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Michel Duprez, Morgan Morancey, Francesco Rossi

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***Abstract*— In this work, we study the minimal time to steer a crowd to a desired configuration. The control is a vector field, representing a perturbation of the crowd displacement, localized on a fixed control set. We give a characterization of the minimal time both for discrete and continuous crowds.**

I. INTRODUCTION AND MAIN RESULTS

In recent years, the study of systems describing a crowd of interacting autonomous agents has drawn a great interest from the control community. A better understanding of such interaction phenomena can have a strong impact in several key applications, such as road traffic and egress problems for pedestrians. For a few reviews about this topic, see *e.g.* [1]–[8]. Beside the description of interactions, it is now relevant to study problems of control of crowds, *i.e.* of controlling such systems by acting on few agents, or on a small subset of the configuration space.

The nature of the control problem relies on the model used to describe the crowd. Two main classes are widely used.

In discrete models, the position of each agent is clearly identified; the crowd dynamics is described by a large dimensional ordinary differential equation, in which couplings of terms represent interactions. For control of such models, a large literature is available, see *e.g.* reviews [9]–[11], as well as applications, both to pedestrian crowds [12], [13] and to road traffic [14], [15].

In continuous models, instead, the idea is to represent the crowd by the spatial density of agents; in this setting, the evolution of the density solves a partial differential equation of transport type. Nonlocal terms (such as convolutions) model the interactions between the agents. To our knowledge, there exist few studies of control of this family of equations. In [16], the authors provide approximate alignment of a crowd described by the continuous Cucker-Smale model [17]. In a similar situation, a stabilization strategy has been established in [18], [19], by generalizing the Jurdjevic-Quinn method to partial differential equations.

In this article, we first study a discrete model, where the crowd is represented by a vector with nd components ($n, d \in \mathbb{N}^*$) representing the positions of n agents in $\mathbb{R}^d$. The natural (uncontrolled) vector field is denoted by $v : \mathbb{R}^d \to \mathbb{R}^d$, assumed Lipschitz and uniformly bounded. We act on the vector field in a fixed portion ω of the space, which will be a nonempty open convex subset of $\mathbb{R}^d$. The admissible controls are thus functions of the form $\mathbb{1}_\omega u(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$. We then consider the following discrete model

$$\begin{cases} \dot{x}_i(t) = (v + \mathbb{1}_\omega u)(x_i(t), t) \text{ for a.e. } t \geqslant 0, \\ x_i(0) = x_i^0 \end{cases} \tag{1}$$

for $i \in \{1, ..., n\}$, where $X^0 := \{x_1^0, ..., x_n^0\}$ is the initial configuration of the crowd.

We also study a continuous model, where the crowd is represented by its density, that is a time-evolving measure $\mu(t)$ defined on the space $\mathbb{R}^d$. We consider the same natural vector field v, control region ω and admissible controls $\mathbb{1}_\omega u$. We then study the following continuous model

$$\begin{cases} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u)\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \tag{2}$$

where μ^0 is the initial density of the crowd. The function $v + \mathbb{1}_\omega u$ represents the vector field acting on μ.

Systems (1) and (2) are first approximations for crowd models, since the uncontrolled vector field v is given, and it does not describe interactions between agents. Nevertheless, it is necessary to understand control properties for such simple equations as a first step, before dealing with vector fields depending on the crowd itself. In a future work, we will study control problems for crowd models with a nonlocal term $v[\mu]$, based on the results for systems presented here.

To a discrete configuration $\{x_1, ..., x_n\}$, we can associate the empirical measures

$$\mu := \sum_{i=1}^n \frac{1}{n} \delta_{x_i}.$$

With this notation, System (1) is a particular case of System (2). This identification will be used several times in the following, namely to approximate continuous crowds with discrete ones.

We now recall the notion of approximate and exact controllability for Systems (1) and (2). We say that they are *approximately controllable* from the initial configuration from μ^0 to the final one μ^1 on the time interval $[0, T]$ if we can steer the solution from μ^0 at time 0 to a configuration at time T as close to the final configuration as we want with a suitable control $\mathbb{1}_\omega u$. Similarly, *exact controllability* means that we can steer the solution from μ^0 at time 0 exactly to μ^1 at time T. In Definition 5 below, we give a

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formal definition of the notion of approximate controllability in terms of Wasserstein distance.

In all this paper, we assume that the following geometric condition is satisfied:

CONDITION *1.1 (Geometric condition):* Let μ^0, μ^1 be two probability measures on $\mathbb{R}^d$ satisfying:

(i) For each $x^0 \in \text{supp}(\mu^0)$, there exists $t^0 > 0$ such that $\Phi^v_{t^0}(x^0) \in \omega$, where Φ^v_t is the *flow* associated to v (see Definition 3 below).

(ii) For each $x^1 \in \text{supp}(\mu^1)$, there exists $t^1 > 0$ such that $\Phi^v_{-t^1}(x^1) \in \omega$.

Condition 1.1 means that particle crosses the control region. It is the minimal condition that we can expect to steer any initial condition to any target. Indeed, if Item (i) of Condition 1.1 is not satisfied, then there exists a whole subpopulation of the measure μ^0 or μ^1 that never intersects the control region, thus we cannot act on it.

We have proved in [20] that if we consider μ^0, μ^1 two probability measures on $\mathbb{R}^d$ compactly supported, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1.1, then there exists T such that System (2) is **approximately controllable** at time T from μ^0 to μ^1 with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time.

For arbitrary continuous measures, one can expect approximate controllability only, since for general measures there exists no homeomorphism sending one to another. Indeed, if we impose the classical Carathéodory condition of $\mathbb{1}_\omega u$ being Lipschitz in space, measurable in time and uniformly bounded, then the flow $\Phi^{v+\mathbb{1}_\omega u}_t$ is an homeomorphism (see [21, Th. 2.1.1]). Similarly, in the discrete case, such control vector field u cannot separate points, due to uniqueness of the solution of (1). We then assume that both the configuration X^0 and X^1 are disjoint, in the following sense.

DEFINITION *1:* A configuration $X = \{x_1, ..., x_n\}$ is said to be disjoint if $x_i \neq x_j$ for all $i \neq j$.

Consider the quantity

$$T^* := \sup\{t^i(x) \text{ s.t. } x \in \text{supp}(\mu^i) \text{ and } i = 0, 1\},$$

where, for all $x \in \mathbb{R}^d$,

$$\begin{cases} t^0(x) := \inf\{t \in \mathbb{R}^+ : \Phi^v_t(x) \in \omega\}, \\ t^1(x) := \inf\{t \in \mathbb{R}^+ : \Phi^v_{-t}(x) \in \omega\}. \end{cases}$$

In this article, we aim to study the minimal time problem. We denote by T_a the minimal time to approximately steer the initial configuration μ^0 to a final one μ^1 in the following sense: it is the infimum of times $T > T^*$ for which there exists a control with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time steering μ^0 arbitrarily close to μ^1. We similarly define the minimal time T_e to exactly steer the initial configuration μ^0 to a final one μ^1. Since the minimal time is not always reached, we will speak about *infimum time*.

A. Infimum time for discrete crowds

We denote by $t^0_i := t^0(x^0_i)$ and $t^1_i := t^1(x^1_i)$, for $i \in \{1, ..., n\}$. We now state our first main result.

THEOREM *1.1: (Main result - discrete crowd)* Let $X^0 := \{x^0_1, ..., x^0_n\}$ and $X^1 := \{x^1_1, ..., x^1_n\}$ be disjoint configurations, satisfying Condition 1.1. Arrange the sequences $\{t^0_i\}_i$ and $\{t^1_j\}_j$ to be increasingly and decreasingly ordered, respectively. Then the infimum time for exact control of System (1) satisfies

$$T_e = M(X^0, X^1) := \max_{i \in \{1,...,n\}} |t^0_i + t^1_i|. \tag{3}$$

We give a proof of Theorem 1.1 in Section III. We only consider the case $T > T^*$, that is when all particles of X^0 has entered in ω, so we can act on them (idem for X^1). When $T \in (0, T^*)$ or $T = M$, System (1) can be controllable in some specific cases only (see [22]).

B. Infimum time for continuous crowds

Introduce the maps $\mathcal{F}_0$ and $\mathcal{F}_1$ defined for all $t \geqslant 0$ by

$$\begin{cases} \mathcal{F}_0(t) := \mu^0(\{x^0 \in \text{supp}(\mu^0) : t^0(x^0) \leqslant t\}), \\ \mathcal{F}_1(t) := \mu^1(\{x^1 \in \text{supp}(\mu^1) : t^1(x^1) \leqslant t\}). \end{cases}$$

The function $\mathcal{F}_0$ (resp. $\mathcal{F}_1$) gives the quantity of mass coming from μ^0 (resp. the quantity of mass coming from μ^1 backward in time) which has entered in ω at time t. Observe that we do not decrease $\mathcal{F}_0$ when the mass eventually leaves ω, and similarly for $\mathcal{F}_1$. Define the generalized inverse functions $\mathcal{F}_0^{-1}$ and $\mathcal{F}_1^{-1}$ by

$$\begin{cases} \mathcal{F}_0^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_0(t) \geqslant m\}, \\ \mathcal{F}_1^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_1(t) \geqslant m\}. \end{cases} \tag{4}$$

The function $\mathcal{F}_0^{-1}$ is increasing, lower semi-continuous and gives the time at which a mass m has entered in ω, and similarly for $\mathcal{F}_1^{-1}$. We then have the following main result about infimum time in the continuous case:

THEOREM *1.2 (Main result - continuous crowd):* Let μ^0 and μ^1 be two probability measures, with compact support, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1.1. Then the infimum time T_a to approximately steer μ^0 to μ^1 is equal to

$$S(\mu^0, \mu^1) := \sup_{m \in [0,1]} \{\mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1 - m)\}. \tag{5}$$

We give a proof of Theorem 1.2 in Section IV. We observe that S in (5) is the continuous equivalent of M in (3). As in the discrete case, for the same reason, we only consider the case $T > T^*$ and $T \neq S$ (see [22, Rem. 5 and 6]).

This paper is organised as follow. In Section II, we recall basic properties of the Wasserstein distance, ordinary differential equations and continuity equations. We prove our main results Theorem 1.1 in Section III and Theorem 1.2 in Section IV.

II. THE WASSERSTEIN DISTANCE

In this section, we recall some properties of the Wasserstein distance and its connections with dynamics (1) and (2). We denote by $\mathcal{P}_c(\mathbb{R}^d)$ the space of probability measures in $\mathbb{R}^d$ with compact support.

DEFINITION *2:* For $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$, we denote by $\Pi(\mu, \nu)$ the set of *transference plans* from μ to ν, *i.e.* the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ and second marginal ν. Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$. Define

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p d\pi \right)^{1/p}, \qquad (6)$$

$$W_\infty(\mu, \nu) := \inf\{\pi - \text{esssup}|x-y| : \pi \in \Pi(\mu,\nu)\}. \qquad (7)$$

This is the idea of *optimal transportation*, consisting in finding the optimal way to transport mass from a given measure to another. For a thorough introduction, see *e.g.* [23]. These distances satisfy some useful properties.

PROPERTY *2.1 (see [23, Chap. 7] and [24]):* For all $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$, the infima in (6) or (7) are achieved by at least one minimizer $\pi \in \Pi(\mu, \nu)$.

For $p \in [1, \infty]$, W_p is a distance on $\mathcal{P}_c(\mathbb{R}^d)$, called the **Wasserstein distance**. Moreover, for $p \in [1, \infty)$, the topology induced by the Wasserstein distance W_p on $\mathcal{P}_c(\mathbb{R}^d)$ coincides with the weak topology.

The Wasserstein distance with $p \in [1, +\infty)$ can be extended to all pairs of measures μ, ν compactly supported with the same mass $|\mu| := \mu(\mathbb{R}^d) = |\nu| \neq 0$, by the formula

$$W_p(\mu, \nu) = |\mu|^{1/p} W_p(\mu/|\mu|, \nu/|\nu|).$$

In the rest of the paper, the following properties of the Wasserstein distance will be helpful.

PROPERTY *2.2 (see [23], [25]):* Let μ, ρ, ν, η be four positive measures compactly supported satisfying $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ and $\rho(\mathbb{R}^d) = \eta(\mathbb{R}^d)$. For each $p \in [1, \infty]$, it holds

$$W_p^p(\mu + \rho, \nu + \eta) \leqslant W_p^p(\mu, \nu) + W_p^p(\rho, \eta).$$

Consider the Cauchy problem

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \qquad (8)$$

where $w : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$. This equation is called the **continuity equation**. We now introduce the flow associated to System (8).

DEFINITION *3:* We define the **flow** associated to the vector field w as the application $(x^0, t) \mapsto \Phi_t^w(x^0)$ such that, for all $x^0 \in \mathbb{R}^d$, $t \mapsto \Phi_t^w(x^0)$ is the solution to

$$\dot{x}(t) = w(x(t), t) \text{ for a.e. } t \geqslant 0, \qquad x(0) = x^0. \qquad (9)$$

We denote by Γ the set of the Borel maps $\gamma : \mathbb{R}^d \to \mathbb{R}^d$. We first recall the definition of the *push-forward* of a measure and of the Wasserstein distance.

DEFINITION *4:* For a $\gamma \in \Gamma$, we define the *push-forward* $\gamma\#\mu$ of a measure μ of $\mathbb{R}^d$ as follows:

$$(\gamma\#\mu)(E) := \mu(\gamma^{-1}(E)),$$

for every subset E such that $\gamma^{-1}(E)$ is μ-measurable.

PROPERTY *2.3 (see [25]):* Let μ, $\nu \in \mathcal{P}_c(\mathbb{R}^d)$ and $w : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be a vector field uniformly bounded, Lipschitz in space and measurable in time with a Lipschitz constant equal to L. For each $t \in \mathbb{R}$ and $p \in [1, \infty)$, it holds

$$W_p(\Phi_t^w \# \mu, \Phi_t^w \# \nu) \leqslant e^{\frac{(p+1)}{p} L|t|} W_p(\mu, \nu). \qquad (10)$$

We denote by "AC measures" the measures which are absolutely continuous with respect to the Lebesgue measure and by $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ the subset of $\mathcal{P}_c(\mathbb{R}^d)$ of AC measures. We now recall a standard result linking (8) and (9), known as the method of characteristics.

THEOREM *2.1 (see [23, Th. 5.34]):* Let $T > 0$, $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ and w a vector field uniformly bounded, Lipschitz in space and measurable in time. Then, System (8) admits a unique solution μ in $\mathcal{C}^0([0,T]; \mathcal{P}_c(\mathbb{R}^d))$, where $\mathcal{P}_c(\mathbb{R}^d)$ is equipped with the weak topology. Moreover:

(i) it holds $\mu(\cdot, t) = \Phi_t^w \# \mu^0$;
(ii) if $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$, then $\mu(\cdot, t) \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$.

We now give the precise notions of approximate controllability for System (2) in terms of the Wasserstein distance.

DEFINITION *5:* We say that System (2) is **approximately controllable** from μ^0 to μ^1 on the time interval $(0,T)$ if for each $\varepsilon > 0$ there exists a control $\mathbb{1}_\omega u$ such that the corresponding solution μ to System (2) satisfies

$$W_p(\mu^1, \mu(T)) \leqslant \varepsilon.$$

All the Wasserstein distances W_p are equivalent for $p \in [1, \infty)$, see [23]. Thus, from now on we study approximate controllability with the Wasserstein distance W_1.

III. INFIMUM TIME IN THE DISCRETE CASE

In this section, we prove Theorem 1.1, *i.e.* the infimum time in the discrete case. We first obtain the following result:

PROPOSITION *3.1:* Let $X^0 := \{x_1^0, ..., x_n^0\} \subset \mathbb{R}^d$ and $X^1 := \{x_1^1, ..., x_n^1\} \subset \mathbb{R}^d$ be disjoint, satisfying Condition 1.1. Then the infimum time T_e for exact control of (1) is

$$T_e = \widetilde{M}(X^0, X^1) := \min_{\sigma \in S_n} \max_{i \in \{1,...,n\}} |t_i^0 + t_{\sigma(i)}^1|. \qquad (11)$$

Proof: Let $T := \widetilde{M}(X^0, X^1) + \delta$ with $\delta > 0$. Consider the sequences $\{t_i^0\}_{i \in \{1,...,n\}}$ and $\{t_i^1\}_{i \in \{1,...,n\}}$ given at the beginning of Section I-A. For all $i \in \{1, ..., n\}$, there exist $s_i^0 \in (t_i^0, t_i^0 + \delta/3)$ and $s_i^1 \in (t_i^1, t_i^1 + \delta/3)$ such that

$$y_i^0 := \Phi_{s_i^0}^v(x_i^0) \in \omega \text{ and } y_i^1 := \Phi_{-s_i^1}^v(x_i^1) \in \omega.$$

For all $i, j \in \{1, ..., n\}$, we define the cost

$$K_{ij} := \begin{cases} \|(y_i^0, s_i^0) - (y_j^1, T - s_j^1)\|_{\mathbb{R}^{d+1}} & \text{if } s_i^0 < T - s_j^1, \\ \infty & \text{otherwise.} \end{cases}$$

Consider the minimization problem:

$$\inf_{\pi \in \mathcal{B}_n} \frac{1}{n} \sum_{i,j=1}^{n} K_{ij} \pi_{ij}, \qquad (12)$$

where $\mathcal{B}_n$ is the set of the bistochastic $n \times n$ matrices, *i.e.* the matrices $\pi := (\pi_{ij})_{1 \leqslant i,j \leqslant n}$ satisfying, for all $i, j \in \{1, ..., n\}$, $\sum_{i=1}^n \pi_{ij} = 1$, $\sum_{j=1}^n \pi_{ij} = 1$, $\pi_{ij} \geqslant 0$. Using the definition of $\widetilde{M}(X^0, X^1)$, the infimum in (12) is attained. It is a linear minimization problem on the closed convex set $\mathcal{B}_n$. Hence, as a consequence of Krein-Milman's Theorem (see [26]), the functional (12) admits a minimum at a extremal point, *i.e.* a permutation matrix. Let σ be a permutation, for which the associated matrix minimizes (12). Consider the linear trajectory $y_i(t)$ steering y_i^0 at time s_i^0

to $y^1_{\sigma(i)}$ at time $T - s^1_{\sigma(i)}$. We now prove by contradiction that these trajectories have no intersection: Assume that there are i and j such that the associated trajectories $y_i(t)$ and $y_j(t)$ intersect. If we steer x^0_i to $x^0_{\sigma(j)}$ and x^0_j to $x^0_{\sigma(i)}$, *i.e.* we consider the permutation $\mathcal{T}_{i,j} \circ \sigma$, where $\mathcal{T}_{i,j}$ is the transposition between the i^{th} and the j^{th} elements, then the associated cost (12) is strictly smaller than the cost associated to σ. This is in contradiction with the fact that σ minimizes (12). We conclude taking a control u around trajectories of class $\mathcal{C}^\infty$ steering X^0 to X^1.

Assume now that System (1) is exact controllable at a time $T > T^*$. Consider σ the permutation satisfying $x_i(T) = x^1_{\sigma(i)}$. Then, using the definition of $\widetilde{M}_e(X^0, X^1)$, it holds $T > \widetilde{M}_e(X^0, X^1)$. ■

Formula (11) leads to the proof of Theorem 1.1.

Proof of Theorem 1.1: Consider $\widetilde{M}(X^0, X^1)$ given in (11). We assume that the sequence $\{t^0_i\}_{i\in\{1,...,n\}}$ is increasingly ordered. Let σ_0 be a minimizing permutation in (11), and k_1 such that $t^1_{\sigma_0(k_1)}$ is a maximiser of $\{t^1_{\sigma_0(1)}, ..., t^1_{\sigma_0(n)}\}$. Since $t^0_1 \le t^0_{k_1}$ and $t^1_{\sigma_0(1)} \le t^1_{\sigma_0(k_1)}$, it holds

$$\max\{t^0_1 + t^1_{\sigma_0(1)}, t^0_1 + t^1_{\sigma_0(k_1)}, t^0_{k_1} + t^1_{\sigma_0(1)}\} \leqslant t^0_{k_1} + t^1_{\sigma_0(k_1)}.$$

We denote by $\sigma_1 := \mathcal{T}_{1,k_1} \circ \sigma_0$; it minimizes (11) too. We build recursively the sequence of permutations $\sigma_{i+1} = T_{i+1,k_{i+1}} \circ \sigma_i$, where k_i is a maximizer of $\{t^1_{\sigma_i(i+1)}, ..., t^1_{\sigma_i(n)}\}$. The sequence $\{t^1_{\sigma_n(1)}, ..., t^1_{\sigma_n(n)}\}$ is then decreasing and σ_n is a minimizing permutation in (11). We deduce that $\widetilde{M}(X^0, X^1) = M(X^0, X^1)$. ■

IV. Infimum time for AC measures

In this section, we prove main Theorem 1.2 about infimum time for AC measures. We first introduce the auxiliary Corollary 4.1 and Proposition 4.1, that are its natural counterparts for discrete measures. We then prove the main theorem by discrete approximation.

Let $M > 0$ be a positive mass, not necessarily 1, and μ^0, μ^1 be two disjoint measures given by

$$\mu^0 := \textstyle\sum_{i=1}^n \frac{M}{n}\delta_{x^0_i} \text{ and } \mu^1 := \sum_{i=1}^n \frac{M}{n}\delta_{x^1_i}. \qquad (13)$$

It is possible to compute the infimum time to steer μ^0 to μ^1 up to a small mass.

Definition 6 ***(Infimum time up to small mass):*** Let $X^0 := \{x^0_1, ..., x^0_n\} \subset \mathbb{R}^d$ and $X^1 := \{x^1_1, ..., x^1_n\} \subset \mathbb{R}^d$ be disjoint, and satisfying Condition 1.1. Let $M > 0$ and the corresponding measures μ^0 and μ^1 defined in (13). Fix $R \in \{1, ..., n\}$ and $\varepsilon := MR/n$. We define the infimum time $T_{e,\varepsilon}$ to exactly steer μ^0 to μ^1 (or X^0 to X^1) up to a mass ε (or R particles) as the infimum of time $T \geqslant T^*$ for which there exists a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ uniformly bounded, Lipschitz in space and measurable in time and $\sigma_0, \sigma_1 \in S_n$ such that for all $i \in \{1, ..., n - R\}$ it holds $x_{\sigma_0(i)}(T) = x^1_{\sigma_1(i)}$.

We use the definition of $\mathcal{F}_0$, $\mathcal{F}_1$, t^0_i and t^1_i, together with applying Theorem 1.1 to suitable subsets of X^0, X^1, to have the following result.

Corollary *4.1:* Let $X^0 := \{x^0_1, ..., x^0_n\} \subset \mathbb{R}^d$ and $X^1 := \{x^1_1, ..., x^1_n\} \subset \mathbb{R}^d$ be disjoint, satisfying Condition 1.1, and μ^0, μ^1 the corresponding measures defined by (13). Fix $\varepsilon := MR/n$ with $R \in \{1, ..., n\}$. The infimum time $T_{e,\varepsilon}$ to exactly steer μ^0 to μ^1 up to a mass ε is equal to

$$S_\varepsilon(\mu^0, \mu^1) := \sup_{m\in[0,1-\varepsilon]} \{\mathcal{F}_0^{-1}(m) + \mathcal{F}_1^{-1}(1-\varepsilon-m)\}, \quad (14)$$

where $\mathcal{F}_0^{-1}$ and $\mathcal{F}_1^{-1}$ are given in (4).

Proof: Remark that if the sequences $\{t^0_i\}_{i\in\{1,...n\}}$ and $\{t^1_i\}_{i\in\{1,...n\}}$ are increasingly and decreasingly ordered respectively, then for $m \in \left(\frac{i-1}{n}, \frac{i}{n}\right)$ it holds $\mathcal{F}_0^{-1}(m) = t^0_i$ and $\mathcal{F}_1^{-1}(1 - m) = t^1_i$. ■

Proposition *4.1:* Consider $\mu^0, \mu^1 \in \mathcal{P}^{ac}_c(\mathbb{R}^d)$ satisfying Condition 1.1, sequences $\{\mu^0_n\}_{n\in\mathbb{N}}$, $\{\mu^1_n\}_{n\in\mathbb{N}}$ of measures compactly supported satisfying Condition 1.1 and two sequences of sets $\{R^0_n\}_{n\in\mathbb{N}}$, $\{R^1_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ such that

$$\begin{cases} r_n := \mu^0(R^0_n) = \mu^1(R^1_n) \underset{n\to\infty}{\longrightarrow} 0, \\ \mu^0_n(\mathbb{R}^d) = \mu^1_n(\mathbb{R}^d) = 1 - r_n, \\ d^0_n := W_\infty(\mu^0_{|(R^0_n)^c}, \mu^0_n) \underset{n\to\infty}{\longrightarrow} 0, \\ d^1_n := W_\infty(\mu^1_{|(R^1_n)^c}, \mu^1_n) \underset{n\to\infty}{\longrightarrow} 0. \end{cases}$$

Consider the quantity S_ε given in (14). Then for all $\varepsilon, \delta > 0$, there exists $N \in \mathbb{N}^*$ such that for all $n \geq N$, it holds

(i) $S_{2\varepsilon}(\mu^0_n, \mu^1_n) \leqslant S_\varepsilon(\mu^0, \mu^1) + \delta$.

(ii) $S_{2\varepsilon}(\mu^0, \mu^1) \leqslant S_\varepsilon(\mu^0_n, \mu^1_n) + \delta$.

Proof: There exists $r^0_n \underset{n\to\infty}{\longrightarrow} 0$ and $r^1_n \underset{n\to\infty}{\longrightarrow} 0$ such that

$$\begin{aligned} & S_{2\varepsilon}(\mu^0_n, \mu^1_n) = \\ & \quad \sup_{m\in[0,1-r_n-2\varepsilon]} \{\mathcal{F}_{0,n}^{-1}(m) + \mathcal{F}_{1,n}^{-1}(1 - r_n - 2\varepsilon - m)\} \\ \leqslant & \quad \sup_{m\in[0,1-r_n-2\varepsilon]} \{\mathcal{F}_0^{-1}(m + r^0_n + r_n) \\ & \qquad\qquad + \mathcal{F}_1^{-1}(1 + r^1_n - 2\varepsilon - m)\} + \delta \\ \leqslant & \quad \sup_{m\in[r^0_n+r_n,1+r^0_n-2\varepsilon]} \{\mathcal{F}_0^{-1}(m) \\ & \qquad + \mathcal{F}_1^{-1}(1 + r^1_n + r^0_n + r_n - 2\varepsilon - m)\} + \delta. \end{aligned}$$

Thus, taking n large enough such that $r^1_n + r^0_n + r_n \leqslant \varepsilon$, we deduce Item (i) by using the fact that $\mathcal{F}_1^{-1}(m_1) \leqslant \mathcal{F}_1^{-1}(m_2)$, for all $m_1 \leqslant m_2$.

We similarly prove Item (ii). ■

We now prove Theorem 1.2.

Proof of Theorem 1.2: We first prove Item (i). Fix $\varepsilon, s > 0$. We prove that we can steer μ^0 to a W_1-neighbourhood of μ^1 of size ε at time $T := S(\mu^0, \mu^1) + s$. We assume that $d := 2$, but the reader will see that the proof can be clearly adapted to any space dimension. The proof is divided into four steps.

Step 1: We first discretize uniformly in space the supports of μ^0 and μ^1. To simplify the presentation, assume $\text{supp}(\mu^0) \subset (0, 1)^2$ and $\text{supp}(\mu^1) \subset (0, 1)^2$. Consider the sequence of uniform meshes $\mathcal{T}_n := \cup_{k\in\{0,...,n-1\}^2} S_{n,k}$ with

$$S_{n,k} := [k_1/n, (k_1+1)/n) \times [k_2/n, (k_2+1)/n), k := (k_1, k_2).$$

Define

$$\begin{aligned} S^0_n := \{ & x \in \text{supp}(\mu^0) : \exists k \in \{1, ..., n\}^2 \\ & \text{and } t^*(x) \in (t^0(x), t^0(x) + s/8) \\ & \qquad \text{s.t. } x \in S_{n,k} \text{ and } \Phi^v_{t^*(x)}(S_{n,k}) \subset\subset \omega\}. \end{aligned}$$

We similarly construct S_n^1. Condition 1.1 implies for $l = 0, 1$

$$\mu^l((S_n^l)^c) \underset{n\to\infty}{\longrightarrow} 0.$$

Without loss of generality, we assume $\mu^0(S_n^0) = \mu^1(S_n^1)$. We now control only $\mu^0{|_{S_n^0}}$, that will be sent close to $\mu^1{|_{S_n^1}}$.

Step 2: To send a measure to another, these measures need to have the same total mass. Thus, for each $n \in \mathbb{N}^*$ and $k \in \{1, ..., n\}^2$ such that $\mu^0(S_{n,k} \cap S_n^0) > 1/n^4$, we discretize measures $\mu^0_{|S_{n,k}\cap S_n^0}$ and $\mu^1_{|S_{n,k}\cap S_n^1}$ with some measures with the same total mass $1/n^4$. As illustrated in Figure 1, we partition $S_{n,k} \cap S_n^0$ into some subsets $\{A^0_{ki}\}_i$ with $A^0_{ki} = [a_i^0, a_{i+1}^0) \times (0, 1)$ such that $\mu^0_{|S_{n,k}\cap S_n^0}(A^0_{ki}) = 1/n^2$ and for all i we partition A^0_{ki} into some subsets $\{A^0_{kij}\}_{ij}$ with $A^0_{kij} = [a_i^0, a_{i+1}^0) \times [a^0_{ij}, a^0_{i(j+1)})$ such that

$$\mu^0_{|S_{n,k}\cap S_n^0}(A^0_{kij}) = 1/n^4.$$

For more details on such discretization, we refer to [20].

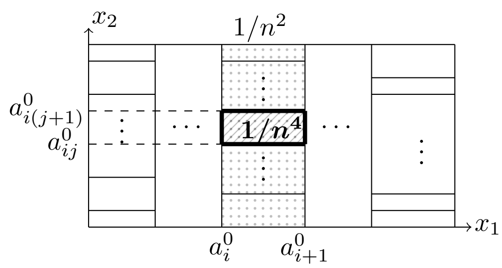

Fig. 1. Example of a partition of $S_{n,k}$ with a cell A^0_{kij} (hashed).

If $\mu^0(S_{n,k} \cap S_n^0)$ is not a multiple of $1/n^4$, it remains a small mass (smaller than $1/n^2$) that we do not control. We discretize similarly the measure μ^1 on some sets A^1_{kij}. As in Figure 2, we then build $B^0_{kij} := [b_i^0, b_{i+1}^0) \times [b^0_{ij}, b^0_{i(j+1)}) \subset\subset A^0_{kij}$ and $B^1_{kij} := [b_i^1, b_{i+1}^1) \times [b^1_{ij}, b^1_{i(j+1)}) \subset\subset A^1_{kij}$ such that $\mu^0(B^0_{kij}) = \mu^1(B^1_{kij}) = (n^2 - 2)^2/n^8$.

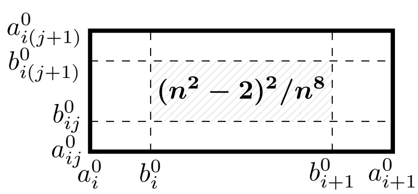

Fig. 2. Example of cells B^0_{kij} (hashed).

Step 3: In this step, we send the mass of μ^0 from each $B^0_{kij} \cap S_n^0$ to each $B^1_{k'i'j'} \cap S_n^1$, while we do not control the rest of the mass outside $B^0_{kij} \cap S_n^0$.

We first explain why this rest is negligible. Consider

$$I_n^0 := \{(k, i, j) : \mu^0(B^0_{kij} \cap S_n^0) > 1/n^4\}.$$

We define similarly I_n^1. Without loss of generality, we can assume that $|I_n^0| = |I_n^1|$. Indeed, for example in the case $n_0 := |I_n^0| - |I_n^1| > 0$, we remove the n_0 last cells in the set of indices I_n^0, the total corresponding removed mass is smaller that $1/n^2$, then negligible when $n \to \infty$. We define for $l = 0, 1$ the sets

$$R_n^l := \mathbb{R}^d \setminus \bigcup_{kij\in I_n^l} (B^l_{kij} \cap S_n^l).$$

We remark that, for $l = 0, 1$, it holds

$$\begin{array}{rcl} \mu^l(R_n^l) & \leqslant & 1 - \frac{(n^2-2)^2}{n^4} + \frac{2}{n^2} + \mu^0((S_n^0)^c) \\ & = & \frac{6n^2-4}{n^4} + \mu^0((S_n^0)^c) \underset{n\to\infty}{\longrightarrow} 0. \end{array}$$

We now approximate the measures μ^l restricted to $(R_n^l)^c$ ($l = 0, 1$) by a sum of Dirac masses μ_n^l defined by

$$\mu_n^l := \sum_{kij\in I_n^l} \frac{(n^2-2)^2}{n^8} \delta_{x^l_{kij}},$$

where the points x^l_{kij} will be chosen later, to obtain suitable properties of the measures μ_n^l. Using the definition of S_n^0, for all $n \in \mathbb{N}^*$ and $k \in \{1, ..., n\}^2$ satisfying $\mu^0(S_{n,k} \cap S_n^0) \neq 0$, there exists a square $Q^0_{n,k} \subset S_{n,k} \cap S_n^0$ and a time $t^0_{n,k}$ such that for all $x \in Q^0_{n,k}$ there exists

$$t^0_{n,k}(x) \in (t^0(x), t^0(x) + s/8)$$

for which $\Phi^v_{t^0_{n,k}(x)}(S_{n,k}) \subset\subset \omega$. We define x^0_{kij} as the homothetic transformation of the center of B^0_{kij} from $S_{n,k}$ to $Q^0_{n,k}$. In particular,

$$t^0_{n,k} \in (t^0(x^0_{kij}), t^0(x^0_{kij}) + s/8). \tag{15}$$

It is important that points x^0_{kij} satisfy (15). Indeed, in Theorem 1.1 for the discrete case we act on the particles at time $t^0(x^0_{kij})$, whereas we want to act on them only at time $t^0_{n,k}$, *i.e.* when the cell B^0_{kij} is completely included in ω, so it will be possible to use the control applied to μ_n^0. Thus, the two times need to be close each other. We similarly build the points x^1_{kij}. By definition of $S^l_{n,k}$, for $l \in \{0, 1\}$, it holds

$$W_\infty(\mu^l_{|(R_n^l)^c}, \mu_n^l) \leqslant \sqrt{2}/n \underset{n\to\infty}{\longrightarrow} 0. \tag{16}$$

We remark that the measures μ^0, μ^1 and the sequences $\{\mu_n^0\}_{n\in\mathbb{N}^*}$, $\{\mu_n^1\}_{n\in\mathbb{N}^*}$, $\{R_n^0\}_{n\in\mathbb{N}^*}$ and $\{R_n^0\}_{n\in\mathbb{N}^*}$ satisfy the hypothesis of Proposition 4.1. Since

$$S_{\varepsilon/4}(\mu^0, \mu^1) \leqslant S(\mu^0, \mu^1),$$

applying Proposition 4.1 for $\delta := s/2$, it holds

$$S_{\varepsilon/2}(\mu_n^0, \mu_n^1) + \frac{s}{2} \leqslant S_{\varepsilon/4}(\mu^0, \mu^1) + s \leqslant S(\mu^0, \mu^1) + s = T. \tag{17}$$

We now explain how to use (15) to build the control acting on the Dirac masses only at time $t^0_{n,k}$. Using Corollary 4.1, if we assume that the sequences $\{t^0(x^0_{kij})\}_{kij}$ and $\{t^1(x^1_{kij})\}_{kij}$ are respectively increasingly and decreasingly ordered, then

$$S_{\varepsilon/2}(\mu_n^0, \mu_n^1) = \max_{kij\in\{1,...,|I_n^0|-M_n^0\}} |t^0(x^0_{kij}) + t^1(x^1_{kij+M_n^0})|, \tag{18}$$

where $M_n^0 = \lceil \varepsilon n^8/(n^2-2)^2 \rceil$. Then, (15) implies

$$t^l_{k,n} \leqslant t^l_{kij}(x^l_{kij}) + s/8.$$

Assume the sequences $\{t^0_{k,n}\}_k$ and $\{t^1_{k,n}\}_k$ are respectively increasingly and decreasingly ordered; then, up to adapt Corollary 4.1, we can prove that the minimal time $\widetilde{T}_{e,\varepsilon/2}(\mu^0_n, \mu^1_n)$ to exactly steer μ^0_n to μ^1_n up to a mass $\varepsilon/2$ but acting on the particle coming from x^0_{kij} only after the time $t^0_{k,n}$ and idem for μ^1_n, is equal to

$$\begin{aligned}\widetilde{T}_{e,\varepsilon/2}(\mu^0_n, \mu^1_n) &= \max_{kij\in\{1,...,|I^0_n|-M^0_n\}} |t^0_{k,n} + t^1_{k,n}| \\ &\leqslant \max_{kij\in\{1,...,|I^0_n|-M^0_n\}} |t^0(x^0_{k,n}) + t^1(x^1_{kij})| + s/4.\end{aligned}$$

Combining this estimate with (17), (18) it holds

$$\widetilde{T}_{e,\varepsilon/2}(\mu^0_n, \mu^1_n) < T.$$

Then, there exists a control u_n such that for the initial data μ^0_n the associated solution μ_n to System (2) satisfies

$$W_1(\mu^1_n, \mu_n(T)) \leqslant \varepsilon/2. \tag{19}$$

Denoting by σ_0 and σ_1 the associated permutations, it holds $\Phi^{v+\mathbb{1}_\omega u_n}_T(x^0_{\sigma_0(kij)}) = x^1_{\sigma_1(kij)}$, for all $kij \in \{1,...,|I^0_n| - M^0_n\}$.

Since we have no intersection of the trajectories $\Phi^{u_n}_\cdot(x^0_{kij})$ (see argument given in the proof of Proposition 3.1), there exist $0 < r < R$ such that for all $t \in (t^0_{n,k}, T - t^1_{n,k})$

$$\Phi^{v+\mathbb{1}_\omega u_n}_t(B_r(x^0_{kij})) \subset \Phi^{v+\mathbb{1}_\omega u_n}_t(B_R(x^0_{kij})) \subset \omega$$

and, for all $t \in (0,T)$, it holds $\cap_{kij\in I^0_n} \Phi^{v+\mathbb{1}_\omega u_n}_t(B_R(x^0_{kij})) = \varnothing$. If necessary, the final control u_n concentrates the mass of $\mu^0_{|B^0_{kij}}$ in

$$\Phi^{v+u_n}_{t^0_{n,k}}(B_{\widetilde{r}}(x^0_{kij}) \cap Q^0_{n,k})$$

in the time interval $(t^0_{n,k}, t^0_{n,k}+\delta)$, with $\delta > 0$ small enough. For details, we refer to [20, Prop. 3.3].

Step 4: We now estimate the Wasserstein distance between $\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0$ and μ^1. Using Property 2.2, it holds

$$\begin{aligned}W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0, \mu^1) &\leqslant W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_{|(R^0_n)^c}, \mu^1_{|(R^1_n)^c}) \\ &\quad + W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_{|R^0_n}, \mu^1_{|R^1_n}).\end{aligned} \tag{20}$$

By triangular inequality, it holds

$$\begin{aligned}&W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_{|(R^0_n)^c}, \mu^1_{|(R^1_n)^c})) \\ &\leqslant W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_{|(R^0_n)^c}, \Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_n) \\ &\quad + W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_n, \mu^1_n) + W_1(\mu^1_n, \mu^1_{|(R^1_n)^c}).\end{aligned} \tag{21}$$

We now estimate each term in the right-hand side in (21). Using inequalities (10) and (16), it holds

$$W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_{|(R^0_n)^c}, \Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_n) \leqslant e^{2LT}\sqrt{2}/n \tag{22}$$

$$W_1(\mu^1_n, \mu^1_{|(R^0_n)^c}) \leqslant W_\infty(\mu^1_n, \mu^1_{|(R^0_n)^c}) \leqslant \sqrt{2}/n. \tag{23}$$

Combining (19), (21), (22) and (23), it holds

$$W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_{|(R^0_n)^c}, \mu^1_{|(R^1_n)^c})) \leqslant \varepsilon/2 + (1 + e^{2LT})\sqrt{2}/n. \tag{24}$$

By Property 2.1, there exists $\pi \in \Pi(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_{|R^0_n}, \mu^1_{|R^1_n})$ such that

$$\begin{aligned}W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0_{|R^0_n}, \mu^1_{|R^1_n}) &= \textstyle\int_{(0,1)^2\times(0,1)^2} |x-y| d\pi(x,y) \\ &\leqslant (2 + T\sup|v|) \times (\tfrac{6n^2-4}{n^4} + \mu^0((S^0_n)^c)).\end{aligned}$$

Combining this inequality with (20), (24), we obtain

$$\begin{aligned}W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \mu^0, \mu^1) &\leqslant \varepsilon/2 + \sqrt{2}(1 + e^{2LT})/n \\ &\quad + (2 + T\sup|v|)(\tfrac{6n^2-4}{n^4} + \mu^0((S^0_n)^c)),\end{aligned}$$

which leads to the conclusion when $n \to \infty$.

We similarly prove Item (ii). ∎

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