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A Power Log-Dagum Distribution: Estimation and Applications

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ABSTRACT

Development and application of probability models in data analysis are of major importance for all sciences. Therefore, we introduce a new model called a power log-Dagum distribution defined on the entire real line. The model contains many new sub-models: power logistic, linear log-Dagum, linear logistic and log-Dagum distributions among them. Some properties of the model including three different estimation procedures are justified. The model exhibits various shapes for the density and hazard rate functions. Moreover, the estimation procedures are compared using simulation studies. Finally, the model with others are fitted to three data sets and it shows a better fit than the compared distributions defined on the real line.

KEYWORDS

Distributions on the real line, Moments, Estimation, Goodness of fit statistics, TTT-plot.

2000 MSC: 60E05, 62E15

1. Introduction

Statistical distributions play a significant role in describing and predicting real world phenomena. In the 1970s, Camilo Dagum developed a statistical distribution to fit empirical income and wealth data that are not satisfied with the classical distributions (Pareto and lognormal distributions). He looked for a model accommodating the heavy tails appear in empirical income and wealth data distributions, where the former distribution is well captured by the Pareto but not by the lognormal and the latter by the lognormal but not the Pareto. Experimenting with a shifted log-logistic distribution [5], Dagum realized that a further parameter was needed to such distribution which led to the Dagum type I and generalizations with three-parameter and four-parameter distributions [6, 7].

In the same era Mielke and Johnson [14] proposed the generalized beta distribution of the second kind abbreviated as GBDII. This distribution is used in the flood fre-

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quency analysis and it has the beta-k distribution as a sub-model. After that various authors have shown that the Dagum distribution [5] and GBDII are identical and they are two different parameterizations of the same distribution (see, for example, [4, 15]). Domma and Perri [8] proposed the log-Dagum (LD) distribution obtained by a logarithmic transformation of the Dagum distribution. The LD distribution is defined on the real line and its shape is leptokurtic. Also, it may be symmetric and asymmetric, and hence shall be useful in modeling skewed and leptokurtic distributions which frequently occur in several areas such as finance, reliability, econometrics, insurance and hydrology.

Interpretations of the real world phenomena needs introducing new statistical distributions, namely ones defined on the whole real line and having bimodal behavior for both density and hazard rate functions. Therefore, we introduce a new model called a power log-Dagum (PLD) distribution with the cumulative distribution function (cdf)

$$F(x) = \left\{ 1 + e^{-(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^\vartheta)} \right\}^{-\zeta}, \quad x, \vartheta \in \mathfrak{R}, \zeta > 0, \varrho \geq 0, \quad (1)$$

where sign is 1 if $x > 0$, 0 if $x = 0$ and -1 if $x < 0$. In addition to, it has the following representation: $F(x) = G[xw(x)]$ where $w(x)$ is the polynomial weight: $w(x) = \vartheta + \frac{\varrho}{\vartheta} |x|^{\vartheta-1}$, satisfying $\lim_{x \rightarrow -\infty} xw(x) = -\infty$ and $\lim_{x \rightarrow +\infty} xw(x) = +\infty$, and $G(x) = \{1 + e^{-x}\}^{-\zeta}$ is a cdf of the LD distribution with parameters $(\zeta, 1, 1)$.

The corresponding probability density function (pdf) and the hazard rate function (hrf) are given as

$$f(x) = \zeta \left(\vartheta + \varrho |x|^{\vartheta-1} \right) e^{-(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^\vartheta)} \left\{ 1 + e^{-(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^\vartheta)} \right\}^{-(\zeta+1)}, \quad (2)$$

$$h(x) = \frac{\zeta \left(\vartheta + \varrho |x|^{\vartheta-1} \right) e^{-(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^\vartheta)} \left\{ 1 + e^{-(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^\vartheta)} \right\}^{-(\zeta+1)}}{1 - \left\{ 1 + e^{-(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^\vartheta)} \right\}^{-\zeta}}, \quad (3)$$

respectively.

Obviously, the PLD distribution is defined on the entire real line and this is one of the important features of it, unlike the Dagum [5] and GBDII [14] distributions, which can only provide support on the positive real line.

The PLD distribution defined by (1) has the following submodels.

- (1) When $\zeta = 1$, then (1) reduces to the power logistic (PLo) distribution with the density

$$f(x) = \frac{(\vartheta + \varrho |x|^{\vartheta-1}) e^{-(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^\vartheta)}}{\left[1 + e^{-(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^\vartheta)} \right]^2}, \quad x, \vartheta \in \mathfrak{R}, \vartheta > 0, \varrho \geq 0. \quad (4)$$

- (2) When $\vartheta = 2$, then (1) reduces to the linear log-Dagum distribution with the density

$$f(x) = \zeta (2 + \varrho |x|) \left[1 + e^{-(2x + \text{sign}(x) \frac{\varrho}{2} x^2)} \right]^{-\zeta-1} e^{-(2x + \text{sign}(x) \frac{\varrho}{2} x^2)}.$$

- (3) When $\vartheta = 2$ and $\zeta = 1$, then (1) gives the linear logistic distribution with the

density

$$f(x) = \frac{(2 + \varrho|x|) e^{-(2x + \text{sign}(x)\frac{\varrho}{2}x^2)}}{\left[1 + e^{-(2x + \text{sign}(x)\frac{\varrho}{2}x^2)}\right]^2}.$$

- (4) When $\vartheta = 2$ and $\varrho = 0$, then (1) reduces to the log-Dagum distribution with parameters $(\zeta, 2)$ and the density

$$f(x) = 2\zeta (1 + e^{-2x})^{-\zeta-1} e^{-2x},$$

which is introduced by [8].

- (5) When $\vartheta = 2, \zeta = 1$ and $\varrho = 0$, then (1) leads to the known logistic distribution with the density

$$f(x) = \frac{2e^{-2x}}{[1 + e^{-2x}]^2}.$$

[Figure 1 about here.]

Figure 1 gives the plots of the cumulative distribution function of the PLD distribution. The plots of this figure shows that for fixed ϑ and ϱ and changing ζ the curve stretch out insignificantly towards right as ζ increases. However, for fixed ζ and ϱ and changing ϑ the curve stretch out towards right significantly as ϑ increases.

[Figure 2 about here.]

Plots of Figure 2 display the density functions of the PLD distribution. Figure 2 portrays that changing ϱ against the fixed ϑ and ζ shift the mode towards left. But in case of changing ζ with fixed ϑ and ϱ shift the curve towards right. However, the interesting feature of the distribution is its bimodal behavior which is frequently used in biomedical and engineering phenomenon like formation of bathtub shapes of the hazard function. Such bimodal behavior is captured while fixing ζ and ϱ and changing ϑ . Further, Figure 2 portrays that bimodality nature of the curve shift towards right as ϑ increases.

[Figure 3 about here.]

Hazard function is an important indicator for observing the deteriorating condition of a product which ranges from increasing, decreasing, bathtub (BT) to inverse bathtub (IBT) shapes. So in this regard Figure 3 speaks out it self and justifies the potential of the model. Moreover, the hazard function plots in Figure 3 also portray the deteriorating conditions of the product as time increases in terms of spontaneous spikes at the end of either increasing or decreasing hazard rate. This implies that the hazard function is sensitive against different combinations of the parameters as time changes, which seems to be a refine image of non stationarity process and hence the hazard curve does not remain stable as times passes. Moreover, Figure 3 displays increasing, decreasing, bimodal and upside down bathtub hazard shapes.

It is worth mentioning that the economic and hydrologic data analysis is based on the assumption that the data are stationary but a number of documented researches showed that these data may be non-stationary. Therefore, identification and use of non-stationary probabilistic models in practice have been recommended (See [16]). In

application section of this paper, Example 2 shows the merit of the PLD model for modeling and analyzing non-stationary time series data as inflation rates with positive and negative values.

The reminder of the paper is outlined as follows. We discuss some statistical properties of the PLD distribution in Section 2, including moments, moment generating function and moments of order statistics. We provide in Section 3 three estimation procedures of the PLD model parameters, namely the maximum likelihood estimation, ordinary and weighted least square estimation, and they are compared using simulations studies. Applications of the PLD model for three practical data sets are discussed in Section 4.

2. Some Statistical Properties

In this section, we study some statistical properties of the PLD distribution, including moments, moment generating function and moments of order statistics.

2.1. Moments and moment generating function

Let X be a PLD random variable, the r^{th} moment of X , say $\mu'_r = E(X^r)$, follows from (2) as

$$\mu'_r = \zeta \int_{-\infty}^{\infty} x^r \left(\vartheta + \varrho |x|^{\vartheta-1} \right) e^{-\left(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^{\vartheta} \right)} \left\{ 1 + e^{-\left(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^{\vartheta} \right)} \right\}^{-\zeta-1} dx.$$

Using the series expansion

$$\left\{ 1 + e^{-\left(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^{\vartheta} \right)} \right\}^{-\zeta-1} = \begin{cases} \sum_{j=0}^{\infty} \binom{-\zeta-1}{j} e^{-j\left(\vartheta x + \frac{\varrho}{\vartheta} x^{\vartheta} \right)}, & \text{if } x > 0, \\ \sum_{j=0}^{\infty} \binom{-\zeta-1}{j} e^{(\zeta+j+1)\left(\vartheta x - \frac{\varrho}{\vartheta} (-x)^{\vartheta} \right)}, & \text{if } x < 0, \end{cases} \quad (5)$$

we obtain the moment as

$$\begin{aligned} \mu'_r = & \zeta \vartheta \sum_{j=0}^{\infty} \binom{-\zeta-1}{j} \left\{ \int_{-\infty}^0 x^r e^{-\left(\vartheta x - \frac{\varrho}{\vartheta} (-x)^{\vartheta} \right)} e^{(\vartheta x - \frac{\varrho}{\vartheta} (-x)^{\vartheta})(\zeta+j+1)} dx \right. \\ & + \left. \int_0^{\infty} x^r e^{-\left(\vartheta x + \frac{\varrho}{\vartheta} x^{\vartheta} \right)} e^{-j\left(\vartheta x + \frac{\varrho}{\vartheta} x^{\vartheta} \right)} dx \right\} \\ & + \zeta \varrho \sum_{j=0}^{\infty} \binom{-\zeta-1}{j} \left\{ \int_{-\infty}^0 x^r (-x)^{\vartheta-1} e^{-\left(\vartheta x - \frac{\varrho}{\vartheta} (-x)^{\vartheta} \right)} e^{(\vartheta x - \frac{\varrho}{\vartheta} (-x)^{\vartheta})(\zeta+j+1)} dx \right. \\ & + \left. \int_0^{\infty} x^{r+\vartheta-1} e^{-\left(\vartheta x + \frac{\varrho}{\vartheta} x^{\vartheta} \right)} e^{-j\left(\vartheta x + \frac{\varrho}{\vartheta} x^{\vartheta} \right)} dx \right\}. \end{aligned}$$

By using the series expansion, we can write

$$\begin{aligned}\mu'_r = & \zeta \vartheta \sum_{j,k=0}^{\infty} \binom{-\zeta-1}{j} \left\{ \frac{(-\frac{\varrho}{\vartheta}(\zeta+j))^k}{k!} \int_{-\infty}^0 x^r (-x)^{\vartheta k} e^{\vartheta(\zeta+j)x} dx \right. \\ & + \left. \frac{(-\frac{\varrho}{\vartheta}(1+j))^k}{k!} \int_0^{\infty} x^{r+\vartheta k} e^{-\vartheta(1+j)x} dx \right\} \\ & + \zeta \varrho \sum_{j,k=0}^{\infty} \binom{-\zeta-1}{j} \left\{ \frac{(-\frac{\varrho}{\vartheta}(\zeta+j))^k}{k!} \int_{-\infty}^0 x^r (-x)^{\vartheta(1+k)-1} e^{\vartheta(\zeta+j)x} dx \right. \\ & + \left. \frac{(-\frac{\varrho}{\vartheta}(1+j))^k}{k!} \int_0^{\infty} x^{r+\vartheta(1+k)-1} e^{-\vartheta(1+j)x} dx \right\}.\end{aligned}$$

Hence

$$\begin{aligned}\mu'_r = & \zeta \vartheta \sum_{j,k=0}^{\infty} \binom{-\zeta-1}{j} \frac{1}{k!} \left[\frac{\{-\frac{\varrho}{\vartheta}(\zeta+j)\}^k (-1)^r}{\{\vartheta(\zeta+j)\}^{1+\vartheta k+r}} + \frac{\{-\frac{\varrho}{\vartheta}(1+j)\}^k}{(\vartheta+\vartheta j)^{1+\vartheta k+r}} \right] \Gamma(1+\vartheta k+r) \\ & + \zeta \varrho \sum_{j,k=0}^{\infty} \binom{-\zeta-1}{j} \frac{1}{k!} \left[\frac{\{-\frac{\varrho}{\vartheta}(\zeta+j)\}^k (-1)^r}{\{\vartheta(\zeta+j)\}^{\vartheta+\vartheta k+r}} + \frac{\{-\frac{\varrho}{\vartheta}(1+j)\}^k}{(\vartheta+\vartheta j)^{\vartheta+\vartheta k+r}} \right] \Gamma(\vartheta+\vartheta k+r).\end{aligned}$$

The moment generating function (mgf) of the PLD distribution is

$$M(t) = \zeta \int_{-\infty}^{\infty} e^{tx} \left(\vartheta + \varrho |x|^{\vartheta-1} \right) \left\{ 1 + e^{-(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^{\vartheta})} \right\}^{-(\zeta+1)} e^{-(\vartheta x + \text{sign}(x) \frac{\varrho}{\vartheta} |x|^{\vartheta})} dx, \quad (6)$$

and making use of (5), we have

$$\begin{aligned}M(t) = & \zeta \vartheta \sum_{j,k=0}^{\infty} \binom{-\zeta-1}{j} \frac{1}{k!} \left[\frac{\{-\frac{\varrho}{\vartheta}(\zeta+j)\}^k}{\{\vartheta(\zeta+j)+t\}^{1+\vartheta k}} + \frac{\{-\frac{\varrho}{\vartheta}(1+j)\}^k}{(\vartheta+\vartheta j-t)^{1+\vartheta k}} \right] \Gamma(1+\vartheta k) \\ & + \zeta \varrho \sum_{j,k=0}^{\infty} \binom{-\zeta-1}{j} \frac{1}{k!} \left[\frac{\{-\frac{\varrho}{\vartheta}(\zeta+j)\}^k}{\{\vartheta(\zeta+j)+t\}^{\vartheta(1+k)}} + \frac{\{-\frac{\varrho}{\vartheta}(1+j)\}^k}{(\vartheta+\vartheta j-t)^{\vartheta(1+k)}} \right] \Gamma(\vartheta+\vartheta k).\end{aligned}$$

2.2. Moments of order statistics

Let $X_1, X_2, \dots, X_n$ be a random sample from the PLD distribution and its order statistics is $X_{1:n}, X_{2:n}, \dots, X_{n:n}$. Let $f_{i:n}(x)$ and $F_{i:n}(x)$ denote the pdf and the cdf of the i^{th} order statistic $X_{i:n}$, respectively. Hence, using the standard expressions of order statistics, we find that

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l [F(x)]^{i-1+l} f(x),$$

and

$$F_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{l=0}^{n-i} \binom{n-i}{l} \frac{(-1)^l}{i+l} [F(x)]^{i+l},$$

where $F(x)$ and $f(x)$ are given by equations (1) and (2), respectively.

Hence, the r^{th} moment of the i^{th} order statistic $X_{i:n}$ is

$$E(X_{i:n}^r) = \int_{-\infty}^{\infty} x^r f_{i:n}(x) dx,$$

and by making use of $f_{i:n}(x)$ and equation (5) with some algebra, one find that

$$\begin{aligned} E(X_{i:n}^r) &= \frac{\zeta n!}{(i-1)!(n-i)!} \sum_{l=0}^{n-i} \sum_{j,k=0}^{\infty} \binom{n-i}{l} \binom{-\zeta(i+l)-1}{j} \frac{(-1)^l}{k!} \\ &\times \left[\vartheta \left\{ \frac{\left\{ \frac{\varrho}{\vartheta}(1+\zeta(i+l)-j) \right\}^k (-1)^r}{\{\vartheta(j-1-\zeta(i+l))\}^{1+\vartheta k+r}} + \frac{\left\{ -\frac{\varrho}{\vartheta}(1+j) \right\}^k}{(\vartheta+\vartheta j)^{1+\vartheta k+r}} \right\} \Gamma(1+\vartheta k+r) \right. \\ &\left. + \varrho \left\{ \frac{\left\{ \frac{\varrho}{\vartheta}(1+\zeta(i+l)-j) \right\}^k (-1)^r}{\{\vartheta(j-1-\zeta(i+l))\}^{\vartheta+\vartheta k+r}} + \frac{\left\{ -\frac{\varrho}{\vartheta}(1+j) \right\}^k}{(\vartheta+\vartheta j)^{\vartheta+\vartheta k+r}} \right\} \Gamma(\vartheta+\vartheta k+r) \right]. \end{aligned}$$

3. Estimation Procedures

In this section we investigate three estimation procedures for the PLD model parameters, namely the maximum likelihood estimation, ordinary and weighted least square estimations. Performance of those procedures is investigated by simulation studies.

3.1. Maximum likelihood estimators

Let $X_1, X_2, \dots, X_n$ be a random sample from the PLD distribution with parameter vector $\Theta = (\vartheta, \zeta, \varrho)$ and $x_1, x_2, \dots, x_n$ are the corresponding observed values, then the joint probability function of $X_1, X_2, \dots, X_n$ as a log-likelihood function can be expressed as

$$\begin{aligned} \ell(\Theta) &= n \log(\zeta) + \sum_{i=1}^n \log \left(\vartheta + \varrho |x_i|^{\vartheta-1} \right) - \sum_{i=1}^n \left(\vartheta x_i + \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^{\vartheta} \right) \\ &\quad - (\zeta + 1) \sum_{i=1}^n \log \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^{\vartheta}} \right). \end{aligned} \quad (7)$$

The associated nonlinear log-likelihood equations $\frac{\partial \ell(\Theta)}{\partial \Theta} = 0$ are given by

$$\begin{aligned}
\frac{\partial \ell(\Theta)}{\partial \vartheta} &= \sum_{i=1}^n \frac{1 + \varrho \log(|x_i|) |x_i|^{-1+\vartheta}}{\vartheta + \varrho |x_i|^{-1+\vartheta}} - \sum_{i=1}^n \left\{ x_i + \frac{\varrho}{\vartheta} \text{sign}(x_i) |x_i|^\vartheta \left(\log(|x_i|) - \frac{1}{\vartheta} \right) \right\} \\
&\quad - (1 + \zeta) \sum_{i=1}^n \frac{e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \left\{ -x_i - \frac{\varrho}{\vartheta} \text{sign}(x_i) |x_i|^\vartheta \left(\log(|x_i|) - \frac{1}{\vartheta} \right) \right\}}{1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}} = 0, \\
\frac{\partial \ell(\Theta)}{\partial \zeta} &= \frac{n}{\zeta} - \sum_{i=1}^n \log \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \right) = 0, \\
\frac{\partial \ell(\Theta)}{\partial \varrho} &= - \sum_{i=1}^n \text{sign}(x_i) \frac{|x_i|^\vartheta}{\vartheta} + (1 + \zeta) \sum_{i=1}^n \frac{e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \text{sign}(x_i) |x_i|^\vartheta}{\left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \right) \vartheta} \\
&\quad + \sum_{i=1}^n \frac{|x_i|^{-1+\vartheta}}{\vartheta + \varrho |x_i|^{-1+\vartheta}} = 0.
\end{aligned}$$

Solving the equations above simultaneously, we obtain the maximum likelihood estimators (MLEs) of the model parameters.

The information matrix is used to establish the confidence intervals for the model parameters. The elements of the 3×3 observed information matrix $J(\Theta) = J_{rs}(\Theta)$ for $r, s = \vartheta, \zeta, \varrho$ are provided in Appendix B.

In the simulation study for the MLEs (Subsection 3.3), it is observed that the bias and mean square error of the MLEs decrease as sample sizes increase which suit the usual norms of asymptotic properties of the MLEs. However, when all the parameters posses values greater than one, a downward bias is only reflected with ϑ and ϱ , and when all the parameters posses values less than one, a downward bias is only reflected with ϑ . In addition to, an upward bias is observed for all other combinations. Those features can be observed in Table 1.

3.2. Ordinary and weighted least-square estimators

Let $x_1, x_2, \dots, x_n$ be an ordered sample of the random sample of size n from a distribution function $F(\cdot)$. Let $F(\cdot)$ follow the PLD distribution, then the least square estimators (LSE) can be obtained by minimizing

$$\mathfrak{L}(\Theta) = \sum_{i=1}^n \left\{ \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \right)^{-\zeta} - \frac{i}{n+1} \right\}^2, \quad (8)$$

with respect to the unknown parameters of the distribution. The associated nonlinear equations $\frac{\partial \mathfrak{L}(\Theta)}{\partial \Theta} = 0$ are given by

$$\begin{aligned}
\frac{\partial \mathfrak{L}(\Theta)}{\partial \vartheta} &= 2\zeta \sum_{i=1}^n e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \left\{ \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \right)^{-\zeta} - \frac{i}{1+n} \right\} \\
&\quad \times \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \right)^{-1-\zeta} \left\{ x_i + \frac{\varrho}{\vartheta} \text{sign}(x_i) |x_i|^\vartheta \left(\log(|x_i|) - \frac{1}{\vartheta} \right) \right\} = 0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathfrak{L}(\Theta)}{\partial \zeta} &= -2 \sum_{i=1}^n \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-\zeta} \left\{ \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-\zeta} - \frac{i}{1+n} \right\} \\
&\quad \times \log \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right) = 0, \\
\frac{\partial \mathfrak{L}(\Theta)}{\partial \varrho} &= \frac{2\zeta}{\vartheta} \sum_{i=1}^n e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-1-\zeta} \\
&\quad \times \left\{ \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-\zeta} - \frac{i}{1+n} \right\} \text{sign}(x_i) |x_i|^\vartheta = 0.
\end{aligned}$$

Solving this system of nonlinear equations simultaneously will yield the LSEs of the distribution parameters.

In the simulation study, a down bias is noted in all of the parameter combination except for ϱ , when it is less than one. Moreover, fluctuation in bias and mean square error is reflected in parameter combinations which obviously approaches to zero. Table 2 shows such reflections.

On the other side, the weighted least square estimators (WLSE) of parameters of the PLD distribution can be obtained by minimizing

$$\mathfrak{W}(\Theta) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left\{ \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-\zeta} - \frac{i}{n+1} \right\}^2, \quad (9)$$

with respect to the unknown parameters. The associated nonlinear equations $\frac{\partial \mathfrak{W}(\Theta)}{\partial \Theta} = 0$ are given by

$$\begin{aligned}
\frac{\partial \mathfrak{W}(\Theta)}{\partial \vartheta} &= 2\zeta \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-1-\zeta} \\
&\quad \times \left\{ \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-\zeta} - \frac{i}{1+n} \right\} \\
&\quad \times \left\{ x_i + \frac{\varrho}{\vartheta} \text{sign}(x_i) |x_i|^\vartheta \left(\log(|x_i|) - \frac{1}{\vartheta} \right) \right\} = 0, \\
\frac{\partial \mathfrak{W}(\Theta)}{\partial \zeta} &= -2 \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left\{ \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-\zeta} - \frac{i}{1+n} \right\} \\
&\quad \times \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-\zeta} \log \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right) = 0, \\
\frac{\partial \mathfrak{W}(\Theta)}{\partial \varrho} &= \frac{2\zeta}{\vartheta} \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta} \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-1-\zeta} \\
&\quad \times \left\{ \left(1 + e^{-\vartheta x_i - \text{sign}(x_i) \frac{\varrho}{\vartheta} |x_i|^\vartheta}\right)^{-\zeta} - \frac{i}{1+n} \right\} \text{sign}(x_i) |x_i|^\vartheta = 0.
\end{aligned}$$

Solving the system of equations above yields the WLSEs of the distribution parameters.

A simulation work, see the next subsection, shows that a downward bias is observed in all parameter combinations except when $\varrho < 1$. In addition, fluctuation in bias and mean square error is remarked in all parameter combinations which approaches to zero. Table 3 exhibits such reflections.

3.3. Simulation studies

In this subsection, we perform a small simulation study to gain some results on the estimators obtained by using the above estimation methods as well as their asymptotic behavior for finite samples. All the numerical computations are performed via Mathematica 8.0 using the random numbers generator code. We consider the following different model parameters: Model-I: $\vartheta = 0.3642$, $\varrho = 2.3532$, $\zeta = 1.2315$, Model-II: $\vartheta = 5.8759$, $\varrho = 5.4314$, $\zeta = 4.5472$, Model-III: $\vartheta = 1.2536$, $\varrho = 0.4721$, $\zeta = 1.1973$ and Model-IV: $\vartheta = 0.0542$, $\varrho = 0.0135$, $\zeta = 0.0432$ for all the above discussed methods. We consider the following sample sizes $n = 15$, $n = 25$ (small), 50 (moderate), and 150 (large). For each model parameters and for each sample size, we compute the MLEs, LSEs and WLSEs of each $\vartheta, \varrho, \zeta$. We repeat this process 1000 times and compute the average bias and mean square error (MSE) for all replications in the relevant sample sizes.

The analysis computes the coming values:

- Average bias of the simulated estimates:

$$\frac{1}{1000} \sum_{i=1}^{1000} (\Theta^{\star} - \Theta).$$

- Average mean square error (MSE) of the simulated estimates:

$$\frac{1}{1000} \sum_{i=1}^{1000} (\Theta^{\star} - \Theta)^2,$$

where $\Theta^{\star} = (\hat{\vartheta}, \hat{\varrho}, \hat{\zeta})$ are estimates of the parameter vector $\Theta = (\vartheta, \zeta, \varrho)$. The results are reported in Tables 1, 2 and 3.

[Table 1 about here.]

[Table 2 about here.]

[Table 3 about here.]

4. Evaluation Measures and Practical Data Examples

In the statistical literature, various distributions are employed for different situations. Some of them are suitable for symmetrical data, some for skewed data and others are for both aspects. In this regard, we study those distributions which are either, positively skewed, negatively skewed or symmetrical and possess increasing, decreasing, bathtub shapes (BTS) and inverse bathtub (IBT) hazard function. Moreover, comparison also requires that random variable should also be defined on the whole real line, i.e. $x \in \mathbb{R}$. The above conditions demand the following known distributions that are: Normal, Cauchy, Gumbel, Dagum. For the probability density functions of the above mentioned distributions, the reader may refer to [12] and [13], noting that the log-dagum (LD) was developed by [8].

4.1. Numerical measures

In order to demonstrate the proposed methodology, we consider three different practical data sets described below with their analysis. They represent different level of skewness ranging from negative skewness to positive skewness along with various demonstration of failure rate pattern, like increasing, decreasing, and bathtub shape.

Moreover, perfection of competing models is also tested via the chi-squared (χ^2), Kolmogorov–Smirnov (K S), the Anderson–Darling (A^*) and the Cramér–von Misses (W^*) statistics. The mathematical expressions for the statistics are given by

$$KS = \max \left\{ \frac{i}{m} - z_i, z_i - \frac{i-1}{m} \right\}, \quad \chi^2 = \sum_{i=1}^m \frac{(o_i - e_i)^2}{e_i},$$

$$A^* = \left(\frac{2.25}{m^2} + \frac{0.75}{m} + 1 \right) \left\{ -m - \frac{1}{m} \sum_{i=1}^m (2i-1) \ln(z_i(1 - z_{m-i+1})) \right\},$$

$$W^* = \sum_{i=1}^m \left(z_i - \frac{2i-1}{2m} \right)^2 + \frac{1}{12m},$$

where m denotes the number of classes, $z_i = F_X(x_i)$, the x_i 's being the ordered observations, o_i and e_i are the observed and expected frequencies of the i^{th} class, respectively.

4.2. Graphical measure

A hazard or failure rate function can be considered for finding the chances of occurring a dangerous event that can lead to an emergency or disaster. It also conveys the meaning of intensity function, or risk rate, among other names. Moreover, such function also portrays different meaning of inherent characteristics of different life-time phenomena, like increasing hazard rate (IHR), constant, decreasing hazard rate (DHR), bathtub (BT) and inverse bathtub (IBT) shapes. For modeling the above mentioned characteristic of the hazard function, researchers generally used a graphical tool known as the total time on test (TTT) plot. The TTT–plot is a graph which mainly serves to discriminate between different types of aging represented in hazard rate shapes, for details the readers are referred to [11]. The TTT plot is drawn

by plotting $T(\frac{i}{n}) = \frac{\sum_{r=1}^i y_{r:n} + (n-i)y_{i:n}}{\sum_{r=1}^n y_{r:n}}$ against $\frac{i}{n}$, where $i = 1, \dots, n$ and $y_{r:n}$,

$r = 1, \dots, n$ are the order statistics of the sample. An empirical TTT–plot that takes its course randomly around the 45° –line indicates a sample from an exponential distribution, TTT–plot that is nearly concave (convex) and is mainly above (below) the 45° –line indicates a sample from an IHR (DHR) distribution, and TTT–plot that takes concave (convex) and then convex (concave) is related to a BT (IBT) hazard rate. The TTT–plots for the next data sets are displayed in Figure 4, which reveal increasing IHRs for the first two data sets while the third data set depicts the BT hazard rate.

[Figure 4 about here.]

Example 1. The first data set represents the strength data originally reported in [3]. It represents the strength measured in GPA for single carbon fibers and impregnated

1000-carbon fiber tows. Carbon fiber tow is a bunch of carbon fibre, like a yarn, that is commonly woven into carbon fibre fabrics. Single fibers were tested under tension at gauge length of 10 mm. Such data have been analyzed previously by [10] and [2]. The data are as follows: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

We fitted all six distributions to the above data by the method of maximum likelihood. Table 4 presents the descriptive statistics. Table 5 portrays parameter estimates, χ^2 , KS with p - values Anderson-Darling and Cramér-von Misses statistics.

[Table 4 about here.]

[Table 5 about here.]

Analysis: From Tables 4, and 5, we see that the proposed PLD model is the most suitable model when the data is positively skewed and underdispersed with approximately equal mean deviation about mean and median as well as posses IFR behavior. Moreover, the goodness of fit statistics also show that the proposed model is the most promising one in such situations too. In addition to, the proposed model also shows promising results while comparing with the models discussed by [2] in terms of the smallest χ^2 value. The compared models are: logistic distribution $L(\mu, \sigma)$ with $\chi^2 = 0.8832$, the type III generalized logistic distribution $GL(\mu, \sigma, \alpha)$ with $\chi^2 = 1.1055$, the skew logistic distribution $SL(\mu, \sigma, \lambda)$ with $\chi^2 = 0.8345$, generalized skew logistic $GSL(\mu, \sigma, \lambda, \alpha)$ with $\chi^2 = 0.2682$ and Azzalini and Capitanio's skew t distribution $ST(\mu, \sigma, \lambda, \alpha)$ with $\chi^2 = 0.2684$. For the density functions of the discussed distributions the reader may refer to [2] and the references therein. Moreover, while comparing with [10] proportional reversed hazard logistic (PRHL) and skew logistic distributions, it is observed that PLD has the smallest χ^2 and KS values.

Example 2. The second data set displays the inflation rate of USA for the period 1965 to 1981, and it is defined as the rate at which prices increase over time, resulting in a fall in the purchasing value of currency. The data is reported by [9], pp. 502. The data values are: -0.4, 0.4, 2.9, 3.0, 1.7, 1.5, 1.8, 0.8, 1.8, 1.6, 1.0, 2.3, 3.2, 2.7, 4.3, 5.0, 4.4, 3.8, 3.6, 7.9, 10.8, 6.0, 4.7, 5.9, 7.9, 9.8, 10.2, 7.3.

We fitted all of the distributions except Dagum to the above data by the method of maximum likelihood. Table 6 identifies the descriptive statistics behavior, Table 7 presents the parameter estimates, χ^2 , p - values, KS, A^* and W^* .

[Table 6 about here.]

[Table 7 about here.]

Analysis: From Tables 6 and 7, we observe that, the proposed model is also suitable when data are overdispersed and based on negative values. Also, the model has positive skewness and demonstrates IFR failure rates which can be visualized in TTT plot of data II. In addition to, the proposed model is also one of the appropriate model from the perspective of goodness of fit statistics. The importance of model is also identified in least χ^2 and the largest p - value.

[Figure 5 about here.]

From econometrics point of view, the sample path of the data given by Figure 5 shows that the data series are non-stationary. Moreover, the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test rejects the stationarity (p -value < 0.05) of the data. Also, following the Box-Ljung test, there is a structure of dependence behind those time series data (p -value < 0.001).

[Figure 6 about here.]

Example 3. The third data set is the famous Aarset data which gives the uncensored practical data set and it refers to the lifetimes of 50 components reported by [1]. The values of the data are: 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 16, 18, 18, 18, 18, 18, 20, 21, 32, 36, 40, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 83, 84, 84, 84, 85, 85, 85, 85, 86, 86.

We fitted all six distributions to the data above by the method of maximum likelihood. Table 8 gives the descriptive statistics of them, Table 9 presents the parameter estimates, χ^2 , p -values, KS, A^* and W^* .

[Table 8 about here.]

[Table 9 about here.]

Analysis: From Table 8, the data are overdispersed and negatively skewed. Moreover, the TTT plot for the Aarset's data indicates BT shapes. In Table 9, we have observed that PLD is the most suitable model based on the goodness of fit statistics with p -value. Further, Figure 6 also demonstrates the flexibility of PLD distribution.

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5. Appendices

Appendix A. Algorithm:

- i Generate U_i , $i = 1, 2, 3, \dots, n$ from uniform distribution on the interval (0,1).
 - ii Generate $\ln \left(U_i^{-\frac{1}{\zeta}} - 1 \right)^{-\vartheta}$, $i = 1, 2, 3, \dots, n$.
 - iii Set $E_i = \frac{\ln \left(U_i^{-\frac{1}{\zeta}} - 1 \right)^{-\vartheta}}{\vartheta + \varrho^2}$, $i = 1, 2, 3, \dots, n$.
 - iv Set $D_i = \frac{\left[\ln \left(U_i^{-\frac{1}{\zeta}} - 1 \right)^{-\vartheta} \right]^{\frac{1}{\vartheta}}}{\vartheta + \varrho^2}$, $i = 1, 2, 3, \dots, n$.
 - v Set $Y_1 = E_i$, $i = 1, 2, 3, \dots, n$.
 - vi Set $Y_2 = D_i$, $i = 1, 2, 3, \dots, n$.
- $\Rightarrow$ If $U_i \leq \frac{\vartheta^2}{\varrho + \vartheta^2}$, then set $X_i = Y_1$ otherwise set $X_i = Y_2$, $i = 1, 2, 3, \dots, n$.

Appendix B. Information Matrix

The 3×3 total information matrix along with its elements are given by

$$J(\Theta) = \begin{pmatrix} J_{\vartheta\vartheta}(\Theta) & J_{\vartheta\zeta}(\Theta) & J_{\vartheta\varrho}(\Theta) \\ J_{\zeta\vartheta}(\Theta) & J_{\zeta\zeta}(\Theta) & J_{\zeta\varrho}(\Theta) \\ J_{\varrho\vartheta}(\Theta) & J_{\varrho\zeta}(\Theta) & J_{\varrho\varrho}(\Theta) \end{pmatrix},$$

$$\begin{aligned}
J_{\vartheta \vartheta}(\Theta) &= \sum_{i=1}^n \left[\frac{\varrho |x_i|^{\vartheta-1} \log(|x_i|)^2}{\vartheta + \varrho |x_i|^{\vartheta-1}} - \frac{\left\{1 + \varrho |x_i|^{\vartheta-1} \log(|x_i|)\right\}^2}{\left(\vartheta + \varrho |x_i|^{\vartheta-1}\right)^2} \right] \\
&\quad - \sum_{i=1}^n \frac{\varrho \left\{2 - 2\vartheta \log(|x_i|) + \vartheta^2 \log(|x_i|)^2\right\} \text{sign}(x_i) |x_i|^\vartheta}{\vartheta^3} \\
&\quad - (1 + \zeta) \sum_{i=1}^n - \left[e^{-2\vartheta x_i - \frac{2\varrho \text{sign}(x_i) |x_i|^\vartheta}{\vartheta}} \left\{ \left(1 + e^{\vartheta x_i + \frac{\varrho \text{sign}(x_i) |x_i|^\vartheta}{\vartheta}}\right) \vartheta \varrho \right. \right. \\
&\quad \times \left(2 - 2\vartheta \log(|x_i|) + \vartheta^2 \log(|x_i|)^2\right) \text{sign}(x_i) |x_i|^\vartheta \\
&\quad + \left(\vartheta^2 x_i - \varrho \text{sign}(x_i) |x_i|^\vartheta (1 - \vartheta \log(|x_i|))\right)^2 \\
&\quad - \left(1 + e^{\vartheta x_i + \frac{\varrho \text{sign}(x_i) |x_i|^\vartheta}{\vartheta}}\right) \\
&\quad \times \left.\left.\left(\vartheta^2 x_i - \varrho \text{sign}(x_i) |x_i|^\vartheta (1 - \vartheta \log(|x_i|))\right)^2\right\} \right] \\
&\quad \times \frac{1}{\left(1 + e^{-\vartheta x_i - \frac{\varrho \text{sign}(x_i) |x_i|^\vartheta}{\vartheta}}\right)^2 \vartheta^4}, \\
J_{\vartheta \zeta}(\Theta) &= \sum_{i=1}^n \frac{\vartheta^2 x_i - \varrho \text{sign}(x_i) |x_i|^\vartheta + \vartheta \varrho \log(|x_i|) \text{sign}(x_i) |x_i|^\vartheta}{\left(1 + e^{\vartheta x_i + \frac{\varrho \text{sign}(x_i) |x_i|^\vartheta}{\vartheta}}\right) \vartheta^2}, \\
J_{\zeta \zeta}(\Theta) &= -\frac{n}{\zeta^2}, \\
J_{\zeta \varrho}(\Theta) &= \sum_{i=1}^n -\frac{e^{-\frac{\varrho |x_i|^\vartheta \text{sign}(x_i)}{\vartheta} - \vartheta x_i} |x_i|^\vartheta \text{sign}(x_i)}{\vartheta \left(1 + e^{-\frac{\varrho |x_i|^\vartheta \text{sign}(x_i)}{\vartheta} - \vartheta x_i}\right)},
\end{aligned}$$

$$\begin{aligned}
J_{\vartheta \varrho}(\Theta) = & - \sum_{i=1}^n \frac{(-1 + \vartheta \log(|x_i|)) \operatorname{sign}(x_i) |x_i|^\vartheta}{\vartheta^2} \\
& + \sum_{i=1}^n \frac{(-1 + \vartheta \log(|x_i|)) \operatorname{sign}(x_i) x_i |x_i|^\vartheta}{(\vartheta x_i + \varrho \operatorname{sign}(x_i) |x_i|^\vartheta)^2} \\
& - (1 + \zeta) \sum_{i=1}^n \left[\operatorname{sign}(x_i) |x_i|^\vartheta \left\{ \vartheta + e^{\vartheta x_i + \frac{\varrho \operatorname{sign}(x_i) |x_i|^\vartheta}{\vartheta}} \vartheta \right. \right. \\
& - \vartheta^2 \log(|x_i|) - e^{\vartheta x_i + \frac{\varrho \operatorname{sign}(x_i) |x_i|^\vartheta}{\vartheta}} \vartheta^2 \log(|x_i|) \\
& + e^{\vartheta x_i + \frac{\varrho \operatorname{sign}(x_i) |x_i|^\vartheta}{\vartheta}} \vartheta^2 x_i - e^{\vartheta x_i + \frac{\varrho \operatorname{sign}(x_i) |x_i|^\vartheta}{\vartheta}} \varrho \operatorname{sign}(x_i) |x_i|^\vartheta \\
& \left. \left. + e^{\vartheta x_i + \frac{\varrho \operatorname{sign}(x_i) |x_i|^\vartheta}{\vartheta}} \vartheta \varrho \log(|x_i|) \operatorname{sign}(x_i) |x_i|^\vartheta \right\} \right] \\
& \times \frac{1}{\vartheta^3 \left(1 + e^{\vartheta x_i + \frac{\varrho \operatorname{sign}(x_i) |x_i|^\vartheta}{\vartheta}} \right)^2}, \\
J_{\varrho \varrho}(\Theta) = & \sum_{i=1}^n - \frac{x_i^{2\vartheta-2}}{\left(\vartheta + \varrho |x_i|^{\vartheta-1} \right)^2} - (1 + \zeta) \sum_{i=1}^n \left[- \frac{e^{-\frac{2\varrho |x_i|^\vartheta \operatorname{sign}(x_i)}{\vartheta} - 2\vartheta x_i} x_i^{2\vartheta}}{\vartheta^2 \left(1 + e^{-\frac{\varrho |x_i|^\vartheta \operatorname{sign}(x_i)}{\vartheta} - \vartheta x_i} \right)^2} \right. \\
& \left. + \frac{e^{-\frac{\varrho |x_i|^\vartheta \operatorname{sign}(x_i)}{\vartheta} - \vartheta x_i} x_i^{2\vartheta}}{\vartheta^2 \left(1 + e^{-\frac{\varrho |x_i|^\vartheta \operatorname{sign}(x_i)}{\vartheta} - \vartheta x_i} \right)} \right],
\end{aligned}$$

the remaining elements follow by symmetry.

Table 1.: Average Bias and MSE Values of MLEs from Simulation of the PLD Distribution

Parameter	Sample Size	Bias($\hat{\vartheta}$)	Bias($\hat{\varrho}$)	Bias($\hat{\zeta}$)	MSE($\hat{\vartheta}$)	MSE($\hat{\varrho}$)	MSE($\hat{\zeta}$)
$\vartheta = 0.3642$ $\varrho = 2.3532$ $\zeta = 1.2315$	$n = 15$	3.2184	2.6517	5.7929	11.6709	13.4664	49.1715
	$n = 25$	3.0733	2.1018	4.6184	10.2525	6.7234	24.3536
	$n = 50$	3.0236	2.0380	3.2913	10.0582	6.1874	11.189
	$n = 150$	2.9609	1.4949	3.1971	9.4931	2.4417	10.4433
$\vartheta = 5.8759$ $\varrho = 5.4314$ $\zeta = 4.5472$	$n = 15$	-3.0702	-3.9940	9.4900	10.0832	16.0354	536.739
	$n = 25$	-3.1834	-3.5111	9.3246	10.0536	13.5562	474.913
	$n = 50$	-3.3497	-3.7332	3.4478	11.2945	13.9725	15.2051
	$n = 150$	-3.2656	-3.6214	3.3609	10.7069	13.1413	13.1649
$\vartheta = 1.2536$ $\varrho = 0.4721$ $\zeta = 1.1973$	$n = 15$	0.3297	1.1955	0.2712	0.2526	1.5412	0.2954
	$n = 25$	0.3081	0.8850	0.2221	0.2371	0.9822	0.1724
	$n = 50$	0.1806	0.8443	0.1125	0.1390	0.9056	0.0798
	$n = 150$	0.1768	0.6552	0.1088	0.1577	0.5455	0.0534
$\vartheta = 0.0542$ $\varrho = 0.0135$ $\zeta = 0.0432$	$n = 15$	-0.0476	0.0856	0.0509	0.0023	0.0080	0.0027
	$n = 25$	-0.0448	0.0750	0.0332	0.0020	0.0058	0.0016
	$n = 50$	-0.0483	0.0647	0.0143	0.0023	0.0042	0.0003
	$n = 150$	-0.0494	0.0629	0.0114	0.0024	0.0039	0.0001

Table 2.: Average Bias and MSE Values of LSEs from Simulation of the PLD Distribution

Parameter	Sample Size	Bias($\hat{\vartheta}$)	Bias($\hat{\varrho}$)	Bias($\hat{\zeta}$)	MSE($\hat{\vartheta}$)	MSE($\hat{\varrho}$)	MSE($\hat{\zeta}$)
$\vartheta = 0.3642$ $\varrho = 2.3532$ $\zeta = 1.2315$	$n = 15$	-0.2599	-0.3483	-0.1233	0.0676	0.1213	0.0152
	$n = 25$	-0.3037	-0.1527	-0.1233	0.0922	0.0233	0.0152
	$n = 50$	-0.3037	-0.1527	-0.1233	0.0922	0.0233	0.0152
	$n = 150$	-0.2993	-0.0525	-0.0206	0.0896	0.0027	0.0004
$\vartheta = 5.8759$ $\varrho = 5.4314$ $\zeta = 4.5472$	$n = 15$	-5.7381	-1.1245	-0.3368	32.9289	1.2645	0.1134
	$n = 25$	-5.7627	-1.1072	-0.3363	33.2108	1.2261	0.1131
	$n = 50$	-5.7738	-1.1013	-0.3363	33.3378	1.2129	0.1131
	$n = 150$	-5.7657	-0.9983	-0.2261	33.2433	0.9967	0.0511
$\vartheta = 1.2536$ $\varrho = 0.4721$ $\zeta = 1.1973$	$n = 15$	-1.2534	-0.3868	-0.9971	1.5711	0.1496	0.9942
	$n = 25$	-1.2522	-0.0672	-0.0890	1.5682	0.0045	0.0079
	$n = 50$	-1.1885	0.4087	-0.1158	1.4252	0.1836	0.0168
	$n = 150$	-1.224	0.4085	-0.1316	1.4984	0.1832	0.0181
$\vartheta = 0.0542$ $\varrho = 0.0135$ $\zeta = 0.0432$	$n = 15$	-0.0517	0.0786	-0.0166	0.0026	0.0213	0.0014
	$n = 25$	-0.0541	0.0255	-0.0015	0.0029	0.0028	0.0020
	$n = 50$	-0.0541	0.0210	-0.0218	0.0029	0.0023	0.0016
	$n = 150$	-0.0541	0.0158	-0.0245	0.0029	0.0020	0.0013

Table 3.: Average Bias and MSE Values of WLSEs from Simulation of the PLD Distribution

Parameter	Sample Size	Bias($\hat{\vartheta}$)	Bias($\hat{\varrho}$)	Bias($\hat{\zeta}$)	MSE($\hat{\vartheta}$)	MSE($\hat{\varrho}$)	MSE($\hat{\zeta}$)
$\vartheta = 0.3642$	$n = 15$	-0.3104	-0.1396	-0.1294	0.0963	0.0195	0.0167
$\varrho = 2.3532$	$n = 25$	-0.3076	-0.0296	-0.1294	0.0947	0.0009	0.0167
$\zeta = 1.2315$	$n = 50$	-0.3140	-0.0311	-0.1293	0.0986	0.0009	0.0167
	$n = 150$	-0.3027	-0.0008	-0.0212	0.0917	0.0000	0.0005
$\vartheta = 5.8759$	$n = 15$	-5.7403	-0.0774	-0.3370	32.9523	0.0060	0.1136
$\varrho = 5.4314$	$n = 25$	-5.7528	-0.0602	-0.3369	33.0951	0.0034	0.1135
$\zeta = 4.5472$	$n = 50$	-5.7544	-0.0792	-0.3369	33.1141	0.0062	0.1135
	$n = 150$	-5.7478	-0.0661	-0.1261	33.0376	0.0043	0.0159
$\vartheta = 1.2536$	$n = 15$	-1.2404	0.3802	0.0028	1.5386	0.1446	0.0000
$\varrho = 0.4721$	$n = 25$	-1.2522	-0.0671	-0.0890	1.5682	0.0045	0.0079
$\zeta = 1.1973$	$n = 50$	-1.1868	0.3061	-0.1087	1.4212	0.1772	0.0191
	$n = 150$	-1.2520	-0.2662	-0.1951	1.5677	0.0714	0.0380
$\vartheta = 0.0542$	$n = 15$	-0.0512	0.0510	-0.0127	0.0026	0.0097	0.0012
$\varrho = 0.0135$	$n = 25$	-0.0541	0.0374	-0.0417	0.0029	0.0024	0.0017
$\zeta = 0.0432$	$n = 50$	-0.0541	0.0174	-0.0316	0.0029	0.0004	0.0019
	$n = 150$	-0.0541	0.0018	-0.0021	0.0029	0.0003	0.0010

Table 4.: Descriptive Statistics for Strength Measured in GPA Data

Sample Size	Mean	Median	Variance	Skewness	Kurtosis	$MD_{\hat{\mu}}$	$MD_{\hat{\mu}}$	Entropy
63	3.0593	2.996	0.3855	0.6328	3.2863	0.5023	0.5023	4.1211

Table 5.: Maximum Likelihood Estimates and Goodness of Fit Statistics for Strength Measured in GPA Data

Distribution	Estimates			χ^2	$p - value$	K S	A*	W*
Dagum(μ, σ, λ)	1.9907	7.20072	2.6203	0.6754	0.879	0.1011	0.1028	0.5378
Cauchy(μ, σ)	2.9902	0.4032		3.1015	0.376	0.1236	0.1804	1.5100
Normal(μ, σ)	3.0593	0.6159		1.1535	0.764	0.1046	0.0885	0.5985
Gumbel(μ, σ)	3.3835	0.6993		7.6350	0.054	0.1376	0.2795	1.9607
LD($\vartheta, \varrho, \zeta$)	2.0640	32.6578	10.0233	0.2820	0.991	0.0948	0.08707	0.4423
PLD($\vartheta, \varrho, \zeta$)	1.5238	0.2525	151.0774	0.2583	0.992	0.0945	0.0822	0.4166

Table 6.: Descriptive Statistics for Return Rate of USA for Period 1965-1981 Data

Sample Size	Mean	Median	Variance	Skewness	Kurtosis	$MD_{\hat{\mu}}$	$MD_{\hat{\mu}}$	Entropy
28	4.1392	3.4	9.4298	0.7134	2.5544	2.4663	2.4663	3.2332

Table 7.: Maximum Likelihood Estimates and Goodness of Fit Statistics for Return Rate of USA for Period 1965-1981 Data

Distribution	Estimates			χ^2	$p - value$	K S	A*	W*
Dagum(μ, σ, λ)	NA	NA	NA	NA	NA	NA	NA	NA
Cauchy(μ, σ)	3.1371	1.7017		3.6640	0.160	0.1427	0.8543	0.1128
Normal(μ, σ)	6.625	20.2921		1.9735	0.373	0.1222	0.7080	0.1119
Gumbel(μ, σ)	5.7403	3.2456		6.9673	0.031	0.1818	1.4275	0.2389
LD($\vartheta, \varrho, \zeta$)	0.4321	0.0592	55.6979	0.2478	0.970	0.0835	0.1901	0.0229
PLD($\varsigma, \varrho, \zeta$)	0.4373	0.0576	4.6582	0.1203	0.982	0.0874	0.2264	0.0286

Table 8.: Descriptive Statistics for Aarset Data

Sample Size	Mean	Median	Variance	Skewness	Kurtosis	$MD_{\hat{\mu}}$	$MD_{\hat{\mu}}$	Entropy
50	43.866	46.5	1079.3	-0.0224	1.3776	30.0594	30.0594	3.1969

Table 9.: Maximum Likelihood Estimates and Goodness of Fit Statistics for Aarset Data

Distribution	Estimates			χ^2	$p - value$	K S	A*	W*
Dagum(μ, σ, λ)	0.2780	2.3915	35416.4785	1973.62	0.000	0.9817	108.6361	10.0757
Cauchy(μ, σ)	45.9498	28.0177		57.9872	0.000	0.1947	1.6702	0.2201
Normal(μ, σ)	43.8660	32.5225		5.5732	0.1343	0.1783	1.0537	0.1499
Gumbel(μ, σ)	59.9630	28.0878		32.0771	0.000	0.1983	1.3545	0.2185
LD($\vartheta, \varrho, \zeta$)	0.0354	0.0698	38.4381	40.005	0.000	0.1344	0.8952	0.1187
PLD($\vartheta, \varrho, \zeta$)	0.0245	0.1572	1710.2743	7.9526	0.103	0.1510	0.6636	0.0829

Figure 1.: Cumulative Distribution plots of PLD

Figure 2.: Density plots of PLD

Figure 3.: Hazard plots of PLD

Figure 4.: TTT plots of the data sets.

Figure 5.: Time Series Plot of Data Set-II

Figure 6.: The fitted PLD densities superimposed on the histogram