On the total variation Wasserstein gradient flow and the TV-JKO scheme

Guillaume Carlier, Clarice Poon

On the total variation Wasserstein gradient flow and the TV-JKO scheme

Guillaume Carlier* Clarice Poon †

February 28, 2017

Abstract

We study the JKO scheme for the total variation, characterize the optimizers, prove some of their qualitative properties (in particular a sort of maximum principle and the regularity of level sets). We study in detail the case of step functions. Finally, in dimension one, we establish convergence as the time step goes to zero to a solution of a fourth-order nonlinear evolution equation.

Keywords: total variation, Wasserstein gradient flows, JKO scheme, fourth-order evolution equations.

MS Classification: 35G31, 49N15.

1 Introduction

Variational schemes based on total variation are extremely popular in image processing for denoising purposes, in particular the seminal work of Rudin, Osher and Fatemi [27] has been extremely influential and is still the object of an intense stream of research, see [10] and the references therein. Continuous-time counterparts are well-known to be related to the L^2 gradient flow of the total variation, see Bellettini, Casselles and Novaga [4] and the mean-curvature flow, see Evans and Spruck [15]. The gradient flow of the total variation for other Hilbertian structures may be natural as well and in

*Ceremade, UMR CNRS 7534, Université Paris Dauphine, Pl. de Lattre de Tassigny, 75775, Paris Cedex 16, France, and MOKAPLAN, INRIA-Paris, E-mail: carlier@ceremade.dauphine.fr

†Centre for Mathematical Sciences, University of Cambridge, Wilberforce Rd, Cambridge CB3 0WA, United Kingdom, Email: C.M.H.S.Poon@maths.cam.ac.uk

particular the H^{-1} case, leads to a singular fourth-order evolution equation studied by Giga and Giga [16], Giga, Kuroda and Matsuoka [17]. In the present work, we consider another metric, namely the Wasserstein one.

Given an open subset Ω of $\mathbb{R}^d$ and $\rho \in L^1(\Omega)$, recall that the total variation of ρ is given by

$$J(\rho) := \sup\Big\{ \int_\Omega \operatorname{div}(z)\rho \ : \ z \in C_c^1(\Omega), \ \|z\|_{L^\infty} \leq 1 \Big\} \tag{1.1}$$

and $\mathrm{BV}(\Omega)$ is by definition the subspace of $L^1(\Omega)$ consisting of those ρ's in $L^1(\Omega)$ such that $J(\rho)$ is finite. The following fourth-order nonlinear evolution equation

$$\partial_t \rho + \operatorname{div}\Big(\rho \ \nabla \operatorname{div}\Big(\frac{\nabla\rho}{|\nabla\rho|}\Big)\Big) = 0, \ \text{ on } (0,T)\times\Omega, \ \rho_{|t=0} = \rho_0, \tag{1.2}$$

supplemented by the zero-flux boundary condition

$$\rho \nabla \operatorname{div}\Big(\frac{\nabla\rho}{|\nabla\rho|}\Big) \cdot \nu = 0 \text{ on } \partial\Omega \tag{1.3}$$

has been proposed in [7] for the purpose of denoising image densities. Numerical schemes for approximating the solutions of this equation have been investigated in [7, 14, 5]. One should of course interpret the nonlinear term $\operatorname{div}(\frac{\nabla\rho}{|\nabla\rho|})$ as the negative of an element of the subdifferential of J at ρ.

At least formally, when ρ_0 is a probability density on Ω, (1.2)-(1.3) can be viewed as the Wasserstein gradient flow of J (we refer to the textbooks of Ambrosio, Gigli, Savaré [2] and Santambrogio [28], for a detailed exposition). Following the seminal work of Jordan, Kinderlehrer and Otto [18] for the Fokker-Planck equation, it is reasonable to expect that solutions of (1.2) can be obtained, at the limit $\tau \to 0^+$, of the JKO Euler implicit scheme:

$$\rho_0^\tau = \rho_0, \ \rho_{k+1}^\tau \in \operatorname{argmin}\Big\{\frac{1}{2\tau} W_2^2(\rho_k^\tau, \rho) + J(\rho), \ \rho \in \mathrm{BV}(\Omega) \cap \mathcal{P}_2(\overline{\Omega})\Big\} \tag{1.4}$$

where $\mathcal{P}_2(\overline{\Omega})$ is the space of Borel probability measures $\overline{\Omega}$ with finite second moment and W_2 is the quadratic Wasserstein distance:

$$W_2^2(\rho_0, \rho_1) := \inf_{\gamma\in\Pi(\rho_0,\rho_1)} \Big\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \mathrm{d}\gamma(x,y)\Big\}, \tag{1.5}$$

$\Pi(\rho_0, \rho_1)$ denoting the set of transport plans between ρ_0 and ρ_1 i.e. the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ having ρ_0 and ρ_1 as marginals. Our aim is to study in detail the discrete TV-JKO scheme (1.4) as well as its connection with (suitable weak solutions) of the PDE (1.2). Although the assertion that (1.2) is the TV Wasserstein gradient flow is central to the numerical schemes described in [7, 14, 5], there has been so far, to the best of our knowledge, no theoretical justification of this fact.

Fourth-order equations which are Wasserstein gradient flows of functionals involving the gradient of ρ, such as the Dirichlet energy or the Fisher information, have been studied by McCann, Matthes and Savaré [24] who found a new method *the flow interchange technique* to prove higher-order compactness estimates, we refer to [19] for a recent reference on this topic. The total variation is however too singular for such arguments to be directly applicable, as far as we know.

The paper is organized as follows. In section 2, we start with the discussion of a few examples. Section 3 is devoted to some properties of solutions of JKO steps and in particular a maximum principle based on a result of [12]. Section 4 establishes optimality conditions for JKO steps thanks to an entropic regularization scheme. Section 5 discusses regularity properties of the boundaries of the level sets of JKO solutions. In section 6, we address in detail the case of step functions in dimension one. Finally, in section 7, we prove convergence of the JKO scheme, as $\tau \to 0^+$, in the case of a strictly positive and bounded initial condition on a bounded interval of the real line.

2 Some examples

We first recall the Kantorovich dual formulation of W_2^2:

$$\frac{1}{2}W_2^2(\mu_0,\mu_1) = \sup\Big\{ \int_{\mathbb{R}^d} \psi \mathrm{d}\mu_0 + \int_{\mathbb{R}^d} \varphi \mathrm{d}\mu_1 \ : \ \psi(x)+\varphi(y) \leq \frac{|x-y|^2}{2}\Big\} \quad (2.1)$$

an optimal pair (ψ,φ) for this problem is called a pair of Kantorovich potentials. The existence of Kantorovich potentials is well-known and such potentials can be taken to be conjugates of each other, i.e. such that

$$\varphi(x) = \inf_{y\in\mathbb{R}^d}\{\frac{1}{2}|x-y|^2 - \psi(y)\},\ \psi(y) = \inf_{x\in\mathbb{R}^d}\{\frac{1}{2}|x-y|^2 - \varphi(x)\},$$

which implies that φ and ψ are semi-concave (more precisely $\frac{1}{2}|.|^2 - \varphi$ is convex). If μ_1 is absolutely continuous with respect to the d-dimensional Lebesgue measure, φ is differentiable μ_1 a.e. and the map $T = \mathrm{id} - \nabla\varphi$ is the gradient of a convex function pushing forward μ_1 to μ_0 which is in fact the optimal transport between μ_0 and μ_1 thanks to Brenier's theorem [6]. In such a case, we will simply refer to φ as a Kantorovich potential between μ_1 and μ_0. We refer the reader to [30] and [28] for details.

In this section, we will consider some explicit examples which rely on the following sufficient optimality condition (details for a rigorous derivation of the Euler-Lagrange equation for JKO steps will be given in section 4) in the case of the whole space i.e. $\Omega = \mathbb{R}^d$.

Lemma 2.1. *Let $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$, $\tau > 0$ and $\Omega = \mathbb{R}^d$ (so J is the total variaton on the whole space), if $\rho_1 \in \mathrm{BV}(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$ is such that*

$$\frac{\varphi}{\tau} + \mathrm{div}(z) \geq 0, \text{ with equality } \rho_1\text{-a.e.} \tag{2.2}$$

where φ is a Kantorovich potential between ρ_1 and ρ_0 and $z \in C^1(\mathbb{R}^d)$, with $\|z\|_{L^\infty} \leq 1$, $\mathrm{div}(z) \in L^d$, and

$$J(\rho_1) = \int_{\mathbb{R}^d} \mathrm{div}(z)\rho_1. \tag{2.3}$$

Then, setting

$$\Phi_{\tau,\rho_0}(\rho) := \frac{1}{2\tau} W_2^2(\rho_0, \rho) + J(\rho), \ \forall \rho \in \mathrm{BV}(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d) \tag{2.4}$$

one has

$$\Phi_{\tau,\rho_0}(\rho_1) \leq \Phi_{\tau,\rho_0}(\rho), \ \forall \rho \in \mathrm{BV}(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d).$$

Proof. For all $\rho \in \mathrm{BV}(\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d)$, $J(\rho) \geq \int_{\mathbb{R}^d} \mathrm{div}(z)\rho = J(\rho_1) + \int_{\mathbb{R}^d} \mathrm{div}(z)(\rho - \rho_1)$, and it follows from the Kantorovich duality formula that

$$\frac{1}{2\tau} W_2^2(\rho_0, \rho) \geq \frac{1}{2\tau} W_2^2(\rho_0, \rho_1) + \int_{\mathbb{R}^d} \frac{\varphi}{\tau}(\rho - \rho_1).$$

The claim then directly follows from (2.2). □

2.1 The case of a characteristic function

A simple illustration of Lemma 2.1 in dimension 1 concerns the case of a uniform ρ_0, (here and in the sequel we shall denote by χ_A the characteristic function of the set A):

$$\rho_0 = \rho_{\alpha_0}, \ \alpha_0 > 0, \ \rho_\alpha := \frac{1}{2\alpha} \chi_{[-\alpha, \alpha]}.$$

It is natural to make the ansatz that the minimizer of Φ_{τ,ρ_0} defined by (2.4) remains of the form $\rho_1 = \rho_{\alpha_1}$ for some $\alpha_1 > \alpha_0$. The optimal transport between ρ_{α_1} and ρ_0 being the linear map $T = \frac{\alpha_0}{\alpha_1}\mathrm{id}$, a direct computation gives

$$\Phi_{\tau,\rho_0}(\rho_{\alpha_1}) = \frac{1}{\alpha_1} + \frac{1}{6\tau}(\alpha_1 - \alpha_0)^2$$

which is minimal when α_1 is the only root in $(\alpha_0, +\infty)$ of

$$\alpha_1^2(\alpha_1 - \alpha_0) = 3\tau. \tag{2.5}$$

To check that this is the correct guess, we shall check that the conditions of Lemma 2.1 are met. First define the Kantorovich potential

$$\varphi(x) = \frac{1}{2\alpha_1}(\alpha_1 - \alpha_0)x^2 - \frac{3\tau}{2\alpha_1}$$

and z_1 by

$$\tau z_1(x) := -\frac{(\alpha_1 - \alpha_0)}{6\alpha_1}x^3 + \frac{3\tau x}{2\alpha_1},\ x \in [-\alpha_1, \alpha_1]$$

extended by 1 on $[1, +\infty)$ and -1 on $(-\infty, -1)$. Then $-1 \leq z_1 \leq 1$ (use the fact that it is odd and nondecreasing on $[0, \alpha_1]$ thanks to (2.5)), also $z_1'(\pm\alpha_1) = 0$ so that $z_1 \in C^1(\mathbb{R})$ and $z_1(\alpha_1) = 1$, $z_1(-\alpha_1) = -1$ hence $J(\rho_1) = -\int_{\mathbb{R}} z_1 D\rho_1 = \int_{\mathbb{R}} z_1'\rho_1$ (here and in the sequel $D\rho_1$ denotes the Radon measure which is the distributional derivative of the BV function ρ_1). Moreover $\tau z_1' + \varphi \geq 0$ with an equality on $[-\alpha_1, \alpha_1]$. The optimality of $\rho_1 = \rho_{\alpha_1}$ then directly follows from Lemma 2.1.

Of course, the argument can be iterated so as to obtain the full TV-JKO sequence:

$$\rho_{k+1}^\tau = \mathrm{argmin}\ \Phi_{\tau,\rho_k^\tau} = \Big(\frac{\alpha_{k+1}^\tau}{\alpha_k^\tau}\mathrm{id}\Big)_{\#}\rho_k^\tau = \Big(\frac{\alpha_{k+1}^\tau}{\alpha_0}\mathrm{id}\Big)_{\#}\rho_0$$

where α_k^τ is defined inductively by

$$(\alpha_{k+1}^\tau - \alpha_k^\tau)(\alpha_{k+1}^\tau)^2 = 3\tau,\ \alpha_0^\tau = \alpha_0$$

which is nothing but the implicit Euler discretization of the ODE

$$\alpha'\alpha^2 = 3,\ \alpha(0) = \alpha_0,$$

whose solution is $\alpha(t) = (\alpha_0^3 + 9t)^{\frac{1}{3}}$. Extending ρ_k^τ in a piecewise constant way: $\rho^\tau(t) = \rho_{k+1}^\tau$ for $t \in (k\tau, (k+1)\tau]$, it is not difficult to check that ρ^τ converges (in $L^\infty((0,T), (\mathcal{P}_2(\mathbb{R}), W_2))$ and in $L^p((0,T)\times\mathbb{R})$ for any $p \in (1, \infty)$ and any $T > 0$) to ρ given by $\rho(t,.) = (\frac{\alpha(t)}{\alpha_0}\mathrm{id})_{\#}\rho_0$. Since $v(t,x) = \frac{\alpha'(t)}{\alpha(t)}x$ is the velocity field associated to $X(t,x) = \frac{\alpha(t)}{\alpha_0}x$, ρ solves the continuity equation

$$\partial_t\rho + (\rho v)_x = 0.$$

In addition, $\rho v = -\rho z_{xx}$ where

$$z(t,x) = \frac{-\alpha'(t)}{6\alpha(t)}x^3 + \frac{3x}{2\alpha(t)},\ x \in [-\alpha(t), \alpha(t)],$$

extended by 1 (respectively -1) on $[\alpha(t), +\infty)$ (respectively $(-\infty, -\alpha(t)]$). The function z is C^1, $\|z\|_{L^\infty} \leq 1$ and $z \cdot D\rho = -|D\rho|$ (in the sense of measures). In other words the limit ρ of ρ^τ satisfies

$$\partial_t \rho - (\rho z_{xx})_x = 0$$

with $|z| \leq 1$ and $z \cdot D\rho = -|D\rho|$ which is the natural weak form of (1.2).

2.2 Instantaneaous creation of discontinuities

We now consider the case where $\rho_0(x) = (1 - |x|)_+$ and will show that the JKO scheme instantaneously creates a discontinuity at the level of ρ_1, the minimizer of Φ_{τ,ρ_0} when τ is small enough. We indeed look for ρ_1 in the form:

$$\rho_1(x) = \begin{cases} 1 - \beta/2 & \text{if } |x| < \beta, \\ (1 - |x|)_+ & \text{if } |x| \geq \beta, \end{cases}$$

for some well-chosen $\beta \in (0,1)$. The optimal transport map T between such a ρ_1 and ρ_0 is odd and given explicitly by

$$T(x) = \begin{cases} 1 - \sqrt{1 - x(2-\beta)} & \text{if } x \in [0, \beta), \\ x & \text{if } x \geq \beta. \end{cases}$$

The Kantorovich potential which vanishes at β (extended in an even way to $\mathbb{R}_-$) is then given by

$$\varphi(x) = \begin{cases} \frac{x^2}{2} - x - \frac{(1-x(2-\beta))^{3/2}}{3(1-\beta/2)} + C & \text{if } x \in [0, \beta), \\ 0 & \text{if } x > \beta, \end{cases}$$

where

$$C = -\frac{\beta^2}{2} + \beta + \frac{2(1-\beta)^3}{3(2-\beta)}.$$

Let us now integrate $\tau z' = -\varphi$ on $[0, \beta]$ with initial condition $z(0) = 0$, i.e. for $x \in [0, \beta]$

$$\begin{aligned} \tau z(x) = &- \frac{x^3}{6} + \frac{x^2}{2} - \frac{4}{15(2-\beta)^2}[1 - (1-2\beta)x]^{\frac{5}{2}} \\ &+ \Big(\frac{\beta^2}{2} - \beta - \frac{2(1-\beta)^3}{3(2-\beta)}\Big)x + \frac{4}{15(2-\beta)^2} \end{aligned}$$

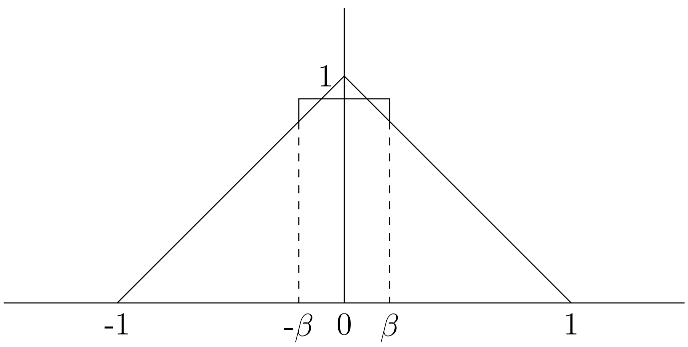

Figure 1: The probablity density functions ρ_0 and ρ_1 from section 2.2

Note that z is nondecreasing on $[0, \beta]$ (because $\varphi(0) < 0$, $\varphi(\beta) = 0$ and φ is convex on $[0, \beta]$ so that $\varphi \leq 0$ on $[0, \beta]$), our aim now is to find $\beta \in (0, 1)$ in such a way that $z(\beta) = 1$ i.e. replacing in the previous formula

$$\tau = \frac{\beta^3}{3} - \frac{\beta^2}{2} + \frac{4(1-(1-\beta)^5)}{15(2-\beta)^2} - \frac{2(1-\beta)^3\beta}{3(2-\beta)}$$

the right hand-side is a continuous function of $\beta \in [0, 1]$ taking value 0 for $\beta = 0$ and $\frac{1}{10}$ for $\beta = 1$, hence as soon as $10\tau < 1$ one may find a $\beta \in (0, 1)$ such that indeed $z(\beta) = 1$. Extend then z by 1 on $[\beta, +\infty)$ and to $\mathbb{R}_-$ in an odd way. We then have built a function z which is C^1 ($\varphi(\beta) = 0$), such that $|z| \leq 1$, $z \cdot D\rho_1 = -|D\rho_1|$ and such that $z' + \frac{\varphi}{\tau} = 0$. Thanks to Lemma 2.1, we conclude that ρ_1 is optimal. This example shows that discontinuities may appear at the very first iteration of the TV-JKO scheme.

3 Maximum principle for JKO steps

Throughout this section, we assume that Ω is a convex open bounded subset of $\mathbb{R}^d$ and denote $\mathcal{P}_{\text{ac}}(\Omega)$ the set of Borel probability measures on Ω that are absolutely continuous with respect to the Lebesgue measure (and will use the same notation for $\mu \in \mathcal{P}_{\text{ac}}(\Omega)$ both for the measure μ and its density). Given $\rho_0 \in \mathcal{P}_{\text{ac}}(\Omega)$ and $\tau > 0$, we consider one step of the TV-JKO scheme:

$$\inf_{\rho \in \mathcal{P}_{\text{ac}}(\Omega)} \left\{ \frac{1}{2\tau} W_2^2(\rho_0, \rho) + J(\rho) \right\}. \tag{3.1}$$

It is easy by the direct method of the calculus of variations to see that (3.1) has at least one solution, moreover J being convex and $\rho \mapsto W_2^2(\rho, \rho_0)$ being strictly convex whenever $\rho_0 \in \mathcal{P}_{\text{ac}}(\Omega)$ (see [28]), the minimizer is in fact unique, and in the sequel we denote it by ρ_1.

3.1 Preliminaries

Our aim is to deduce some bounds on ρ_1 from bounds on ρ_0. To do so, we shall combine some convexity arguments and a remarkable BV estimate due to De Philippis et al. [12]. First we recall the notion of generalized geodesic from Ambrosio, Gigli and Savaré [2]. Given $\overline{\mu}$, μ_0 and μ_1 in $\mathcal{P}_{\rm ac}(\Omega)$, and denoting by T_0 (respectively T_1) the optimal transport (Brenier) map between $\overline{\mu}$ and μ_0 (respectively μ_1), the *generalized geodesic with base* $\overline{\mu}$ joining μ_0 to μ_1 is by definition the curve of measures:

$$\mu_t := ((1-t)T_0 + tT_1)_{\#}\overline{\mu},\ t \in [0,1]. \tag{3.2}$$

A key property of these curves introduced in [2] is the strong convexity of the squared distance estimate:

$$W_2^2(\overline{\mu}, \mu_t) \leq (1-t)W_2^2(\overline{\mu}, \mu_0) + tW_2^2(\overline{\mu}, \mu_1) - t(1-t)W_2^2(\mu_0, \mu_1). \tag{3.3}$$

It is well-known that if $G : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous (l.s.c.) internal energy density, bounded from below such that $G(0) = 0$ and which satisfies McCann's condition (see [25])

$$\lambda \in \mathbb{R}_+ \to \lambda^d G(\lambda^{-d}) \text{ is convex nonincreasing} \tag{3.4}$$

then defining the generalized geodesic curve $(\mu_t)_{t\in[0,1]}$ by (3.2), one has

$$\int_\Omega G(\mu_t(x))\mathrm{d}x \leq (1-t)\int_\Omega G(\mu_0(x))\mathrm{d}x + t\int_\Omega G(\mu_1(x))\mathrm{d}x. \tag{3.5}$$

In particular L^p and uniform bounds are stable along generalized geodesics:

$$\|\mu_t\|_{L^p}^p \leq (1-t)\|\mu_0\|_{L^p}^p + t\|\mu_0\|_{L^p}^p,\ \|\mu_t\|_{L^\infty} \leq \max(\|\mu_0\|_{L^\infty}, \|\mu_1\|_{L^\infty}), \tag{3.6}$$

and

$$\int_\Omega \mu_t(x)\log(\mu_t(x))\mathrm{d}x \leq (1-t)\int_\Omega \mu_0(x)\log(\mu_0(x))\mathrm{d}x + t\int_\Omega \mu_1(x)\log(\mu_1(x))\mathrm{d}x \tag{3.7}$$

An immediate consequence of (3.3) (see chapter 4 of [2] for general contraction estimates) is the following

Lemma 3.1. *Let K be a nonempty subset of $\mathcal{P}_{\rm ac}(\Omega)$, let $\mu_0 \in K$, $\mu_1 \in \mathcal{P}_{\rm ac}(\Omega)$, if $\hat{\mu}_1 \in \mathrm{argmin}_{\mu\in K} W_2^2(\mu_1, \mu)$ and if the generalized geodesic with base μ_1 joining μ_0 to $\hat{\mu}_1$ remains in K then*

$$W_2^2(\mu_0, \hat{\mu}_1) \leq W_2^2(\mu_0, \mu_1) - W_2^2(\mu_1, \hat{\mu}_1). \tag{3.8}$$

Proof. Since $\mu_t \in K$ we have $W_2^2(\mu_1, \hat{\mu}_1) \leq W_2^2(\mu_1, \mu_t)$, applying (3.3) to the generalized geodesics with base μ_1 joining μ_0 to $\hat{\mu}_1$ we thus get

$$(1-t)W_2^2(\mu_1, \hat{\mu}_1) \leq (1-t)W_2^2(\mu_1, \mu_0) - t(1-t)W_2^2(\mu_0, \hat{\mu}_1),$$

dividing by $(1-t)$ and then taking $t = 1$ therefore gives the desired result.

$\square$

The other result we shall use to derive bounds is a BV estimate of De Philippis et al. [12], which states that given $\mu, \in \mathcal{P}_{\mathrm{ac}}(\Omega) \cap \mathrm{BV}(\Omega)$, and $G : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$, proper convex l.s.c., the solution of

$$\inf_{\rho \in \mathcal{P}_{\mathrm{ac}}(\Omega)} \left\{ \frac{1}{2} W_2^2(\mu, \rho) + \int_\Omega G(\rho(x)) \mathrm{d}x \right\} \tag{3.9}$$

is BV with the bound

$$J(\rho) \leq J(\mu). \tag{3.10}$$

Taking in particular,

$$G(\rho) := \begin{cases} 0 \text{ if } \rho \leq M, \\ +\infty \text{ otherwise,} \end{cases}$$

this implies that the Wasserstein projection of μ onto the set defined by the constraint $\rho \leq M$ has a smaller total variation than μ.

3.2 Maximum and minimum principles

Theorem 3.2. *Let $\rho_0 \in \mathcal{P}_{\mathrm{ac}}(\Omega) \cap L^\infty(\Omega)$ and let ρ_1 be the solution of (3.1), then $\rho_1 \in L^\infty(\Omega)$ with*

$$\|\rho_1\|_{L^\infty(\Omega)} \leq \|\rho_0\|_{L^\infty(\Omega)}. \tag{3.11}$$

Proof. Thanks to (3.6) the set $K := \{\rho \in \mathcal{P}_{\mathrm{ac}}(\Omega) \cap L^p(\Omega) \ : \ \rho \leq \|\rho_0\|_{L^\infty(\Omega)} \text{ a.e.}\}$ has the property that the generalized geodesics (with any base) joining two of its points remains in K. Let then $\hat{\rho}_1$ be the W_2 projection of ρ_1 onto K i.e. the solution of $\inf_{\rho \in K} W_2^2(\rho_1, \rho)$. Thanks to Lemma 3.1 we have $W_2^2(\rho_0, \hat{\rho}_1) \leq W_2^2(\rho_0, \rho_1) - W_2^2(\rho_1, \hat{\rho}_1)$ and thanks to Theorem 1.1 of [12], $J(\hat{\rho}_1) \leq J(\rho_1)$. The optimality of ρ_1 for (3.1) therefore implies $W_2(\rho_1, \hat{\rho}_1) = 0$ i.e. $\rho_1 \leq \|\rho_0\|_{L^\infty(\Omega)}$.

$\square$

Remark 3.3. In section 4, we shall use an approximation of (3.1) with an additional small entropy term, the same bound as in Theorem 3.2 will remain valid in this case. Indeed, consider a proper convex l.s.c. and bounded from below internal energy density G and consider given $h \geq 0$, the variant of (3.1)

$$\inf_{\rho \in \mathcal{P}_{\mathrm{ac}}(\Omega)} \Big\{ \frac{1}{2\tau} W_2^2(\rho_0, \rho) + J(\rho) + h \int_\Omega G(\rho(x)) \mathrm{d}x \Big\}. \tag{3.12}$$

Then we claim that the solution ρ_h still satisfies $\rho_h \leq \|\rho_0\|_{L^\infty(\Omega)}$. Indeed we have seen in the previous proof that the Wasserstein projection $\hat{\rho}_h$ of ρ_h onto the constraint $\rho \leq \|\rho_0\|_{L^\infty(\Omega)}$ both diminishes J and the Wasserstein distance to ρ_0. It turns out that it also diminishes the internal energy. Indeed, thanks to Proposition 5.2 of [12], there is a measurable set A such that $\hat{\rho}_h = \chi_A \rho_h + \chi_{\Omega \setminus A} \|\rho_0\|_{L^\infty}$, it thus follows that $|\Omega \setminus A| \|\rho_0\|_{L^\infty} = \int_{\Omega \setminus A} \rho_h$. So, from the convexity of G and Jensen's inequality,

$$\int G(\hat{\rho}_h) = \int_A G(\rho_h) + |\Omega \setminus A| G \left(|\Omega \setminus A|^{-1} \int_{\Omega \setminus A} \rho_h \right) \leq \int G(\rho_h),$$

thus yielding the same conclusion as above.

In dimension one, it turns out that we can similarly obtain bounds from below:

Proposition 3.4. *Assume that $d = 1$, that Ω is a bounded interval and that $\rho_0 \geq \alpha > 0$ a.e. on Ω then the solution ρ_1 of (3.1) also satifies $\rho_1 \geq \alpha > 0$ a.e. on Ω.*

Proof. The proof is similar to that of Theorem 3.2 but using the Wasserstein projection on the set $K := \{\rho \in \mathcal{P}_{\mathrm{ac}}(\Omega) \ : \ \rho \geq \alpha\}$, the only thing to check to be able to use Lemma 3.1 is that for any basepoint $\overline{\mu}$ and any μ_0 and μ_1 in K, the generalized geodesic with base point $\overline{\mu}$ joining μ_0 to μ_1 remains in K. The optimal transport maps T_0 and T_1 from $\overline{\mu}$ to μ_0 and μ_1 respectively are nondecreasing and continuous and setting $T_t := (1-t)T_0 + tT_1$, one has

$$\overline{\mu} = \mu_t(T_t)T_t' = \mu_0(T_0)T_0' = \mu_1 T_1' = (1-t)\mu_0(T_0)T_0' + t\mu_1(T_1)T_1' \geq \alpha T_t'$$

which is easily seen to imply that $\mu_t \geq \alpha$ a.e.. □

4 Euler-Lagrange equation for JKO steps

The aim of this section is to establish optimality conditions for (3.1). Despite the fact that it is a convex minimization problem, it involves two nonsmooth terms J and $W_2^2(\rho_0, .)$, so some care should be taken of to justify rigorously the arguments. In the next section, we introduce an entropic regularization approximation, the advantage of this strategy is that the minimizer will be positive everywhere, giving some differentiability of the transport term.

4.1 Entropic approximation

In this whole section we assume that Ω is an open bounded connected subset of $\mathbb{R}^d$ with Lipschitz boundary and that $\rho_0 \in \mathcal{P}_{\rm ac}(\Omega)$. Given $h > 0$ we consider the following approximation of (3.1):

$$\inf_{\rho \in \mathcal{P}(\Omega)} \Big\{ \mathcal{F}_h(\rho) := \frac{1}{2\tau} W_2^2(\rho_0, \rho) + J(\rho) + h\mathcal{E}(\rho) \Big\} \tag{4.1}$$

where

$$\mathcal{E}(\rho) := \int_\Omega \rho(x) \log(\rho(x)) \mathrm{d}x.$$

It is easy to see that (4.1) admits a unique solution ρ_h and since $J(\rho_h)$ is bounded, up to a subsequence of vanishing h's, one may assume that ρ_h converges as $h \to 0$ a.e. and strongly in $L^p(\Omega)$ for every $p \in [1, \frac{d}{d-1})$ to ρ_1 the solution of (3.1).

We first have a bound from below on ρ_h:

Lemma 4.1. *There is an $\alpha_h > 0$ such that $\rho_h \geq \alpha_h$ a.e..*

Proof. Assume on the contrary that $|\rho \leq \alpha| > 0$ for every $\alpha > 0$. For small $\varepsilon \in (0, 1)$ set $\mu_{\varepsilon,h} := \max((1 - \sqrt{\varepsilon})\rho_h + \varepsilon, \rho_h)$ that is $(1 - \sqrt{\varepsilon})\rho_h + \varepsilon$ on $A_{\varepsilon,h} := \{\rho_h \leq \sqrt{\varepsilon})$ and ρ_h elsewhere. Define

$$c_{\varepsilon,h} := \int_\Omega (\mu_{\varepsilon,h} - \rho_h)$$

and observe that $c_{\varepsilon,h} \leq \varepsilon|\Omega| \leq \sqrt{\varepsilon}|\Omega|$. Now chose $M_h > 0$ such that $V_h := \{\rho_h > M_h\}$ has positive Lebesgue measure and finite perimeter (recall that ρ_h is BV) and chose ε small enough so that

$$\sqrt{\varepsilon} \leq \frac{M_h|V_h|}{2|\Omega|}. \tag{4.2}$$

Note that (4.2) implies that $c_{\varepsilon,h} \leq \frac{1}{2} M_h|V_h|$ and $M_h > \sqrt{\varepsilon}$ (so that $A_{\varepsilon,h}$ and V_h are disjoint). Finally, define

$$\rho_{\varepsilon,h} := \mu_{\varepsilon,h} - c_{\varepsilon,h} \frac{\chi_{V_h}}{|V_h|}.$$

By construction $\rho_{\varepsilon,h} \in \mathcal{P}(\Omega)$ hence $0 \leq \mathcal{F}_h(\rho_{\varepsilon,h}) - \mathcal{F}_h(\rho_h)$, in this difference we have four terms, namely

- the Wasserstein term, which, using the Kantorovich duality formula (2.1) and the fact that Ω is bounded can be estimated in terms of $\|\rho_{\varepsilon,h} - \rho_h\|_{L^1} = 2c_{\varepsilon,h}$:

$$\frac{1}{2\tau} W_2^2(\rho_{\varepsilon,h}, \overline{\rho}) - \frac{1}{2\tau} W_2^2(\rho_h, \overline{\rho}) \leq \frac{C}{\tau} c_{\varepsilon,h}. \tag{4.3}$$

  for a constant C that depends on Ω but neither on ε nor h,

- the TV term: $J(\rho_{\varepsilon,h}) - J(\rho_h)$: outside V_h we have replaced ρ_h by a 1-Lipschitz function of ρ_h which decreases the TV semi-norm, on V_h on the contrary we have created a jump of magnitude $c_{\varepsilon,h}/|V_h|$ so

$$J(\rho_{\varepsilon,h}) - J(\rho_h) \leq c_{\varepsilon,h} \frac{\mathrm{Per}(V_h)}{|V_h|} \tag{4.4}$$

  where $\mathrm{Per}(V_h) = J(\chi_{V_h})$ denotes the perimeter of V_h (in Ω),

- the entropy variation on $A_{\varepsilon,h}$, on this set both $\rho_{\varepsilon,h}$ and ρ_h are less than $\sqrt{\varepsilon}$ so that $(1 + \log(t)) \leq (1 + \log(\sqrt{\varepsilon}))$ whenever $t \in [\rho_h, \rho_{\varepsilon,h}]$ which by the mean value theorem yields

$$\int_{A_{\varepsilon,h}} (\rho_{\varepsilon,h} \log(\rho_{\varepsilon,h}) - \rho_h \log(\rho_h)) \leq (1 + \log(\sqrt{\varepsilon})) c_{\varepsilon,h} \tag{4.5}$$

- the entropy variation on V_h, but on V_h, if $\rho_{\varepsilon,h} \geq \frac{1}{e}$ then $(\rho_{\varepsilon,h} \log(\rho_{\varepsilon,h}) - \rho_h \log(\rho_h)) \leq 0$, we then observe that the remaining set $V_h \cap \{\rho_{\varepsilon,h} \leq \frac{1}{e}\} \subset \{\rho_h \leq \frac{1}{e} + \frac{M_h}{2}\}$ so that both $\rho_{\varepsilon,h}$ and ρ_h are bounded away from 0 and infinity on this set so remain in an interval where $t\log(t)$ is Lipschitz with Lipschitz constant at most

$$C_h(M_h) := \max\Big\{ |1 + \log(t)| \ : \ \frac{M_h}{2} \leq t \leq \frac{1}{e} + \frac{M_h}{2} \Big\}, \tag{4.6}$$

  we thus have

$$\int_{V_h} (\rho_{\varepsilon,h} \log(\rho_{\varepsilon,h}) - \rho_h \log(\rho_h)) \leq C_h(M_h) c_{\varepsilon,h}. \tag{4.7}$$

Putting together (4.3)-(4.4)-(4.5)-(4.7), we arrive at

$$0 \leq \Big(\frac{C}{\tau} + \frac{\mathrm{Per}(V_h)}{|V_h|} + hC_h(M_h) + h\log(\sqrt{\varepsilon}) + h \Big) c_{\varepsilon,h}$$

which for small enough ε is possible only when $c_{\varepsilon,h} = 0$ i.e. $|A_{\varepsilon,h}| = 0$. More precisely, either we have the lower bound:

$$h \log(\rho_h) \geq -\frac{C}{\tau} - hC_h(M_h) - \frac{\mathrm{Per}(V_h)}{|V_h|} - h \tag{4.8}$$

or (4.2) is impossible i.e.

$$\rho_h \geq \frac{M_h |V_h|}{2|\Omega|}. \tag{4.9}$$

□

We actually also have uniform bounds with respect to h:

Lemma 4.2. *The family $\theta_h := -h\log(\rho_h)$ is (up to a subsequence) uniformly bounded from above. Moreover, θ_h is bounded in $L^p(\Omega)$ for any $p > 1$.*

Proof. In view of (4.6), (4.8) and (4.9), it is enough to show that we can find a family M_h, bounded and bounded away from 0, such that setting $V_h := \{\rho_h > M_h\}$, $|V_h|$ remains bounded away from 0, and $\mathrm{Per}(V_h)$ is uniformly bounded from above as $h \to 0$. First note that, since $J(\rho_h)$ is bounded, there exists ρ such that $\rho_h \to \rho$ in L^1 and a.e. up to a subsequence, note also that $\rho \in \mathrm{BV}$ and ρ is a probability density. Setting $F_t^h := \{\rho_h > t\}$ and $F_t := \{\rho > t\}$, it is easy to deduce from Fatou's Lemma that when $s > t$, $\liminf_h |F_t^h| \geq |F_s|$, hence choosing $0 < \beta_1 < \beta_2 < \beta$ so that $|F_\beta| > 0$ we have that there exists $h_0 > 0$ and $c_1 > 0$ such that for all $t \in [\beta_1, \beta_2]$

$$c_1 \leq |F_t^h| \leq |\Omega|$$

whenever $0 < h < h_0$. Also, since $J(\rho_h) \leq C$, by the co-area formula

$$\int_{\beta_1}^{\beta_2} \mathrm{Per}(F_t^h)\mathrm{d}t \leq J(\rho_h) \leq C.$$

So, there exists $t_h \in [\beta_1, \beta_2]$ such that $\mathrm{Per}(F_{t_h}^h) \leq C/(\beta_2 - \beta_1)$. Therefore, it suffices to choose $M_h = t_h$ and $V_h = F_{t_h}^h$.

We may assume that $\rho_h \leq \phi$ for some $\phi \in L^1$, then by Dominated convergence and since $\log(\max(\phi, 1)) \in L^p(\Omega)$ for every $p > 1$, we have that $\log(\max(\rho_h, 1))$ converges a.e. and in L^p, in particular this implies that $\max(0, -\theta_h)$ converges to 0 strongly in $L^p(\Omega)$, and we have just seen that $\max(0, \theta_h)$ is bounded in $L^\infty(\Omega)$.

□

Let us also recall some well-known facts (see [9]) about the total variation functional J viewed as a convex l.s.c. and one-homogeneous functional on $L^{\frac{d}{d-1}}(\Omega)$. Define

$$\Gamma_d := \Big\{\xi \in L^d(\Omega) \ : \ \exists z \in L^\infty(\Omega, \mathbb{R}^d), \ \|z\|_{L^\infty} \le 1, \ \mathrm{div}(z) = \xi, \ \ z\cdot\nu = 0 \text{ on } \partial\Omega\Big\} \tag{4.10}$$

where $\mathrm{div}(z) = \xi, \ z \cdot \nu = 0$ on $\partial\Omega$ are to be understood in the weak sense

$$\int_\Omega \xi u = -\int_\Omega z \cdot \nabla u, \ \forall u \in C^1(\overline{\Omega}).$$

Note that Γ_d is closed and convex in $L^d(\Omega)$ and J is its support function:

$$J(\mu) = \sup_{\xi\in\Gamma_d} \int_\Omega \xi\mu, \ \forall \mu \in L^{\frac{d}{d-1}}(\Omega). \tag{4.11}$$

As for the Wasserstein term, recalling Kantorovich dual formulation (2.1), the derivative of the Wasserstein term $\rho \mapsto W_2^2(\rho_0, \rho)$ term will be expressed in terms of a Kantorovich potential between ρ and ρ_0.

We then have the following characterization for ρ_h:

Proposition 4.3. *There exists $z_h \in L^\infty(\Omega, \mathbb{R}^d)$ such that $\mathrm{div}(z_h) \in L^p(\Omega)$ for every $p \in [1, +\infty)$, $\|z_h\|_{L^\infty} \le 1$, $z_h \cdot \nu = 0$ on $\partial\Omega$, $J(\rho_h) = \int_\Omega \mathrm{div}(z_h)\rho_h$ and*

$$\frac{\varphi_h}{\tau} + \mathrm{div}(z_h) + h\log(\rho_h) = 0, \ \textit{a.e. in } \Omega \tag{4.12}$$

where φ_h is the Kantorovich potential between ρ_h and ρ_0.

Proof. Let $\mu \in L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ such that $\int_\Omega \mu = 0$. Thanks to Lemma 4.1, we know that ρ_h is bounded away from 0 hence for small enough $t > 0$, $\rho_h + t\mu$ is positive hence a probability density. Also, as a consequence of Theorem 1.52 in [28], we have that

$$\lim_{t\to 0^+} \frac{1}{2t}[W_2^2(\rho_0, \rho_h + t\mu) - W_2^2(\rho_0, \rho_h)] = \int_\Omega \varphi_h\mu \tag{4.13}$$

where φ_h is the (unique up to an additive constant) Kantorovich potential between ρ_h and ρ_0, in particular φ_h is Lipschitz and semi concave ($D^2\varphi_h \le \mathrm{id}$ in the sense of measures and $\mathrm{id} - \nabla\varphi_h$ is the optimal transport between ρ_h and ρ_1). By the optimality of ρ_h and the fact that J is a semi-norm, we get

$$J(\mu) \ge J(\rho_h + \mu) - J(\rho_h) \ge \lim_{t\to 0^+} t^{-1}(J(\rho_h + t\mu) - J(\rho_h)) \ge \int_\Omega \xi_h\mu, \tag{4.14}$$

where

$$\xi_h := -\frac{\varphi_h}{\tau} - h\log(\rho_h).$$

Since φ_h is defined up to an additive constant, we may chose it in such a way that ξ_h has zero mean, doing so, (4.14) holds for any $\mu \in L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ (not necessarily with zero mean). Being Lipschitz, φ_h is bounded, also observe that $h(\log(\rho_h))_+ = h\log(\max(1,\rho_h))$ is in $L^p(\Omega)$ for every $p \in [1,+\infty)$ since $\rho_h \in L^{\frac{d}{d-1}}(\Omega)$ and $h\log(\rho_h)_- = -h\log(\min(1,\rho_h))$ is $L^\infty(\Omega)$ thanks to Lemma 4.1, hence we have $\xi_h \in L^p(\Omega)$ for every $p \in [1,+\infty)$.

By approximation and observing that $\xi_h \in L^d(\Omega)$, (4.14) extends to all $\mu \in L^{\frac{d}{d-1}}(\Omega)$. In particular, we have

$$\sup_{\xi \in \Gamma_d} \int_\Omega \xi\mu \geq \int_\Omega \xi_h \mu$$

but since Γ_d is convex and closed in $L^d(\Omega)$, it follows from Hahn-Banach's separation theorem that $\xi_h \in \Gamma_d$. Finally, getting back to (4.14) (without the zero mean restriction on μ) and taking $\mu = -\rho_h$ gives $J(\rho_h) \leq \int_\Omega \xi_h \rho_h$, and we then deduce that this should be an equality.

□

4.2 Euler-Lagrange equation

We are now in position to rigorously establish the Euler-Lagrange equation for (3.1):

Theorem 4.4. *If ρ_1 solves (3.1), there exists φ a Kantorovich potential between ρ_0 and ρ_1 (in particular $\mathrm{id} - \nabla\varphi$ is the optimal transport between ρ_1 and ρ_0), $\beta \in L^\infty(\Omega)$, $\beta \geq 0$ and $z \in L^\infty(\Omega, \mathbb{R}^d)$ such that*

$$\frac{\varphi}{\tau} + \mathrm{div}(z) = \beta,\ z \cdot \nu = 0 \text{ on } \partial\Omega, \tag{4.15}$$

and

$$\beta\rho_1 = 0,\ \ \|z\|_{L^\infty} \leq 1,\ J(\rho_1) = \int_\Omega \mathrm{div}(z)\rho_1. \tag{4.16}$$

Proof. As in section 4.1, we denote by ρ_h the solution of the entropic approximation (4.1). Up to passing to a subsequence (not explicitly written), we may assume that ρ_h converges a.e. and strongly in $L^p(\Omega)$ (for any $p \in [1, \frac{d}{d-1})$) to ρ_1. We then rewrite the Euler-Lagrange equation from Proposition 4.3 as

$$\frac{\varphi_h}{\tau} + \mathrm{div}(z_h) + \beta_h^+ = \beta_h^-, \tag{4.17}$$

where $\beta_h^+ := h\log(\max(\rho_h, 1))$, $\beta_h^- := -h\log(\min(\rho_h, 1))$, and

$$\|z_h\|_{L^\infty} \leq 1,\ z_h \cdot \nu = 0 \text{ on } \partial\Omega \text{ and } J(\rho_h) = \int_\Omega \operatorname{div}(z_h)\rho_h. \tag{4.18}$$

It is easy to see that β_h^+ converges to 0 strongly in any L^q, $q \in [1, +\infty)$ and it follows from Lemma 4.2 that β_h^- is bounded in L^∞. Up to subsequences, we may therefore assume that z_h and β_h^- weakly-$*$ converge in L^∞ respectively to some z and β with $\|z\|_{L^\infty} \leq 1$, $z \cdot \nu = 0$ on $\partial\Omega$ and $\beta \geq 0$. As for φ_h, it is an equi-Lipschitz family and $\int_\Omega \varphi_h = \tau \int_\Omega (\beta_h^- - \beta_h^+)$ which remains bounded, hence we may assume that φ_h converges uniformly to some potential φ and it is well-known (see [28]) that φ is a Kantorovich potential between ρ_1 and ρ_0. Letting h tends to 0 gives (4.15).

Since ρ_h converges strongly in L^1 to ρ_1 and β_h^- converges weakly-$*$ to β in L^∞ we have

$$\int_\Omega \rho_1 \beta = \lim_h \int_\Omega \rho_h \beta_h^- = \lim_h h \int_\Omega \rho_h |\log(\min(1, \rho_h))| = 0,$$

hence $\beta\rho_1 = 0$. Thanks to (4.11), we obviously have $J(\rho_1) \geq \int_\Omega \operatorname{div}(z)\rho_1$, for the converse inequality, it is enough to observe that

$$J(\rho_1) \leq \liminf_h J(\rho_h) = \liminf_h \int_\Omega \operatorname{div}(z_h)\rho_h$$

and that $\operatorname{div}(z_h) = -\frac{\varphi_h}{\tau} - \beta_h^+ + \beta_h^-$ converges to $\operatorname{div}(z)$ weakly in L^q for every $q \in [1, +\infty)$. Since ρ_h converges strongly to ρ_1 in L^q when $q \in [1, \frac{d}{d-1})$ we deduce that $J(\rho_1) = \int_\Omega \operatorname{div}(z)\rho_1$ which completes the proof of (4.16).

□

Remark 4.5. It is not difficult (since (3.1) is a convex problem) to check that (4.15)-(4.16) are also sufficient optimality conditions. The main point here is that the right hand side β in (4.15) which is a multiplier associated with the nonnegativity constraint is better than a measure, it is actually an L^∞ function.

In dimension 1, we can integrate the Euler-Lagrange equation and then deduce higher regularity for the dual variable z:

Corollary 4.6. *Assume that $d = 1$ and Ω is a bounded interval. If ρ_1 solves (3.1) and z is as in Theorem 4.4 then $z \in W_0^{1,\infty}(\Omega)$. If in addition $\rho_0 \geq \alpha > 0$ a.e. on Ω, then $z \in W^{3,\infty}(\Omega)$.*

Proof. The first claim is obvious because both φ and β are bounded hence so is z'. As for the second one when $\rho_0 \geq \alpha > 0$, thanks to Proposition 3.4, we also have $\rho_1 \geq \alpha$ hence $\beta = 0$ in (4.15) and in this case $\operatorname{div}(z) = z' = -\frac{\varphi}{\tau}$ is Lipschitz i.e. $z \in W^{2,\infty}$. One can actually go one step further because $x - \varphi'(x) = T(x)$ where T is the optimal (monotone) transport between ρ_1 and ρ_0. This map is explicit in terms of the cumulative distribution function of ρ_1, F_1, and F_0^{-1} the inverse of F_0, the cumulative distribution function of ρ_0, namely $T = F_0^{-1} \circ F_1$. But F_1 is Lipschitz since its derivative is ρ_1 which is BV hence bounded and F_0^{-1} is Lipschitz as well since $\rho_0 \geq \alpha > 0$. This gives that $\varphi \in W^{2,\infty}$ hence $z \in W^{3,\infty}$. □

5 Regularity of level sets

We discuss in this section how the fact that $\operatorname{div}(z) \in L^\infty$ in Theorem 4.4 allows for conclusions about the regularity of the level sets of ρ_1, the solution of (3.1). A first consequence of the high integrability of $\operatorname{div}(z)$ is that one can give a meaning to $z \cdot \nabla u$ for any $u \in \mathrm{BV}(\Omega)$. Indeed, following Anzellotti [3], if $u \in \mathrm{BV}(\Omega)$ and $\sigma \in L^\infty(\Omega, \mathbb{R}^d)$ is such that $\operatorname{div}(\sigma) \in L^d(\Omega)$, one can define the distribution $\sigma \cdot Du$ by

$$\langle \sigma \cdot Du, v \rangle = -\int_\Omega \operatorname{div}(\sigma)\, uv - \int_\Omega u\, \sigma \cdot \nabla v,\ \forall v \in C_c^1(\Omega).$$

Then $\sigma \cdot Du$ is a Radon measure which satisfies $|\sigma \cdot Du| \leq \|\sigma\|_{L^\infty} |Du|$ (in the sense of measures) hence is absolutely continuous with respect to $|Du|$. Moreover one can also define a weak notion of normal trace of σ, $\sigma \cdot \nu \in L^\infty(\partial\Omega)$ such that the following integration by parts formula holds

$$\int_\Omega \sigma \cdot Du = -\int_\Omega \operatorname{div}(\sigma) u + \int_{\partial\Omega} u(\sigma \cdot \nu).$$

We refer to [3] for proofs. These considerations of course apply to $\sigma = z$ and $u = \rho_1 \in \mathrm{BV}(\Omega)$ and in particular enable one to see $z \cdot D\rho_1$ as a measure and to interpret the optimality condition $J(\rho_1) = \int_\Omega \operatorname{div}(z)\rho_1$ as $|D\rho_1| = -z \cdot D\rho_1$ in the sense of measures.

It now follows from Proposition 3.3 of [10], that every (not only almost every) level set $F_t = \{\rho_1 > t\}$ with $t > 0$ satisfies

$$\operatorname{Per}(F_t) = \int_{F_t} \operatorname{div}(z) \text{ and } F_t \in \underset{G \subset \Omega}{\operatorname{argmin}} \Big\{ \operatorname{Per}(G) - \int_G \operatorname{div}(z) \Big\}. \tag{5.1}$$

This means that $-\operatorname{div}(z)$ is the *variational mean curvature* of F_t. Indeed, recall, following Gonzalez and Massari [21], that a set of finite perimeter

$E \subset \Omega \subset \mathbb{R}^d$ is said to have variational mean curvature $g \in L^1(\Omega)$ precisely when E minimizes

$$\min_{F \subset \Omega} \mathrm{Per}(F) + \int_F g. \tag{5.2}$$

Regularity of sets with an L^p variational mean curvature, in connection with the so-called quasi-minimizers of the perimeter has been extensively studied, see Tamanini [29], Massari [22, 23], Theorem 3.6 of [21] and Maggi's book [20]. It follows from the results of [29] that if E has variational mean curvature $g \in L^p(\Omega)$ with $p \in (d, +\infty]$, then its reduced boundary (see [1]) $\partial^* E$ is a $(d-1)$-dimensional manifold of class $C^{1,\frac{p-d}{2p}}$ and $\mathcal{H}^s((\partial E \setminus \partial^* E) \cap \Omega) = 0$ for all $s > d - 8$. We thus deduce from Theorem 4.4:

Theorem 5.1. *If ρ_1 solves* (3.1)*, then for every $t > 0$, the level set $F_t = \{\rho_1 > t\}$ has the property that its reduced boundary, $\partial^* F_t$ is a $C^{1,\frac{1}{2}}$ hypersurface and $(\partial F_t \setminus \partial^* F_t) \cap \Omega$ has Hausdorff dimension less than $d - 8$.*

Finally, the question of whether one can assign a pointwise geometric meaning to $z \cdot D\chi_E$ was addressed by Chambolle, Goldman and Novaga in [11]. In dimensions $d = 2$ and $d = 3$, it is indeed proved in [11] that if $g = -\mathrm{div}(z) \in L^d(\Omega)$ and E minimizes (5.2), then any point $x \in \partial^* E$ is a Lebesgue point of z and $z(x) = \nu_E(x)$ where ν_E is the unit outward normal to $\partial^* E$.

6 The case of step functions

As another illustration of the results of section 4, we have the following result concerning step-functions in dimension one:

Theorem 6.1. *Let $d = 1$, $\Omega = (a, b)$ and ρ_0 be a step function with at most N-discontinuities i.e.:*

$$\rho_0 := \sum_{j=0}^{N} \alpha_j \chi_{[a_j,\, a_{j+1})},\ a_0 = a < a_1 \cdots < a_N < a_{N+1} = b, \tag{6.1}$$

then the solution ρ_1 of (3.1) is also a step function with at most N discontinuities.

Proof. **Step 1: reduction to the positive case** We first claim that we may reduce ourselves to the case where $\rho_0 \geq \alpha > 0$ (so that $\rho_1 \geq \alpha > 0$ as well by virtue of Proposition 3.4). Indeed, assume that the statement of Theorem 6.1 holds under the additional assumption that $\alpha := \min(\alpha_0, \cdots, \alpha_N) > 0$.

Then, setting for every integer $n \geq 1$, $\rho_0^n := \frac{1}{n} + (1 - \frac{1}{n})\rho_0$, the corresponding solution of (3.1), ρ_1^n will also be a step function with at most N discontinuities. It is clear that up to a subsequence, ρ_1^n converges strongly in L^1 as $n \to \infty$ and a.e. to ρ_1 which thus also has to be a step function with at most N discontinuities. We therefore assume from now on that ρ_0 and ρ_1 are everywhere positive.

Step 2 : ρ_1 is a jump function. Thanks to Theorem 4.4 and Corollary 4.6, there is a $z \in W^{3,\infty}$ such that $z(a) = z(b) = 0$, $|z| \leq 1$ and a Kantorovich potential φ such that

$$z' + \frac{\varphi}{\tau} = 0, \ \varphi'(x) = x - T(x), \tag{6.2}$$

where T is the optimal (monotone nondecreasing) transport between ρ_1 and ρ_0:

$$F_0 \circ T = F_1, \ F_0(x) := \int_a^x \rho_0, \ F_1(x) := \int_a^x \rho_1, \tag{6.3}$$

(note that T is a bi-Lipschitz homeomorphism) and

$$J(\rho_1) = \int_a^b z'\rho_1 = -\int_a^b z \cdot D\rho_1 = |D\rho_1|(a,b)$$

where $D\rho_1$ is the (signed measure) distributional derivative of ρ_1. Observe also that from (6.2) points at which z'' vanish are fixed points of T.

We then perform a Hahn-Jordan decomposition of $D\rho_1$:

$$D\rho_1 = \mu^+ - \mu^-, \ \mu^+ \geq 0, \ \mu^- \geq 0, \ \mu^+ \perp \mu^-, \tag{6.4}$$

and set

$$A := \operatorname{spt}(|D\rho_1|) = A^+ \cup A^- \text{ with } A^+ := \operatorname{spt}(\mu^+), \ A^- := \operatorname{spt}(\mu^-). \tag{6.5}$$

Next, noting that $|D\rho_1| = \mu^+ + \mu^- = -z(\mu^+ - \mu^-)$, we deduce that $z = -1$ μ^+-a.e and since z is continuous we should have $z = -1$ on $A^+ = \operatorname{spt}(\mu^+)$. In a similar way, $z = 1$ on $A^- := \operatorname{spt}(\mu^-)$, it implies in particular that the compact sets A^+ and A^- are disjoint so that the distance between A^+ and A^- is positive. Note also that since z is C^2, minimal on A^+ and maximal on A^- we have (also see [10] for a similar discussion):

$$z' = 0 \text{ on } A, \ z'' \geq 0 \text{ hence } T \geq \text{id on } A^+, \text{ and } z'' \leq 0 \text{ hence } T \leq \text{id on } A^-. \tag{6.6}$$

Since $z' = 0$ on A, it follows from Rolle's Theorem that if $a < x < y < b$ with $x, y \in A \times A$, there exists $c \in (x, y)$ such that $z''(c) = 0$ i.e. $T(c) = c$. In particular $T = \mathrm{id}$ on the set of limit points of A. We now further decompose $\mu^\pm$ in its purely atomic and nonatomic parts:

$$\mu^\pm = \sum_{x\in J^\pm} \mu^\pm(\{x\})\delta_x + \widetilde{\mu}^\pm, \tag{6.7}$$

where $J^\pm$ is the (finite or countable) set of atoms of $\mu^\pm$ and $\widetilde{\mu}^\pm$ has no atom. Our aim is to show that the sets

$$\widetilde{A}^\pm := \mathrm{spt}(\widetilde{\mu}^\pm), \tag{6.8}$$

are empty. Assume on the contrary that $\widetilde{A}^+ \neq \emptyset$, then since all points of $\widetilde{A}^+$ are limit points of A^+, $T = \mathrm{id}$ on $\widetilde{A}^+$. In particular this implies that

$$\chi_{\widetilde{A}^+}\rho_1 = T_\#(\chi_{\widetilde{A}^+}\rho_1) = \chi_{T(\widetilde{A}^+)}T_\#\rho_1 = \chi_{\widetilde{A}^+}\rho_0,$$

i.e. $\rho_0 = \rho_1$ on $\widetilde{A}^+$. Now if $x \in \widetilde{A}^+ \setminus \{a_0, \cdots a_{N+1}\}$, we may find $\delta > 0$ such that ρ_0 is constant on $[x-\delta, x+\delta]$ and $[x-\delta, x+\delta] \cap A^- = \emptyset$, so that ρ_1 is nondecreasing on $[x-\delta, x+\delta]$. Define then

$$x_1 := \inf \widetilde{A}^+ \cap [x-\delta, x+\delta],\ x_2 := \sup \widetilde{A}^+ \cap [x-\delta, x+\delta],$$

since both x_1 and x_2 lie in $\widetilde{A}^+$ and $D\rho_1 = \mu^+ \geq \widetilde{\mu}^+$ on $[x-\delta, x+\delta]$ we have

$$\rho_1(x_2) - \rho_1(x_1) = \rho_0(x_2) - \rho_0(x_1) = 0 \geq \widetilde{\mu}^+([x_1, x_2]) = \widetilde{\mu}^+([x-\delta, x+\delta])$$

which contradicts $x \in \widetilde{A}^+$. This proves that μ^+ (and μ^- likewise) are purely atomic (i.e. ρ_1 is a jump function in the terminology of [1]):

$$D\rho_1 = \sum_{x\in J^+} \mu^+(\{x\})\delta_x - \sum_{x\in J^-} \mu^-(\{x\})\delta_x.$$

Step 3: the jump sets J^+ and J^- are finite. Recall from the previous step that $A^+ = \overline{J^+}$ and $A^- = \overline{J^-}$ are disjoint sets. In particular, there cannot be points which are both limit points of J^+ and J^-. We argue by contradiction that J^+ is a finite set (a similar argument can be applied for J^-). Suppose that J^+ is not finite so that for some $x \in J^+$, every neighbourhood of x contains an element of J^+. Then, there exists $x_1 \in J^+$ with $x_1 \neq x$ ($x_1 > x$ say) such that $[x, x_1] \cap J^- = \emptyset$ (which implies that F_1 is convex on $[x, x_1]$). If $x_2 \in (x, x_1) \cap J^+$, then we know from the previous step that $T(x_2) \geq x_2$ and there exist $c_1 \in (x, x_2)$ and $c_2 \in (x_2, x_1)$ which are fixed points of T.

We then have

$$F_1(x_2) - F_1(c_1) = F_0(T(x_2)) - F_0(c_1) \geq F_0(x_2) - F_0(c_1)$$

and similarly

$$F_1(c_2) - F_1(x_2) = F_0(c_2) - F_0(T(x_2)) \leq F_0(c_2) - F_0(x_2)$$

but since ρ_1 has an upward jump at x_2 we have

$$\frac{F_1(x_2) - F_1(c_1)}{x_2 - c_1} < \frac{F_1(c_2) - F_1(x_2)}{c_2 - x_2}$$

hence

$$\frac{F_0(x_2) - F_0(c_1)}{x_2 - c_1} < \frac{F_0(c_2) - F_0(c_1)}{c_2 - x_2}$$

implying that ρ_0 has a discontinuity point in $[c_1, c_2]$ hence in $[x, x_1]$, since there are only finitely many such points this shows that J^+ is finite.

Step 4: ρ_1 has no more than N jumps. We know from the previous steps that ρ_1 can be written as

$$\rho_1 = \sum_{k=0}^{K} \beta_k \chi_{[b_k, b_{k+1})},\ b_0 = a < b_1 \cdots < b_K < b_{K+1} = b, \beta_k \neq \beta_{k+1}$$

If $\beta_{k+1} > \beta_k$ arguing exactly as in the previous step, we find two fixed-points of T, $c_k \in (b_k, b_k + 1)$ and $c_{k+1} \in (b_{k+1}, b_{k+2})$ such that ρ_0 has a discontinuity in (c_k, c_{k+1}), the case of a downward jump $\beta_k > \beta_{k+1}$ can be treated similarly (using $T(b_{k+1}) \leq b_{k+1}$ in this case). This shows that ρ_0 has at least K jumps so that $N \geq K$.

□

7 Convergence of the TV-JKO scheme in dimension one

We are now interested in the convergence of the TV-JKO scheme to a solution of the fourth-order nonlinear equation (1.2) in dimension 1, as the time step τ goes to 0. Throughout this section, we assume that $\Omega = (0, 1)$ and that the initial condition ρ_0 satisfies

$$\rho_0 \in \mathcal{P}_{\rm ac}((0,1)) \cap BV((0,1)),\ \rho_0 \geq \alpha > 0 \text{ a.e. on } (0,1). \tag{7.1}$$

We fix a time horizon T, and for small $\tau > 0$, define the sequence ρ_k^τ by

$$\rho_0^\tau = \rho_0,\ \rho_{k+1}^\tau \in \operatorname{argmin}\Big\{\frac{1}{2\tau} W_2^2(\rho_k^\tau, \rho) + J(\rho),\ \rho \in \mathrm{BV} \cap \mathcal{P}_{\mathrm{ac}}((0,1))\Big\} \quad (7.2)$$

for $k = 0, \dots N_\tau$ with $N_\tau := [\frac{T}{\tau}]$. Thanks to Proposition 3.4, (7.1) ensures that the JKO-iterates ρ_k^τ defined by (7.2) also remain bounded from below by α. We also extend this discrete sequence by piecewise constant interpolation i.e.

$$\rho^\tau(t,x) = \rho_{k+1}^\tau(x),\ t \in (k\tau, (k+1)\tau],\ k = 0, \dots N_\tau,\ x \in (0,1). \quad (7.3)$$

We shall see that ρ^τ converges to a solution ρ of

$$\partial_t \rho + \Big(\rho\Big(\frac{\rho_x}{|\rho_x|}\Big)_{xx}\Big)_x = 0,\ (t,x) \in (0,T) \times (0,1),\ \rho_{|t=0} = \rho_0, \quad (7.4)$$

with the no-flux boundary condition

$$\rho\Big(\frac{\rho_x}{|\rho_x|}\Big)_{xx} = 0, \quad \text{on } (0,T) \times \{0,1\}. \quad (7.5)$$

Since ρ is no more than BV in x, one has to be slightly cautious in the meaning of $\frac{\rho_x}{|\rho_x|}$ which be conveniently done by interpreting this term as the negative of a suitable z in the subdifferential of J (in the L^2 sense for instance):

$$z \in H_0^1((0,1)),\ \|z\|_{L^\infty} \leq 1 \text{ and } J(\rho) = \int_0^1 z_x \rho. \quad (7.6)$$

This leads to the following definition

Definition 7.1. *A weak solution of (7.4)-(7.1) is a* $\rho \in L^\infty((0,T), \mathrm{BV}((0,1))) \cap C^0((0,T), (\mathcal{P}, W_2))$ *such that there exists* $z \in L^\infty((0,T) \times (0,1)) \cap L^2((0,T), H^2 \cap H_0^1((0,1)))$ *with*

$$\|z(t,.)\|_{L^\infty} \leq 1 \textit{ and } J(\rho(t,.)) = \int_0^1 z_x(t,x)\rho(x)\,dx, \textit{ for a.e. } t \in (0,T), \quad (7.7)$$

and ρ *is a weak solution of*

$$\partial_t \rho - (\rho z_{xx})_x = 0,\ \rho_{|t=0} = \rho_0,\ \rho z_{xx} = 0 \textit{ on } (0,T) \times \{0,1\}. \quad (7.8)$$

i.e. for every $u \in C_c^1([0,T) \times [0,1])$

$$\int_0^T \int_0^1 (\partial_t u\, \rho - (\rho z_{xx}) u_x)\, dx\, dt = -\int_0^1 u(0,x)\rho_0(x)\, dx.$$

We then have

Theorem 7.2. *If ρ_0 satisfies (7.1), there exists a vanishing sequence of time steps $\tau_n \to 0$ such that the sequence ρ^{τ_n} constructed by (7.2)-(7.3) converges strongly in $L^p((0,T)\times(0,1))$ for any $p \in [1,+\infty)$ and in $C^0((0,T),(\mathcal{P}([0,1]),W_2))$ to $\rho \in L^\infty((0,T),\mathrm{BV}((0,1))) \cap C^0((0,T),(\mathcal{P}([0,1]),W_2))$, a weak solution of (7.4)-(7.1).*

Proof. First, ρ_0 being BV it is bounded on $(0,1)$ which gives uniform bounds on ρ^τ thanks to Theorem 3.2, moreover we know from (7.1) and Proposition 3.4 that we also have a uniform bound from below

$$M := \|\rho_0\|_{L^\infty} \geq \rho^\tau(t,x) \geq \alpha,\ t \in [0,T], \quad \text{a.e. } x \in [0,1]. \tag{7.9}$$

Moreover by construction of the TV-JKO scheme (7.2), one has

$$\frac{1}{2\tau}\sum_{k=0}^{N_\tau} W_2^2(\rho_k^\tau, \rho_{k+1}^\tau) \leq J(\rho_0), \quad \sup_{t\in[0,T]} J(\rho^\tau(t,.)) \leq J(\rho_0) \tag{7.10}$$

By using an Aubin-Lions type compactness Theorem of Savaré and Rossi (Theorem 2 in [26]), the fact that the imbedding of $\mathrm{BV}((0,1))$ into $L^p((0,1))$ is compact for every $p \in [1,+\infty)$ as well as a refinement of Arzèla-Ascoli Theorem (Proposition 3.3.1 in [2]), one obtains (see section 4 of [13] or section 5 of [8] for details) that, up to taking suitable sequence of vanishing times steps $\tau_n \to 0$, we may assume that

$$\rho^\tau \to \rho \text{ a.e. in } (0,T)\times(0,1) \text{ and in } L^p((0,T)\times(0,1)),\ \forall p \in [1,+\infty) \tag{7.11}$$

and

$$\sup_{t\in[0,T]} W_2(\rho^\tau(t,.), \rho(t,.)) \to 0 \text{ as } \tau \to 0, \tag{7.12}$$

for some limit curve $\rho \in C^{0,\frac{1}{2}}((0,T),(\mathcal{P}([0,1]),W_2)) \cap L^p((0,T)\times(0,1))$. From (7.9) and (7.10), one also deduces $\rho \in L^\infty((0,T),\mathrm{BV}((0,1))$ and from (7.9) $M \geq \rho \geq \alpha$.

We deduce from the fact that $\rho_k^\tau \geq \alpha > 0$ and Theorem 4.4 that for each $k = 0,\dots,N_\tau$, there exists $z_k^\tau \in W^{2,\infty}((0,1))$ such that

$$\|z_k^\tau\|_{L^\infty} \leq 1,\ z_k^\tau(0) = z_k^\tau(1) = 0,\ J(\rho_k^\tau) = \int_0^1 (z_k^\tau)_x \rho_k^\tau, \tag{7.13}$$

and the optimal (backward) optimal transport T_{k+1}^τ from ρ_{k+1}^τ to ρ_k^τ is related to z_{k+1}^τ by

$$\mathrm{id} - T_{k+1}^\tau = -\tau (z_{k+1}^\tau)_{xx}. \tag{7.14}$$

We extend z_k^τ in a piecewise constant way i.e. set

$$z^\tau(t,x) = z_{k+1}^\tau(x),\ t \in (k\tau, (k+1)\tau],\ k = 0,\dots,N_\tau,\ x \in (0,1). \tag{7.15}$$

We then observe that

$$\begin{aligned} W_2^2(\rho_k^\tau, \rho_{k+1}^\tau) &= \int_0^1 (x - T_{k+1}^\tau(x))^2 \rho_{k+1}^\tau(x)\mathrm{d}x \\ &= \tau^2 \int_0^1 (z_{k+1}^\tau)_{xx}^2 \rho_{k+1}^\tau(x)\mathrm{d}x \\ &\geq \alpha\tau^2 \int_0^1 (z_{k+1}^\tau)_{xx}^2 \mathrm{d}x \end{aligned}$$

Thanks to (7.10) we thus get an $L^2((0,T), H^2((0,1))$ bound

$$\|z^\tau\|_{L^2((0,T),H^2((0,1)))} \leq C. \tag{7.16}$$

We may therefore assume (up to further suitable extractions) that there is some $z \in L^\infty((0,T)\times(0,1)) \cap L^2((0,T), H^2((0,1)))$ such that z^τ converges weakly $*$ in $L^\infty((0,T)\times(0,1))$ and weakly in $L^2((0,T), H^2((0,1)))$ to z. Of course $\|z\|_{L^\infty} \leq 1$ and $z \in L^2((0,T), H_0^1((0,1))$. Note also that $\rho^\tau z_{xx}^\tau$ converges weakly to ρz_{xx} in $L^1((0,T)\times(0,1))$.

The limiting equation can now be derived using standard computations (see the proof of Theorem 5.1 of the seminal work [18], or chapter 8 of [28]): Let $u \in C_c^1([0,T)\times[0,1])$ and observe that

$$\int_0^T \int_0^1 \partial_t u\ \rho^\tau \mathrm{d}x\mathrm{d}t = \sum_{k=1}^{N_\tau} \left(\int_0^1 u(k\tau,x)(\rho_k^\tau(x) - \rho_{k+1}^\tau(x))\mathrm{d}x \right) - \int_0^1 u(0,x)\rho_1^\tau(x)\mathrm{d}x.$$

Recalling that $\rho_k^\tau = T_{k+1\#}^\tau \rho_{k+1}^\tau$, and applying Taylor's theorem, we have

$$\begin{aligned} &\sum_{k=1}^{N_\tau} \left(\int_0^1 u(k\tau,x)(\rho_k^\tau(x) - \rho_{k+1}^\tau(x))\mathrm{d}x \right) \\ &= \sum_{k=1}^{N_\tau} \left(\int_0^1 ((T_{k+1}^\tau(x) - x)u_x(k\tau,x) + \tilde{R}_\tau(x))\rho_{k+1}^\tau \mathrm{d}x \right) \\ &= \sum_{k=1}^{N_\tau} \left(\int_0^1 (\tau(z_{k+1}^\tau)_{xx} u_x(k\tau,x) + \tilde{R}_\tau(x))\rho_{k+1}^\tau \mathrm{d}x \right), \end{aligned}$$

where $|\tilde{R}_\tau(x)| \leq C\|u_{xx}(k\tau,\cdot)\|_{L^\infty}|T_{k+1}^\tau(x) - x|^2$. Note also that for $t \in (k\tau,(k+1)\tau]$, $|u_x(k\tau,\cdot) - u_x(t,\cdot)| \leq \tau\|u_{xt}\|_{L^\infty}$. Therefore,

$$\int_0^T \int_0^1 (\partial_t u\ \rho^\tau - \rho^\tau z_{xx}^\tau u_x)\mathrm{d}x\mathrm{d}t = -\int_0^1 u(0,x)\rho_1^\tau(x)\mathrm{d}x + R_\tau(u) \tag{7.17}$$

with

$$|R_\tau(u)| \le C \max\{\|u_{xx}\|_{L^\infty}, \|u_{xt}\|_{L^\infty}\} \sum_{k=0}^{N_\tau} W_2^2(\rho_k^\tau, \rho_{k+1}^\tau) \le C\tau. \tag{7.18}$$

Passing to the limit τ to 0 in (7.17) yields that ρ is a weak solution to

$$\partial_t \rho - (\rho z_{xx})_x = 0,\ \rho_{|t=0} = \rho_0,\ \rho z_{xx} = 0 \text{ on } (0,T) \times \{0,1\}.$$

It remains to prove that $J(\rho(t,.)) = \int_0^1 z_x(t,x)\rho(x)\mathrm{d}x$, for a.e. $t \in (0,T)$. The inequality $J(\rho(t,.)) \ge \int_0^1 z_x(t,x)\rho(x)\mathrm{d}x$ is obvious since $z(t,.) \in H_0^1((0,1))$ and $\|z(t,.)\|_{L^\infty} \le 1$. To prove the converse inequality, we use Fatou's Lemma, the lower semi-continuity of J, (7.13) and the weak-convergence in $L^1((0,T)\times(0,1))$ of $z_x^\tau \rho^\tau$ to $z_x \rho$:

$$\begin{aligned}
\int_0^T J(\rho(t,.))\mathrm{d}t &\le \int_0^1 \liminf_\tau J(\rho^\tau(t,.))\mathrm{d}t \\
&\le \liminf_\tau \int_0^1 J(\rho^\tau(t,.))\mathrm{d}t \\
&= \liminf_\tau \int_0^T \int_0^1 z_x^\tau(t,x)\rho^\tau(t,x)\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_0^1 z_x(t,x)\rho(t,x)\mathrm{d}x\mathrm{d}t
\end{aligned}$$

which concludes the proof.

□

Acknowledgements: The authors wish to thank Vincent Duval and Gabriel Peyré for suggesting the TV-Wasserstein problem to them as well as for fruitful discussions. They also thank Maxime Laborde and Filippo Santambrogio for helpful remarks in particular regarding the maximum principle.

References

[1] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford university press, 2000.

[2] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows: in metric spaces and in the space of probability measures.* Springer Science & Business Media, 2008.

[3] G. Anzellotti. Pairing between measures and bounded functions and compensated compactness. *Ann. di Matematica Pura ed Appl.*, IV(135):293–318, 1983.

[4] Giovanni Bellettini, Vicent Caselles, and Matteo Novaga. The total variation flow in $\mathbb{R}^n$. *Journal of Differential Equations*, 184(2):475–525, 2002.

[5] Martin Benning, Luca Calatroni, Bertram Düring, and Carola-Bibiane Schönlieb. A primal-dual approach for a total variation Wasserstein flow. In *Geometric Science of Information*, pages 413–421. Springer, 2013.

[6] Yann Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.

[7] Martin Burger, Marzena Franek, and Carola-Bibiane Schönlieb. Regularized regression and density estimation based on optimal transport. *Applied Mathematics Research eXpress*, 2012(2):209–253, 2012.

[8] Guillaume Carlier and Maxime Laborde. A splitting method for nonlinear diffusions with nonlocal, nonpotential drifts. *Nonlinear Anal.*, 150:1–18, 2017.

[9] Antonin Chambolle, Vicent Caselles, Daniel Cremers, Matteo Novaga, and Thomas Pock. An introduction to total variation for image analysis. In *Theoretical foundations and numerical methods for sparse recovery*, volume 9 of *Radon Ser. Comput. Appl. Math.*, pages 263–340. Walter de Gruyter, Berlin, 2010.

[10] Antonin Chambolle, Vincent Duval, Gabriel Peyré, and Clarice Poon. Geometric properties of solutions to the total variation denoising problem. *Inverse Problems*, 33(1):015002, 2016.

[11] Antonin Chambolle, Michael Goldman, and Matteo Novaga. Fine properties of the subdifferential for a class of one-homogeneous functionals. *Advances in Calculus of Variations*, 8(1):31–42, 2015.

[12] Guido De Philippis, Alpár Richárd Mészáros, Filippo Santambrogio, and Bozhidar Velichkov. BV estimates in optimal transportation and applications. *Archive for Rational Mechanics and Analysis*, 219(2):829–860, 2016.

[13] Marco Di Francesco and Daniel Matthes. Curves of steepest descent are entropy solutions for a class of degenerate convection-diffusion equations. *Calc. Var. Partial Differential Equations*, 50(1-2):199–230, 2014.

[14] Bertram Düring and Carola-Bibiane Schönlieb. A high-contrast fourth-order pde from imaging: numerical solution by ADI splitting. *Multiscale and High-Contrast Partial Differential Equations, H. Ammari et al.(eds.)*, pages 93–103, 2012.

[15] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. II. *Trans. Amer. Math. Soc.*, 330(1):321–332, 1992.

[16] Mi-Ho Giga and Yoshikazu Giga. Very singular diffusion equations: second and fourth order problems. *Jpn. J. Ind. Appl. Math.*, 27(3):323–345, 2010.

[17] Yoshikazu Giga, Hirotoshi Kuroda, and Hideki Matsuoka. Fourth-order total variation flow with Dirichlet condition: characterization of evolution and extinction time estimates. *Adv. Math. Sci. Appl.*, 24(2):499–534, 2014.

[18] Richard Jordan, David Kinderlehrer, and Felix Otto. The variational formulation of the Fokker–Planck equation. *SIAM journal on mathematical analysis*, 29(1):1–17, 1998.

[19] Daniel Loibl, Daniel Matthes, and Jonathan Zinsl. Existence of weak solutions to a class of fourth order partial differential equations with Wasserstein gradient structure. *Potential Anal.*, 45(4):755–776, 2016.

[20] Francesco Maggi. *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*, volume 135. Cambridge University Press, 2012.

[21] E Gonzalez-U Massari. Variational mean curvatures. *Rend. Sem. Mat. Univ. Pol. Torino*, 52:1–28, 1994.

[22] Umberto Massari. Esistenza e regolarita delle ipersuperfici di curvatura media assegnata in $\mathbb{R}^n$. *Arch. Ration. Mech. Anal.*, 55:357–382, 1974.

[23] Umberto Massari. Frontiere orientate di curvatura media assegnata in L^p. *Rendiconti del Seminario Matematico della Università di Padova*, 53:37–52, 1975.

[24] Daniel Matthes, Robert J. McCann, and Giuseppe Savaré. A family of nonlinear fourth order equations of gradient flow type. *Comm. Partial Differential Equations*, 34(10-12):1352–1397, 2009.

[25] Robert J. McCann. A convexity principle for interacting gases. *Adv. Math.*, 128(1):153–179, 1997.

[26] Riccarda Rossi and Giuseppe Savaré. Tightness, integral equicontinuity and compactness for evolution problems in Banach spaces. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 2(2):395–431, 2003.

[27] Leonid I. Rudin, Stanley Osher, and Emad Fatemi. Nonlinear total variation based noise removal algorithms. *Phys. D*, 60(1-4):259–268, 1992. Experimental mathematics: computational issues in nonlinear science (Los Alamos, NM, 1991).

[28] Filippo Santambrogio. *Optimal transport for applied mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling.

[29] Italo Tamanini. Boundaries of Caccioppoli sets with Hölder-continuous normal vector. *Journal für die reine und angewandte Mathematik*, 334:27–39, 1982.

[30] Cédric Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.