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Abstract. In this paper we consider a stochastic model of perpetuity-type. In contrast to the classical affine perpetuity model of Kesten [12] and Goldie [8] all discount factors in the model are mutually independent. We prove that the tails of the distribution of this model are regularly varying both in the univariate and multivariate cases. Due to the additional randomness in the model the tails are not pure power laws as in the Kesten-Goldie setting but involve a logarithmic term.

1. Problem description

Consider an array (X_{ni}) of iid random variables with generic sequence $(X_i) = (X_{1i})$ and $X = X_1$. We define a stochastic *perpetuity* in the following way:

$$\begin{aligned} \widetilde{Y}_1 &= X_{11}\,, \\ \widetilde{Y}_2 &= X_{11}X_{12} + X_{21}\,, \\ \widetilde{Y}_3 &= X_{11}X_{12}X_{13} + X_{21}X_{22} + X_{31}\,, \ldots . \end{aligned}$$

At any time i, each of the investments in the previous and current periods $j = 1, \ldots, i$ gets discounted by an independent factor X_{ij}. Therefore $(\widetilde{Y}_n)$ can be interpreted as the dynamics of a perpetuity stream. Obviously, $\widetilde{Y}_n$ has the same distribution as

$$Y_n = X_{11} + X_{21}X_{22} + X_{31}X_{32}X_{33} + \cdots + X_{n1}\cdots X_{nn}\,, \quad n \geqslant 1\,,$$

and, under mild conditions, the sequence (Y_n) has the a.s. limit

$$Y = \sum_{n=1}^{\infty} \Pi_n \text{ where } \Pi_n = \textstyle\prod_{i=1}^{n} X_{ni} \text{ for } n \geqslant 1. \tag{1.1}$$

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We assume that the infinite series in (1.1) converges a.s. Since $\Pi_n \stackrel{\text{a.s.}}{\to} 0$ is a necessary condition for this convergence to hold we need that

$$\log|\Pi_n| = \sum_{i=1}^n \log|X_{ni}| \stackrel{\text{a.s.}}{\to} -\infty\,, \qquad n\to\infty\,.$$

Hence the random walk $(\log|\Pi_n|)$ has a negative drift, i.e., $\mathbb{E}[\log|X|] < 0$, possibly infinite.

Throughout this paper we assume that there exists a positive number α such that

$$h(\alpha) = \mathbb{E}[|X|^\alpha] = 1\,. \tag{1.2}$$

Assume for the moment, that $X \geqslant 0$ a.s. By convexity of the function $h(s)$, $h(\alpha+\varepsilon) > 1$ and $h(\alpha-\varepsilon) < 1$ for small $\varepsilon \in (0,\alpha)$ where we assume that $h(s)$ is finite in some neighborhood of α. Then for positive ε,

$$\mathbb{E}[|Y|^{\alpha+\varepsilon}] \geqslant \mathbb{E}[|\Pi_n|^{\alpha+\varepsilon}] = (h(\alpha+\varepsilon))^n\,.$$

The right-hand side diverges to infinity as $n\to\infty$, hence $\mathbb{E}[|Y|^{\alpha+\varepsilon}] = \infty$. We also have for $\alpha \leqslant 1$ and $\varepsilon\in(0,\alpha)$,

$$\mathbb{E}[|Y|^{\alpha-\varepsilon}] \leqslant \sum_{i=1}^n \mathbb{E}[|\Pi_n|^{\alpha-\varepsilon}] = \sum_{i=1}^n (h(\alpha-\varepsilon))^n < \infty\,.$$

For $\alpha > 1$ a similar argument with the Minkowski inequality shows that $\mathbb{E}[|Y|^{\alpha-\varepsilon}] < \infty$.

These observations on the moments indicate that $|Y|$ has some heavy tail in the sense that certain moments are infinite. In this paper we will investigate the precise asymptotic behavior of $\mathbb{P}(\pm Y > x)$ as $x\to\infty$. It will turn out that, under (1.2) and some additional mild assumptions,

$$\mathbb{P}(Y > x) \sim \begin{cases} \dfrac{2}{m(\alpha)}\dfrac{\log x}{x^\alpha}\,, & \text{if } X \geqslant 0 \text{ a.s.,} \\ \dfrac{1}{m(\alpha)}\dfrac{\log x}{x^\alpha}\,, & \text{if } \mathbb{P}(X<0)>0, \end{cases} \qquad x\to\infty\,, \tag{1.3}$$

where $m(\alpha) = \mathbb{E}[|X|^\alpha \log|X|]$ is a positive constant. In the case $\mathbb{P}(X<0) > 0$ we also have $\mathbb{P}(Y > x) \sim \mathbb{P}(Y < -x)$ as $x\to\infty$.

An inspection of (1.1) shows that the structure of Y is in a sense close to

$$Y' = 1 + \sum_{n=1}^\infty \Pi_n' \text{ where } \Pi_n' = \textstyle\prod_{i=1}^n X_i \text{ for } n \geqslant 1.$$

This structure has attracted a lot of attention; see the recent monograph Buraczewski et al. [4] and the references therein. Indeed, assuming X and Y' independent, it is easy to see that the following fixed point equation holds:

$$Y' \stackrel{d}{=} X\,Y' + 1\,. \tag{1.4}$$

If this equation has a solution Y' for given X it is not difficult to see that the stationary solution (Y'_t) to the stochastic recurrence equation

$$Y'_t = X_t Y'_{t-1} + 1\,, \qquad t \in \mathbb{Z}\,, \tag{1.5}$$

satisfies (1.4) for $Y' = Y'_t$, and if Y' solves (1.4) it has the stationary distribution of the Markov chain decribed in (1.5).

One of the fascinating properties of (1.4) and (1.5) is that, under condition (1.2), these equations generate power-law tail behavior. Indeed, if $X \geqslant 0$ a.s.

$$\mathbb{P}(Y' > x) \sim \frac{\mathbb{E}[(X\,Y' + 1)^\alpha - (X\,Y')^\alpha]}{\alpha\, m(\alpha)}\,\frac{1}{x^\alpha}, \qquad x \to \infty\,, \tag{1.6}$$

and if $\mathbb{P}(X < 0) > 0$ then

$$\mathbb{P}(\pm Y' > x) \sim \frac{\mathbb{E}[|X\,Y' + 1|^\alpha - |X\,Y'|^\alpha]}{2\alpha\, m(\alpha)}\,\frac{1}{x^\alpha}, \qquad x \to \infty\,, \tag{1.7}$$

This follows from Kesten [12] who also proved (1.6) and (1.7) for the linear combinations of solutions to multivariate analogs of (1.5). Goldie [8] gave an alternative proof of (1.6) and (1.7) and also derived the scaling constants for the tails.

We will often make use of Kesten's [12] Theorems A and B, and Theorem 4.1 in Goldie [8]; cf. Theorem 2.4.4 and 2.4.7 in Buraczewski et al. [4]. For the reader's convenience, we formulate these results here, tailored for our particular setting. In the case $\mathbb{P}(X < 0) > 0$ we did not find a result of type (1.8) in the literature. Therefore we give an independent proof in Appendix A.

Theorem 1.1. *Assume the following conditions:*

1. *The conditional law of* $\log|X|$ *given* $\{X \neq 0\}$ *is non-arithmetic.*
2. *There exists* $\alpha > 0$ *such that* $\mathbb{E}[|X|^\alpha] = 1$ *and* $\mathbb{E}[|X|^\alpha \log|X|] < \infty$.
3. $\mathbb{P}(X\,x + 1 = x) < 1$ *for every* $x \in \mathbb{R}$.

If either $X \geqslant 0$ *a.s. or* $\mathbb{P}(X < 0) > 0$ *hold then* (1.6) *or* (1.7) *hold, respectively. In both cases, there is a constant* $c > 0$ *such that*

$$\mathbb{P}\big(\max_{n\geqslant 1} \Pi'_n > x\big) \sim c\, x^{-\alpha}\,, \qquad x \to \infty\,. \tag{1.8}$$

Here and in what follows, $c, c', \ldots$ stand for any positive constants whose values are not of interest.

We have a corresponding result for the arithmetic case, i.e., when the law of $\log|X|$ conditioned on $\{X \neq 0\}$ is arithmetic. This means that the support of $\log X$ (excluding zero if $\mathbb{P}(X = 0) > 0$) is a subset of $a\mathbb{Z}$ for some non-zero a.

Theorem 1.2. *Assume conditions* (2), (3) *of Theorem 1.1 and*

(1') *the law of* $\log|X|$ *conditioned on* $\{X \neq 0\}$ *is arithmetic.*

Then there exist constants $0 < c < c' < \infty$ *such that for large* x*,*

$$x^\alpha\,\mathbb{P}\big(\max_{n\geqslant 1}\Pi_n' > x\big) \;\in\; [c,c']\,, \tag{1.9}$$

$$x^\alpha\,\mathbb{P}\big(Y'' > x\big) \;\in\; [c,c']\,, \tag{1.10}$$

where $Y" = \sum_{n=1}^\infty |\Pi_n'|$

For $X \geqslant 0$, (1.9) is part of the folklore on ruin probability in the arithmetic case; see Asmussen [1], Remark 5.4, Section XIII. For the general case $\mathbb{P}(X < 0) > 0$ we refer to the proof in Appendix A. Relation (1.10) can be found in Grincevičius [9], Theorem 2b.

This paper has two main goals:

(1) We want to show that the function $\mathbb{P}(Y > x)$ is regularly varying with index $-\alpha$ under the condition $\mathbb{E}[|X|^\alpha] = 1$. More precisely, we will show (1.3).
(2) We want to show that

$$\mathbb{P}(Y > x) \sim \sum_{n=1}^\infty \mathbb{P}(\Pi_n > x) =: p(x)\,, \qquad x\to\infty\,. \tag{1.11}$$

Relation (1.11) reminds one of similar results for sums of independent regularly varying or subexponential random variables; see for example Chapter 2 in Embrechts et al. [6]. The crucial difference between (1.11) and these results is that the summands Π_n of Y can be light-tailed for every fixed n; the heavy tail of Y builds up only for Π_n with an index n close to $\log x/m(\alpha)$.

Positive solutions to these two problems are provided in Theorem 2.1 and Corollary 2.3. They also show that $\mathbb{P}(Y > x)/\mathbb{P}(Y' > x) \sim c\log x$ for some positive constant. The proof in Section 3 makes use of Theorems 1.1 and 1.2 as auxilary results. We use classical exponential bounds for sums of independent random variables and change-of-measure techniques; see Petrov's classic [15] for an exposition of these results and techniques.

We also make an attempt to understand the tails of a vector-valued version of Y when $\Pi_n = \mathbf{X}_{n1}\cdots\mathbf{X}_{nn}$ is the product of iid $d\times d$ matrices $(\mathbf{X}_{ni})$ with non-negative entries and a generic element $\mathbf{X}$ satisfies an analog of (1.2) defining the value $\alpha > 0$; see Section 4.1 for details. We define $\mathbf{Y} = \mathbf{Y}(\mathbf{u}) = \sum_{n=1}^\infty \Pi_n^\top\mathbf{u}$ for some unit vector $\mathbf{u}$ with non-negative components and show that $\mathbb{P}(|\mathbf{Y}| > x)$ is of the order $\log x/x^\alpha$. This approximation does not depend on the choice of $\mathbf{u}$ when $|\mathbf{u}| = 1$. We prove this result by showing the asymptotic equivalence between $\mathbb{P}(|\mathbf{Y}| > x)$ and $p_{\mathbf{u}}(x) = \sum_{n=1}^\infty \mathbb{P}(|\Pi_n^\top\mathbf{u}| > x)$. Of course, the tail of $\mathbf{Y}$ is not characterized by the tail of the norm. Therefore we also consider linear combinations $\mathbf{v}^\top\mathbf{Y}$ for any unit vector $\mathbf{v}$ with positve components and show that $\mathbb{P}(\mathbf{v}^\top\mathbf{Y} > x)$ is also of the asymptotic order $\log x/x^\alpha$.

This paper is structured as follows. In Section 2 we present the main results in the univariate case (Theorem 2.1 and Corollary 2.3) followed by a discussion of the results. Proofs are given in Section 3. In Appendix A we provide proofs of relations (1.8) and (1.9) in the case when $\mathbb{P}(X < 0) > 0$; we did not find a corresponding result in the literature. The multivariate case is treated in Section 4; Theorem 4.3 is a multivariate analog of Theorem 2.1 and Corollary 2.3.

2. Main results

We formulate one of the main results of this paper.

Theorem 2.1. *Assume the conditions of Theorems 1.1 or 1.2, in particular there exists* $\alpha > 0$ *such that* $h(\alpha) = \mathbb{E}[|X|^\alpha] = 1$. *In addition, we assume that* $\mathbb{E}[|X|^\alpha(\log|X|)^2] < \infty$, *or* $\mathbb{E}[|X|^\alpha(\log|X|)^2] = \infty$ *and* $\mathbb{E}[|X|^\alpha \mathbb{1}(\log|X| > x)]$ *is regularly varying with index* $\kappa \in (1, 2]$,

(1) *If* $X \geqslant 0$ *a.s. then*

$$p(x) \sim \frac{2}{m(\alpha)} \frac{\log x}{x^\alpha}, \qquad x \to \infty.$$

(2) *If* $\mathbb{P}(X < 0) > 0$ *then*

$$p(x) \sim \frac{1}{m(\alpha)} \frac{\log x}{x^\alpha}, \qquad x \to \infty.$$

Remark 2.2. In the course of the proof of Theorem 2.1 we show that for $X \geqslant 0$ a.s.

$$p(x) \sim \sum_{n=1}^{[\log x/m(\alpha)]} \mathbb{P}(\Pi_n > x)$$

$$\sim 2x^{-\alpha} \sum_{n=1}^{[\log x/m(\alpha)]} \Phi\big((\log x - n\, m(\alpha))/\sqrt{\sigma^2(\alpha) n}\big), \qquad x \to \infty, \tag{2.1}$$

where Φ is the standard normal distribution function and $\sigma^2(\alpha) = \mathbb{E}[X^\alpha(\log X)^2] - (m(\alpha))^2$ is assumed finite.

The following result is an immediate consequence of Theorem 2.1 and Proposition 3.1.

Corollary 2.3. *Assume the conditions of Theorem 2.1. If* $X \geqslant 0$ *a.s. then*

$$\mathbb{P}(Y > x) \sim \sum_{n=1}^{\infty} \mathbb{P}(\Pi_n > x) \sim \frac{2}{m(\alpha)} \frac{\log x}{x^\alpha}, \qquad x \to \infty. \tag{2.2}$$

If $\mathbb{P}(X < 0) > 0$ *then*

$$\mathbb{P}(\pm Y > x) \sim \sum_{n=1}^{\infty} \mathbb{P}(\Pi_n > x) \sim \frac{1}{m(\alpha)} \frac{\log x}{x^\alpha}, \qquad x \to \infty. \tag{2.3}$$

In contrast to the distinct results for $\mathbb{P}(Y' > x)$ in Theorems 1.1 and 1.2 for the non-arithmetic and arithmetic cases, respectively, relations (2.2) and (2.3) hold in both cases. In particular, in contrast to Theorem 1.2 for $\mathbb{P}(Y' > x)$, we get precise asymptotics for $\mathbb{P}(Y > x)$ in the arithmetic case. Corollary 2.3 and Kesten's Theorem 1.1 in the general and in the non-arithmetic cases, respectively, show that $\mathbb{P}(Y > x)$ and $\mathbb{P}(Y' > x)$ are regularly varying functions with index $-\alpha$. However, we have $\mathbb{P}(Y > x)/\mathbb{P}(Y' > x) \to \infty$ as $x \to \infty$, accounting for the additional independence of (Π_n) in the structure of Y. In the non-arithmetic case we can even compare the scaling constants in the tails. For example, for $X \geqslant 0$ a.s. we have (see (1.6))

$$\frac{\mathbb{P}(Y > x)}{\mathbb{P}(Y' > x)} \sim \frac{2\alpha}{\mathbb{E}[(X\,Y' + 1)^\alpha - (X\,Y')^\alpha]} \log x\,.$$

We proved (2.2) under conditions implying that $\mathbb{E}[X^\alpha (\log X)^{1+\delta}] < \infty$ for some $\delta > 0$ which is slightly stronger than the condition $m(\alpha) < \infty$ in Kesten's theorem.

We observe the similarity of the results in Theorem 1.1 and Corollary 2.3 as regards the asymptotic symmetry of the tails in the case when $\mathbb{P}(X < 0) > 0$. In both cases, we have $\mathbb{P}(Y' > x) \sim \mathbb{P}(Y' < -x)$ and $\mathbb{P}(Y > x) \sim \mathbb{P}(Y < -x)$ as $x \to \infty$. Moreover, in this case we also have

$$\mathbb{P}(|Y| > x) \sim \mathbb{P}\Big(\sum_{n=1}^\infty |\Pi_n| > x\Big)\,, \qquad x \to \infty\,.$$

2.1. **Implications and discussion of the results.** The tail behavior of $\mathbb{P}(Y > x)$ described by Corollary 2.3 immediately ensures limit theory for the extremes and partial sums of an iid sequence (Y_i) with generic element Y. Assuming the conditions of Theorem 2.1 and $X \geqslant 0$ a.s., choose $a_n = (2n\,\log n/(\alpha\, m(\alpha)))^{1/\alpha}$. Then we know from classical theory that

$$a_n^{-1} \max_{i=1,\dots,n} Y_i \stackrel{d}{\to} \xi_\alpha\,, \tag{2.4}$$

$$a_n^{-1}\Big(\sum_{i=1}^n Y_i - c_n\Big) \stackrel{d}{\to} S_\alpha\,; \tag{2.5}$$

see for example Chapters 2 and 3 in Embrechts et al. [6]. Relation (2.4) holds for any $\alpha > 0$ and the distribution of ξ_α is Fréchet with parameter α. Relation (2.5) holds only for $\alpha \in (0, 2)$ and the distribution of S_α is α-stable. The centering constants c_n can be chosen as $n\,\mathbb{E}[Y]$ for $\alpha > 1$, $n\,\mathbb{E}[|Y|\mathbb{1}(Y \leqslant a_n)]$ for $\alpha = 1$ and $c_n = 0$ for $\alpha \in (0, 1)$.

One can introduce the stationary time series

$$Y_n = \sum_{i=-\infty}^n \prod_{j=n-i+1}^n X_{ij}\,, \qquad n \in \mathbb{Z}\,.$$

We observe that $Y_n \stackrel{d}{=} Y$. Unfortunately, Y_n cannot be derived via an affine stochastic recurrence equation as in the Kesten case; see (1.5). Therefore its dependence structure is less straightforward. However, it is another example of a time series whose power-law tails do not result from heavy-tailed input variables X_{ni}.

Now assume for the sake of argument that $X \geqslant 0$ and $\log X$ has a non-arithmetic distribution. Write $S_n' = \log \Pi_n' = \sum_{i=1}^n \log X_i$. As a byproduct from Theorem 2.1 and (1.8) we conclude that

$$\sum_{n=1}^{\infty} \mathbb{P}\big(S_n' > x \mid \max_{j\geqslant 1} S_j' > x\big) = \frac{\sum_{n=1}^{\infty} \mathbb{P}(S_n' > x)}{\mathbb{P}\big(\max_{j\geqslant 1} S_j' > x\big)} \sim c\,x\,.$$

From (2.1) and the latter relation we also obtain

$$\begin{aligned}&\frac{1}{x}\sum_{n=1}^{[x/m(\alpha)]} \mathbb{P}\big(S_n' > x \mid \max_{j\geqslant 1} S_j' > x\big)\\ &= \frac{1}{x}\,\mathbb{E}\Big[\#\{n \leqslant [x/m(\alpha)] : S_n' > x\} \mid \max_{j\geqslant 1} S_j' > x\Big] \to c\,, \qquad x \to \infty\,.\end{aligned}$$

The constant c can be calculated explicitly. Indeed, it has a nice interpretation in terms of a so-called *extremal index;* see Section 8.1 in Embrechts et al. [6] and Leadbetter et al. [14] for its definition and properties.

Notice that the maxima of (S_t') have the same distribution as those of the *Lindley process* given by

$$S_t^+ = \max(S_{t-1}^+ + \log X_t, 0), \qquad t \geqslant 1, \qquad S_0^+ = 0\,; \tag{2.6}$$

see Asmussen [1], Section III.6. As $\mathbb{E}[\log X_0] < 0$ the existence of the stationary solution $\tilde S_0^+$ to (2.6) is ensured since $\{0\}$ is an atom. The extremal behavior of the Lindley process is well studied: its extremal index θ exists, is positive and satisfies

$$\mathbb{E}\Big[\#\{n \leqslant [c'\log(x)] : \tilde S_n^+ > x\} \mid \max_{1\leqslant j\leqslant [c'\log x]} \tilde S_j^+ > x\Big] \to \frac{1}{\theta}\,, \qquad x \to \infty\,,$$

for some $c' > 0$ depending on the exponential moments of the return time to the atom; see Rootzén [18]. The extremal index can be expressed by using the Cramér constant for the associated ruin problem, i.e., the constant in (1.8); see Collamore and Vidyashankar [5]. From the previous discussion, we obtain

$$\frac{1}{x}\,\mathbb{E}\Big[\#\{n \leqslant [x/m(\alpha)] : S_n^+ > x\} \mid \max_{1\leqslant j\leqslant [x/m(\alpha)]} S_j^+ > x\Big] \to \frac{2\alpha}{\theta}\,, \qquad x \to \infty\,.$$

Surprisingly, under certain conditions the tail decay rate in (2.2) is the same as for the solution to the fixed point equation

$$\widetilde Y \stackrel{d}{=} \sum_{i=1}^{N} X_i\,\widetilde Y_i\,,$$

where $(\widetilde Y_i)$ are iid copies of $\widetilde Y$, (X_i) is an iid positive sequence and N is positive integer-valued. Moreover, $(\widetilde Y_i)$, (X_i) are mutually independent. In this case, the tail index $\alpha > 0$ is given as the unique solution to the equation $\widetilde m(\alpha) = \mathbb{E}\big[\sum_{i=1}^N X_i^\alpha\big] = 1$. The decay rate in (2.2) is the same as for $\mathbb{P}(\widetilde Y > x)$ if $\alpha \in (0,1)$ and $\widetilde m'(\alpha) = 0$. Results of this type appear in the context of smoothing transforms, branching and telecommunication models; see Buraczewski et al. [4], in particular Theorem 5.2.8(2), and the references therein.

2.2. **Examples.** In this section we illustrate our theory by considering various examples.

Example 2.4. We assume that (X_i) is an iid lognormal sequence, where $\log X$ has an $N(\mu,1)$ distribution with negative μ. Then for $s>0$,

$$\log(\mathbb{E}[X^s]) = \mu\, s + s^2/2\,,\quad m(\alpha)=\alpha/2,,\quad \alpha=-2\mu,\quad \sigma^2(\alpha)=1\,.$$

Notice that $x^\alpha\, p(x)/2 = \sum_{n=1}^\infty \Phi\big((\log x - n\,\alpha/2)/\sqrt{n}\big)$.

Example 2.5. Assume that X has a $\Gamma(\gamma,\beta)$-density given by

$$f_X(x) = \frac{\beta^\gamma x^{\gamma-1}\,\mathrm{e}^{-x\beta}}{\Gamma(\gamma)}\,,\quad \beta,\gamma,x>0\,. \tag{2.7}$$

Since X has unbounded support the equation $\mathbb{E}[X^\alpha]=1$ always has a unique positive solution. For given values α and γ we can determine suitable values β such that

$$\mathbb{E}[X^\alpha] = \frac{\Gamma(\gamma+\alpha)}{\Gamma(\alpha)\beta^\alpha} = 1\,.$$

We also have

$$\begin{aligned} m(\alpha) &= \frac{\Gamma(\gamma+\alpha)}{\beta^\alpha\Gamma(\gamma)}\Big(\frac{\Gamma'(\gamma+\alpha)}{\Gamma(\gamma+\alpha)} - \log\beta\Big)\,,\\ \mathbb{E}[X^\alpha\,(\log X)^2] &= \frac{\Gamma(\gamma+\alpha)}{\beta^\alpha\Gamma(\gamma)}\Big((\log\beta)^2 - 2\log\beta\frac{\Gamma'(\gamma+\alpha)}{\Gamma(\gamma+\alpha)} + \frac{\Gamma''(\gamma+\alpha)}{\Gamma(\gamma+\alpha)}\Big)\,. \end{aligned}$$

Example 2.6. Assume that $X=\mathrm{e}^{Z-\mu}$ for some positive $\mu>0$ and a $\Gamma(\gamma,\beta)$-distributed random variable Z, i.e., X has a loggamma distribution. For $\alpha<\beta$ we can calculate

$$\mathbb{E}[X^\alpha] = \mathrm{e}^{-\alpha\mu}\,\big(1-\frac{\alpha}{\beta}\big)^{-\gamma}\,.$$

The equation $\mathbb{E}[X^\alpha]=1$ has a positive solution if and only if $\beta\mu>\gamma$. Under this assumption,

$$\begin{aligned} m(\alpha) &= \mathrm{e}^{-\alpha\mu}\big(1-\frac{\alpha}{\beta}\big)^{-\gamma-1}\frac{1}{\beta}\big[\gamma+\mu(\alpha-\beta)\big]\,,\\ \sigma^2(\alpha) &= \mathrm{e}^{-\alpha\mu}\,\big(1-\frac{\alpha}{\beta}\big)^{-\gamma-2}\frac{1}{\beta^2}\big[\gamma+(\gamma+\mu(\alpha-\beta))^2\big]\,. \end{aligned}$$

Consider iid copies (Z_i) of Z. Then

$$p(x) = \sum_{n=1}^\infty \mathbb{P}\big(\sum_{j=1}^n Z_j - n\mu > \log x\big)\,,$$

where $\sum_{j=1}^n Z_j$ is $\Gamma(n\gamma,\beta)$-distributed. In principle, this formula could be evaluated exactly by using the gamma distribution functions. However, the events $\{\sum_{j=1}^n Z_j > \log x\}$ are very rare for large x. Therefore one needs change-of-measure techniques to evaluate $p(x)$ or suitable approximation techniques. In the top Figure 1 we plot the ratio of the normal approximation of $x^\alpha p(x)$ given in (2.1) and $2\log x/m(\alpha)$ for $\mu = 5$, $\gamma = 4$ and $\beta = 1$. For the same parameter set, in the bottom figure we plot the ratio of

$$x^\alpha p(x) = \sum_{n=1}^{\infty} \mathbb{P}^\alpha\Big(\sum_{j=1}^{n} Z_j > \log x + n\mu\Big) = \sum_{n=1}^{\infty} F_n(\log x + n\mu)\,,$$

where we changed the measure from $\mathbb{P}$ to $\mathbb{P}^\alpha(Z \in dx) = \mathrm{e}^{\alpha x}\mathbb{P}(Z \in dx)$ resulting in the $\Gamma(n(\gamma+\alpha),\beta)$-distribution F_n. The rationale for this change of measure is explained in the proof of Theorem 2.1. A comparison of the two graphs shows the (not unexptected) result that the precise approximation of $x^\alpha p(x)$ via change of measure is better than the approximation via the normal law.

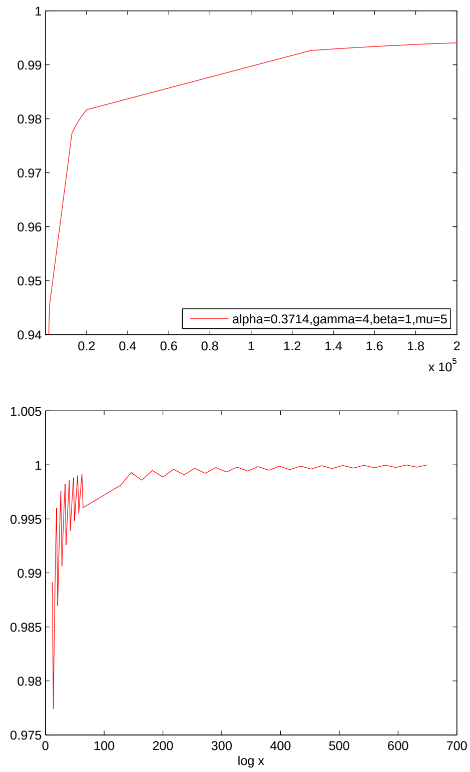

Figure 1. Two approximations of the ratio $x^\alpha p(x)/(2\log x/m(\alpha))$ for loggamma-distributed X with parameters β, γ, μ and the resulting α. The top figure shows the results of the normal approximation, the bottom figure a more precise approximation via change of measure.

3. Proof of Theorem 2.1

3.1. **First approximations.** Recall the definition of $p(x)$ from (1.11).

Proposition 3.1. *Assume the conditions of Theorem 2.1 and that* $p(x) \sim c \log x/x^\alpha$*. Then*

$$\mathbb{P}(Y > x) \sim p(x)\,, \qquad x \to \infty\,. \tag{3.1}$$

A proof of the fact that $p(x) \sim c \log x/x^\alpha$ will be given in Section 3.3.

Proof. Since $\sup_{i\geqslant 1} \mathbb{P}(\Pi_i > x) \leqslant p(x) \to 0$ we have

$$\begin{aligned}
\mathbb{P}(Y > x) &\geqslant \sum_{n=1}^\infty \mathbb{P}(\Pi_n > x)\,\mathbb{P}(\Pi_i \leqslant x\,, i \neq n)\\
&= \sum_{n=1}^\infty \mathbb{P}(\Pi_n > x) \prod_{i\neq n} \mathbb{P}(\Pi_i \leqslant x)\\
&= \sum_{n=1}^\infty \mathbb{P}(\Pi_n > x) \exp\Big(\sum_{i\neq n}^\infty \log(1 - \mathbb{P}(\Pi_i > x)\Big)\\
&= \sum_{n=1}^\infty \mathbb{P}(\Pi_n > x) \exp\Big(-(1+o(1))\sum_{i\neq n}^\infty \mathbb{P}(\Pi_i > x)\Big)\,.
\end{aligned}$$

Hence

$$\mathbb{P}(Y > x) \;\; \geqslant \;\; p(x)\,\mathrm{e}^{-(1+o(1))p(x)} = (1+o(1))p(x)\,,$$

and the liminf-part in (3.1) follows.

Next we consider an upper bound for $\mathbb{P}(Y > x)$. We have for $\varepsilon \in (0,1)$,

$$\mathbb{P}(Y > x) \;\; \leqslant \;\; p((1-\varepsilon)x) + \mathbb{P}\big(C\big)\,.$$

where $C = \{Y > x\,, \max_{n\geqslant 1} \Pi_n \leqslant x(1-\varepsilon)\}$. We write for small $\delta > 0$,

$$\begin{aligned}
B_1 &= \bigcup_{1\leqslant i<j} \{|\Pi_i| > \delta x, |\Pi_j| > \delta x\}\,,\\
B_2 &= \bigcup_{i=1}^\infty \{|\Pi_i| > \delta x\,, \max_{j\neq i} |\Pi_j| \leqslant \delta x\}\,,\\
B_3 &= \{\max_{i\geqslant 1} |\Pi_i| \leqslant \delta x\}\,.
\end{aligned}$$

Observe that we have by Markov's inequality for $\gamma \in (0,\alpha)$,

$$\widetilde{p}(x) = \sum_{n=1}^\infty \mathbb{P}(|\Pi_n| > x) \leqslant x^{-\gamma} \sum_{n=1}^\infty (h(\gamma))^n \leqslant c\,x^{-\gamma}\,. \tag{3.2}$$

Hence for $\gamma \in (\alpha/2, \alpha)$

$$\frac{\mathbb{P}(C \cap B_1)}{p((1-\varepsilon)x)} \;\; \leqslant \;\; \frac{\big(\widetilde{p}(\delta x)\big)^2}{p((1-\varepsilon)x)} \leqslant c\,x^{\alpha-2\gamma} \to 0\,, \quad x \to \infty\,.$$

Similarly, by independence of $Y-\Pi_n$ and Π_n for any $n \geqslant 1$, and since (3.2) holds,

$$\begin{aligned}\frac{\mathbb{P}(C\cap B_2)}{p((1-\varepsilon)x)} &\leqslant \frac{1}{p((1-\varepsilon)x)}\sum_{n=1}^{\infty}\mathbb{P}(|Y-\Pi_n|>\varepsilon x)\,\mathbb{P}(|\Pi_n|>\delta x)\\ &\leqslant \frac{(\widetilde{p}(\min(\varepsilon,\delta)x))^2}{p((1-\varepsilon)x)}\leqslant c\,x^{\alpha-2\gamma}\to 0\,,\qquad x\to\infty\,.\end{aligned}$$

We also have

$$\begin{aligned}\mathbb{P}(C\cap B_3) &\leqslant \mathbb{P}\Big(\sum_{n=1}^{\infty}|\Pi_n|>x\,,\max_i|\Pi_i|\leqslant \delta x\Big)\\ &\leqslant \mathbb{P}\Big(\sum_{n=1}^{\infty}|\Pi_n|\,\mathbb{1}(|\Pi_n|\leqslant \delta\,x)>x\Big)\,.\end{aligned}$$

Choose $\xi\in(0,1)$ and write $g(x)=[c_0\log x]$ for some positive constant c_0 to be chosen later. Then

$$\begin{aligned}\mathbb{P}(C\cap B_3) &\leqslant \mathbb{P}\Big(\sum_{n=1}^{g(x)}|\Pi_n|\mathbb{1}(|\Pi_n|\leqslant\delta x)>x(1-\xi)\Big)\\ &\quad+\sum_{n=g(x)+1}^{\infty}\mathbb{P}\big(|\Pi_n|\mathbb{1}(|\Pi_n|\leqslant\delta x)>x\,\xi^{n-g(x)}\,(1-\xi)\big)\\ &= P_1(x)+P_2(x)\,.\end{aligned}$$

We have by Markov's inequality with $\gamma\in(\alpha/2,\alpha)$,

$$\begin{aligned}\frac{P_2(x)}{p((1-\varepsilon)x)} &\leqslant c\,x^{\alpha-\gamma}\sum_{n=g(x)+1}^{\infty}\xi^{-(n-g(x))\gamma}\,\mathbb{E}[|\Pi_n|^{\gamma}]\\ &= c\,x^{\alpha-\gamma}(h(\gamma))^{g(x)}\sum_{n=0}^{\infty}\Big(\frac{h(\gamma)}{\xi^{\gamma}}\Big)^n\\ &\leqslant c\,x^{\alpha-\gamma}(h(\gamma))^{g(x)}(1-\phi)^{-1}\,.\end{aligned}$$

Here we choose ξ and γ such that $\phi=h(\gamma)/\xi^{\gamma}<1$. Then the right-hand side converges to zero if we choose $c_0>0$ sufficiently large.

Next we find a bound for $P_1(x)$. We apply Prohorov's inequality; see Petrov [15], p. 77. For this reason, we need bounds on the first and second moments of $S(x)=\sum_{n=1}^{g(x)}|\Pi_n|\mathbb{1}(|\Pi_n|\leqslant\delta x)$.

Lemma 3.2. *We have the following bounds*

$$\mathbb{E}[S(x)] \leqslant \begin{cases} h(1)\,(1-h(1))^{-1} < \infty\,, & \alpha > 1\,,\\ c\,(\log x)^2\,, & \alpha = 1\,,\\ c\,\log x\, x^{1-\alpha}\,, & \alpha \in (0,1)\,. \end{cases}$$

$$\operatorname{var}(S(x)) \leqslant \begin{cases} h(2)(1-h(2))^{-1} < \infty\,, & \alpha > 2\,,\\ c\,(\log x)^2\,, & \alpha = 2\,,\\ c\,\log x\; x^{1-\alpha/2}\,, & \alpha \in (0,2)\,. \end{cases}$$

Proof. We start with the bounds for $\mathbb{E}[S(x)]$. (1) If $\alpha > 1$, $\mathbb{E}[|\Pi_n|] = (h(1))^n < 1$. Hence

$$\mathbb{E}[S(x)] \leqslant \sum_{n=1}^{\infty} (h(1))^n = h(1)(1-h(1))^{-1}\,.$$

(2) If $\alpha = 1$ we use a domination argument. Indeed, we have

$$\mathbb{E}[|\Pi_n| \mathbb{1}(|\Pi_n| \leqslant z)]| = \int_0^z \mathbb{P}(|\Pi_n| > y)\,dy \leqslant \int_0^z \mathbb{P}(Y'' > y)\,dy\,, \qquad z > 0\,,$$

where $Y" = \sum_{n=1}^{\infty} |\Pi_n'|$. By (1.6)–(1.7) and (1.10), respectively, we have $x\,\mathbb{P}(Y" > x) \in [c, c']$ for constants $0 < c < c' < \infty$ and large x. Hence

$$\mathbb{E}[S(x)] \leqslant g(x)\mathbb{E}[Y''\mathbb{1}(Y'' \leqslant \delta x)] \leqslant c\,(\log x)^2\,.$$

(3) A similar argument in the case $\alpha \in (0,1)$ shows that $x^{\alpha}\mathbb{P}(Y" > x) \in [c, c']$ and $\mathbb{E}[Y"\mathbb{1}(Y" \leqslant x)] \leqslant c\,x^{1-\alpha}$ for large x. Hence $\mathbb{E}[S(x)] \leqslant c\,x^{1-\alpha}\,\log x$.

Our next goal is to find bounds for $\operatorname{var}(S(x))$. (1) If $\alpha > 2$ then

$$\operatorname{var}(S(x)) \leqslant \sum_{n=1}^{g(x)} \mathbb{E}[\Pi_n^2] \leqslant h(2)(1-h(2))^{-1} < \infty\,.$$

(2) Now assume $\alpha = 2$. Then we have

$$\operatorname{var}(S(x)) \leqslant \sum_{n=1}^{g(x)} \mathbb{E}[\Pi_n^2\,\mathbb{1}(|\Pi_n| \leqslant \delta x)]\,.$$

The same domination argument as for $\mathbb{E}[S(x)]$ in the case $\alpha = 1$ yields

$$\begin{aligned} \mathbb{E}[\Pi_n^2\,\mathbb{1}(|\Pi_n| \leqslant z)] &= \int_0^z \mathbb{P}(\Pi_n^2 > y)dy \\ &\leqslant \int_0^z \mathbb{P}((Y'')^2 > y)\,dy \\ &= \mathbb{E}[(Y'')^2\mathbb{1}((Y'')^2 \leqslant z)]\,, \end{aligned}$$

Hence

$$\operatorname{var}(S(x)) \leqslant g(x)\,\mathbb{E}[(Y'')^2\mathbb{1}((Y'')^2 \leqslant \delta x)] \leqslant c\,(\log x)^2\,.$$

(3) Assume $\alpha \in (0,2)$. In this case, similar arguments as for $\alpha = 2$ yield $\mathbb{P}((Y")^2 > x) \leqslant c x^{-\alpha/2}$ and

$$\operatorname{var}(S(x)) \leqslant c\, x^{1-\alpha/2} \log x\,.$$

□

From Lemma 3.2 we conclude that for large x,

$$P_1(x) \quad \leqslant \quad \mathbb{P}\big(S(x) - \mathbb{E}[S(x)] > 0.5\, x(1-\xi)\big)\,.$$

Now an application of Prohorov's inequality to the right-hand side yields

$$\begin{aligned} P_1(x) &\leqslant \exp\Big(-\frac{0.5\,x(1-\xi)}{2(2\,\delta x)} \operatorname{arsinh} \frac{(2\delta x)(0.5\,x(1-\xi))}{2\operatorname{var}(S(x))}\Big) \\ &= \exp\Big(-\frac{1-\xi}{8\,\delta} \operatorname{arsinh}\big(0.5x^2\,\delta\,(1-\xi)/\operatorname{var}(S(x))\big)\Big)\,, \end{aligned}$$

where $\operatorname{arsinh} y = \log(y + \sqrt{y^2+1}) \geqslant \log(2y)$ for positive y.

Now assume $\alpha > 2$. Choose δ so small that $(1-\xi)/(8\,\delta) > \alpha$ and apply Lemma 3.2 for large x,

$$\begin{aligned} \operatorname{arsinh}\big(0.5x^2\,\delta/\operatorname{var}(S(x))\big) &\geqslant 0.5 \log\big(x^2\,\delta\,(1-\xi)(1-h(2))/h(2)\big) \\ &\sim \log x\,, \qquad x \to \infty\,, \end{aligned}$$

Hence we may conclude that

$$\frac{P_1(x)}{p((1-\varepsilon)x)} \quad \leqslant \quad c\, x^{\alpha}\, P_1(x) \to 0\,, \qquad x \to \infty\,. \tag{3.3}$$

We proved the limsup-part in (3.1) for $\alpha > 2$. For $\alpha = 2$, using $\operatorname{var}(S(x)) \leqslant c\,(\log x)^2$, a slight modification of the Prohorov bound yields the same result. For $\alpha \in (0,2)$, we conclude from Lemma 3.2 that for large x,

$$\begin{aligned} \operatorname{arsinh}\big(0.5x^2\,\delta(1-\xi)/\operatorname{var}(S(x))\big) &\geqslant 0.5 \log\big(x^{1-\alpha/2}\,\delta\,(1-\xi)\big) \\ &\sim 0.5\,(1-\alpha/2)\, \log x\,, \end{aligned}$$

Now choose $\delta > 0$ so small that $(1-\alpha/2)(1-\xi)/(16\,\delta) > \alpha$ and then (3.3), hence (3.1) follows. □

3.2. **Preliminaries.** Our next goal is to show that $p(x) \sim c \log x / x^{\alpha}$. Since we will treat the cases $X \geqslant 0$ a.s. and $\mathbb{P}(X < 0) > 0$ in a similar way we will follow an idea of Goldie [8]. We define

$$N_0 = 0\,, \quad N_i = \inf\{k > N_{i-1} : \Pi_k > 0\}\,, \qquad i \geqslant 1\,, \quad I = \{N_i\,, i \geqslant 1\}\,,$$

If $X \geqslant 0$ a.s. we have $N_i = i$ a.s. We introduce the non-negative variables $\tilde{X}_i = \prod_{j=N_{i-1}+1}^{N_i} |X_i|$, $i \geqslant 1$, and their products $\tilde{\Pi}_n = \prod_{i=1}^n \tilde{X}_i$ so that

$$\begin{aligned} p(x) &= \sum_{n=1}^{\infty} \mathbb{P}(\Pi_n > x) = \mathbb{E}\Big[\sum_{n=1}^{\infty} \mathbb{1}(\Pi_n > x)\Big] \\ &= \mathbb{E}\Big[\sum_{n=1}^{\infty} \mathbb{1}(\tilde{\Pi}_n > x)\Big] = \sum_{n=1}^{\infty} \mathbb{P}(\tilde{\Pi}_n > x)\,. \end{aligned}$$

By independence of (X_i), $(\tilde X_i)$ are iid as well. Under $\mathbb{E}[|X|^\alpha]=1$, the process $\big(\prod_{j=1}^t |X_i|^\alpha\big)_{t\geqslant 1}$ is a martingale adapted to the filtration $\mathcal{F}_t=\sigma(X_i, i\leqslant t)$. As N_1 is a stopping time with respect to this filtration we derive that $\mathbb{E}[\tilde X_1^\alpha]=1$ by an application of the stopping time theorem for martingales.

Write $\tilde X$ for a generic element of $(\tilde X_i)$. We will use the following notation for $s>0$, assuming these moments are finite:

$$\begin{aligned}
\tilde M(s) &= \log \mathbb{E}[\tilde X^s]\,,\\
\tilde m(s) &= \tilde M'(s)=\frac{\mathbb{E}[|\tilde X|^s\,\log|\tilde X|]}{\mathbb{E}[|\tilde X|^s]}\,,\\
\tilde\sigma^2(\alpha) &= \tilde m'(s)=\frac{\mathbb{E}[|\tilde X|^s(\log|\tilde X|)^2]\,\mathbb{E}[|\tilde X|^s]-(\mathbb{E}[|\tilde X|^s\log|\tilde X|])^2}{(\mathbb{E}[|\tilde X|^s])^2}\,.
\end{aligned}$$

The expression for the distribution of $\log\tilde X$ given $\{\tilde X>0\}$ can be derived by mimicing the arguments of Goldie [8]. Denote $p=\mathbb{P}(X>0)=1-q$ and

$$\gamma_\pm(dy)=\frac{\mathbb{P}(\pm X>0,\log|X|\in dy)}{\mathbb{P}(\pm X>0)}\,.$$

Then we have

$$\mathbb{P}(\log\tilde X\in\cdot)=p\,\gamma_+(\cdot)+\sum_{n=2}^\infty q^2\,p^{n-2}\,\gamma_-^{(2)}(\cdot)*\gamma_+(\cdot)^{(n-2)}\,, \tag{3.4}$$

where $*$ denotes the convolution operator.

We introduce the tilted measure $\mathbb{P}^\alpha$:

$$d\mathbb{P}^\alpha(\log|X|\leqslant y)=\mathrm{e}^{\alpha y}d\mathbb{P}(\log|X|\leqslant y)\,,$$

and denote expectation and variance with respect to $\mathbb{P}^\alpha$ by $\mathbb{E}^\alpha$ and var^α, respectively. Under the assumption $\mathbb{E}[|X|^\alpha(\log|X|)^2]<\infty$ the following moments are finite and positive

$$\begin{aligned}
\mathbb{E}^\alpha[\log|X|] &= \mathbb{E}[|X|^\alpha\log|X|]=m(\alpha)\,,\\
\mathrm{var}^\alpha(\log|X|) &= \mathbb{E}^\alpha[(\log|X|)^2]-(\mathbb{E}^\alpha[\log|X|])^2\,,\\
&= \mathbb{E}[|X|^\alpha\,(\log|X|)^2]-(m(\alpha))^2=\sigma^2(\alpha)\,.
\end{aligned}$$

Since the tilted measure $\mathbb{P}^\alpha$ preserves the sum structure one has the identity

$$d\mathbb{P}^\alpha(\log|\tilde X|\leqslant y)=\mathrm{e}^{\alpha y}d\mathbb{P}(\log|\tilde X|\leqslant y)\,.$$

Then one can also check the existence of the quantities

$$\begin{aligned}
\mathbb{E}^\alpha[\log|\tilde X|] &= \mathbb{E}[|\tilde X|^\alpha\log|\tilde X|]=\tilde m(\alpha)\,,\\
\mathrm{var}^\alpha(\log|\tilde X|) &= \mathbb{E}^\alpha[(\log|\tilde X|)^2]-(\mathbb{E}^\alpha[\log|\tilde X|])^2\,,\\
&= \mathbb{E}[|\tilde X|^\alpha\,(\log|\tilde X|)^2]-(\tilde m(\alpha))^2=\tilde\sigma^2(\alpha)\,.
\end{aligned}$$

These quantities coincide with $m(\alpha)$ and $\sigma^2(\alpha)$ when $X \geqslant 0$ a.s. Otherwise, calculation yields

$$\begin{cases} \tilde{m}(\alpha) = & 2m(\alpha)\,, \\ \tilde{\sigma}^2(\alpha) = & 2\sigma^2(\alpha) + \dfrac{p}{q}(2m(\alpha))^2\,. \end{cases}$$

Finally, notice that $\mathbb{P}(N_1 = 1) = p$ and $\mathbb{P}(N_1 = n) = q^2\, p^{n-2}$ so that N_1 admits finite moments of any order. Moreover, $\mathbb{E}[N_1] = 1$ if $p = \mathbb{P}(X > 0) = 1$ and 2 else. If $\mathbb{P}^\alpha(\log|X| > x)$ is regularly varying with index $\kappa \in (1, 2]$, one can apply Case (b3) on p. 130 of Resnick [16] to the stopped random walk $\tilde{X}$ to obtain the equivalence

$$\mathbb{P}^\alpha(\log|\tilde{X}| > x) \sim \mathbb{E}[N_1]\,\mathbb{P}^\alpha(\log|X| > x), \qquad x \to \infty. \tag{3.5}$$

Hence $\mathbb{P}^\alpha(\log|\tilde{X}| > x)$ is also regularly varying with index $\kappa \in (1, 2]$.

3.3. **More precise asymptotics.** On the set $\{\tilde{\Pi}_n > 0\}$ we may write

$$\tilde{S}_n = \log \tilde{\Pi}_n = \sum_{i=1}^n \log \tilde{X}_i\,, \qquad n \geqslant 1\,.$$

We will show that

$$p(x) = \sum_{n=1}^\infty \mathbb{P}(\tilde{S}_n - n\,\mathbb{E}[\log \tilde{X}] > \log x - n\,\mathbb{E}[\log \tilde{X}]\,, \tilde{\Pi}_n > 0) \sim x^{-\alpha}\,L(x)$$

for a suitable slowly varying function L. Since $\mathbb{E}[\log \tilde{X}] = c\,\mathbb{E}[\log|X|] < 0$, where $c = 1$ for $X \geqslant 0$ and $c = 2$ otherwise, the random walk $(\tilde{S}_n)$ has negative drift. We will exploit the fact that, after the change of measure via $\mathbb{P}^\alpha$, the random walk $(\tilde{S}_n)$ has a positive drift $(n\,\tilde{m}(\alpha))$.

In what follows, we will get bounds for sums of $\mathbb{P}(\tilde{\Pi}_n > x)$ over different n-regions. It will be convenient to use the following notation

$$g_\xi(x) = [(1+\xi)\log x/\tilde{m}(\alpha)] \quad \text{for real } \xi.$$

Lemma 3.3. *For any small $\varepsilon > 0$ we can find $\delta > 0$ such that for sufficiently large x,*

$$\sum_{n=g_\varepsilon(x)}^\infty \mathbb{P}(\tilde{\Pi}_n > x) \leqslant c\,x^{-(\alpha+\delta)}\,. \tag{3.6}$$

Proof. Denote $\tilde h(s)=\mathbb{E}[\tilde X^s]$ for $s\leqslant\alpha$ and notice that $\tilde M(s)=\log\tilde h(s)$. By Markov's inequality for small $\varepsilon\in(0,\alpha)$,

$$\begin{aligned}
&x^\alpha\sum_{n=g_\varepsilon(x)}^\infty \mathbb{P}(\tilde\Pi_n>x)\\
&\leqslant\ x^\varepsilon\sum_{n=g_\varepsilon(x)}^\infty(\tilde h(\alpha-\varepsilon))^n=x^\varepsilon\,(\tilde h(\alpha-\varepsilon))^{g_\varepsilon(x)}(1-\tilde h(\alpha-\varepsilon))^{-1}\\
&=\ \exp\Big(\log x\,\big(\varepsilon+\frac{[(1+\varepsilon)\log x/\tilde m(\alpha)]}{\log x}\tilde M(\alpha-\varepsilon)\big)\Big)(1-h(\alpha-\varepsilon))^{-1}\,.
\end{aligned}$$

By a Taylor expansion, $\tilde M(\alpha-\varepsilon)=\tilde M(\alpha-\varepsilon)-\tilde M(\alpha)\sim-\tilde m(\alpha)\,\varepsilon$ as $\varepsilon\downarrow 0$. This proves (3.6) for small ε. □

We apply (3.6) and the fact that $\sum_{n=g_0(x)+1}^{g_\varepsilon(x)}\mathbb{P}(\tilde\Pi_n>x)\leqslant c\,\varepsilon\log x$ to show that

$$p(x)=\sum_{n=1}^\infty\mathbb{P}(\tilde\Pi_n>x)=\sum_{n=1}^{g_0(x)}\mathbb{P}(\tilde\Pi_n>x)+o(\log x)\,,\qquad x\to\infty\,.$$

Next define

$$\nu(x)=\mathrm{e}^{\alpha\log x}\sum_{i=1}^{g_0(x)}\mathbb{P}(\tilde\Pi_n>x)=\mathrm{e}^{\alpha\log x}\sum_{n=1}^{g_0(x)}\mathbb{P}(\tilde S_n>\log x).$$

Then we have

$$\begin{aligned}
\nu(x)&=\mathrm{e}^{\alpha\log x}\sum_{n=1}^{g_0(x)}\int_{\log x}^\infty d\mathbb{P}(\tilde S_n\leqslant t)\\
&=\int_{\log x}^\infty\mathrm{e}^{-\alpha(t-\log x)}d\Big(\sum_{n=1}^{g_0(x)}\mathbb{P}^\alpha(\tilde S_n\leqslant t)\Big)\\
&=\int_0^\infty\mathrm{e}^{-\alpha s}d\nu_\alpha(s+\log x)\,,
\end{aligned}\tag{3.7}$$

where $\nu_\alpha(y)=\sum_{n=1}^{g_0(x)}\mathbb{P}^\alpha(\tilde S_n\leqslant y)$. We want to show that $\ell(x)=\nu_\alpha(\log x)$ is a slowly varying function. More precisely, we want to show that $\ell(x)\sim g_0(x)$. We have $\ell(x)\leqslant g_0(x)$.

First assume $\mathbb{E}^\alpha[(\log|X|)^2]=\mathbb{E}[(\log|X|)^2|X|^\alpha]<\infty$, then $\mathbb{E}^\alpha[(\log\tilde X)^2]<\infty$; see the expression (3.5). By the central limit theorem under the measure

$\mathbb{P}^\alpha$ we have

$$\begin{aligned}\ell(x) &= \sum_{n=1}^{g_0(x)} \mathbb{P}^\alpha\Big(\frac{\tilde S_n - n\,\tilde m(\alpha)}{\sqrt{n}\tilde\sigma(\alpha)} \leqslant \frac{\log x - n\,\tilde m(\alpha)}{\sqrt{n}\tilde\sigma(\alpha)}\Big)\\ &= g_0(x) - \sum_{n=1}^{g_0(x)} \overline{\Phi}\Big(\frac{\log x - n\,\tilde m(\alpha)}{\sqrt{n}\tilde\sigma(\alpha)}\Big) + o(\log x)\\ &=: g_0(x) - T(x) + o(\log x)\,,\end{aligned}$$

where $\overline{\Phi} = 1 - \Phi$ denotes the right tail of the standard normal distribution. We have

$$\begin{aligned}T(x) &= O(K_0) + \sum_{n=1}^{g_0(x)-K_0} \overline{\Phi}\Big(\frac{\log x - n\,\tilde m(\alpha)}{\sqrt{n}\tilde\sigma(\alpha)}\Big)\\ &\leqslant O(K_0) + (g_0(x) - K_0)\overline{\Phi}(K)\,,\end{aligned}$$

where for a given $K > 0$ we choose an integer K_0 so large that $(\log x - n\,\tilde m(\alpha))/(\sqrt{n}\tilde\sigma(\alpha)) > K$. Since we can choose K as large as we wish we finally proved that $\ell(x) \sim g_0(x)$.

Now assume that $\mathbb{P}^\alpha(\log|X| > x)$ is regularly varying for some $\kappa \in (1,2]$ and if $\kappa = 2$ also assume that $\mathbb{E}^\alpha[(\log|X|)^2] = \infty$. Then we also have that $\mathbb{P}^\alpha(\log \tilde X > x)$ is regularly varying for some $\kappa \in (1,2]$ and if $\kappa = 2$ that $\mathbb{E}^\alpha[(\log \tilde X)^2] = \infty$; see the discussion at the end of Section 3.2. Choose (a_n) such that

$$n\left[\mathbb{P}^\alpha(\log \tilde X > a_n) + a_n^{-2}\mathbb{E}^\alpha[(\log \tilde X)^2 \mathbb{1}(\log \tilde X \leqslant a_n)]\right] = 1\,.$$

Then under $\mathbb{P}^\alpha$,

$$a_n^{-1}(\tilde S_n - n\,\tilde m(\alpha)) \xrightarrow{d} S_\kappa\,,$$

where S_κ has a κ-stable distribution Φ_κ. The same arguments as above yield

$$\begin{aligned}\ell(x) &= \sum_{n=1}^{g_0(x)} \mathbb{P}^\alpha\Big(\frac{\tilde S_n - n\,\tilde m(\alpha)}{a_n} \leqslant \frac{\log x - n\,\tilde m(\alpha)}{a_n}\Big)\\ &= g_0(x) - \sum_{n=1}^{g_0(x)} \overline{\Phi}_\kappa\Big(\frac{\log x - n\,\tilde m(\alpha)}{a_n}\Big) + o(\log x)\\ &=: g_0(x) - T(x) + o(\log x)\,.\end{aligned}$$

and also $\ell(x) \sim g_0(x)$.

Finally, integrating by parts, we obtain from (3.7),

$$\begin{aligned}\nu(x) &= \int_0^\infty \mathrm{e}^{-\alpha t} d\ell(x\mathrm{e}^{t}) = \ell(x) + \alpha \int_0^\infty \mathrm{e}^{-\alpha t}\ell(x\mathrm{e}^{t})dt\\ &\sim g_0(x) + \alpha \int_0^\infty \mathrm{e}^{-\alpha t} g_0(x\mathrm{e}^{t})\,dt\\ &\sim \frac{2\log x}{\tilde m(\alpha)} = \begin{cases} \dfrac{2}{m(\alpha)}\log x & \text{if } X \geqslant 0\,,\\ \dfrac{1}{m(\alpha)}\log x & \text{otherwise}\,.\end{cases}\end{aligned}$$

This finishes the proof of the desired bounds for $p(x)$ and concludes the proof of Theorem 2.1.

4. The multivariate case

4.1. **Kesten's multivariate setting.** In this section we will work under the conditions of the multivariate setting of Kesten [12]; see Section 4.4 in Buraczewski et al. [4].

We consider iid $d \times d$ matrices $(\mathbf{X}_t)$ with a generic element $\mathbf{X}$ such that $\mathbb{P}(\mathbf{X} \geqslant \mathbf{0}) = 1$ and $\mathbf{X}$ does not have a zero row with probability 1. Here and in what follows, vector inequalities like $\mathbf{x} \geqslant \mathbf{y}$, $\mathbf{x} > \mathbf{y}, \ldots$, in $\mathbb{R}^d$ are understood componentwise. We write $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ for the unit sphere in $\mathbb{R}^d$ and $\mathbb{S}_+^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\,, \mathbf{x} \geqslant \mathbf{0}\}$. We always use the Euclidean norm $|\cdot|$ and write $\|\cdot\|$ for the corresponding operator norm. Here and in what follows, all vectors are column vectors. We write $\Pi_n' = \mathbf{X}_1 \cdots \mathbf{X}_n$, $n \geqslant 1$.

Assume the following conditions.

(1) The top Lyapunov exponent γ is negative:

$$\gamma = \inf_{n\geqslant 1} n^{-1}\mathbb{E}[\log \|\Pi_n'\|] < 0\,.$$

(2) Consider

$$h(s) = \inf_{n\geqslant 1} \big(\mathbb{E}[\|\Pi_n'\|^s]\big)^{1/n} = \lim_{n\to\infty} \big(\mathbb{E}[\|\Pi_n'\|^s]\big)^{1/n}$$

and assume that there is $\alpha > 0$ such that $h(\alpha) = 1$.

(3) $\mathbb{E}\big[\|\mathbf{X}\|^\alpha \log^+ \|\mathbf{X}\|\big] < \infty$.

(4) The additive subgroup of $\mathbb{R}$ generated by the numbers $\log \lambda_{\mathbf{s}}$ is dense in $\mathbb{R}$, where $\lambda_{\mathbf{s}}$ is the dominant eigenvalue of $\mathbf{s} = \mathbf{x}_1 \cdots \mathbf{x}_n$ for $\mathbf{x}_i$, $i = 1, \ldots, n$, for some $n \geqslant 1$, in the support of $\mathbf{X}$ and such that $\mathbf{s} > \mathbf{0}$.

Let the $\mathbb{R}^d$-dimensional column vector $\mathbf{Y}'$ be independent of $\mathbf{X}$. Under the conditions above, the fixed point equation $\mathbf{Y}' \stackrel{d}{=} \mathbf{X}\mathbf{Y}' + \mathbf{u}$ has a unique solution $\mathbf{Y}'$, where $\mathbf{u} \in \mathbb{S}^{d-1}$ is a deterministic vector. Then one has the representation

$$\mathbf{Y}'^{\top} \stackrel{d}{=} \mathbf{u}^{\top} \sum_{n=1}^{\infty} \Pi'_{n-1} = \mathbf{u}^{\top} + \mathbf{u}^{\top} \sum_{n=2}^{\infty} \Pi'_{n-1} \,.$$

The next result follows from Theorems 4 and A in Kesten [12]; cf. Theorem 4.4.5 in Buraczewski et al. [4], where the conditions below are also compared with those in the original paper.

Theorem 4.1. *Under the conditions above, there exists a finite function* e_α *on* $\mathbb{S}^{d-1}$ *such that*

$$\lim_{x\to\infty} x^{\alpha}\, \mathbb{P}(\mathbf{v}^{\top}\mathbf{Y}' > x) = e_\alpha(\mathbf{v})\,, \qquad \mathbf{v} \in \mathbb{S}^{d-1}\,, \tag{4.1}$$

and $e_\alpha(\mathbf{v}) > 0$ *for* $\mathbf{v} \in \mathbb{S}^{d-1}_{+}$*. Moreover, the limits*

$$\lim_{x\to\infty} x^{\alpha}\, \mathbb{P}\big(\max_{n\geqslant 1} |\Pi_n'^{\top}\mathbf{u}| > x\big) = \widetilde{e}_\alpha(\mathbf{u})\,, \qquad \mathbf{u} \in \mathbb{S}^{d-1}\,, \tag{4.2}$$

exist, are finite and positive for $\mathbf{u} \in \mathbb{S}^{d-1}_{+}$*.*

We notice that this result is analogous to Theorem 1.1 in the non-arithmetic case. Indeed, it is a special case for $d = 1$.

Remark 4.2. Under the condition of non-negativity on $\mathbf{X}$ and $\mathbf{u}$, (4.1) implies regular variation of $\mathbf{Y}'$ (see Buraczewski et al. [4], Theorem C.2.1), i.e., there exists a Radon measure ν on $\overline{\mathbb{R}}^d_0 = (\mathbb{R} \cup \{\pm\infty\})^d \backslash \{\mathbf{0}\}$ such that

$$x^{\alpha}\, \mathbb{P}(x^{-1}\mathbf{Y}' \in \cdot) \stackrel{v}{\to} \nu(\cdot)\,.$$

Here $\stackrel{v}{\to}$ denotes vague convergence in $\overline{\mathbb{R}}^d_0$ and ν has the property $\nu(t\cdot) = t^{-\alpha}\nu(\cdot)$, $t > 0$. In particular, for fixed $\mathbf{u}$ and any $\mathbf{v} \in \mathbb{S}^{d-1}_{+}$ there exist positive constants $c_{\mathbf{u}}$ and $c_{\mathbf{u},\mathbf{v}}$ such that as $x \to \infty$,

$$x^{\alpha}\, \mathbb{P}(|\mathbf{Y}'| > x) \to c_{\mathbf{u}} \quad \text{and} \quad x^{\alpha}\, \mathbb{P}(\mathbf{v}^{\top}\mathbf{Y}' > x) \to c_{\mathbf{u},\mathbf{v}}\,.$$

Under non-negativity of $\mathbf{X}$, $\mathbf{u}$ and $\mathbf{v}$, and if $e_\alpha(\mathbf{v}) \neq 0$ for some $\mathbf{v}$, (4.1) still implies regular variation of $\mathbf{Y}'$ for non-integer-valued α; see the comments after Theorem C.2.1 in [4].

4.2. **Main results.** In what follows, we provide an analog of the univariate theory built in the previous sections. For this reason, consider an iid array $(\mathbf{X}_{ni})_{n,i=1,2,\ldots}$ with generic element $\mathbf{X}$. Assume the conditions on $\mathbf{X}$ and $\mathbf{u}$ from Kesten's Theorem 4.1 and define

$$\mathbf{Y}^{\top} = \mathbf{Y}^{\top}(\mathbf{u}) = \mathbf{u}^{\top} \sum_{n=1}^{\infty} \Pi_n\,, \quad \text{where } \Pi_n = \textstyle\prod_{j=1}^{n} \mathbf{X}_{nj}.$$

For any unit vectors $\mathbf{u}, \mathbf{v} \in \mathbb{S}^{d-1}_{+}$, we define

$$p_{\mathbf{u}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(|\Pi_n^{\top}\mathbf{u}| > x) \quad \text{and} \quad p_{\mathbf{u},\mathbf{v}}(x) = \sum_{n=1}^{\infty} \mathbb{P}(\mathbf{v}^{\top}\Pi_n^{\top}\mathbf{u} > x)\,.$$

The following result is an analog of Theorem 2.1.

Theorem 4.3. *Assume the Kesten conditions of Section 4.1, in particular there exists $\alpha > 0$ such that $h(\alpha) = 1$. In addition, we assume that $\mathbb{E}[\|\mathbf{X}\|^\alpha (\log \|\mathbf{X}\|)^2] < \infty$. Then we have for $\mathbf{Y} = \mathbf{Y}(\mathbf{u})$ and any $\mathbf{u}, \mathbf{v} \in \mathbb{S}_+^{d-1}$*

$$\mathbb{P}(|\mathbf{Y}| > x) \sim p_{\mathbf{u}}(x) \sim \frac{2}{m(\alpha)} \frac{\log x}{x^\alpha}, \qquad x \to \infty, \tag{4.3}$$

$$\mathbb{P}(\mathbf{v}^\top \mathbf{Y} > x) \sim p_{\mathbf{u},\mathbf{v}}(x) \sim \frac{2(\mathbf{v}^\top \mathbf{u})^\alpha}{m(\alpha)} \frac{\log x}{x^\alpha}, \qquad \text{if also } \mathbf{u} > \mathbf{0}, \tag{4.4}$$

where $m(\alpha) = h'(\alpha)$ is independent of $\mathbf{u}, \mathbf{v}$.

The proof of this result is given in Section 4.3.

Remark 4.4. As in the univariate case it is possible to relax the condition $\mathbb{E}[\|\mathbf{X}\|^\alpha (\log \|\mathbf{X}\|)^2] < \infty$ by a regular variation condition of order $\kappa \in (1, 2]$, assuming for $\kappa = 2$ that $\mathbb{E}[\|\mathbf{X}\|^\alpha (\log \|\mathbf{X}\|)^2] = \infty$. This regular variation condition has to be required under the probability measure $\mathbb{P}^\alpha$ which will be explained in the course of the proof of the theorem. Write $Z = \sum_{ij} X_{ij}$ and $V = \min_{i=1,\ldots,d} \sum_{j=1}^d X_{ij}$, where X_{ij} are the entries of $\mathbf{X}$. Then one needs to assume that as $x \to \infty$,

$$\mathbb{P}^\alpha(\pm \log Z > x) \sim c_\pm \frac{L(x)}{x^\kappa} \quad \text{and} \quad \mathbb{P}^\alpha(\log V \leqslant -x) = O(\mathbb{P}^\alpha(|\log Z| > x)), \tag{4.5}$$

where $c_\pm$ are non-negative constants such that $c_+ + c_- = 1$ and L is a slowly varying function. In the case when $\mathbb{E}[\|\mathbf{X}\|^\alpha (\log \|\mathbf{X}\|)^2] < \infty$ we use a central limit theorem with Gaussian limit of Hennion [10]. Under (4.5) and $\mathbb{P}^\alpha$, one can instead apply a corresponding result with a κ-stable limit. The corresponding results can be found in Hennion and Hervé [11]; see their Theorem 1.1 (replacing Theorem 3 in [10]) and Lemma 2.1 (replacing Lemma 5.1 in [10]). Thus, as in the univariate case, the moment condition $\mathbb{E}[\|\mathbf{X}\|^\alpha (\log \|\mathbf{X}\|)^2] < \infty$ can be slightly relaxed.

In [10] and [11] the condition

$$\mathbb{P}\big(\Pi_n > \mathbf{0} \text{ for some } n \geqslant 1\big) > 0. \tag{4.6}$$

is assumed. This condition follows under the conditions of Section 4.1; see p. 171 in [4].

Remark 4.5. We observe that

$$\frac{p_{\mathbf{u},\mathbf{v}}(x)}{p_{\mathbf{u}}(x)} \to (\mathbf{v}^\top \mathbf{u})^\alpha, \qquad x \to \infty.$$

The right-hand side is smaller than one unless $\mathbf{u} = \mathbf{v}$. In particular, for $\mathbf{u} = \mathbf{v} > \mathbf{0}$ we have

$$\begin{aligned} \mathbb{P}(|\mathbf{Y}(\mathbf{u})| > x) &\sim \mathbb{P}(\mathbf{u}^\top \mathbf{Y}(\mathbf{u}) > x), \\ \mathbb{P}(|\mathbf{Y}(\mathbf{u})| > x, \mathbf{u}^\top \mathbf{Y}(\mathbf{u}) \leqslant x) &= o(\log x / x^\alpha). \end{aligned}$$

This means that $\mathbf{Y}(\mathbf{u})$ puts most tail mass in the direction of $\mathbf{u}$.

Following Remark 4.2, we also have full regular variation of the vector $\mathbf{Y}(\mathbf{u})$ if $\mathbf{u} > 0$ since (4.4) holds for $\mathbf{v} \geqslant \mathbf{0}$.

Corollary 4.6. *Assume the conditions of Theorem 4.3 and that α is not an integer. Then $\mathbf{Y}(\mathbf{u})$ is regularly varying with index α. In particular, there is a Radon measure ν on $\overline{\mathbb{R}}_0^d$ such that*

$$\frac{m(\alpha)}{2}\frac{x^\alpha}{\log x}\mathbb{P}(x^{-1}\mathbf{Y}(\mathbf{u}) \in \cdot) \stackrel{v}{\to} \nu(\cdot)\,, \quad x \to \infty\,,$$

and ν is uniquely determined by its values on the sets $A_{\mathbf{v}} = \{\mathbf{y} \in \mathbb{R}^d : \mathbf{v}^\top \mathbf{y}\}$, $\mathbf{v} \geqslant 0$, i.e., $\nu(A_{\mathbf{v}}) = (\mathbf{v}^\top \mathbf{u})^\alpha$.

4.3. **Proof of Theorem 4.3.** The proofs of (4.3) and (4.4) are very much alike. We focus on (4.3) and only indicate the differences with the proof of (4.4). We will follow the lines of the proof of Theorem 2.1 and Corollary 2.3.

With start with an analog of Proposition 3.1.

Proposition 4.7. *Assume the conditions of Section 4.1 and that $p_{\mathbf{u}}$, $p_{\mathbf{u},\mathbf{v}}$ are regularly varying. Then for any $\mathbf{u}, \mathbf{v} \in \mathbb{S}_+^{d-1}$ and $\mathbf{Y} = \mathbf{Y}(\mathbf{u})$,*

$$\frac{\mathbb{P}(|\mathbf{Y}| > x)}{p_{\mathbf{u}}(x)} \sim \frac{\mathbb{P}(\mathbf{v}^\top \mathbf{Y} > x)}{p_{\mathbf{u},\mathbf{v}}(x)} \sim 1\,, \qquad x \to \infty\,. \tag{4.7}$$

Proof. We follow the lines of the proof of Proposition 3.1. Since $h(\alpha) = 1$, we have by convexity of h for $\gamma \in (0, \alpha)$, $h(\gamma) < 1$, hence for sufficiently large n,

$$\mathbb{E}[\|\Pi_n\|^\gamma] < c_0^n\,, \tag{4.8}$$

for some $c_0 \in (0,1)$. By Markov's inequality, with $\widetilde{p}(x) = \sum_{n=1}^\infty \mathbb{P}(\|\Pi_n\| > x)$,

$$x^\gamma p_{\mathbf{v},\mathbf{u}}(x) \leqslant x^\gamma p_{\mathbf{u}}(x) \quad \leqslant \quad x^\gamma \sum_{n=1}^\infty \mathbb{P}(\|\Pi_n\| > x) =: \widetilde{p}(x) \leqslant c\,. \tag{4.9}$$

In particular, $p_{\mathbf{u}}(x) \to 0$. By (4.2) we have

$$\sup_{i \geqslant 1} \mathbb{P}(|\Pi_i^\top \mathbf{u}| > x) \leqslant \widetilde{p}(x) \to 0\,,$$

Using this fact and the same arguments as in the univariate case, we derive

$$\begin{aligned} \mathbb{P}(|\mathbf{Y}| > x) &\geqslant \sum_{n=1}^\infty \mathbb{P}(|\Pi_n^\top \mathbf{u}| > x)\,\mathbb{P}(|\Pi_i^\top \mathbf{u}| \leqslant x\,, i \neq n) \\ &= p_{\mathbf{u}}(x)(1 + o(1))\,, \\ \mathbb{P}(\mathbf{v}^\top \mathbf{Y} > x) &\geqslant p_{\mathbf{u},\mathbf{v}}(x)(1 + o(1))\,. \end{aligned}$$

This proves the liminf-part of (4.7).

Next we prove the limsup-part. We have for $\varepsilon \in (0,1)$,

$$\mathbb{P}(|\mathbf{Y}| > x) \quad \leqslant \quad p_{\mathbf{u}}((1-\varepsilon)x) + \mathbb{P}(C)\,,$$

where $C=\{|\mathbf{Y}|>x\,,\max_{n\geqslant 1}|\Pi_n^\top\mathbf{u}|\leqslant x(1-\varepsilon)\}$. We write for small $\delta>0$,

$$\begin{aligned}B_1&=\bigcup_{1\leqslant i<j}\{|\Pi_i^\top\mathbf{u}|>\delta x,|\Pi_j^\top\mathbf{u}|>\delta x\}\,,\\ B_2&=\bigcup_{i=1}^\infty\{|\Pi_i^\top\mathbf{u}|>\delta x\,,\max_{j\neq i}|\Pi_j^\top\mathbf{u}|\leqslant\delta x\}\,,\\ B_3&=\{\max_i|\Pi_i^\top\mathbf{u}|\leqslant\delta x\}\,.\end{aligned}$$

By (4.9) we have for $\gamma\in(0,\alpha)$ and large x, $p_{\mathbf{u}}(x)\leqslant x^{-\gamma}$. Therefore we may proceed as in the univariate case and obtain for $\gamma\in(\alpha/2,\alpha)$,

$$\frac{\mathbb{P}(C\cap B_1)}{p_{\mathbf{u}}((1-\varepsilon)x)}\leqslant c\,x^{\alpha-2\gamma}\to 0\,,\quad x\to\infty\,.$$

Similarly,

$$\frac{\mathbb{P}(C\cap B_2)}{p_{\mathbf{u}}((1-\varepsilon)x)}\leqslant\sum_{n=1}^\infty\frac{\mathbb{P}(|\mathbf{Y}-\Pi_n^\top\mathbf{u}|>\varepsilon x)\,\mathbb{P}(|\Pi_n^\top\mathbf{u}|>\delta\,x)}{p_{\mathbf{u}}((1-\varepsilon)x)}\leqslant cx^{\alpha-2\gamma}\to 0\,,$$

and for $\xi\in(0,1)$ and $g(x)=[c_0\log x]$, $c_0>0$,

$$\begin{aligned}\mathbb{P}(C\cap B_3)&\leqslant\mathbb{P}\Big(\sum_{n=1}^{g(x)}|\Pi_n^\top\mathbf{u}|\mathbb{1}(|\Pi_n^\top\mathbf{u}|\leqslant\delta x)>x(1-\xi)\Big)\\ &\quad+\sum_{n=g(x)+1}^\infty\mathbb{P}\big(|\Pi_n^\top\mathbf{u}|\mathbb{1}(|\Pi_n^\top\mathbf{u}|\leqslant\delta x)>x\,\xi^{n-g(x)}\,(1-\xi)\big)\\ &=P_1(x)+P_2(x)\,.\end{aligned}$$

The proof of $P_2(x)/p_{\mathbf{u}}((1-\varepsilon)x)\to 0$ is analogous to the univariate case. Write

$$S(x)=\sum_{n=1}^{g(x)}|\Pi_n^\top\mathbf{u}|\,\mathbb{1}(|\Pi_n^\top\mathbf{u}|\leqslant\delta x).$$

We have similar bounds for $\mathbb{E}[S(x)]$ and $\mathrm{var}(S(x))$ as in the univariate case; see Lemma 3.2. The key to this fact is domination via the inequalities

$$\begin{aligned}\mathbb{E}\big[|\Pi_n^\top\mathbf{u}|\,\mathbb{1}(|\Pi_n^\top\mathbf{u}|\leqslant z)\big]&=\int_0^z\mathbb{P}(|\Pi_n^\top\mathbf{u}|>y)\,dy\\ &\leqslant\int_0^z\mathbb{P}\big(\max_{i\geqslant 1}|{\Pi'_i}^\top\mathbf{u}|>y\big)\,dy\,,\\ \mathbb{E}[|\Pi_n^\top\mathbf{u}|^2\mathbb{1}(|\Pi_n^\top\mathbf{u}|\leqslant z)]&=\int_0^z\mathbb{P}(|\Pi_n^\top\mathbf{u}|>\sqrt{y})\,dy\\ &\leqslant\int_0^z\mathbb{P}\big(\max_{i\geqslant 1}|{\Pi'_i}^\top\mathbf{u}|>\sqrt{y}\big)\,dy\,.\end{aligned}$$

Now exploit the result for the tails in (4.2) and the same domination argument as in the univariate case. Finally, Prohorov's inequality applies to show that $P_1(x)/p_{\mathbf{u}}((1-\varepsilon)x) \to 0$.

The proof of $\limsup_{x\to\infty} \mathbb{P}(\mathbf{v}^\top \mathbf{Y} > x)/p_{\mathbf{u},\mathbf{v}}((1-\varepsilon)x) \leqslant 1$ is analogous. □

Our next goal is to show that $p_{\mathbf{u}}$ and $p_{\mathbf{u},\mathbf{v}}$ are regularly varying functions. For $\varepsilon > 0$, we write $g_\varepsilon(x) = [(1+\varepsilon)\log x/m(\alpha)]$ where $m(\alpha) = h'(\alpha)$. We continue with the following analog of Lemma 3.3.

Lemma 4.8. *Assume the conditions of Section 4.1. Then*

$$x^\alpha \sum_{n=g_\varepsilon(x)}^{\infty} \mathbb{P}(|\Pi_n^\top \mathbf{u}| > x) = o(\log x)\,, \qquad x \to \infty\,.$$

Proof. By Markov's inequality and a Taylor expansion of $\log h(\gamma) - \log h(\alpha)$ at α for $\gamma < \alpha$ close to α, since $\log h(\gamma) < 0$,

$$\begin{aligned} x^\alpha \sum_{n=g_\varepsilon(x)}^{\infty} \mathbb{P}(|\Pi_n^\top \mathbf{u}| > x) &\leqslant x^{\alpha-\gamma} \sum_{n=g_\varepsilon(x)}^{\infty} \mathrm{e}^{n\,\log\left(\inf_{k\geqslant 1}(\mathbb{E}[\|\Pi_k\|^\gamma])^{1/k}\right)} \\ &= c\,\mathrm{e}^{\log x\left((\alpha-\gamma)+(g_\varepsilon(x)/\log x)(\log h(\gamma)-\log h(\alpha))\right)} \\ &\leqslant x^{-\delta} \to 0\,, \qquad x \to \infty\,, \end{aligned}$$

for some $\delta > 0$, depending on γ and ε. □

It follows immediately that

$$x^\alpha \sum_{n=g_\varepsilon(x)}^{\infty} \mathbb{P}(\mathbf{v}^\top \Pi_n^\top > x) = o(\log x)\,, \qquad x \to \infty\,.$$

As in the univariate case we proceed with an exponential change of measure. However, the change of measure cannot be done on the marginal distribution but on the transition kernel of the Markov chain $({\Pi'_n}^\top \mathbf{u})$. It is indeed a homogeneous Markov chain as its kernel is given by the expression

$$\mathbb{P}(d\mathbf{y}|{\Pi'_n}^\top \mathbf{u} = \mathbf{x}) = \mathbb{P}(\mathbf{X}_{n+1}^\top \mathbf{x} \in d\mathbf{y}) = \mathbb{P}(\mathbf{X}^\top \mathbf{x} \in d\mathbf{y}),$$

that does not depend on $n \geqslant 1$. The change of kernel is given by

$$\mathbb{P}^\alpha(d\mathbf{y} \mid {\Pi'_n}^\top \mathbf{u} = \mathbf{x}) = \mathrm{e}^{\alpha\,\log(|\mathbf{y}|/|\mathbf{x}|)}\,\mathbb{P}(\mathbf{X}^\top \mathbf{x} \in d\mathbf{y}).$$

Since this change is difficult to justify we refer the reader for details to Buraczewski et al. [3]; see also Section 4.4.6 in the book Buraczewski et al. [4]. In contrast to Kesten's condition in Section 4.1, [3] require a moment condition slightly stronger than in Kesten [12]; see the discussion on p. 181 in [4]. This condition is satisfied because we require $\mathbb{E}[\|\mathbf{X}\|^\alpha(\log \|\mathbf{X}\|)^{1+\delta}] < \infty$ for some positive δ.

Next we notice that $\log |{\Pi'_n}^\top \mathbf{u}|$ has the structure of a Markov random walk associated with $({\Pi'_n}^\top \mathbf{u})$ under the change of measure. Indeed, since

$|\mathbf{u}|=1$ we have the following identity for $n=2$,

$$
\begin{aligned}
&\mathbb{P}^\alpha(\log|\Pi_2'^\top\mathbf{u}|\in dx)\\
&= \int_{\mathbb{R}^d}\mathbb{P}^\alpha\big(\log|\Pi_2'^\top\mathbf{u}|\in dx\mid \Pi_1'^\top\mathbf{u}=\mathbf{y}\big)\,\mathbb{P}^\alpha\big(\Pi_1'^\top\mathbf{u}\in d\mathbf{y}\big)\\
&= \int_{\mathbb{R}^d}\mathrm{e}^{\alpha\,(x-\log|\mathbf{y}|)}\,\mathbb{P}\big(\log|\Pi_2'^\top\mathbf{u}|\in dx\mid \Pi_1'^\top\mathbf{u}=\mathbf{y}\big)\,\mathrm{e}^{\alpha\,\log|\mathbf{y}|}\,\mathbb{P}\big(\Pi_1'^\top\mathbf{u}\in d\mathbf{y}\big)\\
&= \mathrm{e}^{\alpha\,x}\,\mathbb{P}\big(\log|\Pi_2'^\top\mathbf{u}|\in dx\big)\,.
\end{aligned}
$$

Thus, the structure of a Markov random walk is recovered thanks to a recursive argument.

From the discussion on p. 181 in [4] we also learn that under the aforementioned additional moment condition, $(\log\|\Pi_n'\|)$ has a positive drift under $\mathbb{P}^\alpha$, i.e., $\log\|\Pi_n'\|/n\to h'(\alpha)$ $\mathbb{P}^\alpha$-a.s. On the other hand, by the multiplicative ergodic theorem this means that $h'(\alpha)$ is the top Lyapunov exponent under $\mathbb{P}^\alpha$, i.e., $n^{-1}\mathbb{E}^\alpha[\log\|\Pi_n'\|]\to h'(\alpha)$.

As in the univariate case, we have

$$
\nu(x) \;=\; x^\alpha\sum_{n=1}^{g_0(x)}\mathbb{P}(|\Pi_n^\top\mathbf{u}|>x)=\int_0^\infty \mathrm{e}^{-\alpha s}d\nu_\alpha(s+\log x)\,,
$$

where

$$
\nu_\alpha(y)=\sum_{n=1}^{g_0(x)}\mathbb{P}^\alpha(|\Pi_n^\top\mathbf{u}|\leqslant y)\,.
$$

Since we assume $\mathbb{E}[\|\mathbf{X}\|^\alpha(\log\|\mathbf{X}\|)^2]<\infty$ we also have $\mathbb{E}^\alpha[(\log\|\mathbf{X}\|)^2]<\infty$. By virtue of Theorem 3 in Hennion [10] we have for any unit vectors $\mathbf{u},\mathbf{v}$,

$$
\sup_z\big|\mathbb{P}^\alpha\big((\log\mathbf{u}^\top\Pi_n\mathbf{v}-n\,m(\alpha))/\sqrt{n}\leqslant z\big)-\Phi_{0,\sigma^2(\alpha)}(z)\big|\to 0\,,\qquad n\to\infty\,,
\tag{4.10}
$$

where $\Phi_{0,\sigma^2(\alpha)}$ is the distribution function of an $N(0,\sigma^2(\alpha))$-distributed random variable, where $\sigma^2(\alpha)\geqslant 0$ is the limiting variance which is independent of $\mathbf{u},\mathbf{v}$. An application of Lemma 5.1 in [10] ensures that

$$
\sup_z\big|\mathbb{P}^\alpha\big((\log|\Pi_n^\top\mathbf{u}|-n\,m(\alpha))/\sqrt{n}\leqslant z\big)-\Phi_{0,\sigma^2(\alpha)}(z)\big|\to 0\,,\qquad n\to\infty\,,
$$

For these two results of Hennion one needs the condition (4.6).

Now the same arguments as in the univariate case apply to show that

$$
\begin{aligned}
\ell(x) &= \nu_\alpha(\log x)\\
&= \sum_{n=1}^{g_0(x)}\mathbb{P}^\alpha\big((\log|\Pi_n^\top\mathbf{u}|-nm(\alpha))/\sqrt{n}\leqslant(\log x-n\,m(\alpha))/\sqrt{n}\big)\\
&= g_0(x)-\sum_{n=1}^{g_0(x)}\overline{\Phi}_{0,\sigma^2(\alpha)}\big((\log x-n\,m(\alpha))/\sqrt{n}\big)+o(\log x)\sim g_0(x)\,.
\end{aligned}
$$

Now we consider $\mathbb{P}(\mathbf{v}^\top \mathbf{Y}(\mathbf{u}) > x)$ for a given $\mathbf{v} \in \mathbb{S}_+^{d-1}$, $\mathbf{v} \geqslant \mathbf{0}$ and $\mathbf{u} > 0$. We will mimic the preceding arguments. Since $\mathbf{X} \geqslant 0$ a.s. with non-zero rows ${\Pi'_n}^\top \mathbf{u} > 0$ a.s. for any $n \geqslant 1$. Thus, considering the Markov chain $({\Pi'_n}^\top \mathbf{u})_{n\geqslant 0}$ on the restricted state space $(0,\infty)^d$, one can change the measure according to

$$\mathbb{P}^{\alpha,\mathbf{v}}(d\mathbf{y} \mid {\Pi'_n}^\top \mathbf{u} = \mathbf{x}) = \mathrm{e}^{\alpha \log(\mathbf{v}^\top \mathbf{y}/\mathbf{v}^\top \mathbf{x})}\, \mathbb{P}(\mathbf{X}^\top \mathbf{x} \in d\mathbf{y})$$

for any $\mathbf{x}$ and $\mathbf{y} > \mathbf{0}$. Under this change of measure, $\log \mathbf{v}^\top {\Pi'_n}^\top \mathbf{u}$ has the structure of a Markov random walk associated with $({\Pi'_n}^\top \mathbf{u})$:

$$\begin{aligned}
&\mathbb{P}^\alpha(\log \mathbf{v}^\top {\Pi_2'}^\top \mathbf{u} \in dx) \\
&= \int_{\mathbb{R}^d} \mathbb{P}^\alpha\big(\log \mathbf{v}^\top {\Pi'_2}^\top \mathbf{u} \in dx \mid {\Pi'_1}^\top \mathbf{u} = \mathbf{y}\big)\, \mathbb{P}^\alpha\big({\Pi'_1}^\top \mathbf{u} \in d\mathbf{y}\big) \\
&= \int_{\mathbb{R}^d} \mathrm{e}^{\alpha\,(x - \log \mathbf{v}^\top \mathbf{y})}\, \mathbb{P}\big(\log \mathbf{v}^\top {\Pi'_2}^\top \mathbf{u} \in dx \mid {\Pi'_1}^\top \mathbf{u} = \mathbf{y}\big) \\
&\quad \times \mathrm{e}^{\alpha \log(\mathbf{v}^\top \mathbf{y}/\mathbf{v}^\top \mathbf{u})}\, \mathbb{P}\big({\Pi'_1}^\top \mathbf{u} \in d\mathbf{y}\big) \\
&= \mathrm{e}^{\alpha\,(x - \log(\mathbf{v}^\top \mathbf{u}))}\, \mathbb{P}\big(\log \mathbf{v}^\top {\Pi'_2}^\top \mathbf{u} \in dx\big)\,.
\end{aligned}$$

Using the relation $c|\mathbf{x}| \leqslant \mathbf{v}^\top \mathbf{x} \leqslant c'|\mathbf{x}|$ for some $c, c' > 0$ and any $\mathbf{x} \in (0,\infty)^d$, we have

$$\lim_{n\to\infty} \big(\mathbb{E}[(\mathbf{v}^\top {\Pi'_n}^\top \mathbf{u})^s]\big)^{1/n} = \lim_{n\to\infty} \big(\mathbb{E}[|{\Pi'_n}^\top \mathbf{u}|^s]\big)^{1/n} = \lim_{n\to\infty} \big(\mathbb{E}[\|\Pi'_n\|^s]\big)^{1/n} = h(s).$$

Here, the second identity is stated in Theorem 6.1 of [3]. From these 3 identities one concludes that the mean of $\mathbf{v}^\top {\Pi'_n}^\top \mathbf{u}$ is $h(\alpha)$, the same than the one of $|{\Pi'_n}^\top \mathbf{u}|$ under their respective changes of measure. For g_0 defined as above (independently of $\mathbf{v}$) we obtain

$$\nu_{\alpha,\mathbf{v}}(\log x) = \sum_{n=1}^{g_0(x)} \mathbb{P}^{\alpha,\mathbf{v}}(\mathbf{v}^\top \Pi_n \mathbf{u} \leqslant x) \sim g_0(x).$$

Similar arguments as above and as in the proof of Corollary 2.3 finish the proof of Theorem 4.3 because

$$\nu_{\mathbf{v}}(x) \;:=\; \frac{x^\alpha}{(\mathbf{v}^\top \mathbf{u})^\alpha} \sum_{n=1}^{g_0(x)} \mathbb{P}(\mathbf{v}^\top \Pi_n^\top \mathbf{u} > x) = \int_0^\infty \mathrm{e}^{-\alpha s} d\nu_{\alpha,\mathbf{v}}(s + \log x)\,.$$

Appendix A

In this section we provide an auxiliary result which is the analog of (1.8) in the case of positive X.

Proposition A.1. *Assume $\mathbb{P}(X < 0) > 0$. If the conditions of Theorem 1.1 hold then for some $c > 0$,*

$$\lim_{x\to\infty} x^\alpha\, \mathbb{P}\big(\max_{n\geqslant 1} \Pi'_n > x\big) = c\,.$$

If the conditions of Theorem 1.2 hold then for some $0 < c < c' < \infty$ *and for* x *large enough,*

$$x^\alpha \, \mathbb{P}\big(\max_{n\geqslant 1} \Pi'_n > x\big) \in [c, c'].$$

Proof. Here and in what follows, we interpret $\Pi'_0 = 1$. We can also work given that $X'_n \neq 0$ as otherwise $\Pi'_t = 0$ for all $t \geqslant n$ and $\max_{1\leqslant t\leqslant n-1} \Pi_t$ has a negligible tail for any fixed n.

The proof follows the arguments of Goldie [8]. One first observes that

$$\mathbb{P}\big(\max_{n\geqslant 0} \Pi'_n > x\big) = \mathbb{E}\big[\mathbb{P}\big(\max_{n\geqslant 0} \Pi'_n > x \mid \mathbf{Z}\big)\big]\,, \tag{A.1}$$

where $\mathbf{Z} = (Z_n)_{n\geqslant 1}$ is a Markov chain on $\{-1, 1\}$ given by $Z_n = \Pi'_n/|\Pi'_n|$. We define

$$N_0 = 0\,, \quad N_i = \inf\{k > N_{i-1} : Z_k = 1\}\,, \qquad i \geqslant 1\ , \quad I = \{N_i\,, i \geqslant 1\}\,.$$

We have

$$\begin{aligned} \mathbb{E}\big[\mathbb{P}\big(\max_{n\geqslant 0} \Pi'_n > x \mid \mathbf{Z}\big)\big] &= \mathbb{E}\big[\mathbb{P}\big(\max_{n\in I} \Pi'_n > x \mid \mathbf{Z}\big)\big] \\ &= \mathbb{E}\big[\mathbb{P}\big(\max_{n\in I} |\Pi'_n| > x \mid \mathbf{Z}\big)\big] \\ &= \mathbb{P}(\tau < \infty)\,, \end{aligned} \tag{A.2}$$

where τ is defined as the stopping time

$$\tau = \inf\Big\{k \in I : \sum_{i=1}^k \log|X_i| > \log x\Big\}\,.$$

We change the measure as follows

$$d\mathbb{P}^\alpha(\log|X_n| \leqslant y, Z_n = \pm 1) = \mathrm{e}^{\alpha\, y}\, d\mathbb{P}(\log|X_n| \leqslant y, Z_n = \pm 1)\,.$$

We notice that the tilted distribution conditionally on $\mathbf{Z}$ is given by

$$d\mathbb{P}^\alpha(\log|X_n| \leqslant y \mid \mathbf{Z}) = \mathbb{1}(Z_n = Z_{n-1})\, \eta_+(dy) + \mathbb{1}(Z_n \neq Z_{n-1})\, \eta_-(dy)\,,$$

where

$$\eta_\pm(dy) = \frac{\mathbb{P}^\alpha(\pm X > 0, \log|X| \in dy)}{\mathbb{P}^\alpha(\pm X > 0)}\,.$$

Then we have the identity

$$\mathbb{P}(\tau < \infty) = \mathbb{E}^\alpha\big[\mathrm{e}^{-\alpha \sum_{i=1}^\tau \log|X_i|}\, \mathbb{1}(\tau < \infty)\big]. \tag{A.3}$$

The proof of (A.3) relies on the fact that $(\mathrm{e}^{\,\alpha \sum_{i=1}^n \log|X_i|})$ is a martingale under $\mathbb{P}$. First notice that it is a product of the independent random variables $|X_i|^\alpha$ given $\mathbf{Z}$. Moreover, noticing that

$$\mathbb{P}(Z_n = Z_{n-1}) = \mathbb{P}^\alpha(X > 0) = \mathbb{E}[|X|^\alpha\, \mathbb{1}(X > 0)]\,,$$

we have

$$\begin{aligned}&\mathbb{E}[\mathrm{e}^{\alpha\log|X_n|}\mid\mathbf{Z}]=\\&\frac{\mathbb{E}[|X|^\alpha\mathbb{1}(X>0)]}{\mathbb{P}^\alpha(X>0)}\mathbb{1}(Z_n=Z_{n-1})+\frac{\mathbb{E}[|X|^\alpha\mathbb{1}(X<0)]}{\mathbb{P}^\alpha(X<0)}\mathbb{1}(Z_n\neq Z_{n-1})=1.\end{aligned}$$

Then $(\mathrm{e}^{\alpha\sum_{i=1}^n\log|X_i|})$ constitutes a martingale. Write $\mathcal{F}_\tau$ for the σ-field generated by τ. Then we have for $n\geqslant 1$,

$$\begin{aligned}&\mathbb{E}^\alpha\big[\mathrm{e}^{-\alpha\sum_{i=1}^\tau\log|X_i|}\mathbb{1}(\tau\leqslant n)\big]\\&=\ \mathbb{E}\big[\mathrm{e}^{\alpha\sum_{i=1}^n\log|X_i|}\mathrm{e}^{-\alpha\sum_{i=1}^\tau\log|X_i|}\mathbb{1}(\tau\leqslant n)]\big]\\&=\ \mathbb{E}\big[\mathbb{E}\big[\mathrm{e}^{\alpha\sum_{i=1}^n\log|X_i|}\mathrm{e}^{-\alpha\sum_{i=1}^\tau\log|X_i|}\mathbb{1}(\tau\leqslant n)\mid\mathbf{Z}\big]\big]\\&=\ \mathbb{E}\big[\mathbb{E}\big[\mathbb{E}[\mathrm{e}^{\alpha\sum_{i=1}^n\log|X_i|}\mid\mathcal{F}_\tau,\mathbf{Z}]\,\mathrm{e}^{-\alpha\sum_{i=1}^\tau\log|X_i|}\mathbb{1}(\tau\leqslant n)\mid\mathbf{Z}\big]\big]\\&=\ \mathbb{E}\big[\mathbb{E}[\mathrm{e}^{\alpha\sum_{i=1}^\tau\log|X_i|}\mathrm{e}^{-\alpha\sum_{i=1}^\tau\log|X_i|}\mathbb{1}(\tau\leqslant n)\mid\mathbf{Z}]\big]\\&=\ \mathbb{P}(\tau\leqslant n).\end{aligned}$$

Now (A.3) follows by letting $n\to\infty$.

In view of (A.1)–(A.3) it suffices to study the asymptotic behavior of $x^\alpha\,\mathbb{P}(\tau<\infty)$. From (A.3) we conclude that

$$x^\alpha\,\mathbb{P}(\tau<\infty)\ =\ \mathbb{E}^\alpha[\mathbb{E}^\alpha[\mathrm{e}^{-\alpha(\sum_{i=1}^\tau\log|X_i|-\log x)}\mathbb{1}(\tau<\infty)\mid\mathbf{Z}]].$$

We have the representation

$$\sum_{i=1}^\tau\log|X_i|=\sum_{i=1}^{\tilde\tau}\sum_{j=N_{i-1}+1}^{N_i}\log|X_i|=:\sum_{i=1}^{\tilde\tau}W_i$$

where (W_i) is iid and has the common tilted distribution η given explicitly in (9.11) of Goldie [8]. Here

$$\tilde\tau=\inf\{t\geqslant 1:\sum_{i=1}^t W_i\geqslant\log x\}\,.$$

It is shown in equations (9.12) and (9.13) of [8] that $\mathbb{E}^\alpha[W_1]=2\,m(\alpha)>0$.

Assume now that the conditions of Theorem 1.1 are satisfied and that the distribution of $\log|X|$ given $X\neq 0$ is non-arithmetic. Then η inherits non-arithmeticity from $\log|X|$ and the overshoot $B(\infty)$ of the random walk associated with η is well defined (positive drift) and

$$\begin{aligned}x^\alpha\,\mathbb{P}(\tau<\infty)&=\mathbb{E}^\alpha\big[\mathbb{E}^\alpha[\mathrm{e}^{-\alpha(\sum_{i=1}^{\tilde\tau}W_i-\log x)}\mathbb{1}(\tilde\tau<\infty)\mid\mathbf{Z}]\big]\\&=\mathbb{E}^\alpha[\mathrm{e}^{-\alpha(\sum_{i=1}^{\tilde\tau}W_i-\log x)}]\\&\to\mathbb{E}^\alpha[\mathrm{e}^{-\alpha B(\infty)}]=\frac{1-h(0,\infty)}{2\,\alpha\,m(\alpha)}>0\,.\end{aligned}$$

Here H is the ladder height (defective) distribution of the random walk associated with the distribution of W_1 under $\mathbb{P}$ (negative drift) and the positivity of the constant on the right-hand side follows by calculations similar to Theorem 5.3, Section XIII, in Asmussen [1]. This finishes the proof of the first assertion.

Considering that the assumptions of Theorem 1.2 are satisfied then the distribution of the overshoot $B(\infty)$ of the random walk associated with η is well defined only on some lattice span only. Then $x^\alpha\,\mathbb{P}(\tau<\infty)$ converges when $x\to\infty$ on this lattice span only (with a different limiting positive constant), see Remark 5.3, Section XIII, in Asmussen [1]. The second assertion follows by using the monotonicity of the tail probability. □

Remark A.2. The distribution of $W_1=\log\tilde X_1$ under $\mathbb{P}$ given in (3.4) heavily depends on the quantities p and q. Therefore we are not convinced that 1. H ever coincides with the ladder height distribution of the random walk with step sizes $(\log|X_n|)$ and 2. the relation

$$2\,x^\alpha\,\mathbb{P}\big(\max_{k\geqslant 0}\Pi'_k>x\big)\sim x^\alpha\,\mathbb{P}\big(\max_{k\geqslant 0}|\Pi'_k|>x\big),\qquad x\to\infty,$$

holds. It is clearly not the case when $p=0$ ($N_1=2$) as shown by an inspection of Spitzer's formula; see [7], p. 416.

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