

$$A_\varepsilon : \begin{array}{ccc} H^1_0(0,L) & \to & L^2(0,L) \\ \varphi_T & \mapsto & \varphi(\cdot,0) \end{array} \quad \textit{and} \quad B_\varepsilon : \begin{array}{ccc} H^1_0(0,L) & \to & L^2(0,T) \\ \varphi_T & \mapsto & \varepsilon\varphi_x(0,\cdot), \end{array}$$

where φ solves (6). The adjoint operators $A_\varepsilon^\star$ and $B_\varepsilon^\star$ of A_ε and B_ε are given by :

$$\begin{array}{rccl} A_\varepsilon^\star : & L^2(0,L) & \to & H^{-1}(0,L) \\ & y_0 & \mapsto & y(T; y_0, 0) \end{array} \quad \text{and} \quad \begin{array}{rccl} B_\varepsilon^\star : & L^2(0,L) & \to & H^{-1}(0,L) \\ & v & \mapsto & y(T; 0, v), \end{array}$$

where $y(t; y_0, v)$ is the solution to (1) at time t for the initial data y_0 and the control v. With these notations, we may rewrite C_{obs} given by (11) as follows

$$\begin{aligned} C_{obs}(\varepsilon, T, M) &= \sup_{\varphi_T \in H_0^1(0,L)} \frac{(A_\varepsilon \varphi_T, A_\varepsilon \varphi_T)_{L^2(0,L)}}{(B_\varepsilon \varphi_T, B_\varepsilon \varphi_T)_{L^2(0,T)}} \\ &= \sup_{\varphi_T \in H_0^1(0,L)} \frac{((-\Delta^{-1}) A_\varepsilon^\star A_\varepsilon \varphi_T, \varphi_T)_{H_0^1(0,L)}}{((-\Delta^{-1}) B_\varepsilon^\star B_\varepsilon \varphi_T, \varphi_T)_{H_0^1(0,L)}} \end{aligned}$$

leading to an eigenvalue problem over $H_0^1(0,L)$.

Remark that the operator $B_\varepsilon^\star B_\varepsilon$ from $H_0^1(0,L)$ to $H^{-1}(0,L)$ associates to the initial state φ_T of (6) the final state $y(\cdot, T)$ of (1) with $y_0 = 0$ and $v = \varepsilon \varphi_x(0, \cdot)$. v is therefore the control of minimal $L^2(0,T)$-norm with drives the state y from 0 to the trajectory $y(\cdot, T)$. $B_\varepsilon^\star B_\varepsilon$ is the so-called HUM operator.

Remark 3 *Actually, the supremum of $\varphi_T \in H_0^1(0,L)$ in (11) can be taken over $\varphi(\cdot, 0) \in L^2(0,L)$ (or even over φ !) leading immediately to*

$$C_{obs}(\varepsilon, T, M) = \sup_{\varphi(\cdot,0) \in L^2(0,L)} \frac{(\varphi(\cdot, 0), \varphi(\cdot, 0))}{(\mathcal{A}_\varepsilon^{-1} \varphi(\cdot, 0), \varphi(\cdot, 0))_{L^2(0,L)}}$$

in full agreement with (9) and the equality $K(\varepsilon, T, M) = \sqrt{C_{obs}(\varepsilon, T, M)}$.

Remark 4 *The sup-inf problem (3) may be solved by a gradient procedure. Let us consider the Lagrangien $\mathcal{L} : L^2(0,L) \times \mathbb{R} \to \mathbb{R}$ defined by*

$$\mathcal{L}(y_0, \mu) := \frac{1}{2} \|v(y_0)\|^2_{L^2(0,T)} + \frac{1}{2} \mu \biggl(\|y_0\|^2_{L^2(0,L)} - 1 \biggr)$$

where $v(y_0)$ is the control of minimal L^2-norm associated to the initial data $y_0 \in L^2(0,L)$ and $\mu \in \mathbb{R}$ a lagrange multiplier to enforce the constraint $\|y_0\|_{L^2(0,L)} = 1$. $v(y_0)$ satisfies (8). The first variation of $\mathcal{L}$ is given by

$$D\mathcal{L}(y_0) \cdot \overline{y_0} = (\mu y_0 - \varphi(\cdot, 0), \overline{y_0})_{L^2(0,L)} = \biggl((\mu\, Id + \mathcal{A}_\varepsilon) y_0, \overline{y_0} \biggr)_{L^2(0,L)} \tag{12}$$

where φ solves (6)-(7). A maximizing sequence $\{y_0^k\}_{k \geq 1}$ can be constructed as follows: given $y_0^0 \in L^2(0,L)$ such that $\|y_0^0\|_{L^2(0,L)} = 1$, compute iteratively

$$y_0^{k+1} = y_0^k + \eta^k (\mu^k y_0^k - \varphi^k(\cdot, 0)), \quad k \geq 0$$

with $\eta^k > 0$ small enough and μ^k such that $\|y_0^{k+1}\|_{L^2(0,L)} = 1$, that is,

$$\mu^k = \frac{\theta^k - 1}{\eta^k}, \quad \theta^k = \eta^k (y_0^k, \varphi^k(\cdot, 0))_{L^2(0,L)} \pm \sqrt{1 + (\eta^k)^2 (y_0^k - \varphi^k(\cdot, 0), \varphi^k(\cdot, 0))_{L^2(0,L)}}.$$

Remark that (12) implies that the optimal initial data y_0 is proportional to the optimal terminal state $\varphi(\cdot, 0)$ of φ solution of (6)-(7). Then, from the characterization (8), the sequence μ^k satisfies $(v^k, v^k) + \mu^k (y_0^k, \varphi^k(\cdot, 0))_{L^2(0,L)} = 0$ and converges toward $-K^2(\varepsilon, T, M)$. Remark that μ^k defined above is always negative.

In order to solve the eigenvalue problem (10) and get the largest eigenvalue of the operator $\mathcal{A}_\epsilon$, we may employ the power iterate method (see [6]), which reads as follows : given $y_0^0 \in L^2(0,L)$ such that $\|y_0^0\|_{L^2(0,L)} = 1$, compute

$$\begin{cases} z_0^{k+1} = \mathcal{A}_\epsilon y_0^k, \quad k \geq 0, \\ y_0^{k+1} = \dfrac{z_0^{k+1}}{\|z_0^{k+1}\|_{L^2(0,L)}}, \quad k \geq 0. \end{cases}$$

The real sequence $\{\|z_0^k\|_{L^2(0,L)}\}_{(k>0)}$ then converges to the eigenvalue with largest modulus of the operator $\mathcal{A}_\varepsilon$, so that

$$\sqrt{\|z_0^k\|_{L^2(0,L)}} \to K(\varepsilon, T, M) \quad \text{as} \quad k \to \infty.$$

The L^2 sequence $\{y_0^k\}_{(k\geq 0)}$ then converges toward the corresponding eigenvector. The first step requires to compute the image of the control operator $\mathcal{A}_\varepsilon$: this is done by determining the control of minimal L^2-norm, i.e. by solving the extremal problem (7) with y_0^k as initial condition for (1).

3 Approximation of the control problem

The generalized eigenvalue problem (10) involves the null control operator $\mathcal{A}_\varepsilon$ associated to (1). At the finite dimensional level, this problem can be solved by the way of the power iterate method, which requires at each iterates, the approximation of the null control of minimal L^2-norm for (1). We discuss in this section such approximation, the initial data y_0 in (1) being fixed.

The numerical approximation of null controls for parabolic equations is a not an easy task and has been first discussed in [4], and then in several works: we refer to the review [23]. Duality theory reduces the problem to the resolution of the unconstrained extremal problem (7). In view of the regularization character of the parabolic operator, the extremal problem (7) is ill-posed as the supremum is not reached in $H_0^1(0,L)$ but in a space, say $\mathcal{H}$, defined as the completion of $H_0^1(0,L)$ for the norm $\|\varphi_T\|_{\mathcal{H}} := \|\varepsilon\varphi_x(0,\cdot)\|_{L^2(0,T)}$, much larger than $H_0^1(0,L)$ and difficult to approximate. We refer to the review paper [23]. The usual "remedy" consists to enforce the regularity H_0^1 and replace (7) by

$$\min_{\varphi_T \in H_0^1(0,L)} J_\beta^\star(\varphi_T) := \frac{1}{2}\|\varepsilon\varphi_x(0,\cdot))\|^2_{L^2(0,T)} + (y_0, \varphi(\cdot,0))_{L^2(0,T)} + \frac{\beta}{2}\|\varphi_T\|^2_{H_0^1(0,L)} \tag{13}$$

for any $\beta > 0$ small. The resulting approximate control $v_\beta = \varepsilon\varphi_{\beta,x}(0,\cdot)$ leads to a state y_β solution of (1) satisfying the property

$$\|y_\beta(\cdot,T)\|_{H^{-1}(0,L)} \leq C\sqrt{\beta}\|y_0\|_{L^2(0,L)} \tag{14}$$

(for a constant $C > 0$ independent of β). This penalty method is discussed in [4] for the boundary controllability of the heat equation (for the distributed case, we refer to [2, 12, 15]). As in [4], problem (13) may be solved using a gradient iterative method: in view of the ill-posedness of (7), such method requires an increasing number of iterates to reach convergence as β goes to zero.

Moreover, in the context of the transport equation (1), it is necessary to take β small enough, in relation with the diffusion coefficient ε. Indeed, if $\beta > 0$ is fixed (independently of ε), then for $\varepsilon > 0$ small enough, the uncontrolled solution of (1) satisfies (14) as soon as $T \geq L/|M|$. In that case, problem (13) leads to the minimizer $\varphi_T = 0$ and then to the null control which is certainly not the optimal control we expect for negatives values of M (in view of (4))!

Therefore, as ε tends to 0, the occurence of the transport term makes the approximation of the null control for (1) a challenging task. Consequently, instead of minimizing the functional $J^\star$ (or $J^\star_\beta$), we adapt [22] (devoted to the inner situation for $M = 0$ and $\varepsilon = 1$) and try to solve directly the corresponding optimality conditions. This leads to a mixed variational formulation (following the terminology used in [22]).

3.1 Mixed variational formulation

We introduce the linear space $\Phi^0 := \{\varphi \in C^2(\overline{Q_T}),\, \varphi = 0 \text{ on } \Sigma_T\}$. For any $\eta > 0$, we define the bilinear form

$$(\varphi,\overline{\varphi})_{\Phi^0} := \int_0^T \varepsilon\varphi_x(0,t)\,\varepsilon\overline{\varphi}_x(0,t)\,dt + \beta\big(\varphi(\cdot,T),\overline{\varphi}(\cdot,T)\big)_{H_0^1(0,L)} + \eta\iint_{Q_T} L^\star\varphi\, L^\star\overline{\varphi}\,dx\,dt, \quad \forall\varphi,\overline{\varphi}\in\Phi^0.$$

From the unique continuation property for the transport equation, this bilinear form defines for any $\beta \geq 0$ a scalar product. Let Φ_β be the completion of Φ^0 for this scalar product. We denote the norm over Φ_β by $\|\cdot\|_{\Phi_\beta}$ such that

$$\|\varphi\|^2_{\Phi_\beta} := \|\varepsilon\varphi_x(0,\cdot)\|^2_{L^2(0,T)} + \beta\|\varphi(\cdot,T)\|^2_{H_0^1(0,L)} + \eta\|L^\star\varphi\|^2_{L^2(Q_T)}, \quad \forall\varphi\in\Phi_\beta. \tag{15}$$

Finally, we define the closed subset W_β of Φ_β by $W_\beta = \{\varphi\in\Phi_\beta : L^\star\varphi = 0 \text{ in } L^2(Q_T)\}$ endowed with the same norm than Φ_β. Then, for any $r \geq 0$, we define the following extremal problem :

$$\min_{\varphi\in W_\beta} \hat{J}^\star_\beta(\varphi) := \frac{1}{2}\|\varepsilon\varphi_x(0,\cdot)\|^2_{L^2(0,T)} + \frac{\beta}{2}\|\varphi(\cdot,T)\|^2_{H_0^1(0,L)} + (y_0,\varphi(\cdot,0))_{L^2(0,L)} + \frac{r}{2}\|L^\star\varphi\|^2_{L^2(Q_T)}. \tag{16}$$

Standard energy estimates for (1) imply that, for any $\varphi \in W_\beta$, $\varphi(\cdot,0) \in L^2(0,L)$ so that the functional $\hat{J}^\star_\beta$ is well-defined over W_β. Moreover, since for any $\varphi\in W_\beta$, $\varphi(\cdot,T)$ belongs to $H_0^1(0,L)$, problem (16) is equivalent to the extremal problem (13). The main variable is now φ submitted to the constraint equality (in $L^2(Q_T)$) $L^\star\varphi = 0$, which is addressed through a Lagrange multiplier.

3.1.1 Mixed formulation

We consider the following mixed formulation : find $(\varphi_\beta,\lambda_\beta)\in\Phi_\beta\times L^2(Q_T)$ solution of

$$\left\{\begin{array}{rcll} a_{\beta,r}(\varphi_\beta,\overline{\varphi}) + b(\overline{\varphi},\lambda_\beta) & = & l(\overline{\varphi}), & \forall\overline{\varphi}\in\Phi_\beta \\ b(\varphi_\beta,\overline{\lambda}) & = & 0, & \forall\overline{\lambda}\in L^2(Q_T), \end{array}\right. \tag{17}$$

where

$$\begin{aligned} &a_{\beta,r} : \Phi_\beta\times\Phi_\beta\to\mathbb{R}, \quad a_{\beta,r}(\varphi,\overline{\varphi}) := (\varepsilon\varphi_x(0,\cdot),\varepsilon\overline{\varphi}_x(0,\cdot))_{L^2(0,T)} + \beta(\varphi(\cdot,T),\overline{\varphi}(\cdot,T))_{H_0^1(0,L)} \\ &\qquad\qquad\qquad\qquad\qquad\qquad + r(L^\star\varphi, L^\star\overline{\varphi})_{L^2(Q_T)} \\ &b : \Phi_\beta\times L^2(Q_T)\to\mathbb{R}, \quad b(\varphi,\lambda) := (L^\star\varphi,\,\lambda)_{L^2(Q_T)} \\ &l : \Phi_\beta\to\mathbb{R}, \quad l(\varphi) := -(y_0,\varphi(\cdot,0))_{L^2(0,L)}. \end{aligned}$$

We have the following result :

THEOREM 3.1 *Assume that $\beta > 0$ and $r \geq 0$.*

1. *The mixed formulation* (17) *is well-posed.*

2. *The unique solution $(\varphi_\beta,\lambda_\beta)\in\Phi_\beta\times L^2(Q_T)$ is the unique saddle-point of the Lagrangian $\mathcal{L}_{\beta,r} : \Phi_\beta\times L^2(Q_T)\to\mathbb{R}$ defined by*

$$\mathcal{L}_{\beta,r}(\varphi,\lambda) := \frac{1}{2}a_{\beta,r}(\varphi,\varphi) + b(\varphi,\lambda) - l(\varphi). \tag{18}$$

3. *The optimal function φ_β is the minimizer of $\hat{J}^\star_\beta$ over W_β while $\lambda_\beta\in L^2(Q_T)$ is the state of* (1) *in the weak sense.*

Proof- The proof is very closed to the proof given in [22], Section 2.1.1. The bilinear form $a_{\beta,r}$ is continuous, symmetric and positive over $\Phi_\beta \times \Phi_\beta$. The bilinear form b is continuous over $\Phi_\beta \times L^2(Q_T)$. Furthermore, for any $\beta > 0$, the continuity of the linear form l over Φ_β is deduced from the energy estimate:

$$\|\varphi(\cdot,0)\|^2_{L^2(0,L)} \leq C \iint_{Q_T} |L^\star \varphi|^2 dx\, dt + \|\varphi(\cdot,T)\|^2_{L^2(0,L)}, \quad \forall \varphi \in \Phi_\beta,$$

for some $C > 0$ so that $\|\varphi(\cdot,0)\|^2_{L^2(0,L)} \leq \max(C\eta^{-1}, \beta^{-1})\|\varphi\|^2_{\Phi_\beta}$. Therefore, the well-posedness of the mixed formulation is a consequence of the following properties (see [3]):

- $a_{\beta,r}$ is coercive on $\mathcal{N}(b)$, where $\mathcal{N}(b)$ denotes the kernel of b :

$$\mathcal{N}(b) := \{\varphi \in \Phi_\beta \ : \ b(\varphi,\lambda) = 0 \text{ for every } \lambda \in L^2(Q_T)\}.$$

- b satisfies the usual "inf-sup" condition over $\Phi_\beta \times L^2(Q_T)$: there exists $\delta > 0$ such that

$$\inf_{\lambda \in L^2(Q_T)} \sup_{\varphi \in \Phi_\beta} \frac{b(\varphi,\lambda)}{\|\varphi\|_{\Phi_\beta}\|\lambda\|_{L^2(Q_T)}} \geq \delta. \tag{19}$$

The first point follows from the definition. Concerning the inf-sup condition, for any fixed $\lambda^0 \in L^2(Q_T)$, we define the (unique) element φ^0 such that $L^\star\varphi^0 = \lambda^0$, $\varphi = 0$ on Σ_T and $\varphi^0(\cdot,T) = 0$ in $L^2(0,L)$. The function φ^0 is therefore solution of the backward transport equation with source term $\lambda^0 \in L^2(Q_T)$, null Dirichlet boundary condition and zero initial state. Moreover, since $\lambda^0 \in L^2(Q_T)$, the following estimate proved in the Appendix A of [9] (more precisely, we refer to the inequality (94))

$$\varepsilon\|\varphi^0_x(0,\cdot)\|_{L^2(0,T)} \leq C_{L,T,M}\|\lambda^0\|_{L^2(Q_T)}$$

for a constant $C_{L,T,M} > 0$ independent of ε, implies that $\varphi^0 \in \Phi_\beta$. In particular, we have $b(\varphi^0,\lambda^0) = \|\lambda^0\|^2_{L^2(Q_T)}$ and

$$\sup_{\varphi \in \Phi_\beta} \frac{b(\varphi,\lambda^0)}{\|\varphi\|_{\Phi_\beta}\|\lambda^0\|_{L^2(Q_T)}} \geq \frac{b(\varphi^0,\lambda^0)}{\|\varphi^0\|_{\Phi_\beta}\|\lambda^0\|_{L^2(Q_T)}} = \frac{\|\lambda^0\|^2_{L^2(Q_T)}}{\left(\|\varepsilon\varphi^0_x(0,\cdot)\|^2_{L^2(0,T)} + \eta\|\lambda^0\|^2_{L^2(Q_T)}\right)^{\frac{1}{2}} \|\lambda^0\|_{L^2(Q_T)}}.$$

Combining the above two inequalities, we obtain

$$\sup_{\varphi 0 \in \Phi_\beta} \frac{b(\varphi^0,\lambda^0)}{\|\varphi^0\|_{\Phi_\beta}\|\lambda^0\|_{L^2(Q_T)}} \geq \frac{1}{\sqrt{C^2_{L,T,M} + \eta}} \tag{20}$$

and, hence, (19) holds with $\delta = \left(C^2_{L,T,M} + \eta\right)^{-1/2}$.

The second point is due to the symmetry and to the positivity of the bilinear form $a_{\beta,r}$. Concerning the third point, the equality $b(\varphi_\beta,\overline{\lambda}) = 0$ for all $\overline{\lambda} \in L^2(Q_T)$ implies that $L^\star\varphi_\beta = 0$ as an $L^2(Q_T)$-function, so that if $(\varphi_\beta,\lambda_\beta) \in \Phi_\beta \times L^2(Q_T)$ solves the mixed formulation, then $\varphi_\beta \in W_\beta$ and $\mathcal{L}_\beta(\varphi_\beta,\lambda_\beta) = \hat{J}^\star_\beta(\varphi_\beta)$. Finally, the first equation of the mixed formulation (taking $r = 0$) reads as follows:

$$\int_0^T \varepsilon(\varphi_\beta)_x(0,t)\, \varepsilon\overline{\varphi}_x(0,t)dt + \beta\big(\varphi_\beta(\cdot,T),\overline{\varphi}(\cdot,T)\big)_{H^1_0(0,L)} - \iint_{Q_T} L^\star\overline{\varphi}\, \lambda_\beta\, dx\, dt = l(\overline{\varphi}), \quad \forall \overline{\varphi} \in \Phi_\beta,$$

or equivalently, since the control is given by $v_\beta := \varepsilon(\varphi_\beta)_x(0,\cdot)$,

$$\int_0^T v_\beta(t)\, \varepsilon\overline{\varphi}_x(0,t)\, dt + \beta(\varphi_\beta(\cdot,T),\overline{\varphi}(\cdot,T))_{H^1_0(0,L)} - \iint_{Q_T} L^\star\overline{\varphi}\, \lambda_\beta\, dx\, dt = l(\overline{\varphi}), \quad \forall \overline{\varphi} \in \Phi_\beta.$$

But this means that $\lambda_\beta \in L^2(Q_T)$ is solution of (1) in the transposition sense. Since $y_0 \in L^2(0,L)$ and $v_\beta \in L^2(0,T)$, λ_β coincides with the unique weak solution to (1) such that $-\Delta^{-1}\lambda_\beta(\cdot,T) + \beta\varphi_\beta(\cdot,T) = 0$. $\square$

3.1.2 Minimization with respect to the Lagrange multiplier

The augmented mixed formulation (17) allows to solve simultaneously the dual variable φ_β, argument of the conjugate functional (16), and the Lagrange multiplier λ_β, qualified as the primal variable of the problem.

Assuming that the augmentation parameter r is strictly positive, we derive the corresponding extremal problem involving only the variable λ_β. For any $r > 0$, let the linear operator $\mathcal{A}_{\beta,r}$ from $L^2(Q_T)$ into $L^2(Q_T)$ be defined by $\mathcal{A}_{\beta,r}\lambda := L^\star\varphi$ where $\varphi = \varphi(\lambda) \in \Phi_\beta$ is the unique solution to

$$a_{\beta,r}(\varphi, \overline{\varphi}) = b(\overline{\varphi}, \lambda), \quad \forall \overline{\varphi} \in \Phi_\beta. \tag{21}$$

For any $r > 0$, the form $a_{\beta,r}$ defines a norm equivalent to the norm on Φ_β (see (15)), so that (21) is well-posed. The following crucial lemma holds true.

LEMMA 3.1 *For any $r > 0$, the operator $\mathcal{A}_{\beta,r}$ is a strongly elliptic, symmetric isomorphism from $L^2(Q_T)$ into $L^2(Q_T)$.*

It allows to get the following proposition which permits to replace the minimization of J_β over W_β to the minimization of the functional $J^{\star\star}_{\beta,r}$ over $L^2(Q_T)$, which is a space much easier to approximate than W_β.

PROPOSITION 3.1 *For any $r > 0$, let $\varphi^0 \in \Phi_\beta$ be the unique solution of*

$$a_{\beta,r}(\varphi^0, \overline{\varphi}) = l(\overline{\varphi}), \quad \forall \overline{\varphi} \in \Phi_\beta$$

and let $J^{\star\star}_{\beta,r} : L^2(Q_T) \to L^2(Q_T)$ be the functional defined by

$$J^{\star\star}_{\beta,r}(\lambda) := \frac{1}{2}(\mathcal{A}_{\beta,r}\lambda,\, \lambda)_{L^2(Q_T)} - b(\varphi^0, \lambda).$$

The following equality holds :

$$\sup_{\lambda \in L^2(Q_T)} \inf_{\varphi \in \Phi_\beta} \mathcal{L}_{\beta,r}(\varphi, \lambda) = - \inf_{\lambda \in L^2(Q_T)} J^{\star\star}_{\beta,r}(\lambda) \quad + \; \mathcal{L}_{\beta,r}(\varphi^0, 0).$$

We refer to [22], section 2.1 for the proof in the case $M = 0$.

Remark 5 *By introducing appropriate weights functions (vanishing at the time $t = T$) leading to optimal L^2-weighted controls vanishing at time T, we may consider the case $\beta = 0$. We refer to [22], section 2.3.*

3.2 Numerical approximation

We now turn to the discretization of the mixed formulation (17) assuming $r > 0$. We follow [22] for which we refer for the details. Let then $\Phi_{\beta,h}$ and $M_{\beta,h}$ be two finite dimensional spaces parametrized by the variable h such that, for any $\beta > 0$,

$$\Phi_{\beta,h} \subset \Phi_\beta, \quad M_{\beta,h} \subset L^2(Q_T), \quad \forall h > 0.$$

Then, we can introduce the following approximated problems : find $(\varphi_h, \lambda_h) \in \Phi_{\beta,h} \times M_{\beta,h}$ solution of

$$\left\{ \begin{array}{rcll} a_{\beta,r}(\varphi_h, \overline{\varphi}_h) + b(\overline{\varphi}_h, \lambda_h) & = & l(\overline{\varphi}_h), & \forall \overline{\varphi}_h \in \Phi_{\beta,h} \\ b(\varphi_h, \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h \in M_{\beta,h}. \end{array} \right. \tag{22}$$

The well-posedness of this mixed formulation is a consequence of two properties : the first one is the coercivity of the form $a_{\beta,r}$ on the subset $\mathcal{N}_h(b) = \{\varphi_h \in \Phi_{\beta,h}; b(\varphi_h, \lambda_h) = 0 \quad \forall \lambda_h \in M_{\beta,h}\}$. Actually, from the relation

$$a_{\beta,r}(\varphi, \varphi) \geq C_{r,\eta} \|\varphi\|^2_{\Phi_\beta}, \quad \forall \varphi \in \Phi_\beta,$$

where $C_{r,\eta} = \min\{1, r/\eta\}$, the form $a_{\beta,r}$ is coercive on the full space Φ_β, and so *a fortiori* on $\mathcal{N}_h(b) \subset \Phi_{\beta,h} \subset \Phi_\beta$. The second property is a discrete inf-sup condition :

$$\delta_{r,h} := \inf_{\lambda_h \in M_{\beta,h}} \sup_{\varphi_h \in \Phi_{\beta,h}} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_{\Phi_{\beta,h}} \|\lambda_h\|_{M_{\beta,h}}} > 0 \quad \forall h > 0. \tag{23}$$

Let us assume that this property holds. Consequently, for any fixed $h > 0$, there exists a unique couple (φ_h, λ_h) solution of (22). The property (23) is in general difficult to prove and strongly depends on the choice made for the approximated spaces $M_{\beta,h}$ and $\Phi_{\beta,h}$. We shall analyze numerically this property in the next section.

Remark 6 *For $r = 0$, the discrete formulation (22) is not well-posed over $\Phi_{\beta,h} \times M_{\beta,h}$ because the form $a_{\beta,r=0}$ is not coercive over the discrete kernel of b: the equality $b(\lambda_h, \varphi_h) = 0$ for all $\lambda_h \in M_{\beta,h}$ does not imply that $L^\star \varphi_h$ vanishes. The term $r\|L^\star \varphi_h\|^2_{L^2(Q_T)}$ is a numerical stabilization term: for any $h > 0$, it ensures the uniform coercivity of the form $a_{\beta,r}$ and vanishes at the limit in h. We also emphasize that this term is not a regularization term as it does not add any regularity to the solution φ_h.*

The finite dimensional and conformal space $\Phi_{\beta,h}$ must be chosen such that $L^\star \varphi_h$ belongs to $L^2(Q_T)$ for any $\varphi_h \in \Phi_{\beta,h}$. This is guaranteed as soon as φ_h possesses second-order derivatives in $L^2(Q_T)$. Any conformal approximation based on standard triangulation of Q_T achieves this sufficient property as soon as it is generated by spaces of functions continuously differentiable with respect to the variable x and spaces of continuous functions with respect to the variable t.

We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K \in \mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. Then, we introduce the space $\Phi_{\beta,h}$ as follows :

$$\Phi_{\beta,h} = \{\varphi_h \in C^1(\overline{Q_T}) : \varphi_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h,\ \varphi_h = 0 \text{ on } \Sigma_T\} \tag{24}$$

where $\mathbb{P}(K)$ denotes an appropriate space of polynomial functions in x and t. In this work, we consider for $\mathbb{P}(K)$ the so-called *Bogner-Fox-Schmit* (BFS for short) C^1-element defined for rectangles. In the one dimensional setting (in space), $\mathbb{P}(K) = (\mathbb{P}_{3,x} \otimes \mathbb{P}_{3,t})(K)$ where $\mathbb{P}_{r,\xi}$ is the space of polynomial functions of order r in the variable ξ.

We also define the finite dimensional space

$$M_{\beta,h} = \{\lambda_h \in C^0(\overline{Q_T}) : \lambda_h|_K \in \mathbb{Q}(K) \quad \forall K \in \mathcal{T}_h\},$$

where $\mathbb{Q}(K)$ denotes the space of affine functions both in x and t on the element K. In the one dimensional setting in space, K is a rectangle and we simply have $\mathbb{Q}(K) = (\mathbb{P}_{1,x} \otimes \mathbb{P}_{1,t})(K)$.

The resulting approximation is conformal: for any $h > 0$, $\Phi_{\beta,h} \subset \Phi_\beta$ and $M_{\beta,h} \subset L^2(Q_T)$.

Let $n_h = \dim \Phi_{\beta,h}, m_h = \dim M_{\beta,h}$ and let the real matrices $A_{\beta,r,h} \in \mathbb{R}^{n_h,n_h}$, $B_h \in \mathbb{R}^{m_h,n_h}$, $J_h \in \mathbb{R}^{m_h,m_h}$ and $L_h \in \mathbb{R}^{n_h}$ be defined by

$$\begin{cases} a_{\beta,r}(\varphi_h, \overline{\varphi_h}) = < A_{\beta,r,h}\{\varphi_h\}, \{\overline{\varphi_h}\} >_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}} & \forall \varphi_h, \overline{\varphi_h} \in \Phi_{\beta,h}, \\ b(\varphi_h, \lambda_h) = < B_h\{\varphi_h\}, \{\lambda_h\} >_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}} & \forall \varphi_h \in \Phi_{\beta,h} \lambda_h \in M_{\beta,h}, \\ \displaystyle\iint_{Q_T} \lambda_h \overline{\lambda_h}\, dx\, dt = < J_h\{\lambda_h\}, \{\overline{\lambda_h}\} >_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}} & \forall \lambda_h, \overline{\lambda_h} \in M_{\beta,h}, \\ l(\varphi_h) = < L_h, \{\varphi_h\} > & \forall \varphi_h \in \Phi_{\beta,h}, \end{cases}$$

where $\{\varphi_h\} \in \mathbb{R}^{n_h}$ denotes the vector associated to φ_h and $< \cdot, \cdot >_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}$ the usual scalar product over $\mathbb{R}^{n_h}$. With these notations, Problem (22) reads as follows : find $\{\varphi_h\} \in \mathbb{R}^{n_h}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} A_{\beta,r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{\varphi_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}.$$

3.2.1 The discrete inf-sup test

Before to discuss some numerical experiments, we numerically test the discrete inf-sup condition (23). Taking $\eta = r > 0$ so that $a_{\beta,r}(\varphi,\overline{\varphi}) = (\varphi,\overline{\varphi})_{\Phi_\beta}$ exactly for all $\varphi,\overline{\varphi} \in \Phi_\beta$, it is readily seen (see for instance [5]) that the discrete inf-sup constant satisfies

$$\delta_{\beta,r,h} = \inf\left\{\sqrt{\delta} : B_h A^{-1}_{\beta,r,h} B_h^T\{\lambda_h\} = \delta\, J_h\{\lambda_h\}, \quad \forall\,\{\lambda_h\} \in \mathbb{R}^{m_h} \setminus \{0\}\right\}. \tag{25}$$

The matrix $B_h A^{-1}_{\beta,r,h} B_h^T$ enjoys the same properties than the matrix $A_{\beta,r,h}$: it is symmetric and positive definite so that the scalar $\delta_{\beta,r,h}$ defined in term of the (generalized) eigenvalue problem (25) is strictly positive. This eigenvalue problem is solved using the power iterate algorithm (assuming that the lowest eigenvalue is simple): for any $\{v_h^0\} \in \mathbb{R}^{n_h}$ such that $\|\{v_h^0\}\|_2 = 1$, compute for any $n \geq 0$, $\{\varphi_h^n\} \in \mathbb{R}^{n_h}, \{\lambda_h^n\} \in \mathbb{R}^{m_h}$ and $\{v_h^{n+1}\} \in \mathbb{R}^{m_h}$ iteratively as follows :

$$\begin{cases} A_{\beta,r,h}\{\varphi_h^n\} + B_h^T\{\lambda_h^n\} = 0 \\ B_h\{\varphi_h^n\} = -J_h\{v_h^n\} \end{cases} , \quad \{v_h^{n+1}\} = \frac{\{\lambda_h^n\}}{\|\{\lambda_h^n\}\|_2}.$$

The scalar $\delta_{\beta,r,h}$ defined by (25) is then given by $\delta_{\beta,r,h} = \lim_{n\to\infty}(\|\{\lambda_h^n\}\|_2)^{-1/2}$.

We now reports some numerical values of $\delta_{\beta,r,h}$ with respect to h for the C^1-finite element introduced in Section 3.2. We use the value $T = 1$ and $\beta = 10^{-16}$. Tables 1, 2 and 3 provides the value of $\delta_{\beta,r,h}$ with respect to h and r for $M = 1$ for $\varepsilon = 10^{-1}, 10^{-2}$ and $\varepsilon = 10^{-3}$ respectively. For a fixed value of the parameter ε, we observe as in [22], that the inf sup constant increases as $r \to 0$ and behaves like $\delta_{\beta,r,h} \approx r^{-1/2}$, and more importantly, is bounded by below uniformly with respect to h. This key property is preserved as the parameter ε decreases, in agreement with the estimate (20) uniform with respect to ε.

r	10.	1.	0.1	h	h^2
$h = 1/80$	0.315	0.919	1.909	2.359	2.535
$h = 1/160$	0.313	0.923	1.94	2.468	2.599
$h = 1/320$	0.313	0.927	1.969	2.548	2.658

Table 1: $\delta_{\beta,r,h}$ w.r.t. h and r; $\varepsilon = 10^{-1}$ - $\beta = 10^{-16}$ - $M = 1$.

r	10.	1.	0.1	h	h^2
$h = 1/80$	0.311	0.961	2.423	3.64	4.473
$h = 1/160$	0.316	0.967	2.492	4.06	4.692
$h = 1/320$	0.316	0.971	2.545	4.406	4.916

Table 2: $\delta_{\beta,r,h}$ w.r.t. h and r; $\varepsilon = 10^{-2}$ - $\beta = 10^{-16}$ - $M = 1$.

r	10.	1.	0.1	h	h^2
$h = 1/80$	0.310	0.942	2.121	3.412	6.012
$h = 1/160$	0.310	0.987	2.435	4.012	5.944
$h = 1/320$	0.310	0.969	2.544	4.561	5.756

Table 3: $\delta_{\beta,r,h}$ w.r.t. h and r; $\varepsilon = 10^{-3}$ - $\beta = 10^{-16}$ - $M = 1$.

The case $M = -1$ is reported in Tables 4, 5 and 6. The same behavior is observed except that we note larger values of the inf-sup constant.

Consequently, we may conclude that the finite approximation we have used "passes" the discrete inf-sup test. Such property together with the uniform coercivity of the form $a_{\beta,r}$ then imply the convergence of the approximation sequence (φ_h, λ_h), unique solution of (22). As the matter of fact, the use of stabilization technics (so as to enrich the coercivity of the saddle point problem) introduced and analyzed in a closed context in [21, 20] is not necessary here. We emphasize that for $\beta = 0$ (or $\beta \to 0$ as $h \to 0$), the convergence of the approximation v_h is still an open issue. For $\beta = 0$, the convergence is guarantees if a vanishing weight is introduced, see [12]. This however leads to a different control and therefore a different definition of the cost of control $K(\varepsilon, T, M)$.

The choice of r affects the convergence of the sequences φ_h and λ_h with respect to h and may be very important here, in view of the sensitivity of the boundary control problem with respect to ε. Recall from Theorem 3.1, that for any $r \geq 0$, the multiplier λ coincides with the controlled solution. At the finite dimensional level of the mixed formulation (22) where r must be strictly positive, this property is lost for any h fixed: the non zero augmentation term $r\|L^\star \varphi_h\|_{L^2(Q_T)}$ introduces a small perturbation and requires to take $r > 0$ small (in order that the approximation λ_h be closed to the controlled solution y). In the sequel, the value $r = h^2$ is used.

r	10.	1.	0.1	h	h^2
$h = 1/80$	0.3161	0.997	2.663	4.358	5.069
$h = 1/160$	0.316	0.9805	2.673	4.69	5.139
$h = 1/320$	0.3162	0.9801	2.653	4.172	5.171

Table 4: $\delta_{\beta,r,h}$ for $\varepsilon = 10^{-1}$ - $\beta = 10^{-16}$ - $M = -1$.

r	10.	1.	0.1	h	h^2
$h = 1/80$	0.316	0.997	3.109	7.562	13.936
$h = 1/160$	0.3161	0.9997	3.086	9.433	14.101
$h = 1/320$	0.316	0.9809	3.086	11.101	14.140

Table 5: $\delta_{\beta,r,h}$ for $\varepsilon = 10^{-2}$ - $\beta = 10^{-16}$ - $M = -1$.

r	10.	1.	0.1	h	h^2
$h = 1/80$	0.302	0.9129	2.887	8.16	39.09
$h = 1/160$	0.301	0.957	3.022	12.14	43.08
$h = 1/320$	0.301	0.981	3.084	16.61	44.29

Table 6: $\delta_{\beta,r,h}$ for $\varepsilon = 10^{-3}$ - $\beta = 10^{-16}$ - $M = -1$.

3.3 Numerical experiments

We discuss some experiments for both $M = 1$ and $M = -1$ respectively and several values of ε. We consider a fixed data, independent of the parameter ε: precisely, we take $y_0(x) = \sin(\pi x)$ for $x \in (0, L)$ and $L = 1$.

We consider regular but non uniform rectangular meshes refined near the four edges of the space-time domain Q_T. More precisely, we refine at the edge $\{x = 1\} \times (0, T)$ to capture the boundary layer of length ε which appear for the variable λ_h when M is positive (see [1]), at the edge $\{x = 0\} \times (0, T)$ to approximate correctly the "control" function given by $v_h := \varepsilon \varphi_{h,x}$, and finally at $(0, L) \times \{0, T\}$ to represent correctly the initial condition and final condition. Precisely, let $p : [0, L] \to [0, L]$ be the polynomial of degree 3 such that $p(0) = 0, p'(0) = \eta_1, p'(L) = \eta_2$ and

$p(L) = L$ for some fixed $\eta_1, \eta_2 > 0$. The $[0, L]$ interval is then discretized as follows :

$$\begin{cases} [0, L] = \cup_{j=0}^{J}[y_j, y_{j+1}], \\ y_0 = 0, y_j - y_{j-1} = p(x_j) - p(x_{x_{j-1}}), \quad j = 1, \cdots, J+1 \end{cases} \tag{26}$$

where $\{x_j\}_{j=0,\cdots,J+1}$ is the uniform discretization of $[0, L]$ defined by $x_j = jh, j = 0, \cdot, J+1$, $h = L/(J+1)$. Small values for η_1, η_2 lead to a refined discretization $\{y_j\}_{j=0,\cdots,J+1}$ at $x = 0$ and $x = L$. The same procedure is used for the time discretization of $[0, T]$. In the sequel, we use $\eta_1 = \eta_2 = 10^{-3}$.

Preliminary, Table 7 gives some values of the H^{-1}-norm of the uncontrolled solution of (1) at time T associated to $y_0(x) = \sin(\pi x)$. We take $L = |M| = 1$. A time-marching approximation scheme is used with a very fine discretization both in time and space. As expected, for T greater than $L/|M|$, the norm $\|y(\cdot, T)\|_{H^{-1}(0,1)}$ decreases as ε goes to zero. For $T = L/M$, we observe that $\|y(\cdot, T)\|_{H^{-1}(0,1)} = O(\varepsilon)$ while for T strictly greater than $L/|M|$, the decrease to zero as $\varepsilon \to 0$ is faster.

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
$T = 0.9L/\|M\|$	2.20×10^{-2}	7.45×10^{-4}	2.76×10^{-3}	2.20×10^{-3}	2.15×10^{-3}
$T = L/\|M\|$	1.58×10^{-2}	2.67×10^{-3}	1.72×10^{-4}	9.76×10^{-6}	3.07×10^{-7}
$T = 1.1L/\|M\|$	1.12×10^{-2}	8.13×10^{-4}	1.15×10^{-6}	1.63×10^{-19}	8.62×10^{-20}

Table 7: Approximation $\|y_h(\cdot, T)\|_{H^{-1}(0,L)}$ w.r.t. T and ε for $y_0(x) = \sin(\pi x)$. $M = L = 1$.

We first discuss the case $M = 1$. As ε goes to 0^+, a boundary layer appears for the approximation λ_h at $x = 1$. The profile of the solution takes along the normal the form $(1 - e^{\frac{-M(1-x)}{\varepsilon}})$ and is captured with a locally refined mesh (we refer to [1]). Tables 8, 9 and 10 reports some numerical norms for $\epsilon = 10^{-1}, 10^{-2}$ and 10^{-3} respectively. These results are obtained by minimizing the functional $J^{\star\star}_{\beta,r}$ over $M_{\beta,h}$ defined in Proposition 3.1. The minimization of $J^{\star\star}_{\beta,r}$ of M_h is performed using the conjugate gradient algorithm: the stopping criterion is $\|g_h^n\|_{L^2(Q_T)} \leq 10^{-6}\|g_h^0\|_{L^2(Q_T)}$ where g_h^n is the residus at the iterate n. The algorithm is initialized with $\lambda_h^0 = 0$. We refer to [22] for the details.

We take $\beta = 10^{-16}$ and $r = h^2$ for the augmentation parameter leading to an appropriate approximation of the controlled solution y by the function λ_h: in particular, the optimality condition $\lambda_h(0, \cdot) - \varepsilon\varphi_{h,x}(0, \cdot) = 0$ is well respected in $L^2(0, T)$. The convergence of $\sqrt{r}\|L^\star\varphi_h\|_{L^2(Q_T)}$ (close to $\|L^\star\varphi_h\|_{L^2(H^{-1})}$ and actually sufficient to describe the solution of (1), see [7]) is also observed. As usual, we observe a faster convergence for the norm $\|\lambda_h\|_{L^2(Q_T)}$ than for the norm $\|v_h\|_{L^2(0,T)}$. From $\varepsilon = 10^{-1}$ to 10^{-3}, we also clearly observe a deterioration of the convergence order with respect to h.

h	1/80	1/160	1/320	1/640
$\sqrt{r}\|L^\star\varphi_h\|_{L^2(Q_T)}$	7.76×10^{-2}	3.01×10^{-2}	1.12×10^{-2}	7.12×10^{-3}
$\frac{\|\varepsilon\varphi_x(0,\cdot)-\lambda_h(0,\cdot)\|_{L^2(0,T)}}{\|\lambda_h(0,\cdot)\|_{L^2(0,T)}}$	1.06×10^{-2}	4.45×10^{-3}	1.97×10^{-3}	7.61×10^{-4}
$\|v_h\|_{L^2(0,T)}$	0.324	0.357	0.3877	0.3912
$\|\lambda_h\|_{L^2(Q_T)}$	0.367	0.366	0.362	0.363
$\|\lambda_h(\cdot, T)\|_{H^{-1}(0,T)}$	4.47×10^{-6}	9.59×10^{-7}	2.03×10^{-7}	1.01×10^{-7}
$\sharp$ CG iterate	76	117	175	231

Table 8: Mixed formulation (17) - $r = h^2$; $\varepsilon = 10^{-1}$; $\beta = 10^{-16}$ - $M = L = 1$.

For $h = 1/320$, Figure 1, 2 and 3 depict the function $\lambda_h(\cdot, t)$, approximation of the control v, for $t \in (0, T)$, $T = 1$ for $\varepsilon = 10^{-1}$, $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$ respectively. For large values of the

h	1/80	1/160	1/320	1/640
$\sqrt{r}\Vert L^\star\varphi_h\Vert_{L^2(Q_T)}$	5.86×10^{-1}	2.43×10^{-1}	1.41×10^{-1}	9.12×10^{-2}
$\frac{\Vert\varepsilon\varphi_x(0,\cdot)-\lambda_h(0,\cdot)\Vert_{L^2(0,T)}}{\Vert\lambda_h(0,\cdot)\Vert_{L^2(0,T)}}$	2.5×10^{-2}	1.24×10^{-2}	6.04×10^{-3}	2.89×10^{-3}
$\Vert v_h\Vert_{L^2(0,T)}$	1.391	2.392	2.929	3.316
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	0.518	0.6001	0.789	0.832
$\Vert\lambda_h(\cdot,T)\Vert_{H^{-1}(0,T)}$	5.46×10^{-6}	3.56×10^{-6}	8.77×10^{-7}	6.12×10^{-8}
$\sharp$ CG iterate	53	93	155	181

Table 9: Mixed formulation (17) - $r=h^2$; $\varepsilon=10^{-2}$; $\beta=10^{-16}$ - $M=L=1$.

h	1/80	1/160	1/320	1/640
$\sqrt{r}\Vert L^\star\varphi_h\Vert_{L^2(Q_T)}$	1.75×10^{-1}	1.01×10^{-1}	8.51×10^{-2}	6.91×10^{-2}
$\frac{\Vert\varepsilon\varphi_x(0,\cdot)-\lambda_h(0,\cdot)\Vert_{L^2(0,T)}}{\Vert\lambda_h(0,\cdot)\Vert_{L^2(0,T)}}$	4.87×10^{-2}	2.43×10^{-2}	1.3×10^{-4}	7.19×10^{-5}
$\Vert v_h\Vert_{L^2(0,T)}$	0.231	0.713	0.855	0.911
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	0.498	0.5015	0.5210	0.5319
$\Vert\lambda_h(\cdot,T)\Vert_{H^{-1}(0,T)}$	1.17×10^{-6}	3.69×10^{-7}	1.20×10^{-7}	8.12×10^{-8}
$\sharp$ CG iterate	29	68	129	151

Table 10: Mixed formulation (17) - $r=h^2$; $\varepsilon=10^{-3}$; $\beta=10^{-16}$ - $M=L=1$.

diffusion coefficient ε, for instance $\varepsilon=10^{-1}$, the transport term has a weak influence: the control of minimal L^2-norm is similar to the corresponding control for the heat equation and oscillates near the controllability time. On the contrary, for ε small, typically $\varepsilon=10^{-3}$, the solution - mainly driven by the transport term - is transported along a direction closed to $(1,1/M)=(1,1)$, so that at time $T=1/M$, is mainly distributed in the neighborhood of $x=1$. Consequently, the control (of minimal L^2-norm) acts mainly at the beginning of the time interval, so as to have an effect, at time T, in the neighborhood of $x=1$. We observe a regular oscillatory and decreasing behavior of the controls.

Let us now discuss the case $M=-1$. This negative case is *a priori* "simpler" since there is no more boundary layer at $x=1$: the solution is somehow "absorbed" by the control at the left edge $x=0$. Tables 11, 12 and 13 give some numerical values with respect to h for $\varepsilon=10^{-1},10^{-2}$ and 10^{-3}. Concerning the behavior of the approximation with respect to h, similar remarks (than for $M=1$) can be made: the notable difference is a lower rate of convergence, probably due to the singularity of the controls we obtain. Precisely, for the same data as in the case $M=1$, Figure 4, 5 and 6 depicts the "control" function $\lambda_h(0,t)$ for $t\in(0,T)$, $T=1$ for $\varepsilon=10^{-1}$, $\varepsilon=10^{-2}$ and $\varepsilon=10^{-3}$ respectively. The behavior of the control is quite different from the previous case. For ε large, typically $\varepsilon=10^{-1}$, the control is again similar to the control we observe for the heat equation, with an oscillatory behavior at the final time. We observe however that the corresponding norm is significantly larger that for the case $M=1$: this is due to the fact, that for $M<0$, the transport term "pushes" the solution toward $x=0$ where the control acts: this reduces the effect of the control which therefore must be stronger. For ε small, the solution is mainly transported along the direction $(1,1/M)=(1,-1)$ so that at time T, the solution is mainly concentrated in the neighborhood of $x=0$. For this reason, the control mainly acts at the end of the time interval: any action of the control not concentrated at the end of the time interval would be useless because pushed back to the edge $x=0$ and will produce a larger L^2-norm. As ε goes to zero, the control is getting concentrated at the terminal time with an oscillatory behavior and large amplitudes. This fact may explain why the behavior of the cost of control with respect to ε observed in [9, 14, 18] is singular for negatives values of M. For $M>0$, the transport term "helps" the control to act on the edge $x=1$ while for $M<0$, the transport term is against the control and reduces its action.

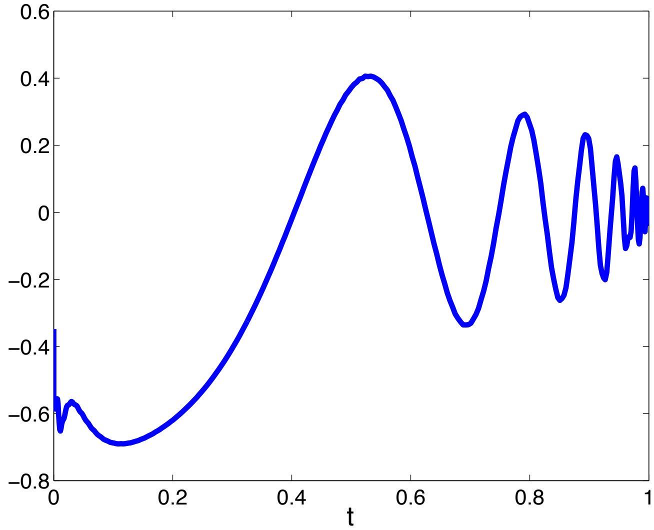

Figure 1: Approximation $\lambda_h(0,t)$ of the control w.r.t. $t \in [0,T]$ for $\varepsilon = 10^{-1}$ and $T = L = M = 1$; $r = h^2$ - $h = 1/320$.

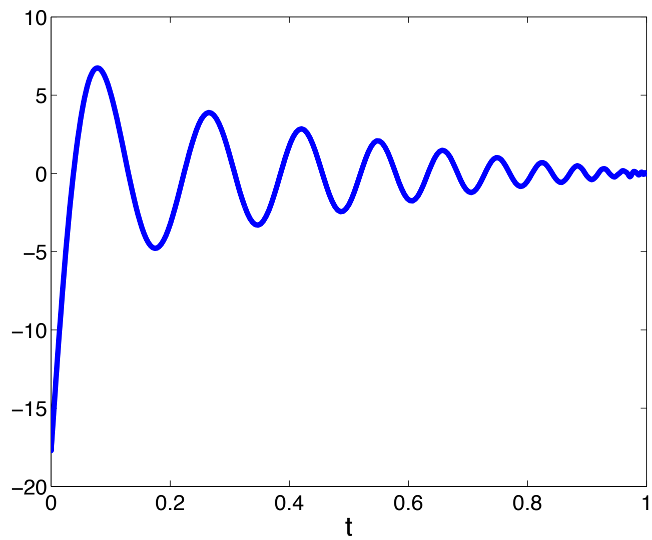

Figure 2: Approximation $\lambda_h(0,t)$ of the control w.r.t. $t \in [0,T]$ for $\varepsilon = 10^{-2}$ and $T = L = M = 1$; $r = h^2$ - $h = 1/320$.

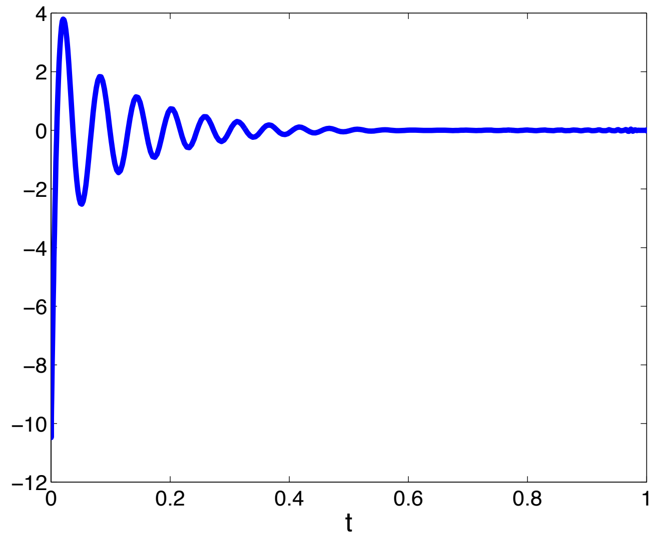

Figure 3: Approximation $\lambda_h(0,t)$ of the control w.r.t. $t \in [0,T]$ for $\varepsilon = 10^{-3}$ and $T = L = M = 1$; $r = h^2$ - $h = 1/320$.

For this reason, the numerical approximation of controls for $M = -1$ is definitively more involved and requires to take a very fine discretization, which will then imply a large number of CG iterates.

h	1/80	1/160	1/320	1/640
$\sqrt{r}\Vert L^\star\varphi_h\Vert_{L^2(Q_T)}$	1.51	0.731	0.231	0.101
$\frac{\Vert\varepsilon\varphi_x(0,\cdot)-\lambda_h(0,\cdot)\Vert_{L^2(0,T)}}{\Vert\lambda_h(0,\cdot)\Vert_{L^2(0,T)}}$	9.19×10^{-3}	3.87×10^{-3}	1.61×10^{-3}	1.12×10^{-3}
$\Vert v_h\Vert_{L^2(0,T)}$	28.16	39.26	49.96	52.03
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	5.74	7.96	9.05	10.12
$\Vert\lambda_h(\cdot,T)\Vert_{H^{-1}(0,T)}$	8.35×10^{-4}	1.82×10^{-4}	3.97×10^{-5}	1.12×10^{-5}
$\sharp$ CG iterate	48	80	129	157

Table 11: Mixed formulation (17) - $r = h^2$; $\varepsilon = 10^{-1}$; $\beta = 10^{-16}$ - $M = -1$.

We also observe, both for $M = 1$ and $M = -1$, that from $\varepsilon = 10^{-2}$ to $\varepsilon = 10^{-3}$, the L^2-norm $\Vert v_\varepsilon\Vert_{L^2(0,T)}$ decreases. Very likely, as ε goes to zero, this norm goes to zero. This does not contradict the theoretical results and is due to the fact that the initial condition we have taken here is independent of ε. In other words, the optimal problem (3) of control is not obtained for $y_0(x) = \sin(\pi x)$ nor by any initial condition independent of the parameter ε. This fact is proven in [1]. We remind that the initial condition $y_0(x) = e^{\frac{Mx}{2\varepsilon}}\sin(\pi x)$ is used in [9, 19].

4 Numerical approximation of the cost of control

We now turn to the numerical approximation of the cost of control $K(\varepsilon, T, M)$ defined by (3). Precisely, we address numerically the resolution of the generalized eigenvalue problem (10):

$$\sup\Big\{\lambda \in \mathbb{R} : \exists\, y_0 \in L^2(0,L), y_0 \neq 0, \text{ s.t. } \mathcal{A}_\varepsilon y_0 = \lambda y_0 \quad \text{in} \quad L^2(0,L)\Big\}.$$

Let V_h be a conformal approximation of the space $L^2(0,L)$ for all $h > 0$. We have then face to

h	$1/80$	$1/160$	$1/320$	$1/640$
$\sqrt{r}\Vert L^\star\varphi_h\Vert_{L^2(Q_T)}$	5.291	2.134	1.213	0.591
$\frac{\Vert\varepsilon\varphi_x(0,\cdot)-\lambda_h(0,\cdot)\Vert_{L^2(0,T)}}{\Vert\lambda_h(0,\cdot)\Vert_{L^2(0,T)}}$	5.27×10^{-4}	2.08×10^{-2}	8.05×10^{-3}	5.01×10^{-3}
$\Vert v_h\Vert_{L^2(0,T)}$	250.54	457.78	666.902	712.121
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	6.76	10.05	13.111	15.301
$\Vert\lambda_h(\cdot,T)\Vert_{H^{-1}(0,T)}$	1.54×10^{-3}	2.08×10^{-3}	1.71×10^{-3}	6.12×10^{-4}
$\sharp$ CG iterate	22	41	79	101

Table 12: Mixed formulation (17) - $r=h^2$; $\varepsilon=10^{-2}$; $\beta=10^{-16}$ - $M=-1$.

h	$1/80$	$1/160$	$1/320$	$1/640$
$\sqrt{r}\Vert L^\star\varphi_h\Vert_{L^2(Q_T)}$	7.12	2.14	1.31	0.59
$\frac{\Vert\varepsilon\varphi_x(0,\cdot)-\lambda_h(0,\cdot)\Vert_{L^2(0,T)}}{\Vert\lambda_h(0,\cdot)\Vert_{L^2(0,T)}}$	2.87×10^{-1}	7.76×10^{-2}	4.31×10^{-2}	2.12×10^{-2}
$\Vert v_h\Vert_{L^2(0,T)}$	0.281×10^{-1}	2.35	18.98	21.23
$\Vert\lambda_h\Vert_{L^2(Q_T)}$	4.97×10^{-1}	5.01×10^{-1}	6.38×10^{-1}	7.23×10^{-1}
$\Vert\lambda_h(\cdot,T)\Vert_{H^{-1}(0,T)}$	2.03×10^{-5}	3.28×10^{-5}	6.01×10^{-5}	8.01×10^{-5}
$\sharp$ CG iterate	7	11	23	26

Table 13: Mixed formulation (17) - $r=h^2$; $\varepsilon=10^{-3}$; $\beta=10^{-16}$ - $M=-1$.

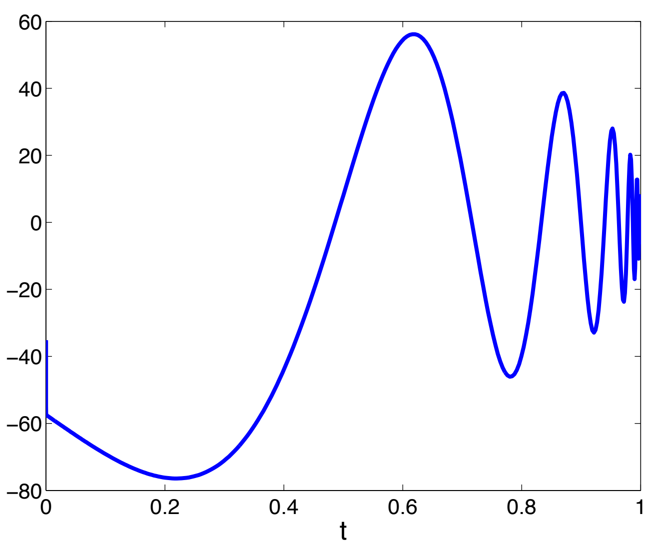

Figure 4: Approximation $\lambda_h(0,t)$ of the control w.r.t. $t\in[0,T]$ for $\varepsilon=10^{-1}$ and $T=L=-M=1$; $r=h^2$ - $h=1/320$.

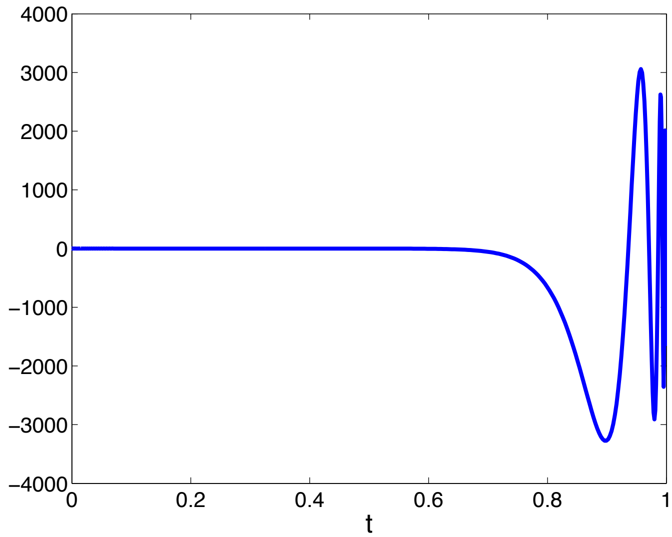

Figure 5: Approximation $\lambda_h(0,t)$ of the control w.r.t. $t \in [0,T]$ for $\varepsilon = 10^{-2}$ and $T = L = -M = 1$; $r = h^2$ - $h = 1/320$.

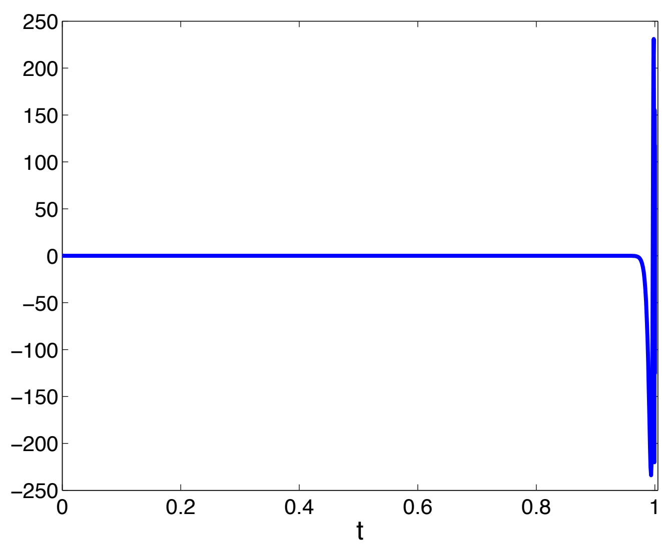

Figure 6: Approximation $\lambda_h(0,t)$ of the control w.r.t. $t \in [0,T]$ for $\varepsilon = 10^{-3}$ and $T = L = -M = 1$; $r = h^2$ - $h = 1/320$.

the following finite dimensional eigenvalues problem:

$$\sup\left\{\lambda \in \mathbb{R} : \exists\ y_{0,h} \in V_h, y_{0,h} \neq 0,\ \text{s.t. } \mathcal{A}_\varepsilon y_{0,h} = \lambda y_{0,h} \quad \text{in} \quad V_h\right\}.$$

$\mathcal{A}_\varepsilon y_{0,h}$ in $L^2(0,L)$ is defined as $-\varphi_h(\cdot,0)$ where $\varphi_h \in \Phi_{\beta,h}$ solves the variational formulation (22). Consequently, from the definition of $\Phi_{\beta,h}$ in (24), the space V_h is the set of C^1-functions and piecewise polynomial of order 3:

$$V_h = \left\{y_{0,h} \in C^1([0,L]) : y_{0,h}|_K \in \mathbb{P}_{3,x} \quad \forall K \in T_h\right\}$$

where T_h is the triangulation of $[0,L]$ defined by (26).

This kind of finite dimensional eigenvalue problems may be solved using the power iterate method (see [6]): the algorithm is as follows: given $y^0_{0,h} \in L^2(0,L)$ such that $\|y^0_{0,h}\|_{L^2(0,L)} = 1$, compute for all $k \geq 0$,

$$\begin{cases} z^k_{0,h} = \mathcal{A}_\epsilon y^k_{0,h}, \quad k \geq 0, \\ y^{k+1}_{0,h} = \dfrac{z^k_{0,h}}{\|z^k_{0,h}\|_{L^2(0,L)}}, \quad k \geq 0. \end{cases}$$

The real sequence $\{\|z^k_{0,h}\|_{L^2(0,L)}\}$ then converges to the eigenvalue with largest modulus of the operator $\mathcal{A}_\varepsilon$, so that

$$\sqrt{\|z^k_{0,h}\|_{L^2(0,1)}} \rightarrow K(\varepsilon,T,M,L) \quad \text{as} \quad k \rightarrow \infty.$$

$\{y^k_{0,h}\}_{k>0}$ converges to the corresponding eigenvectors. The first step requires to compute the image of the control operator $\mathcal{A}_\varepsilon$: this is done by solving the mixed formulation (22) taking $y^k_{0,h}$ as initial condition for (1).

The algorithm is stopped as soon as the sequence $\{z^k_{0,h}\}_{k\geq 0}$ satisfies

$$\frac{\big|\|z^k_{0,h}\|_{L^2(0,L)} - \|z^{k-1}_{0,h}\|_{L^2(0,L)}\big|}{\|z^{k-1}_{0,h}\|_{L^2(0,1)}} \leq 10^{-3}, \tag{27}$$

for some $k > 0$.

We now report the numerical values for $L = 1$ and $M = \pm 1$. We initialize the algorithm with

$$y^0_0(x) = \frac{e^{-\frac{Mx}{2\epsilon}} \sin(\pi x)}{\|e^{-\frac{Mx}{2\epsilon}} \sin(\pi x)\|_{L^2(0,L)}}, \quad x \in (0,L).$$

4.1 Cost of control in the case $M = 1$

Table 14 in the annexe section reports the approximations obtained of the cost of control $K(\varepsilon,T,M)$ for $M = 1$ with respect to T and ε. They corresponds to the discretisation $h = 1/320$. As expected, for T strictly lower than $L/M = 1$, here $T = 0.95$ and $T = 0.99$, we obtain that the cost $K(\varepsilon,T,M)$ blows up as ε goes to zero. This is in agreement with the fact, that for $T < L/M$, the system (1) is not uniformly controllable with respect to the initial data y_0 and ε. Figure 7 displays the approximations with respect to ε for $T = 0.95$. On the other hand, for T larger than $L/M = 1$, we observe that the numerical approximation of $K(\varepsilon,T,M)$ is bounded with respect to ε. More precisely, the cost is not monotonous with respect to ε as it reaches a maximal value for $\varepsilon \approx 1.75 \times 10^{-3}$ for $T = 1$ and $\varepsilon \approx 6 \times 10^{-3}$ for $T = 1.05$ (see Figures 8 and 10). Figure 9 is a zoom in the case $T = 1$ for the smallest values of the diffusion coefficient ε.

Figure 11 displays the approximation of the initial data $y_0 \in L^2(0,L)$ solution of the optimal problem (9) for $T = 1$ and $\varepsilon = 10^{-1}, 10^{-2}$ and 10^{-3}. As ε decreases, the optimal initial condition y_0 with $\|y_0\|_{L^2(0,L)} = 1$ gets concentrated as $x = 0$. Again, this is in agreement

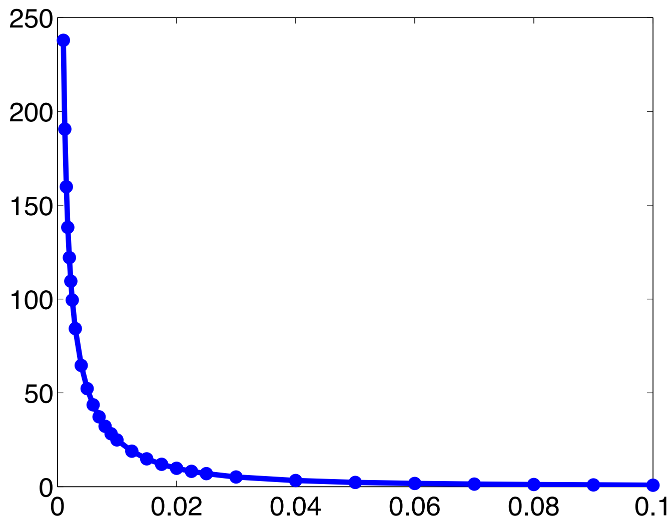

Figure 7: Cost of control $K(\varepsilon, T, M)$ w.r.t. $\varepsilon \in [10^{-3}, 10^{-1}]$ for $T = 0.95L/M$ and $L = M = 1$; $r = h^2$ - $h = 1/320$.

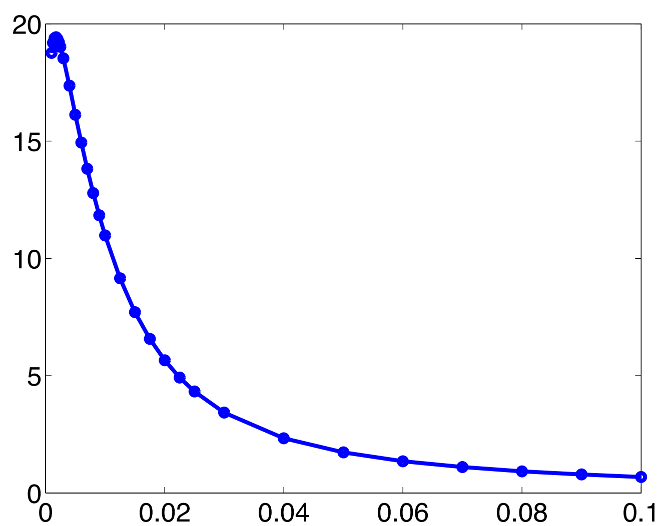

Figure 8: Cost of control $K(\varepsilon, T, M)$ w.r.t. $\varepsilon \in [10^{-3}, 10^{-1}]$ for $T = L/M$ and $L = M = 1$; $r = h^2$ - $h = 1/320$.

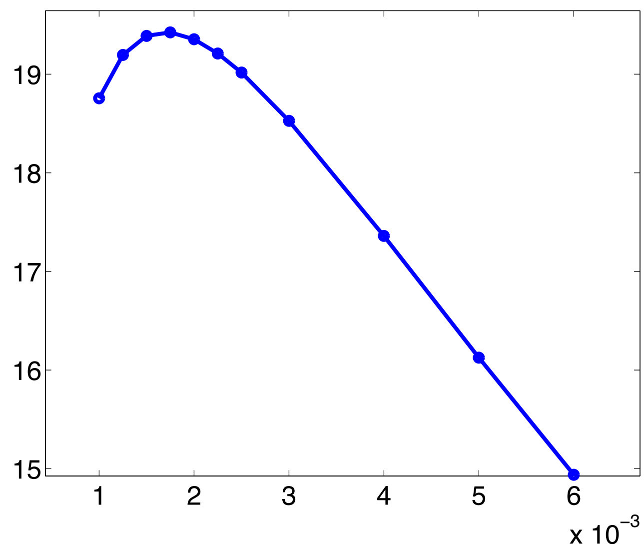

Figure 9: Cost of control $K(\varepsilon, T, M)$ w.r.t. $\varepsilon \in [10^{-3}, 6 \times 10^{-3}]$ for $T = 0.95L/M$ and $L = M = 1$; $r = h^2$ - $h = 1/320$.

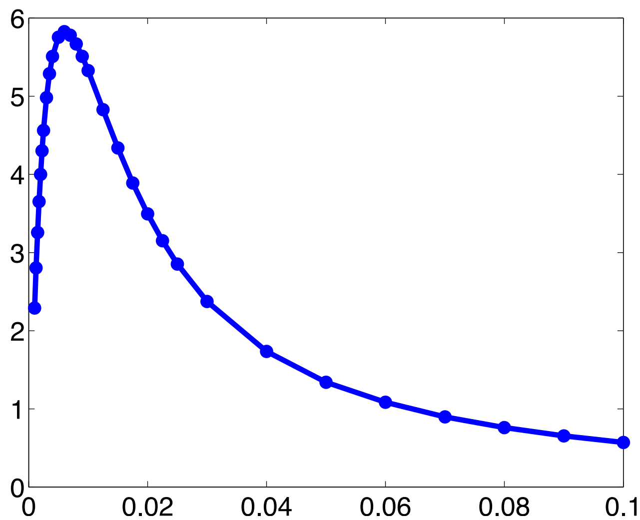

Figure 10: Cost of control $K(\varepsilon, T, M)$ w.r.t. $\varepsilon \in [10^{-3}, 10^{-1}]$ for $T = 1.05L/M$ and $L = M = 1$; $r = h^2$ - $h = 1/320$.

with the intuition since such condition produces (in the uncontrolled situation) larger values of $\|y(\cdot,T)\|_{H^{-1}(0,L)}$. It should be noted however that the solutions we get are different from $e^{-\frac{Mx}{2\epsilon}}\sin(\pi x)/\|e^{-\frac{Mx}{2\epsilon}}\sin(\pi x)\|_{L^2(0,L)}$. Moreover, they are apparently independent of the controllability time T (at least for the values of T closed to $1/M$ we have used). Remark also that the initial data $y_0(x) = e^{\frac{Mx}{2\epsilon}}\sin(\pi x)/\|e^{\frac{Mx}{2\epsilon}}\sin(\pi x)\|_{L^2(0,L)}$ highlighted in [9, 19] leads to a lower numerical value of $\|v_h\|_{L^2(0,L)}$.

For each values of ε and T, the convergence of the power iterate algorithm is fast: the stopping criterion (27) is reached in less than 5 iterates.

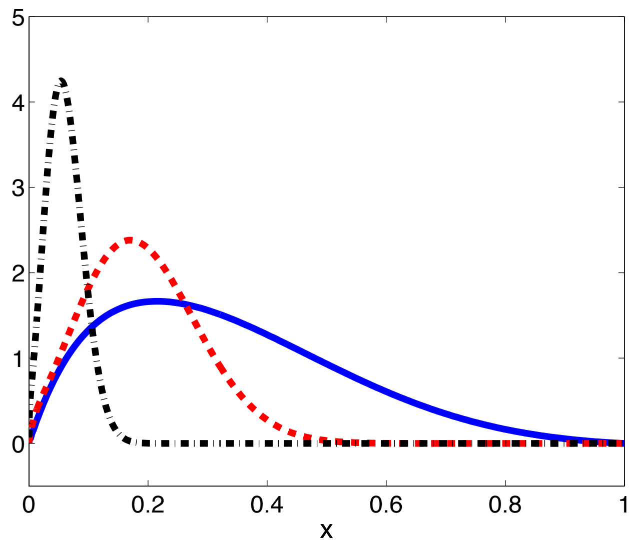

Figure 11: The optimal initial condition y_0 in $(0,L)$ for $\varepsilon = 10^{-1}$ (full line), $\varepsilon = 10^{-2}$ (dashed line) and $\varepsilon = 10^{-3}$ (dashed-dotted line) and $T = M = L = 1$; $r = h^2$ - $h = 1/320$.

Remark 7 *In [9], Theorem 2, the following estimate is obtained for all* $(\varepsilon,T,M) \in]0,\infty[$ *and* $L = 1$*:*

$$K(\varepsilon,T,M) \geq C_1 \frac{\varepsilon^{-3/2}T^{-1/2}M^2}{1+M^3\varepsilon^{-3}} \exp\left(\frac{M}{2\varepsilon}(1-TM) - \pi^2\varepsilon T\right) := C_1\, f(\varepsilon,T,M)$$

for a positive constant C_1*. This estimate is in agreement with the behavior we observe with respect to* ε *and* T *in the previous figures. For* $T = 0.95/M$*, the function* f *increases as* $\varepsilon \to 0$*, while for* $T \geq 1/M$*,* f *increases, reaches a unique maximum and then decreases to* 0 *as* ε *goes to zero.*

4.2 Controllability cost in the case $M = -1$

Table 15 in the annexe section reports the approximation obtained of the cost of control $K(\varepsilon,T,M)$ for $M = -1$ and $T = 1/|M|$ with respect to $\varepsilon \in [10^{-3},10^{-1}]$. With respect to the positive case, the notable difference is the amplitude of the cost, as expected much larger, since the transport term now acts "against" the control. For instance, for $\varepsilon = 10^{-3}$, we obtain $K(\varepsilon,T,M) \approx 18.7555$ for $M = 1$ and $K(\varepsilon,T,M) \approx 1.0718\times 10^4$ for $M = -1$. Moreover, the corresponding optimal initial condition y_0 is supported as $\varepsilon \to 0$ at the right extremity $x = 1$ (see figure 12) leading to a corresponding control localized at $t = T = 1/|M|$, with very large amplitude and oscillations, as shown on figure 13 for $\varepsilon = 10^{-3}$. Such oscillations are difficult to capture numerically and are very sensitive to the discretization used. On the other hand, we observe, as for $M = 1$, that the cost

$K(\varepsilon, T, M)$ does not blow up as $\varepsilon \to 0$, in contradiction with the theoretical results from [9, 19]. The discretization used is not fine enough here to capture the highly oscillatory behavior of the control near the controllability time T (in contrast to the positive case) and very likely leads to an uncorrect approximation of the controls. For T lower than $1/|M|$, as expected, we observe that the cost blows up, while for T strictly greater than $1/|M|$, the cost decreases to zero with ε.

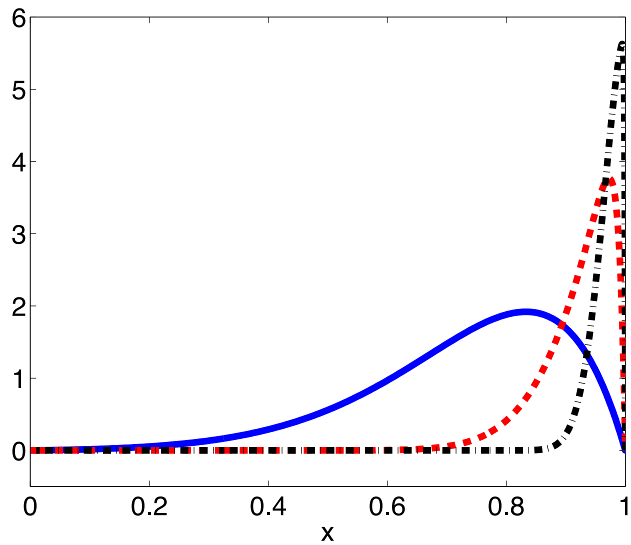

Figure 12: The optimal initial condition y_0 in $(0, L)$ for $\varepsilon = 10^{-1}$ (full line), $\varepsilon = 10^{-2}$ (dashed line) and $\varepsilon = 10^{-3}$ (dashed-dotted line) and $T = -M = L = 1$; $r = h^2$ - $h = 1/320$.

5 Concluding remarks and perspectives

We have presented a direct method to approximate the cost of control associated to the equation $y_t - \varepsilon y_{xx} + M y_x = 0$. For $M > 0$, the "worst" initial data we observe are concentrated at $x = 0$ leading to a control distributed at the beginning of the time interval, and vanishing as $t \to T$. In this case, controls v are smooth and easily approximated. Vanishing exponentially weighs as considered in [22] leading to strong convergent results (w.r.t. h) are not necessary here. Consequently, for $M > 0$, we are confident with the numerical approximation obtained and may conjecture that the minimal time of uniform controllability w.r.t. ε is $T_M = L/M$. The situation is much more singular for $M < 0$ for which the transport term acts "against" the control. The optimal initial data are now concentrated as the right extremity leading to a highly singular controls at the end of the time interval. Such controls, similar to the controls we observed for the heat equation (see [23]) are difficult to approximate. The strong convergent approximation of controls w.r.t. h is still open in such situations. Let us comment possible perspectives to improve the resolution of this singular controllability problem.

a) A way to recover a strong convergent approximation with respect to h is to force the control to vanish exponentially as time T of the form $v(t) := \varepsilon\rho^{-2}(t)\varphi_x(0, t)$, with $\rho(t) := O(e^{1/(T-t)})$. Remark that this modifies the cost of control as follows:

$$K_\rho(\varepsilon, T, M) := \sup_{\|y_0\|_{L^2(0,L)}=1} \left\{ \min_{u \in \mathcal{C}(y_0, T, \varepsilon, M)} \|\rho\, u\|_{L^2(0,T)} \right\},$$

larger than $K(\varepsilon, T, M)$ leading a priori to an upper bound $T_{M,\rho}$ of T_M. Since ρ^{-1} vanishes only

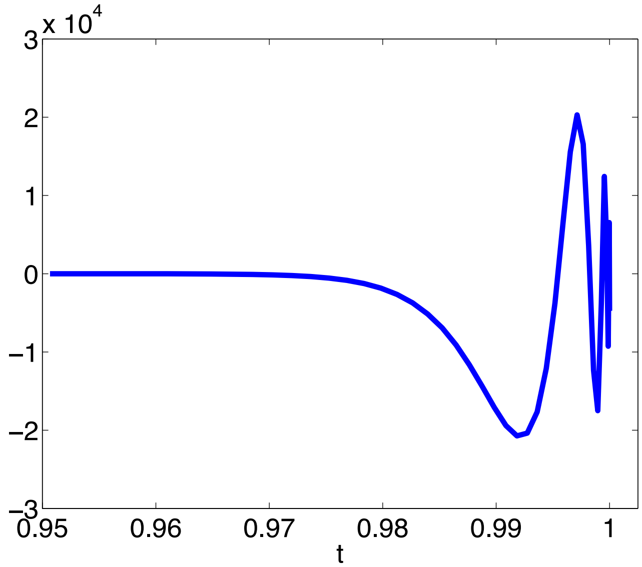

Figure 13: Approximation $\lambda_h(0,t)$ of the corresponding control w.r.t. $t \in [0,T]$ for $\varepsilon = 10^{-3}$ and $T = L = -M = 1$; $r = h^2$ - $h = 1/320$.

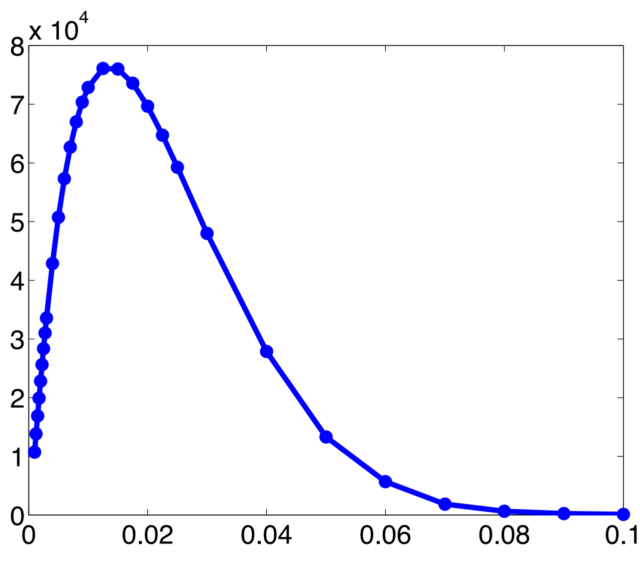

Figure 14: Cost of control $K(\varepsilon, T, M)$ w.r.t. $\varepsilon \in [10^{-3}, 10^{-1}]$ for $T = L/M$ and $L = -M = 1$; $r = h^2$ - $h = 1/320$.

at time T, we suspect that the minimal time of uniform controllability $T_{M,\rho}$ coincides with T_M.

b) Even if the introduction of weights like ρ improves the numerical stability of the mixed formulation (22), it seems quite impossible to consider values of T far from $L/|M|$: for instance, for $T = 2\sqrt{2}$ exhibited in [19] (see (4)), the norm $\|y(\cdot,T)\|_{H^{-1}(0,L)}$ is the uncontrolled situation, is for $\varepsilon = 10^{-2}$, about 3.33×10^{-17}. Consequently, when the double precision is used, we achieve "numerically" zero. Resolution of (22) would then lead to $v := 0$ on $(0,T)$! A possible way to avoid such pathologies is to preliminary consider a change of variables. We may write the solution y as follows, for any $\alpha, \gamma \in \mathbb{R}$,

$$y(x,t) = e^{\frac{M\alpha x}{2\epsilon}} e^{-\frac{\gamma M^2 t}{4\varepsilon}} z(x,t)$$

leading to

$$L_\varepsilon y := e^{\frac{M\alpha x}{2\epsilon}} e^{-\frac{\gamma M^2 t}{4\varepsilon}} \bigg(z_t - \varepsilon z_{xx} + M(1-\alpha) z_x - \frac{M^2}{4\varepsilon}(\gamma + \alpha^2 - 2\alpha) z \bigg).$$

Remark that $y(\cdot,T) = 0$ if and only if $z(\cdot,T) = 0$. Taking $1-\alpha$ small and $\frac{M^2}{4\varepsilon}(\gamma + \alpha^2 - 2\alpha) \geq 0$ allows to reduce the dissipation of the solution at time T as $\varepsilon \to 0$ and therefore avoid the zero numeric effect. For instance, for $\alpha = \gamma = 1$, z solves $z_t - \varepsilon z_{xx} = 0$. Within this change of variable, the cost of control is

$$K^2(\varepsilon, T, M) = \sup_{z_0 \in L^2(0,L)} \frac{(\mathcal{A}_\varepsilon z_0, z_0)}{(e^{\frac{M\alpha x}{\varepsilon}} z_0, z_0)}$$

where $\mathcal{A}_\varepsilon$ is the control operator defined by $\mathcal{A}_\varepsilon : z_0 \to -w(\cdot,0) \in L^2(0,L)$; here w solves the adjoint problem

$$\begin{cases} -w_t - \varepsilon w_{xx} - M(1-\alpha) w_x - \frac{M^2}{4\varepsilon}(\gamma + \alpha^2 - 2\alpha) w = 0 & \text{in} \quad Q_T, \\ w(0,\cdot) = w(L,\cdot) = 0 & \text{on} \quad (0,T), \\ w(\cdot,T) = w_T & \text{in} \quad (0,L), \end{cases}$$

with $w_T \in H_0^1(0,L)$ the minimizer of the functional

$$J^\star(w_T) := \frac{1}{2} \int_0^T \varepsilon^2 e^{\frac{\gamma M^2 t}{2\varepsilon}} w_x^2(0,t) dt + (z_0, w(\cdot,0))_{L^2(0,L)}.$$

The corresponding control of minimal $L^2(e^{-\frac{\gamma M^2 t}{4\varepsilon}})$ norm for the variable z is given by $v_{\varepsilon,z} := \varepsilon e^{\frac{\gamma M^2 t}{2\varepsilon}} w_x(\cdot,t)$. The optimality conditions for $J^\star$ lead to a mixed formulation similar to (17). The introduction of appropriate parameters α and γ allows to avoid the effect of the transport term; on the other hand, the change of variables make appear explicitly in the formulation exponential functions which may leads to numerical overflow for small values of ε.

c) Another numerical strategy, employed in [23], is to use a spectral expansion of the adjoint solution φ of (6):

$$\varphi(x,t) = e^{-\frac{Mx}{2\varepsilon}} \sum_{k>0} \alpha_k e^{-\lambda_{\varepsilon,k}(T-t)} \sin(k\pi x), \quad \lambda_{\varepsilon,k} := \varepsilon k^2 \pi^2 + \frac{M^2}{4\varepsilon}$$

with $\{\alpha_k\}_{k>0} \in L(\varepsilon, M, T)$ such that $\varphi(\cdot,0)$ is in $L^2(0,L)$, equivalently

$$L(\varepsilon, M, T) := \bigg\{ \{\alpha_p\}_{p>0} \in \mathbb{R}, \sum_{p,q \geq 0} \alpha_p \alpha_q e^{-(\lambda_{\varepsilon,k} + \lambda_{\varepsilon,p})T} \frac{32\varepsilon^3 M (p\pi)(q\pi)(1 - e^{-\frac{M}{\varepsilon}}(-1)^{p+q})}{(a_{p,q}^2 - b_{p,q}^2)}) < \infty \bigg\}$$

with $a_{p,q} := 4(M^2 + \varepsilon^2((p\pi)^2 + (q\pi)^2))$ and $b_{p,q} := 8\varepsilon^2(p\pi)(q\pi)$. The characterization (8) of the control with $v_\varepsilon = \varepsilon \varphi_x(0,\cdot)$ then rewrites as follows: find $\{\alpha_k\}_{k\geq 1} \in L(\varepsilon, M, T)$ such that

$$\varepsilon^2 \sum_{k,p\geq 1} \alpha_k \overline{\alpha}_p (k\pi)(p\pi) \frac{1-e^{-(\lambda_{\varepsilon,p}+\lambda_{\varepsilon,k})T}}{\lambda_{\varepsilon,p}+\lambda_{\varepsilon,k}} + \sum_{k\geq 1} \overline{\alpha}_k e^{-\lambda_{\varepsilon,k}T} \sum_{p\geq 1} \beta_p M_{p,k} = 0, \quad \forall \{\overline{\alpha_k}\}_{k\geq 1} \in L(\varepsilon, M, T), \tag{28}$$

with $y_0(x) := \sum_{p>0} \beta_p \sin(p\pi x)$ and $M_{p,q} := \int_0^1 e^{-\frac{Mx}{2\varepsilon}} \sin(p\pi x)\sin(q\pi x)dx$. The use of symbolic computations with large digit numbers may allow to solve (28) with robustness.

d) At last, it seems interesting to perform as well an asymptotic analysis of the system of optimality (17) with respect to ε, in the spirit of [17]. This may allow to replace the direct resolution of (17) by the resolution of a sequel of simpler optimality systems independent of ε. This analysis is investigated in [1].

Eventually, we also mention that similar methods can be used to consider the case $M = 0$ in (3) in order to examine precisely the evolution of the cost of control for the heat equation when the controllability time T goes to zero. Precisely, the change of variable $\tilde{t} := \varepsilon t$ in (1) leads to the equation $\tilde{y}_{tt} - \tilde{y}_{xx} = 0$ over $(0, L) \times (0, \varepsilon T)$. This case, easier than the case considered in this work, is still open in the literature and is numerically discussed in [10].

6 Annexe

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ε	$T = 0.95$	$T = 0.99$	$T = 1.$	$T = 1.05$
10^{-3}	237.877	30.4972	18.7555	2.2915
1.25×10^{-3}	190.574	29.7622	19.1953	2.8028
1.5×10^{-3}	159.813	29.0015	19.3883	3.2556
1.75×10^{-3}	138.166	28.2446	19.4234	3.6529
2×10^{-3}	122.044	27.4997	19.3540	4.0005
2.25×10^{-3}	109.519	26.7745	19.2093	4.3013
2.5×10^{-3}	99.476	26.0722	19.0163	4.5623
3×10^{-3}	84.250	24.7318	18.5275	4.9814
4×10^{-3}	64.648	22.3060	17.3600	5.5078
5×10^{-3}	52.289	20.1837	16.1269	5.7530
6×10^{-3}	43.650	18.3289	14.9392	5.8259
7×10^{-3}	37.213	16.6883	13.8166	5.7787
8×10^{-3}	32.198	15.2461	12.7839	5.6683
9×10^{-3}	28.210	13.9660	11.8380	5.5099
10^{-2}	24.934	12.8331	10.9763	5.3276
1.25×10^{-2}	18.898	10.5015	9.1493	4.8282
1.5×10^{-2}	14.810	8.7281	7.7087	4.3378
1.75×10^{-2}	11.913	7.3526	6.5694	3.8897
2×10^{-2}	9.784	6.2780	5.6566	3.4943
2.25×10^{-2}	8.176	5.4196	4.9210	3.1506
2.5×10^{-2}	6.937	4.7293	4.3237	2.8534
3×10^{-2}	5.180	3.7047	3.4240	2.3744
4×10^{-2}	3.264	2.4895	2.3297	1.7350
5×10^{-2}	2.294	1.8261	1.7304	1.3416
6×10^{-2}	1.736	1.4209	1.3522	1.0848
7×10^{-2}	1.376	1.1510	1.1030	0.8978
8×10^{-2}	1.113	0.9596	0.9223	0.7612
9×10^{-2}	0.0952	0.8130	0.7865	0.6554
10^{-1}	0.8175	0.7075	0.6808	0.5711

Table 14: Cost of control $K(\varepsilon, T, M)$ for $L = M = 1$ with respect to T and ε; - $h = 1/320$ - $r = h^2$ - $\beta = 10^{-16}$.

ε	$T = 1.$
10^{-3}	10718.0955936799
1.25×10^{-3}	13839.4039394749
1.5×10^{-3}	16903.9918205099
1.75×10^{-3}	19898.1360771887
2×10^{-3}	22812.2634798022
2.25×10^{-3}	25638.7601386909
2.5×10^{-3}	28375.3693789053
2.75×10^{-3}	31021.5479842987
3×10^{-3}	33575.948263826
4×10^{-3}	42871.1424334121
5×10^{-3}	50751.4443114544
6×10^{-3}	57316.7716579456
7×10^{-3}	62692.7273334616
8×10^{-3}	66997.3602057935
9×10^{-3}	70350.3966144308
10^{-2}	72862.0738060569
1.25×10^{-2}	76089.8839137614
1.5×10^{-2}	75988.4041456468
1.75×10^{-2}	73579.1022138189
2×10^{-2}	69647.3042543371
2.25×10^{-2}	64735.7778969391
2.5×10^{-2}	59254.0430977822
3×10^{-2}	47994.1519570731
4×10^{-2}	27872.8642664892
5×10^{-2}	13312.4452504554
6×10^{-2}	5687.69600914237
7×10^{-2}	1864.72524997867
8×10^{-2}	648.702980070232
9×10^{-2}	264.559407164062
10^{-1}	123.306947646919

Table 15: Cost of control $K(\varepsilon, T, M)$ for $L = -M = 1$ with respect to T and ε; $h = 1/320$ - $r = h^2$ - $\beta = 10^{-16}$.

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