# On the (di)graphs with (directed) proper connection number two

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**Abstract.** A *properly connected* coloring of a given graph $G$ is one that ensures that every two vertices are the ends of a properly colored path. The *proper connection number* of $G$ is the minimum number of colors in such a coloring. We study the proper connection number for edge and vertex colorings, in undirected and directed graphs, respectively. More precisely, we initiate the study of the *complexity* of computing these four parameters. First we disprove some conjectures of Magnant et al. (2016) on characterizing the strong digraphs with *proper arc connection number* at most two. We prove that deciding whether a given digraph has proper arc connection number at most two is NP-complete. Furthermore, we show there are infinitely many such digraphs with no even-length dicycle. To the best of our knowledge, the *proper vertex connection number* of digraphs has not been studied before. We initiate the study of proper vertex connectivity in digraphs and we prove similar results as for the arc version. Finally, on a more positive side we present polynomial-time recognition algorithms for *bounded-treewidth* graphs and *bipartite* graphs with *proper edge connection number* at most two.

## 1 Introduction

There is a broad literature on frequency assignment problems in radio networks and their modeling by colorings problems on (di)graphs [20,10,9]. We study here a relaxed variant of proper colorings, introduced by Borozan et al. [6], where we only impose for every two vertices to be the ends of a properly colored (di)path – the (di)graph itself may not be properly colored. The latter concept is sometimes called *proper connectivity*. We motivate the study of proper connectivity next.

Motivation. Properly colored paths have applications in many fields like genetics [16,15,14], social sciences [12], radio communication [26], etc. We present a concrete application of *proper connectivity* related to wireless networks [26]. Indeed in any network the natural requirement is to

have all the parties connected and the goal is to maximize throughput. In wireless and communication networks, it is sometimes impossible to transmit a signal directly between, say, two communication towers A and B. Thus, the signal should go through some other towers. In order to maximize throughput, it is desirable to avoid interference by ensuring that the incoming and the outgoing signal from a tower should be on a different frequency. Suppose that we assign a vertex to each signal tower, an edge between two vertices if the corresponding signal towers are directly connected by a signal and a color to each edge corresponding to the frequency used for the communication. Then, the number of frequencies needed to assign the connections between towers so that there is always a path avoiding interference between each pair of towers is precisely the proper connection number of the corresponding graph.

Another example of proper connectivity is password sharing, where we aim at exchanging information between, say, secret agencies, while not repeating the same password for any two consecutive communications.

Related work. The proper edge connection number in undirected graphs was first defined in [6] by Borozan et al. In the same paper [6], the authors relate the proper connection number with the graph connectivity. Since then the problem was intensively studied from the combinatorial point of view [24,2,6,11,19,22,25,23,29,27]. More recently, Magnant et al. studied the proper *arc* connection number for digraphs [29]. They proved that this number is always at most three and they asked to characterize the digraphs with proper connection number at most two. In particular, they conjectured that all such digraphs must contain an even dicycle.

For more details, see the following surveys on the proper edge connection number: [26,3] and Chapter 16 of [4]. Nevertheless, no complexity results have been proved for this problem, until our work. Our goal is to fill in this gap in the literature, as discussed in Sections 3-5.

Furthermore, the notion of proper connectivity was generalized and strenghtened in many ways [6,22,2,28,24]. In the current paper, we also study some notions of *vertex proper connection*, *i.e.*, vertex-coloring versions of the proper connection number (see [11,25]). To the best of our knowledge, there are no complexity results for any of these variants either.

Proper connection number is related to the rainbow connection number, defined in [9]. The rainbow connection number of a given graph $G$ is the minimum number of colors necessary to edge-color $G$ such that there exists a rainbow path (i.e., a path with all the edges colored with a distinct color) between every two vertices in $G$. Since computing the

rainbow connection number is NP-hard and not FPT for any fixed $k \geq 2$ [7,1] it is natural to study the complexity of computing the proper edge connectivity – which can be seen a relaxed variant of rainbow connectivity.

Our results In this paper we study the complexity of computing the proper edge (resp., vertex, resp., arc) connection number of a given (di)graph. This gives four parameters to study, that are respectively denoted by $pc_e(G)$ and $pc_v(G)$ for undirected graphs $G$; $\overrightarrow{pc_e}(D)$ and $\overrightarrow{pc_v}(D)$ for directed graphs $D$.

First we study in Section 3 the computational complexity of computing $\overrightarrow{pc_e}(D)$ for a given digraph $D$. In [29] Magnant et al. prove that $\overrightarrow{pc_e}(D) \leq 3$ for every strongly connected digraph $D$. We prove here that deciding whether $\overrightarrow{pc_e}(D) \leq 2$ is NP-complete (Theorem 1). Furthermore, we disprove a conjecture of Magnant et al. from [29], claiming that if $D$ is a strongly connected digraph with no even dicycle, then $\overrightarrow{pc_e}(D) = 3$. Indeed we show, via an exhaustive search using a computer program, that $D_7$, the only strongly biconnected graph with no even dicycle, has a properly connected 2-coloring. Based on the coloring of $D_7$, we construct an infinite family of digraphs with no even dicycles that also have properly connected 2-colorings (Proposition 1).

Then we turn our attention, in Section 4, to the computational complexity of computing $\overrightarrow{pc_v}(D)$ for a given digraph $D$. To the best of our knowledge, the proper vertex connection number of directed graphs has not been studied before. We initiate the study of proper vertex connection number for digraphs and we prove on the positive side that every strongly connected digraph admits a properly connected 3-coloring (Theorem 2). However, deciding whether $\overrightarrow{pc_v}(D) \leq 2$ is also NP-complete (Theorem 3). Although the proofs of Theorems 1 and 3 are similar, notice that none of the two results is implied by the other.

Finally, we study in Section 5 the proper connection numbers of undirected graphs. On the one hand, $pc_v(G) = 2$ for every graph $G$. Indeed, given a graph $G$, it suffices to observe that any of its spanning trees can be properly colored with two colors (representing the two sides of its bipartition). On the other hand the proper edge connectivity number of undirected graphs is much more difficult to understand – though it has been extensively studied in the literature. We propose advances in the study of proper edge connectivity by presenting some tractable cases. In particular, Borozan et al. show that the trees with proper (edge) connection number two are exactly the paths [6]. We extend their result and

prove that for some graph classes generalizing the trees, namely graphs with bounded treewidth and bipartite graphs, we can decide if the proper (edge) connection number is less than or equal to two. Our result for bipartite graph is especially interesting since, the $pc_e(G)$ for bipartite graphs has been extensively studied in the literature and several sufficient conditions have been given for these graphs to have proper connection number equal to two [6,8,17,21,22]. We use in our proof the notion of bridge-block tree introduced by Tarjan [32]. The complexity of deciding whether $pc_e(G) \leq 2$ for general graphs is left as an interesting open question.

## 2 Preliminaries

We follow notations and terminology from [4,5]. (Di)graphs considered are finite and (strongly) connected. We write $[u = x_0, x_1, \ldots, x_l = v]$ for an $uv$-(di)path.

Let $G_c$ be an edge-colored graph, resp. a vertex-colored graph. A path in $G_c$ is called *proper* if no two incident edges, resp. no two adjacent vertices, are assigned the same color. We call $G_c$ *properly connected* if every two vertices are the ends of a proper path. The *proper edge connection number* and *proper vertex connection number* of a given unweighted graph $G$, denoted by $pc_e(G)$ and $pc_v(G)$, respectively, are the minimum number of colors to assign on its edges, resp. on its vertices, in order to make $G$ properly connected. Furthermore, we name a coloring of $G$ properly connected if it makes $G$ properly connected.

Proper dipaths in arc-colored and vertex-colored digraphs are defined similarly as for undirected graphs. We write $\overrightarrow{pc_e}(D)$ and $\overrightarrow{pc_v}(D)$ for the directed proper connection numbers of a given unweighted digraph $D$.

## 3 Proper arc connection number

This section is devoted to the complexity of computing $\overrightarrow{pc_e}(D)$ for a given digraph $D$. We prove in Section 3.1 that the corresponding decision problem is NP-complete. This further motivates the study of (polynomial-time computable) sufficient conditions for keeping this parameter as small as possible. In Section 3.2 we disprove a conjecture from [29]: namely, we show that there are infinitely many digraphs $D$ such that $\overrightarrow{pc_e}(D) = 2$ and $D$ has no even dicycle.

### 3.1 NP-hardness

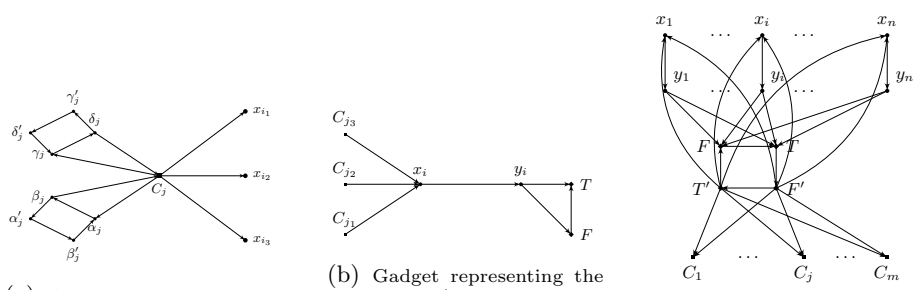
As proved in [29], for every (strongly connected) digraph $D$ that is not a complete symmetric digraph we have $\overrightarrow{pc_e}(D) \in \{2, 3\}$. Therefore, computing $\overrightarrow{pc_e}(D)$ is equivalent to deciding whether $\overrightarrow{pc_e}(D) \leq 2$. In order to prove that this decision problem is hard, we reduce from the following well-known (NP-complete) variation of the Boolean Satisfiability problem.

*Problem 1 (*Positive NAE-SAT*).*

**Input:** A propositional formula $\Phi$ in conjunctive normal form, with unnegated variables.
**Question:** Does there exist a truth assignment satisfying $\Phi$ in which no clause has all its literals valued 1 ?

**Theorem 1.** *Deciding whether $\overrightarrow{pc_e}(D) \leq 2$ for a given digraph $D$ is NP-complete.*

*Proof.* The problem is in NP. In order to prove the NP-hardness, let $\Phi$ be any formula in conjunctive normal form, with unnegated variables. The incidence graph of $\Phi$ is a bipartite graph $B_\Phi$, with one side being the $m$ clauses $C_1, \ldots, C_m$ of $\Phi$ and the other side being the $n$ variables $x_1, \ldots, x_n$ that are contained in at least one clause; for every $1 \leq j \leq m$, there exists an edge between $C_j$ and every variable $x_i$ that is contained in $C_j$. In what follows, we describe the transformation of $B_\Phi$ to a given digraph $D_\Phi$ such that $\overrightarrow{pc_e}(D_\Phi) = 2$ if and only if $\Phi$ is a yes-instance of Positive NAE-SAT. We illustrate the reduction in Figure 1.




(a) Gadget representing the clause $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$.

(b) Gadget representing the variable $x_i$ (contained in the clauses $C_{j_1}$, $C_{j_2}$, $C_{j_3}$).

(c) Truth gadget.

Fig. 1: The reduction.

*Clause gadget.* (see Fig. 1a). Let $1 \leq j \leq m$ be fixed. We replace every edge $\{C_j, x_i\}$ in $B_\Phi$ by an arc $(C_j, x_i)$ (oriented from $C_j$ to $x_i$). Furthermore, we add eight vertices $\alpha_j, \beta_j, \gamma_j, \delta_j$ and $\alpha'_j, \beta'_j, \gamma'_j, \delta'_j$ such that: the triples $[C_j, \alpha_j, \beta_j]$ and $[C_j, \gamma_j, \delta_j]$ induce directed cycles of length three; while the 4-tuples $[\alpha_j, \beta_j, \alpha'_j, \beta'_j]$ and $[\gamma_j, \delta_j, \gamma'_j, \delta'_j]$ induce directed cycles of length four.

Our construction will ensure that there is no other arc incident to either $\alpha_j, \beta_j, \gamma_j, \delta_j$ or $\alpha'_j, \beta'_j, \gamma'_j, \delta'_j$. Note that it will imply that the only $\alpha_j\delta_j$-dipath is $[\alpha_j, \beta_j, C_j, \gamma_j, \delta_j]$, and similarly the only $\gamma_j\beta_j$-dipath is $[\gamma_j, \delta_j, C_j, \alpha_j, \beta_j]$. Hence, in order to obtain proper dipaths with two colors, we need to assign the same color to the arcs $(C_j, \alpha_j)$, $(\beta_j, C_j)$ and $(\gamma_j, \delta_j)$ and the other color to the arcs $(\alpha_j, \beta_j)$, $(C_j, \gamma_j)$ and $(\delta_j, C_j)$.

*Variable gadget.* (see Fig. 1b). We add two vertices $F$ and $T$, along with an arc $(F, T)$. We show later how to use these two vertices in order to obtain a truth assignment satisfying $\Phi$. Let $1 \leq i \leq n$ be fixed. We add a vertex $y_i$, along with an arc $(x_i, y_i)$ and two more arcs $(y_i, F)$ and $(y_i, T)$.

Our construction will ensure that $y_i$ is the only out-neighbour of $x_i$. This will imply that any dipath ending at $x_i$ can be made longer only by passing through the arc $(x_i, y_i)$. The color assigned to this arc $(x_i, y_i)$ will intuitively represent the truth assignment of the variable $x_i$.

*Truth gadget.* (see Fig. 1c). We add two vertices $T', F'$ such that $[F, T, F', T']$ induces a directed cycle of length four. Finally, for every $1 \leq i \leq n$ we add the arcs $(T', x_i)$ and $(F', x_i)$, and for every $1 \leq j \leq m$, we add the arcs $(T', C_j)$ and $(F', C_j)$.

Observe that by construction, for every vertex $v \in V(D_\Phi) \setminus \{T', F'\}$, any $vF'$-dipath (resp., any $vT'$-dipath) must pass through the arc $(T, F')$ (resp., by the dipath $[T, F', T']$). The color assigned to this arc $(T, F')$ will intuitively represent the value "True". Furthermore, the arcs $(T', x_i)$ and $(F', x_i)$, for every $1 \leq i \leq n$, and the arcs $(T', C_j)$ and $(F', C_j)$, for every $1 \leq j \leq m$, are added to the construction simply to ensure the existence of a properly connected arc-coloring of $D_\Phi$ if $\Phi$ is a yes-instance of Positive NAE-SAT (this will be made more precise in what follows).

The resulting digraph $D_\phi$ can be constructed from $\Phi$ in $\mathcal{O}(\sum_{j=1}^{m} |C_j|)$-time, that is linear in the size of the formula. In what follows, we prove $\overrightarrow{pc_e}(D_\Phi) = 2$ if and only if $\Phi$ is a yes-instance of Positive NAE-SAT. Since Positive NAE-SAT is NP-complete, it proves the theorem.

In one direction, suppose that $\overrightarrow{pc_e}(D_\Phi) = 2$. We call Blue and Red the two colors assigned to the arcs in a properly connected arc-coloring

of $D_\Phi$. Without loss of generality, the arc $(T, F')$ is assigned color Blue. Furthermore, recall that by construction, for every $1 \leq j \leq m$, the three arcs $(C_j, \alpha_j)$, $(\beta_j, C_j)$ and $(\gamma_j, \delta_j)$ are assigned the same color, while the three arcs $(\alpha_j, \beta_j)$, $(C_j, \gamma_j)$ and $(\delta_j, C_j)$ are assigned the other color (otherwise, the arc-coloring could not be properly connected). By symmetry, let us assume that $(C_j, \alpha_j)$, $(\beta_j, C_j)$ and $(\gamma_j, \delta_j)$ are assigned color Red (and so, $(\alpha_j, \beta_j)$, $(C_j, \gamma_j)$ and $(\delta_j, C_j)$ are assigned color Blue).

We claim that there must exist a variable $x_i$ contained in $C_j$ such that the arc $(x_i, y_i)$ is assigned color Red. Indeed, we have on one hand that any $\beta_j T$-dipath must start with a subdipath $[\beta_j, C_j, x_i, y_i]$, for some $x_i$ contained in $C_j$. In particular, since the arc-coloring is assumed to be properly connected there must exist a proper $\beta_j T$-dipath, and so, there must exist a dipath $[\beta_j, C_j, x_i, y_i]$ that is proper. Since on the other hand, the arc $(\beta_j, C_j)$ is assigned color Red, therefore the arc $(x_i, y_i)$ in this proper dipath must be also colored Red. In the same way (up to replacing $\beta_j$ by $\delta_j$), there must exist a variable $x_i$ contained in $C_j$ such that the arc $(x_i, y_i)$ is assigned color Blue. Therefore, by assigning 1 to all the variables $x_i$ such the arc $(x_i, y_i)$ is colored Blue, and by assigning 0 to all the remaining variables (variables $x_{i'}$ such the arc $(x_{i'}, y_{i'})$ is colored Red), one obtains a truth assignment satisfying $\Phi$ in which no clause has all its literals valued 1. This implies that $\Phi$ is a yes-instance of Positive NAE-SAT.

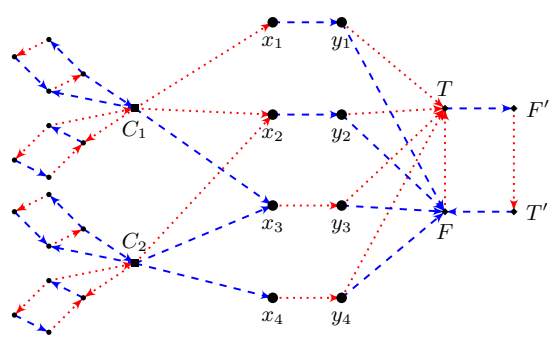



Fig. 2: A properly connected arc-coloring of $D_\Phi$, with $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4)$. For ease of readability, the arcs $(T', x_i)$ and $(F', x_i)$, for every $1 \leq i \leq n$, and the arcs $(T', C_j)$ and $(F', C_j)$, for every $1 \leq j \leq m$, are not drawn.

In the other direction, suppose that $\Phi$ is a yes-instance of Positive NAE-SAT. Let us consider a truth assignment satisfying $\Phi$ in which no

clause has all its litterals valued 1. We color the arcs of $D_\Phi$ as follows (see Figure 2 for an illustration).

- For every $1 \leq i \leq n$, the arc $(x_i, y_i)$ is assigned color Blue if the variable $x_i$ is valued 1, and it is assigned color Red otherwise. Furthermore, the arcs $(y_i, T)$ and $(F', x_i)$ are both assigned color Red while the arcs $(y_i, F)$ and $(T', x_i)$ are both assigned color Blue.
- For every $1 \leq j \leq m$, the arcs $(C_j, \alpha_j)$, and $(\beta_j, C_j)$ are assigned color Red and the arcs $(C_j, \gamma_j)$ and $(\delta_j, C_j)$ are assigned color Blue. The arcs $(\alpha_j, \beta_j)$, $(\alpha'_j, \beta'_j)$, $(\delta_j, \gamma'_j)$ and $(\delta'_j, \gamma_j)$ are assigned color Blue, whereas the arcs $(\beta_j, \alpha'_j)$, $(\beta'_j, \alpha_j), (\gamma_j, \delta_j)$ and $(\gamma'_j, \delta'_j)$ are assigned color Red. Furthermore, for every variable $x_i$ contained in $C_j$, the arc $(C_j, x_i)$ is assigned color Red if $x_i$ is valued 1 and it is assigned color Blue otherwise. The arcs $(T', C_j)$ and $(F', C_j)$ are assigned color Blue and Red, respectively.
- Finally, the arcs $(F, T)$ and $(F', T')$ are both assigned color Red, while the arcs $(T, F')$ and $(T', F)$ are both assigned color Blue.

Let us prove that the resulting arc-coloring of $D_\Phi$ is properly connected. In order to do so, we first notice that all the dipaths in the following collection $\mathcal{P}^-$ (ending in $T$) are proper:

- $[F', T', F, T]$;
- $[\alpha'_j,\ \beta'_j,\ \alpha_j,\ \beta_j,\ C_j,\ x_i,\ y_i,\ F,\ T]$, with $x_i$ being contained in $C_j$ and valued 0;
- $[\gamma'_j,\ \delta'_j,\ \gamma_j,\ \delta_j,\ C_j,\ x_i,\ y_i,\ T]$, with $x_i$ being contained in $C_j$ and valued 1.

Similarly, we notice that all the dipaths in the following collection $\mathcal{P}^+$ (starting in $T$) are proper:

- $[T, F', T', F]$;
- $[T, F', C_j, \gamma_j, \delta_j, \gamma'_j, \delta'_j]$ and $[T, F', T', C_j, \alpha_j, \beta_j, \alpha'_j, \beta'_j]$, for every $1 \leq j \leq m$;
- $[T, F', x_i, y_i]$ with $x_i$ being valued 1;
- $[T, F', T', x_i, y_i]$ with $x_i$ being valued 0.

Let us finally prove that for every $u, v \in V(D_\Phi)$, there exists a proper $uv$-dipath. We consider three different cases.

1. If some dipath of $\mathcal{P}^- \cup \mathcal{P}^+$ contains a $uv$-subdipath then the statement holds true.

2. Otherwise, if there exist a proper $uT$-dipath (subdipath of a dipath in $\mathcal{P}^-$) and a proper $Tv$-dipath (subdipath of a dipath in $\mathcal{P}^+$) that only intersect in $T$, then we claim that the statement is also true. Indeed, both proper dipaths can be merged into one since by construction, every proper $uT$-dipath ends with an arc colored Red whereas every proper $Tv$-dipath starts with an arc colored Blue.
3. Furthermore, it can be checked that any two vertices $u, v$ fall in one of the two above cases, unless $u, v \in \{\alpha_j, \beta_j, \gamma_j, \delta_j, C_j, \alpha'_j, \beta'_j, \gamma'_j, \delta'_j\}$ for some $1 \leq j \leq m$. In the latter case, there is a proper $uv$-dipath that is a subdipath of either: $[\alpha'_j, \beta'_j, \alpha_j, \beta_j, C_j, \gamma_j, \delta_j, \gamma'_j, \delta'_j]$;
or $[\gamma'_j, \delta'_j, \gamma_j, \delta_j, C_j, \alpha_j, \beta_j, \alpha'_j, \beta'_j]$; or of one of the two directed cycles $[\alpha_j, \beta_j, \alpha'_j, \beta'_j]$ or $[\gamma_j, \delta_j, \gamma'_j, \beta'_j]$. So, a path also exists in this final case.

□

### 3.2 Strongly-connected digraphs with no even dicycle

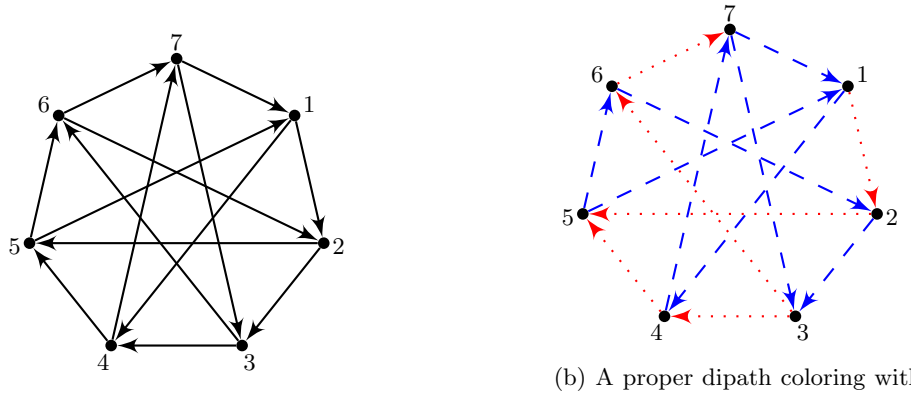
In [29], Magnant et al. conjecture that if $D$ is a strongly connected digraph with no even dicycle, then $\overrightarrow{pc_e}(D) = 3$. Note that in light of the hardness result of Theorem 1, their conjecture is all the more interesting than even-dicycle free digraphs can be recognized in polynomial-time [31]. However, we disprove the conjecture and we construct an infinite collection of digraphs that disprove their claim.

For that, we first consider the case of strongly 2-connected digraphs. There is only one strongly 2-connected digraph with no even dicycle (up to an isomorphism), namely the digraph $D_7$ from Figure 3a [30]. The following Lemma can be deduced from the (properly connected) arc-coloring indicated in Figure 3b.

**Lemma 1.** $\overrightarrow{pc_e}(D_7) = 2$. *Moreover, there exists such a coloring with the property that both outgoing arcs from vertex* 1 *have the same color* $c_1$ *and both ingoing arcs to vertex* 3 *have the same color* $c_2$*, with* $c_1 \neq c_2$.

Observe that there exist at least two such colorings (as stated in Lemma 1) since we can always exchange the colors of all the arcs while keeping the property to be properly connected.

Furthermore, based on Lemma 1, we can construct infinitely many other counter examples to the conjecture in [29]. In order to do that, let $H_1, H_2$ be two digraphs and $v_1, v_2$ two fixed vertices from $H_1$ and $H_2$, respectively. We denote by $(H_1, v_1) \odot (H_2, v_2)$ the digraph obtained from the union of graphs $H_1$ and $H_2$ by identifying vertices $v_1$ and $v_2$.




(a) $D_7$.

(b) A proper dipath coloring with 2 colors for $D_7$.

Fig. 3: Digraph $D_7$.

We recursively define an infinite family of strongly connected digraphs $(SD_n)_n$ and fix a vertex $v_n$ in every digraph $SD_n$ as follows, starting from $D_7$ and using vertices $x_1 = 1$ and $x_2 = 3$ from Lemma 1.

1. $SD_1 = D_7$; we fix in $S_1$ vertex $v_1 = x_1$;
2. $SD_n = (SD_{n-1}, v_{n-1}) \odot (D_7, x_1)$ and we set $v_n = x_2$.

**Proposition 1.** *For every $n \geq 1$, $SD_n$ is a strongly connected digraph with no even dicycle having $\overrightarrow{pc_e}(SD_n) = 2$. Moreover, there is such a coloring of $SD_n$ where all ingoing arcs to $v_n$ have the same color.*

*Proof.* We prove the result by induction on $n$. If $n = 1$ then the result follows from Lemma 1. Otherwise $n \geq 2$ and we assume for the induction hypothesis that $SD_{n-1}$ have the stated properties. By construction, $SD_n$ is a strongly connected digraph. Furthermore, it has no even dicycle since every dicycle of $SD_n$ is either contained in $SD_{n-1}$ or in $D_7$.

Finally, by the induction hypothesis, there exists a (properly connected) 2-arc coloring of $SD_{n-1}$ with the property that all ingoing arcs to $v_{n-1}$ have the same color. Assume w.l.o.g. that this color is Blue. By Lemma 1, there exists a (properly connected) 2-arc coloring of $D_7$ such that both outgoing arcs from $x_1$ have color Red and all ingoing arcs to $v_n = x_2$ have color Blue. Since $E(SD_n) = E(SD_{n-1}) \dot\cup E(D_7)$, the two colorings of $SD_{n-1}$ and $D_7$ determine an edge-coloring of $SD_n$. To prove that this coloring is properly connected it suffices to consider pairs of vertices $u, v$ with $u \in V(SD_{n-1})$ and $v \in V(D_7)$ and to prove that there

exist a proper $uv$-dipath and a proper $vu$-dipath. Indeed, since the coloring restricted to $SD_{n-1}$ is properly connected, there exists a proper $uv_{n-1}$-dipath $P$ in $SD_{n-1}$. Similarly, there exists a proper $x_1v$-dipath $Q$ in $D_7$. By merging these two dipaths in $v_{n-1} = x_1$ we obtain an $uv$-dipath in $SD_n$. The obtained dipath is proper since the ingoing arc to $v_{n-1}$ in $P$ has color Blue and the outgoing arc from $x_1$ contained in $Q$ has color Red. Note that using a similar argument, we can obtain a proper $vu$-dipath. Therefore, this coloring of $SD_n$ is properly connected. □

## 4 Proper vertex connection number

In this section, we initiate the study of the (directed) proper vertex connection number in (di)graphs. We observe this number must be at least two. The bound is tight for *undirected* graphs: indeed, every connected graph has a spanning tree, and we can choose as our two color classes the two sides of its bipartition. However, the proper vertex connection number is more difficult to study in directed graphs than in undirected graphs.

We prove in Section 4.1 that for every strongly connected digraph $D$, $\overrightarrow{pc_v}(D) \leq 3$. Then, we prove that computing $\overrightarrow{pc_v}(D)$ for a given digraph $D$ is NP-hard (Section 4.2). Interestingly, we so obtain similar results for proper *vertex* connection number as for proper *arc* connection number in digraphs.

### 4.1 Upper-bound

In what follows we prove an upper bound for the proper vertex connection number of a digraph. Unlike the proof for the arc version from [29], our proof is not based on the existence of an ear decomposition of a strongly connected digraph.

**Theorem 2.** *If $D$ is a strongly connected digraph, then $\overrightarrow{pc}_v(D) \leq 3$.*

*Proof.* We will proceed by induction on the number of vertices in $D$. If $D$ is a strongly connected digraph with at most three vertices, then a coloring that assigns a different color to all vertices is properly connected, hence $\overrightarrow{pc}_v(D) \leq 3$. Otherwise let $n \geq 4$. Suppose by the induction hypothesis that for every strongly connected digraph with $n-1$ vertices, there exists a vertex-coloring with at most 3 colors that is properly connected. Let $D$ be a strongly connected digraph with $n$ vertices. It suffices to prove that $D$ has a spanning strongly digraph $D'$ with $\overrightarrow{pc}_v(D') \leq 3$.

In particular, let $D'$ be a minimally strongly connected spanning digraph of $D$. Since $D'$ is minimally strongly connected, it has a vertex with in-degree and out-degree equal to 1 [18]. Let $v$ be such vertex, with $u$ and $w$ being its unique in-neighbor and out-neighbor, respectively. We remove $v$ from $D'$ and we add an arc from $u$ to $w$ if this arc is not already present. Clearly, the resulting digraph $D''$ is still strongly connected and it has order $n-1$. Furthermore by the induction hypothesis, $\overrightarrow{pc}_v(D'') \leq 3$. Consider such a (properly connected) coloring for $D''$. We can obtain from this coloring a properly connected coloring for $D'$ as follows. We color all vertices of $D' \setminus v$ as in $D''$, then finally we color $v$ with any color not assigned to $u$ nor $w$. Hence $\overrightarrow{pc}_v(D) \leq \overrightarrow{pc}_v(D') \leq 3$. □

### 4.2 NP-completeness

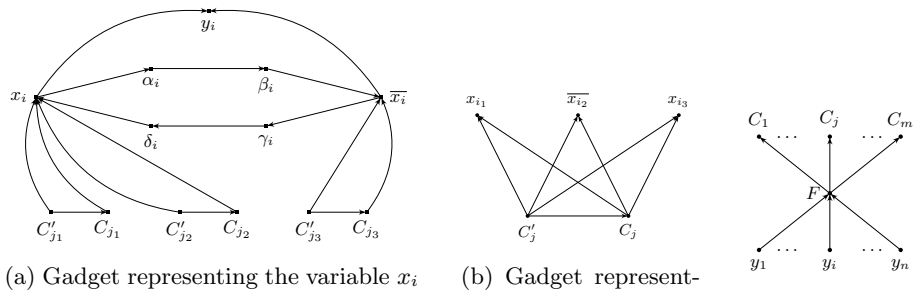
Since, by Theorem 2, we have $2 \leq \overrightarrow{pc}_v(D) \leq 3$ for every (strongly connected) digraph $D$, one important problem is to decide when it is the case that $\overrightarrow{pc}_v(D) = 2$. We show that this decision problem is in fact NP-hard by reducing it from 3-SAT.

**Theorem 3.** *Deciding whether $\overrightarrow{pc_v}(D) \leq 2$ for a given digraph $D$ is NP-complete.*

*Proof.* The problem is in NP. Furthermore as in the proof of Theorem 1, let $\Phi$ be any formula in 3-conjunctive normal form and $B_\Phi$ the incidence graph of $\Phi$. Assume that there is no clause in $\Phi$ containing both a variable and its negation. Next we describe the transformation of the incident graph $B_\Phi$ to a given digraph $D_\Phi$ such that $\overrightarrow{pc_v}(D_\Phi) = 2$ if and only if $\Phi$ is a yes-instance of 3-SAT. This will prove the NP-hardness, and so, the result.

*Variable gadget.* (see Fig. 4a). Let $1 \leq i \leq n$ be fixed. We add a vertex $\overline{x_i}$ corresponding to the negation of variable $x_i$. Furthermore, we add four new vertices: $\alpha_i, \beta_i, \gamma_i, \delta_i$ and the necessary arcs such that
$[x_i, \alpha_i, \beta_i, \overline{x_i}, \gamma_i, \delta_i, x_i]$ is a dicycle. This dicycle is used to ensure that vertices $x_i$ and $\overline{x_i}$ are assigned different colors in any properly connected coloring of $D_\Phi$ with two colors. We also add a vertex $y_i$, along with arcs $(x_i, y_i)$, $(\overline{x_i}, y_i)$.

*Clause gadget.* (see Fig. 4b). Let $1 \leq j \leq m$ be fixed. We replace every edge $\{C_j, x_i\}$ in $B_\Phi$ by an arc $(C_j, l_i)$ (directed from $C_j$ to $l_i$), where $l_i$ is the literal corresponding to variable $x_i$ contained in clause $C_j$ ($l_i$




(a) Gadget representing the variable $x_i$ ($x_i$ is contained in the clauses $C_{j_1}$, $C_{j_2}$ and $\overline{x_i}$ in $C_{j_3}$).

(b) Gadget representing the clause $C_j = x_{i_1} \vee \overline{x_{i_2}} \vee x_{i_3}$.

(c) Truth gadget.

Fig. 4: The reduction.

is either $x_i$ or $\overline{x_i}$). Furthermore we add a new vertex $C'_j$, an arc from this vertex to $C_j$ and arcs from $C'_j$ to all the out-neighbors of $C_j$, that is to all the literals contained in clause $C_j$. Thus, vertex $C'_j$ can reach the dicycle corresponding to $x_i$ through dipaths $[C'_j, l_i]$ or $[C'_j, C_j, l_i]$. Our construction will ensure that $(C'_j, C_j)$ is the only in-arc for $C_j$. Therefore, vertices $C_j$ and $C'_j$ must have distinct colors in any properly connected coloring of $D_\Phi$. In particular, at least one of these two above $C'_j l_i$-dipaths must be proper.

*Truth gadget.* (see Fig. 4c). We add a new vertex $F$. For every $1 \leq j \leq m$ we add an arc $(F, C'_j)$. In the same way, for every $1 \leq i \leq n$ we add an arc $(y_i, F)$.

For every $1 \leq j \leq m$, note that since we have that $[F, C'_j, C_j]$ is the only $FC_j$-dipath, vertices $F$ and $C_j$ must be assigned the same color in any properly connected 2-coloring of $D_\Phi$. In particular, there must be, in such coloring, an out-neighbor of $C_j$ with a different color than $F$, that will correspond to a literal $l_i \in C_j$ valued by "True".

Furthermore for every $1 \leq i \leq n$, vertices $y_i$ together with the dicycle containing vertices $x_i$ and $\overline{x_i}$ will ensure that if any of the two vertices $x_i$ or $\overline{x_i}$ can be reached from a vertex corresponding to a clause by a proper dipath, then $F$ can be reached.

The digraph $D_\phi$ can be constructed from $\Phi$ in $\mathcal{O}(m)$-time, hence linear in the number of clauses. Next we prove that $\overrightarrow{pc_v}(D_\Phi) = 2$ if and only if $\Phi$ is a yes-instance of 3-SAT.

First suppose that $\overrightarrow{pc_v}(D_\Phi) = 2$. Consider such a (properly connected) coloring $c$ of $D_\Phi$ with two colors, which we will call Blue and Red. W.l.o.g. assume that $c(F) = Red$. On the one hand, let $1 \leq j \leq m$ be fixed.

Since the only $FC_j$-dipath is $[F, C'_j, C_j]$, we must have $c(C'_j) = Blue$ and $c(C_j) = Red$. In this situation, at least one of the outgoing arcs from $C_j$ must have its other end colored with Blue. The latter corresponds to a literal $l$ in $C_j$. On the other hand, for every $1 \leq i \leq n$, since $[x_i, \alpha_i, \beta_i]$ and $[\alpha_i, \beta_i, \overline{x_i}]$ are the only $x_i\beta_i$-dipath and $\alpha_i\overline{x_i}$-dipath, respectively, we have that $c(x_i) \neq c(\overline{x_i})$. Then we can assign value 1 to all literals with the corresponding vertex having color Blue, and value 0 to the other literals. Since every vertex $C_j$ is adjacent with a literal with color Blue, in this way we obtain a truth assignment satisfying $\Phi$.

Suppose now that $\Phi$ is a yes-instance of 3-SAT. Consider a truth assignment satisfying $\Phi$. We define a coloring $c$ for the vertices of $D_\Phi$ as follows (see Fig. 5).

- $c(F) = Red$.
- For every $1 \leq j \leq m$, $c(C_j) = Red$ and $c(C'_j) = Blue$.
- If $x_i$ is assigned value 1 $c(x_i) = c(\beta_i) = c(\gamma_i) = Blue$ and $c(\overline{x_i}) = c(\delta_i) = c(\alpha_i) = Red$.
  Otherwise $c(x_i) = c(\beta_i) = c(\gamma_i) = Red$ and $c(\overline{x_i}) = c(\delta_i) = c(\alpha_i) = Blue$.

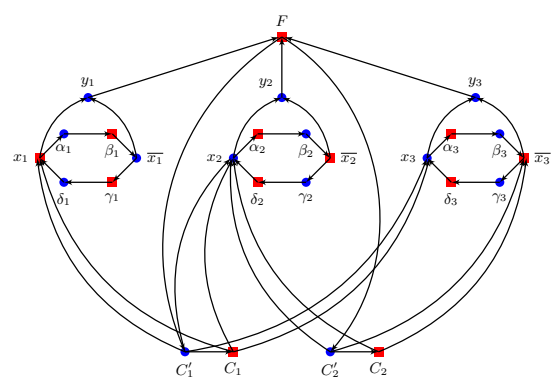



Fig. 5: A properly connected vertex-coloring of $D_\Phi$, with $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \overline{x_3})$ corresponding to the truth assignment $x_1 = 0$, $x_2 = 1$, $x_3 = 1$

We shall prove that $c$ is properly connected. First note that there exists a proper dipath from every vertex to $F$. More precisely the following dipaths are proper:

- $[\gamma_i, \delta_i, x_i, \alpha_i, \beta_i, \overline{x_i}, y_i, F]$ if $x_i$ is a variable valued 1;

- $[\alpha_i, \beta_i, \overline{x_i}, \gamma_i, \delta_i, x_i, y_i, F]$ if $x_i$ is a variable valued 0;
- $[C'_j, C_j, x_i, \alpha_i, \beta_i, \overline{x_i}, y_i, F]$ if $x_i$ is valued 1 and is contained in $C_j$;
- $[C'_j, C_j, \overline{x_i}, \gamma_i, \delta_i, x_i, y_i, F]$ if $\overline{x_i}$ is valued 1 and is contained in $C_j$;
- $[C'_j, x_i, y_i, F]$ if $x_i$ is valued 0 and is contained in $C_j$;
- $[C'_j, \overline{x_i}, y_i, F]$ if $\overline{x_i}$ is valued 0 and is contained in $C_j$.

Let us denote by $\mathcal{P}^-$ this above collection of dipaths. Also, the following dipaths from vertex $F$ are proper:

- $[F, C'_j, C_j, x_i, \alpha_i, \beta_i, \overline{x_i}, y_i]$ and $[F, C'_j, C_j, x_i, \alpha_i, \beta_i, \overline{x_i}, \gamma_i, \delta_i]$ if $x_i$ is a variable valued 1 in clause $C_j$;
- $[F, C'_j, C_j, \overline{x_i}, \gamma_i, \delta_i, x_i, y_i]$ and $[F, C'_j, C_j, \overline{x_i}, \gamma_i, \delta_i, x_i, \alpha_i, \beta_i]$ if $\overline{x_i}$ is valued 1 and is contained in clause $C_j$;
- $[F, C'_j, x_i, y_i]$ and $[F, C'_j, x_i, \alpha_i, \beta_i, \overline{x_i}, \gamma_i, \delta_i]$ if $x_i$ is a variable valued 0 in clause $C_j$;
- $[F, C'_j, \overline{x_i}, y_i]$ and $[F, C'_j, \overline{x_i}, \gamma_i, \delta_i, x_i, \alpha_i, \beta_i]$ if $\overline{x_i}$ is valued 0 and is contained in clause $C_j$.

Let us denote by $\mathcal{P}^+$ this above collection of dipaths.

We shall prove that for every two distinct vertices there exists a proper dipath of which they are the respective ends. This follows from the fact that every pair of distinct vertices $u, v$ is in one of the two following cases:

1. $\mathcal{P}^- \cup \mathcal{P}^+$ contains an $uv$-subdipath;
2. or there exist a proper $uF$-dipath (subdipath of a dipath in $\mathcal{P}^-$) and a proper $Fv$-dipath (subdipath of a dipath in $\mathcal{P}^+$) that only intersect in $F$. By merging these two dipaths in $F$ we obtain a proper $uv$-dipath.

Hence the coloring $c$ is properly connected.

□

## 5 Proper edge connection number

The last section is devoted to partial results on the proper *edge* connection number in graphs. The complexity of computing this parameter remains open, but we propose advances in this direction by presenting some tractable cases. More precisely, we provide some classes of graphs generalizing trees for which there are polynomial-time algorithms to decide if the proper (edge) connection number is two.

**Lemma 2.** *If $G = (V, E)$ has bounded-treewidth then it can be decided in linear-time whether $pc_e(G) \leq 2$.*

*Proof.* By Courcelle's Theorem [13], it suffices to prove that the problem can be written as a formula in $MSO_2$ logic. Such a formula can be obtained from a slight variation of the corresponding formula for the Hamiltonian Path problem. More precisely, $pc_e(G) \leq 2$ if and only if there exists a bipartition $E_0, E_1$ of the edge-set such that, for every $u, v \in V$, there is some set of edges $F$ that induces a connected subgraph, with the properties that $u, v \in V(G[F])$ and every vertex is incident to at most one edge in $F \cap E_i$ for every $i \in \{0, 1\}$. The latter can be written as an $MSO_2$ formula. □

*Bipartite graphs.* The special case of bipartite graphs has been extensively studied in the literature [6,8,17,21,22]. Several sufficient conditions have been given for these graphs to have proper connection number equal to two. Before concluding this section, we base on one of these conditions in order to provide a complete characterization of the bipartite graphs with proper connection number two.

We introduce additional terminology. Let $G = (V, E)$ be a connected graph. We remind that a *bridge* of $G$ is any edge $e \in E$ such that $G \setminus e = (V, E \setminus e)$ is disconnected. A *bridge-block* of $G$ is a connected component of $G \setminus F = (V, E \setminus F)$, with $F$ being the set of bridges of $G$. It is well-known that if we take as nodes the bridge-blocks of $G$, and for every bridge we add an edge between the two bridge-blocks that contain its ends, the resulting graph is a tree, sometimes called the *bridge-block tree* of $G$, that is linear time computable [32].

**Theorem 4.** *Let $G = (V, E)$ be a connected bipartite graph. We have $pc_e(G) \leq 2$ if and only if the bridge-block tree of $G$ is a path. Furthermore, if $pc_e(G) \leq 2$, then such a coloring can be computed in linear-time.*

As a byproduct of Theorem 4, we retrieve a known characterization of trees with proper connection number at most two.

**Corollary 1.** *([6]) Let $T$ be a tree. We have $pc_e(T) \leq 2$ if and only if $T$ is a path.*

The remaining of the section is devoted to the proof of Theorem 4. There are two main ingredients in the proof of Theorem 4 that will be stated in Lemmas 3 and 4. For that we need to introduce additional notions. We say that an edge-coloring of $G$ has the *strong property* if, for every two vertices $u$ and $v$, there are at least two $uv$-paths (not necessarily disjoint) with the two edges being incident to $u$ (resp., to $v$) in these two paths being of different colors.

**Lemma 3 ([22]).** *If $G$ is a connected bipartite bridgeless graph, then $pc_e(G) \leq 2$. Furthermore, such a coloring can be produced with the strong property.*

Hence, by Lemma 3, it remains to study bipartite graphs with bridges. Furthermore, it has been proved in [2] that if a vertex in $G$ is incident to at least $k$ bridges then $pc_e(G) \geq k$. So, we can deduce from this lower-bound that there can be no more than 2 bridges that are incident to every vertex. However, the latter is not sufficient, even for bipartite graphs, in order to ensure that $pc_e(G) \leq 2$. We provide a stronger lower-bound as follows.

**Lemma 4.** *Let $G = (V, E)$ be a connected graph, $B$ be a bridge-block of $G$ that is bipartite. If $B$ is incident to at least three bridges then $pc_e(G) \geq 3$.*

*Proof.* Let $v_0, v_1, v_2 \notin B$ be the three ends of bridges incident to $B$. For every $i \neq j$, all the $v_iv_j$-paths in $G$ have their internal vertices in $B$. In particular, since $B$ is bipartite, either all these paths have odd length or all of them have even length. It implies that there exists some index $i$ such that all the $v_iv_{i+1}$-paths and $v_iv_{i+2}$-paths in $G$ have the same parity (indices are taken modulo 3). W.l.o.g., let $i = 0$. Let $u_0, u_1, u_2 \in B$ be respectively adjacent to $v_0, v_1, v_2$ (they may not be pairwise different).

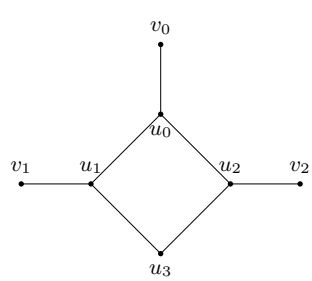



Fig. 6: A bipartite graph $G$ with $pc_e(G) = 3$.

Suppose for the sake of contradiction that $pc_e(G) \leq 2$. Let us fix such a coloring. Note that all the $v_0v_1$-paths and $v_0v_2$-paths start with the same edge $\{v_0, u_0\}$. Furthermore, since all these paths have the same parity, edges $\{u_1, v_1\}$ and $\{u_2, v_2\}$ must be assigned the same color. However, we claim that these two edges should be assigned a different color. Indeed, by taking the union of an $u_1u_0$-path with an $u_0u_2$-path we obtain an

$u_1u_2$-walk of even length in $B$. Since $B$ is bipartite, it implies that all the $u_1u_2$-paths have even length. In particular, all the $v_1v_2$-paths have even length, with the extremal edges on the paths being $\{u_1, v_1\}$ and $\{u_2, v_2\}$. The latter proves the contradiction since, in any proper path with 2 colors, the colors of the edges must alternate As a result, $pc_e(G) \geq 3$. $\square$

*Proof of Theorem 4.* On the one direction, suppose that the bridge-block tree of $G$ is not a path. Edges in this tree are in one-to-one correspondance with bridges in $G$. So, there is a bridge-block of $G$ that is incident to at least three bridges. Since such a component must be bipartite (because $G$ is), it follows from Lemma 4 that $pc_e(G) \geq 3$.

On the other direction, suppose that the bridge-block tree of $G$ is a path. This gives us a linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

We claim that for every $0 \leq i \leq l$ there exists a coloring of $G[\bigcup_{j \leq i} B_j]$ with the following two properties:

1. $G[\bigcup_{j \leq i} B_j]$ is properly connected;
2. the coloring restricted to $B_i$ has the strong property.

We divide the proof in three cases.

Suppose for the first base case $i = 0$. By Lemma 3 we have $pc_e(B_0) \leq 2$ and such a coloring can be produced with the strong property. So, the claim is proved for this base case.

Consider now $i = 1$ for the second base case. Again, by Lemma 3, we can color the edges of $B_0$, resp. the edges of $B_1$, with 2 colors, such that $B_0$, resp. $B_1$, is properly connected and the coloring for every bridge-block has the strong property. Then assigning an arbitrary color to the unique bridge between $B_0$ and $B_1$ results in having $G[B_0 \cup B_1]$ properly connected. This proves the claim for this base case.

Finally, assume $i > 1$. Suppose by the induction hypothesis that the claim holds for $i - 1$. In this situation the unique bridge between $B_{i-2}$ and $B_{i-1}$ is already assigned a color $c$. We first color the edges of $B_i$ with two colors so that the coloring is proper connected and it has the strong property; such a coloring is guaranteed to exist by Lemma 3. Then there are two subcases to be distinguished. Either all the paths between the two bridges incident in $B_{i-1}$ have even length, in which case we color the bridge between $B_{i-1}$ and $B_i$ with a color different than $c$. Or all these paths have odd length, in which case we assign to the bridge between $B_{i-1}$ and $B_i$ color $c$. Note that it can be easily checked in which case we are by looking for the end-vertices of the two bridges that are on the same side of the bipartition of $G$. In doing so, we extend the coloring in such

a way that $G[\bigcup_{j \leq i} B_j]$ is properly connected. Therefore, the claim holds for $i$.

In particular, by taking $i = l$, we obtain that $pc_e(G) \leq 2$. □

## 6 Conclusions

We give the first known complexity results on proper connectivity, in undirected graphs and in directed graphs. In particular, computing the proper connection numbers of digraphs is NP-hard. Computing the proper *vertex* connection number of undirected graphs is trivial. The only remaining case is proper *edge* connectivity. We leave the complexity of computing the proper edge connection number of a given graph as an interesting open question. An intermediate problem will be to characterize the graphs with proper edge connection number at most two. We also intend to study the complexity of computing the proper edge connection number of *bipartite* graphs, thereby generalizing the positive result of Theorem 4.

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