Uniform infinite planar quadrangulations with a boundary

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Abstract

We introduce and study the uniform infinite planar quadrangulation (UIPQ) with a boundary via an extension of the construction of [14]. We then relate this object to its simple boundary analog using a pruning procedure. This enables us to study the aperture of these maps, that is, the maximal graph distance between two points on the boundary, which in turn sheds new light on the geometry of the UIPQ. In particular we prove that the self-avoiding walk on the UIPQ is at most diffusive.

Introduction

Motivated by the theory of 2D quantum gravity, the probabilistic theory of random planar maps has been considerably growing over the last few years. In this paper we continue the study of the geometry of random maps and focus in particular on random quadrangulations with a *boundary*.

Recall that a planar map is a proper embedding of a finite connected planar graph into the two-dimensional sphere seen up to orientation-preserving homeomorphisms. The faces are the connected components of the complement of the union of the edges, and the degree of a face is the number of edges that are incident to it, where it should be understood that an edge is counted twice if it lies entirely in the face. A map is a quadrangulation if all its faces have degree 4. All the maps considered in this work are rooted, meaning that an oriented edge is distinguished and called the *root edge*. The face lying to the right of the root edge is called the *root face*.

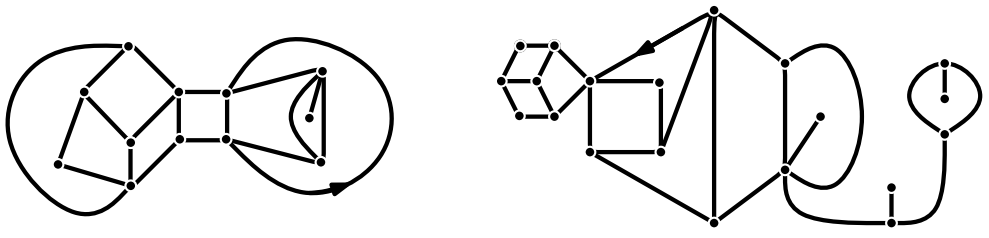


Figure 1: A quadrangulation with a simple boundary and a quadrangulation with a general boundary.

A planar map q is a *quadrangulation with a boundary* if all its faces have degree four except possibly the root face, which can have an arbitrary even degree. This face is called the *external face* although it is sometimes represented "inside" the map. The degree of

the external face is called the *perimeter* of the map, and the boundary is said to be simple if during the contour of the external face all the vertices on the boundary are visited only once (i.e. there is no cut vertex on the boundary). By contrast, we speak of quadrangulation with a general boundary when we do not require the boundary to be simple. The size of q is its number of faces minus one.

Uniform quadrangulations of size n with a boundary of perimeter p have recently been studied from a combinatorial and a probabilistic point of view [8, 11]. Three different regimes have been exhibited for the large scale structure: if $p \ll n^{1/2}$, then these maps converge, in the scaling limit, towards the Brownian map introduced in [21, 25]. If $p \asymp n^{1/2}$, then the boundary becomes macroscopic and Bettinelli [8] introduced the natural candidate for the scaling limit of these objects, which is a sort of Brownian map with a hole. When $p \gg n^{1/2}$ these random quadrangulations fold on themselves and become tree-like [8, 11]. In this work, we shall take a different approach and study infinite local limits of quadrangulations with a boundary as the size tends to infinity. Let us specify the setting.

In a pioneering work [7], Benjamini & Schramm initiated the study of local limits of maps. If m, m' are two rooted maps, the local distance between m and m' is

$$\mathrm{d}_{\mathrm{map}}(m, m') \quad = \quad \big(1 + \sup\{r \geqslant 0 : B_r(m) = B_r(m')\}\big)^{-1},$$

where $B_r(m)$ denotes the map formed by the edges of m whose incident vertices are all at graph distance smaller than or equal to r from the origin of the root edge. Let Q_n be uniformly distributed over the set of all rooted quadrangulations with n faces. Krikun [19] proved that

$$Q_n \quad \xrightarrow[n\to\infty]{(d)} \quad Q_\infty, \tag{1}$$

in distribution in the sense of $\mathrm{d}_{\mathrm{map}}$. The object Q_∞ is a random infinite rooted quadrangulation called the Uniform Infinite Planar Quadrangulation (UIPQ) (see also [2, 4] for previous works concerning triangulations). The UIPQ (and its related triangulation analog) has been the subject of active research in recent years, see [13, 14, 15, 19, 20, 22]. Despite these progresses, the geometry of the UIPQ remains quite mysterious. The purpose of this article is to provide some new geometric understanding of the UIPQ via the study of the UIPQ *with a boundary*.

We will show that the convergence (1) can be extended to quadrangulations with a boundary. More precisely, for any $p \geqslant 1$, we let $Q_{n,p}$ (resp. $\widetilde{Q}_{n,p}$) be a uniform quadrangulation with a general (resp. simple) boundary, of size n and with perimeter $2p$, then we have

$$\widetilde{Q}_{n,p} \quad \xrightarrow[n\to\infty]{(d)} \quad \widetilde{Q}_{\infty,p},$$
$$Q_{n,p} \quad \xrightarrow[n\to\infty]{(d)} \quad Q_{\infty,p},$$

in distribution for the metric $\mathrm{d}_{\mathrm{map}}$. The random map $Q_{\infty,p}$ (resp. $\widetilde{Q}_{\infty,p}$) is called the Uniform Infinite Planar Quadrangulation with a general (resp. simple) boundary of perimeter $2p$. The first convergence is an easy consequence of (1) (see the discussion around (7) below) whereas the second convergence requires an adaptation of the techniques of [14]: in

Theorem 2, we construct $Q_{\infty,p}$ from a labeled "treed bridge" and extend the main result of [14] to our setting. This construction is yet another example of the power of the bijective technique triggered by Schaeffer [27], which has been one of the key tool for studying random planar maps [10, 12, 13, 24].

We then turn to the study of the UIPQ's with a boundary and their relationships. Although well-suited for the definition and the study of $Q_{\infty,p}$, the techniques "à la Schaeffer" seem much harder to develop in the case of simple boundary because of the topological constraint imposed on the external face. In order to bypass this difficulty we use a pruning decomposition to go from general to simple boundaries. More precisely, we prove that $Q_{\infty,p}$ has a unique infinite irreducible component, that is, a core made of an infinite quadrangulation with a simple boundary together with finite quadrangulations hanging off from this core, see Fig. 2.

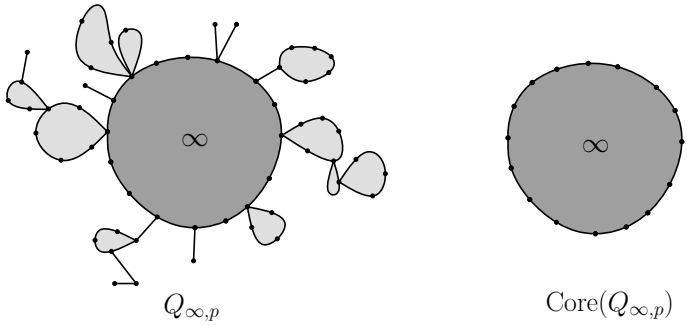


Figure 2: Pruning $Q_{\infty,p}$.

We show that if we remove these finite components then the core of $Q_{\infty,p}$ has a (random) perimeter $|\partial\mathrm{Core}(Q_{\infty,p})|$, which is roughly a third of the original one. More precisely we prove in Proposition 4 the following convergence in distribution

$$\frac{|\partial\mathrm{Core}(Q_{\infty,p})| - 2p/3}{p^{2/3}} \xrightarrow[p\to\infty]{(d)} \mathcal{Z},$$

where $\mathcal{Z}$ is a spectrally negative stable random variable of parameter $3/2$. Furthermore, conditionally on its perimeter, the core of $Q_{\infty,p}$ is distributed as a UIPQ with a simple boundary (Theorem 4). This confirms and sharpens a phenomenon already observed in a slightly different context by Bouttier & Guitter, see [11, Section 5]

As an application of our techniques we study the *aperture* of these maps: if q is a quadrangulation with a boundary, we denote the maximal graph distance between two vertices on the boundary of q by $\mathrm{aper}(q)$ and call it the aperture of q. We prove that the aperture of the UIPQ with a simple boundary of perimeter p cannot exceed the order $\sqrt{p}$, in the sense of the following statement.

Theorem 1. *There exists $c, c' > 0$ such that for all $p \geqslant 1$ and $\lambda > 0$ the aperture of the uniform infinite planar quadrangulation with a simple boundary of perimeter $2p$ satisfies*

$$P\left(\mathrm{aper}(\widetilde{Q}_{\infty,p}) \geqslant \lambda\sqrt{p}\right) \leqslant c\,p^{2/3}\exp\left(-c'\lambda^{2/3}\right).$$

This result is first established for the UIPQ with a general boundary using the construction from a treed bridge (Theorem 3) and then transferred to the simple boundary

case using the pruning procedure. This theorem provides a new tool for studying the UIPQ itself via the technique of *peeling*, see [2, 6]. In particular, Theorem 1 is one of the key estimates of [6] used to prove that the simple random walk on the UIPQ is subdiffusive with exponent less than $1/3$, that is, the graph distance after n steps of the walk is less than $n^{1/3+o(1)}$.

Let us finish this introduction with one more motivation. There is an obvious bijective correspondence between, on the one hand, quadrangulations of size n with a self-avoiding path of length p starting at the root edge and, on the other hand, quadrangulations with a simple boundary of perimeter $2p$ and size n: simply consider the self-avoiding walk as a zipper. See Fig. 3. Hence, the UIPQ with a simple boundary of perimeter $2p$ can be seen as an annealed model of UIPQ endowed with a self-avoiding path of length p.

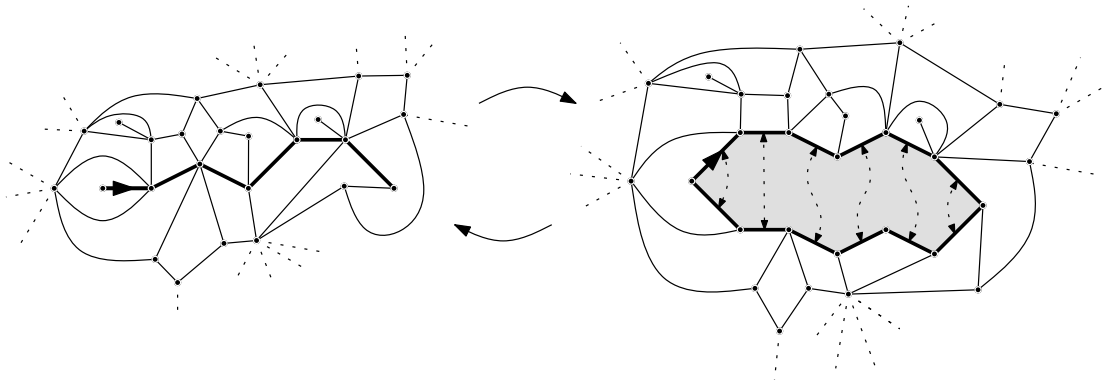


Figure 3: Zip the external face and unzip the self-avoiding walk.

With this correspondence, the aperture of the map obviously bounds from above the maximal graph distance of any point of the self-avoiding walk to the origin of the map. The estimates of Theorem 1 then show that, when p is large, the maximal graph distance displacement of the self-avoiding walk with respect to the root of the map is at most of order $\sqrt{p}$. This contrasts with the Euclidean case where a displacement of order $p^{3/4}$ is conjectured.

Let us remark, however, that the aperture of a quadrangulation with a boundary only gives an upper bound on the maximal displacement of the self-avoiding walk obtained after zipping. We plan to study the lower bound in a future work.

The paper is organized as follows: the first section contains some background on quadrangulations with a boundary. In the second section, we present the bijective techniques adapted from [11] that we apply in Section 3 to define the UIPQ with a general boundary and study its aperture. The fourth section is devoted to the pruning decomposition and its applications and contains the proof of Theorem 1. Finally, the last section is devoted to extensions and comments. In particular, we define the UIPQ of the half-plane (with an infinite boundary) with a general and a simple boundary and propose some open questions.

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1 Quadrangulations with a boundary

1.1 Definitions

Recall that all the maps we consider here are *rooted*, that is, given with one distinguished oriented edge $\vec{e}$.

A planar map q is a *quadrangulation with a boundary* if all its faces have degree four, with the possible exception of the root face (also called external face). Since the number of odd-degree faces in a map is necessarily even, the degree of the external face has to be even. The boundary of the external face is denoted by ∂q and its degree by $|\partial q|$. We say that q has a *perimeter* $|\partial q|$ and its *size* $|q|$ is the number of faces minus 1.

A quadrangulation has a *simple* boundary if there is no cut vertex on the boundary, that is, if ∂q is a cycle with no self-intersection. By convention, all the notation involving a quadrangulation with a simple boundary will be decorated with a "$\sim$" to avoid confusion.

We denote by $\mathcal{Q}_{n,p}$ (resp. $\widetilde{\mathcal{Q}}_{n,p}$) the set of all rooted quadrangulations with a general (resp. simple) boundary with $n+1$ faces and such that the external face has degree $2p$, and by $q_{n,p}$ (resp. $\widetilde{q}_{n,p}$) its cardinality. By convention, the set $\mathcal{Q}_{0,0} = \widetilde{\mathcal{Q}}_{0,0}$ contains a unique "vertex" map denoted by †. Note also that $\mathcal{Q}_{0,1} = \widetilde{\mathcal{Q}}_{0,1}$ is composed of the map with one oriented edge (which has a simple boundary). Note that any quadrangulation with a boundary of perimeter 2 can be seen as a rooted quadrangulation without a boundary, by contracting the external face of degree two.

1.2 Enumeration

We gather here a few enumeration results that will be useful. We refer to [11] for the derivations of these formulas. We start with some simple combinatorial decompositions.

Let q be a quadrangulation with a boundary. If the boundary of q is not simple (if it has some cut vertices) we can decompose q unambiguously into quadrangulations with a simple boundary attached by the cut vertices of the boundary of q: these quadrangulations are called *the irreducible components* of q. The root edge is carried by a unique irreducible component, and all the other irreducible components have a unique boundary vertex that is closest to the component of the root. By convention, we root each component at the oriented edge that immediately precedes this particular vertex in counterclockwise order. See Fig. 4.

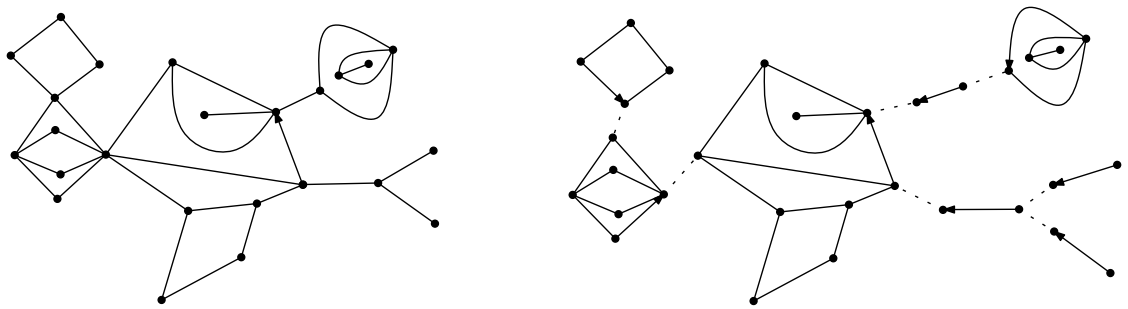


Figure 4: Decomposition of a quadrangulation with a boundary into irreducible components.

If q is a quadrangulation with a general boundary we can also decompose q according

to the irreducible component that contains the root edge and other quadrangulations a with general boundary attached to it. This decomposition yields an identity relating the bi-variate generating functions of quadrangulations with a simple and a general boundary: for $g, z \geqslant 0$, let W (resp. $\widetilde{W}$) be the bi-variate generating function of $\mathcal{Q}_{n,p}$ (resp. $\widetilde{\mathcal{Q}}_{n,p}$) with weight g per internal face and $\sqrt{z}$ per oriented edge on the boundary, that is

$$W(g,z) := \sum_{n,p\geqslant 0} q_{n,p} g^n z^p, \quad \text{and} \quad \widetilde{W}(g,z) := \sum_{n,p\geqslant 0} \widetilde{q}_{n,p} g^n z^p.$$

The last decomposition translates into the identity

$$\widetilde{W}\big(g, zW^2(g,z)\big) \;=\; W(g,z). \tag{2}$$

The exact expression of W can be found in [11] and reads

$$W(g,z) = \omega(1 - gR^2(\omega-1)), \quad \text{where} \quad \omega = \frac{1-\sqrt{1-4zR}}{2zR} \quad \text{and} \quad R = \frac{1-\sqrt{1-12g}}{6g}. \tag{3}$$

From this we can deduce

$$q_{n,p} \;=\; \frac{(2p)!}{p!(p-1)!} 3^n \frac{(2n+p-1)!}{n!(n+p+1)!},$$

for $n \geqslant 0$ and $p \geqslant 1$. Note the asymptotics

$$\begin{aligned} q_{n,p} &\underset{n\to\infty}{\sim} C_p 12^n n^{-5/2}, \\ C_p &= \frac{(2p)!}{p!(p-1)!} 2^{p-1}\pi^{-1/2} \underset{p\to\infty}{\sim} (2\pi)^{-1} 8^p \sqrt{p}. \end{aligned} \tag{4}$$

Moreover (2) enables us to find the expressions for $\widetilde{q}_{n,p}$ (see [11] for more details):

$$\begin{aligned} \widetilde{q}_{n,p} &= 3^{-p}\frac{(3p)!}{p!(2p-1)!} 3^n \frac{(2n+p-1)!}{(n-p+1)!(n+2p)!}, \quad \text{for } n \geqslant 1 \text{ and } p \geqslant 1, \\ \widetilde{q}_{n,p} &\underset{n\to\infty}{\sim} \widetilde{C}_p 12^n n^{-5/2}, \\ \widetilde{C}_p &= \frac{(3p)!}{p!(2p-1)!} 3^{-p} 2^{p-1}\pi^{-1/2} \underset{p\to\infty}{\sim} \frac{\sqrt{3p}}{2\pi}\left(\frac{9}{2}\right)^p. \end{aligned} \tag{5}$$

To simplify notation we introduce $W_c(z) = W(12^{-1}, z)$ (resp. $\widetilde{W}_c(z) = \widetilde{W}(12^{-1}, z)$) the generating function of quadrangulations with a general (resp. simple) boundary taken at the critical point $g = \frac{1}{12}$ for the size. From (3) we get that

$$W_c(z) \;=\; \frac{(1-8z)^{3/2} - 1 + 12z}{24z^2}. \tag{6}$$

Remark 1. The $n^{-5/2}$ and $\sqrt{p}$ polynomial corrections in the asymptotics of $q_{n,p}$ and $\widetilde{q}_{n,p}$ are common features in planar structures with a boundary, in particular it holds for other "reasonable" classes of maps with a boundary such as triangulations, see [4].

For all $n, p \geqslant 0$, we denote by $Q_{n,p}$ and $\widetilde{Q}_{n,p}$ random variables with uniform distributions over $\mathcal{Q}_{n,p}$ and $\widetilde{\mathcal{Q}}_{n,p}$ respectively. In the next section we recall the definition of the UIPQ and construct the UIPQ with a simple boundary from it.

1.3 The UIPQ with a simple boundary

Recall the metric $\mathrm{d}_{\mathrm{map}}$ presented in the Introduction. The set of all finite rooted planar quadrangulations with a boundary is not complete for this metric and we will have to work in its completion $\mathcal{Q}$. The additional elements of this set are called infinite quadrangulations with a boundary. Formally they can be seen as sequences $(q_1, \ldots, q_n, \ldots)$ of finite rooted quadrangulations with a boundary such that for any $r \geqslant 0$, $B_r(q_n)$ is eventually constant. See [14] for more details. Recall from (1) that the UIPQ is the weak limit in the sense of $\mathrm{d}_{\mathrm{map}}$ of uniform rooted quadrangulations whose size tends to infinity.

We can already use (1) to deduce a similar convergence result for rooted quadrangulations with a simple boundary. Indeed, notice that a rooted quadrangulation with n faces and perimeter $2p$ can be turned into a rooted quadrangulation with $n+p$ faces that has a special neighborhood around the origin. More precisely, if q_{n+p} is a rooted quadrangulation with $n+p$ faces and no boundary, and such that the neighborhood of the root edge is composed of p squares arranged like a star around the origin of the root edge as depicted in Fig. 5 (note that the vertices on the boundary of the star must be pairwise distinct), then we can remove this star from q_{n+p} and move the root edge in a deterministic way to obtain a quadrangulation with a boundary of perimeter $2p$ with n internal faces. This operation is reversible.

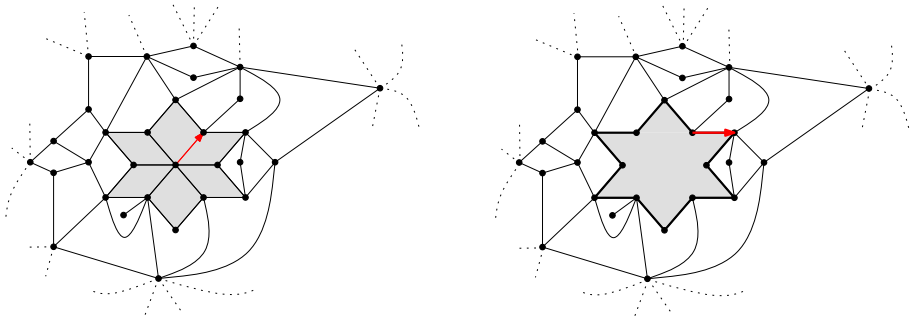


Figure 5: A fragment of a rooted quadrangulation with a special neighborhood of the root and the rooted quadrangulation with a simple boundary obtained by removing this neighborhood and transferring the root edge.

Hence the uniform distribution over $\widetilde{\mathcal{Q}}_{n,p}$ can be seen as the uniform distribution over $\mathcal{Q}_{n+p,1}$ conditioned on having a "starred neighborhood" of the root composed of p squares. For $p \geqslant 1$, we condition a uniform infinite planar quadrangulation Q_∞ to have a "starred neighborhood" with p squares (which is an event of positive probability) and denote by $\widetilde{Q}_{\infty,p}$ the complement of this neighborhood, which is an infinite quadrangulation with a simple boundary of perimeter $2p$ rooted as explained before. The convergence (1) together with the preceding remarks then yields

$$\widetilde{Q}_{n,p} \xrightarrow[n\to\infty]{} \widetilde{Q}_{\infty,p}, \tag{7}$$

in distribution in the sense of $\mathrm{d}_{\mathrm{map}}$. The random variable $\widetilde{Q}_{\infty,p}$ is called the uniform infinite planar quadrangulation (UIPQ) with a simple boundary of perimeter $2p$.

Remark 2. It is not easy to deduce from (1) a similar convergence result for quadrangulations with a general boundary: this is due to the fact that quadrangulations with a

general boundary are not *rigid* in the sense of [4, Definition 4.7] and thus not well suited for this type of surgery. We shall take a different approach to define the UIPQ with a general boundary in the next sections.

2 Bijective representation

In this section we extend the bijective approach of the UIPQ developed in [14] to the case of quadrangulations with a boundary using the tools of [11]. For technical reasons, we will have to consider pointed quadrangulations: a (rooted) quadrangulation with a boundary is *pointed* if it is given with a distinguished vertex, which we denote by ρ. We let $\mathcal{Q}^\bullet_{n,p}$ (resp. $\widetilde{\mathcal{Q}}^\bullet_{n,p}$) be the set of all rooted pointed quadrangulations with a general (resp. simple) boundary and $q^\bullet_{n,p}$ (resp. $\widetilde{q}^\bullet_{n,p}$) its cardinality.

2.1 Trees

We use the same notation as in [14]. Let $\mathcal{U} = \cup_{n=0}^{\infty} \mathbb{N}^n$, where $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}^0 = \{\varnothing\}$ by convention. An element u of $\mathcal{U}$ is thus a finite sequence of positive integers. If $u, v \in \mathcal{U}$, uv denotes the concatenation of u and v. If v is of the form uj with $j \in \mathbb{N}$, we say that u is the *parent* of v or that v is a *child* of u. More generally, if v is of the form uw for $u, w \in \mathcal{U}$, we say that u is an *ancestor* of v or that v is a *descendant* of u. A *plane tree* τ is a (finite or infinite) subset of $\mathcal{U}$ such that

1. $\varnothing \in \tau$ ($\varnothing$ is called the *root* of τ),
2. if $v \in \tau$ and $v \neq \varnothing$, the parent of v belongs to τ,
3. for every $u \in \mathcal{U}$ there exists $k_u(\tau) \geqslant 0$ such that $uj \in \tau$ if and only if $j \leqslant k_u(\tau)$.

A plane tree can be seen as a graph, in which an edge links two vertices u, v such that u is the parent of v or vice-versa. This graph is of course a tree in the graph-theoretic sense, and has a natural embedding in the plane, in which the edges from a vertex u to its children $u1, \ldots, uk_u(\tau)$ are drawn from left to right. We let $|u|$ be the length of the word u. The integer $|\tau|$ denotes the number of edges of τ and is called the size of τ. A *corner* of a tree is an angular sector formed by two consecutive edges in the clockwise contour. A *spine* in a tree τ is an infinite sequence $u_0, u_1, u_2, \ldots$ in τ such that $u_0 = \varnothing$ and u_i is the parent of u_{i+1} for every $i \geqslant 0$. In this work, unless explicitly mentioned, all the trees considered are plane trees.

The uniform infinite plane tree. For any plane tree τ and any $h \geqslant 0$ we define the tree $[\tau]_h = \{u \in \tau : |u| \leqslant h\}$ as the tree τ restricted to the first $h+1$ generations. If τ and τ' are two plane trees, we set

$$\mathrm{d}_{\mathrm{tree}}(\tau, \tau') \quad = \quad \big(1 + \sup\{h \geqslant 0 : [\tau]_h = [\tau']_h\}\big)^{-1}.$$

Obviously, $\mathrm{d}_{\mathrm{tree}}$ is a distance on the set of all plane trees. In the following, for every $n \geqslant 0$, we denote by T_n a random variable uniformly distributed over the set of all rooted plane trees with n edges. It is standard (see [17, 23]) that there exists a random infinite plane

tree T_∞ with one spine called the uniform infinite plane tree, or critical geometric Galton–Watson tree conditioned to survive, such that we have the convergence in distribution for $\mathrm{d}_{\mathrm{tree}}$

$$T_n \xrightarrow[n\to\infty]{(d)} T_\infty. \tag{8}$$

The tree T_∞ can be informally described as follows. Start with a semi-infinite line of vertices (which will be the unique spine of the tree), then on the left and right hand sides of each vertex of the spine, graft independent critical geometric Galton–Watson trees with parameter $1/2$. The resulting plane tree has the same distribution as T_∞. See [5, 14] for more details.

Labeled trees. A *rooted labeled tree* (or spatial tree) is a pair $\theta = (\tau, (\ell(u))_{u\in\tau})$ that consists of a plane tree τ and a collection of integer labels assigned to the vertices of τ, such that if $u, v \in \tau$ are neighbors then $|\ell(u) - \ell(v)| \leqslant 1$. Unless otherwise mentioned, the label of the root vertex is 0. If $\theta = (\tau, \ell)$ is a labeled tree, $|\theta| = |\tau|$ is the size of θ. Obviously, the distance $\mathrm{d}_{\mathrm{tree}}$ can be extended to labeled trees by taking into account the labels, and we keep the notation $\mathrm{d}_{\mathrm{tree}}$ for this distance.

Let τ be a random plane tree and, conditionally on τ, consider a sequence of independent random variables uniformly distributed over $\{-1, 0, 1\}$ carried by each edge of τ. For any vertex u of τ, we define the label of u as being the sum of the variables carried by the edges along the unique path from the root $\varnothing$ to u. This labeling is called the *uniform* labeling of τ. When the tree τ is uniform infinite plane tree, we will speak of the uniform infinite labeled plane tree. Using the notation of [14], we denote by $\mathscr{S}$ the set of all labeled infinite trees with only one spine such that the infimum of the labels along the spine is $-\infty$. Note that if τ is an infinite tree with one spine and $\theta = (\tau, \ell)$ is a uniform labeling of τ then $\theta \in \mathscr{S}$ almost surely.

2.2 Treed bridges

A *bridge* of length $2p$ is a sequence of integers $x_0, x_1, \ldots, x_{2p-1}$ such that $x_0 = 0$ and for every $i \in \{0, \ldots, 2p-1\}$ we have $|x_i - x_{i+1}| = 1$, where by convention we let $x_{2p} = x_0 = 0$. Note that in any bridge of length $2p$, there are exactly p *down-steps*, which are the indices $i \in \{0, 1, \ldots, 2p-1\}$ such that $x_{i+1} = x_i - 1$. A *labeled treed bridge* of size n and length $2p$ is a bridge $\mathrm{b}_p = (x_0, \ldots, x_{2p-1})$ together with p labeled plane trees $\theta_1, \ldots, \theta_p$ (with root label 0) such that the sum of the sizes of the trees $\theta_1, \ldots, \theta_p$ is n. We denote by $\mathscr{B}_p$ the union set of all labeled finite treed bridges of length $2p$ and of all infinite labeled treed bridges $(\mathrm{b}_p; \theta_1, \ldots, \theta_p)$ such that *one and only one* of the labeled trees θ_i belongs to $\mathscr{S}$, all others being finite. In the following, unless explicitly mentioned, all labeled treed bridges considered belong to $\mathscr{B}_p$ for some $p \geqslant 1$.

Representation. Let $\mathbf{b} = (\mathrm{b}_p; \theta_1, \ldots, \theta_p)$ be a treed bridge of $\mathscr{B}_p$. If $\mathrm{b}_p = (x_0, \ldots, x_{2p-1})$ we denote by $(x_{i_1}, x_{i_1+1}), \ldots, (x_{i_p}, x_{i_p+1})$ its p down-steps. We construct *a representation* of $\mathbf{b}$ in the plane as follows. Let $\mathcal{C}$ be a proper embedding in the plane of a cycle of length $2p$. We label the vertices of $\mathcal{C}$ starting from a distinguished vertex in the clockwise order by the values of the bridge b_p. Now we graft (proper embeddings of) the trees $\theta_1, \ldots, \theta_p$ in the infinite component of $\mathbb{R}^2 \backslash \mathcal{C}$ in such a way that the tree θ_k is grafted on the i_kth point of $\mathcal{C}$ and we shift all the labels of this tree by x_{i_k}. This representation can be constructed

in such a way that the embedding is proper (no edges are crossing except possibly at their endpoints) and such that the set of vertices of the embedding has no accumulation point in $\mathbb{R}^2$. See Fig. 6 below.

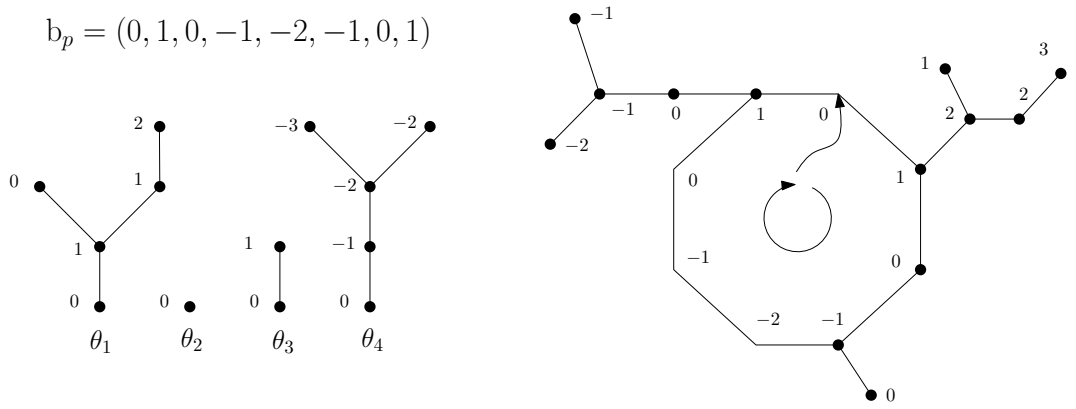


Figure 6: A labeled finite treed bridge and a representation of it. The black dots represent the vertices of the trees of the treed bridge.

The vertex set of this representation is thus formed by the union of the vertices of the trees $\theta_1, \dots, \theta_p$ and of the vertices of the cycle that are not down-steps. The labeling of these vertices, which is given by the bridge on the cycle and the shifted labelings of the trees is denoted by $\ell_{\mathbf{b}}$. We will often abuse notation and write $u \in \theta_i$ for a vertex in the representation that belongs to the embedding of the tree θ_i.
Recall that a corner of a proper embedding of a graph in the plane is an angular sector formed by two consecutive edges in clockwise order. In the case of a representation of a labeled treed bridge we can consider the set of corners of the infinite component of the plane minus the embedding. This set, although possibly infinite, inherits a cyclic order from the clockwise order of the plane. The label of a corner is that of its attached vertex.

The uniform infinite labeled treed bridge. Let $p \geqslant 1$. We say that a sequence of labeled treed bridges $(\mathrm{b}_p^{(n)}; \theta_1^{(n)}, \dots, \theta_p^{(n)})$ of length $2p$ converges to $(\mathrm{b}_p; \theta_1, \dots, \theta_p)$, if eventually $\mathrm{b}_p^{(n)} = \mathrm{b}_p$ and $\theta_i^{(n)}$ converges towards θ_i for any $i \in \{1, \dots, p\}$ with respect to $\mathrm{d}_{\mathrm{tree}}$. This convergence is obviously metrizable by the metric

$$\mathrm{d}_{\mathrm{bridge}}\big((\mathrm{b}_p; \theta_1, \dots, \theta_p), (\mathrm{b}'_p; \theta'_1, \dots, \theta'_p)\big) \quad = \quad \mathbf{1}_{\mathrm{b}_p = \mathrm{b}'_p} \sup_{1 \leqslant i \leqslant p} \mathrm{d}_{\mathrm{tree}}(\theta_i, \theta'_i) + \mathbf{1}_{\mathrm{b}_p \neq \mathrm{b}_{p'}}.$$

In the rest of this work, $\mathbf{B}_{n,p} = (B_p; \Theta_1^{(n)}, \dots, \Theta_p^{(n)})$ is a uniform labeled treed bridge of size n and length $2p$. Note that B_p is uniformly distributed over the set of bridges of length $2p$. We introduce the analog of the uniform infinite labeled plane tree in the setting of treed bridges. The *uniform infinite labeled treed bridge* denoted by $\mathbf{B}_{\infty,p} = (B_p, \Theta_1, \dots, \Theta_p)$ is constructed as follows. Firstly, B_p is a uniform bridge of length $2p$. Then choose $i_0 \in \{1, \dots, p\}$ uniformly and independently of B_p. Conditionally on B_p and i_0, the trees $\Theta_1, \dots, \Theta_p$ are independent, Θ_{i_0} being a uniform infinite labeled plane tree and all other Θ_j for $j \neq i$ are uniformly labeled critical geometric Galton–Watson trees. Notice that $\mathbf{B}_{\infty,p}$ almost surely belongs to $\mathscr{B}_p$. Then the analog of (8) becomes

Proposition 1. *We have the following convergence in distribution for* $\mathrm{d}_{\mathrm{bridge}}$

$$\mathbf{B}_{n,p} \xrightarrow[n\to\infty]{(d)} \mathbf{B}_{\infty,p}. \tag{9}$$

Proof. Since conditionally on the structure of the trees, the bridge and the labels are uniform they do not play any crucial role in the convergence. We just have to prove that if $\tau_1^{(n)}, \dots, \tau_p^{(n)}$ are p plane trees chosen uniformly among all p-uples of plane trees such that $|\tau_1^{(n)}| + \dots + |\tau_p^{(n)}| = n$ then we have the following weak convergence

$$(\tau_1^{(n)}, \dots, \tau_p^{(n)}) \xrightarrow[n\to\infty]{} (\tau_1, \dots, \tau_p),$$

where the distribution of $(\tau_1, \dots, \tau_p)$ is described as follows: choose i_0 uniformly among $\{1, \dots, p\}$, then conditionally on i_0, let the τ_ks be independent, with $\tau_{i_0} = T_\infty$ in distribution and the other trees being critical geometric Galton–Watson trees. This fact is standard and could be derived directly from (8) but we include a pedestrian proof, which serves as a warmup for the manipulations of Section 4. For $p \geqslant 1$ and $n \geqslant 0$, we let $\mathrm{Cat}(n, p)$ be the number of finite sequences ("forests") of p trees with n edges in total, so that by a well-known formula (see e.g. [26])

$$\mathrm{Cat}(n,p) \quad = \quad \frac{p}{2n+p}\binom{2n+p}{n} \quad \underset{n\to\infty}{\sim} \quad 4^n n^{-3/2}\frac{2^{p-1}p}{\sqrt{\pi}}. \tag{10}$$

We also let $\mathrm{Cat}(n) = \mathrm{Cat}(n, 1)$. Let $f_1, \dots, f_p$ be bounded continuous functions for $\mathrm{d_{tree}}$. By definition of the distribution of $(\tau_1^{(n)}, \dots, \tau_p^{(n)})$ we have

$$E\left[\prod_{i=1}^p f_i(\tau_i^{(n)})\right] \quad = \quad \frac{1}{\mathrm{Cat}(n,p)} \sum_{n_1+\dots+n_p=n} \prod_{i=1}^p \mathrm{Cat}(n_i) E[f_i(T_{n_i})], \tag{11}$$

where T_{n_i} denotes a uniform plane tree on n_i edges. We first estimate the probability that two of the trees $\tau_1^{(n)}, \dots, \tau_p^{(n)}$ have a size larger than some large constant $a > 0$. For that purpose we recall a classical lemma whose proof is very similar to [4, Lemma 2.5] and is left to the reader.

Lemma 1. *For any $\beta > 1$, and any $p \geqslant 0$ there exists a constant $c(\beta, p)$ such that for any $a > 0$ and any $n \geqslant 0$ we have*

$$\sum_{\substack{n_1+\dots+n_p=n \\ n_1,n_2>a}} \left(\prod_{i=1}^p (n_i+1)^{-\beta}\right) \quad \leqslant \quad c(\beta,p) n^{-\beta} a^{-\beta+1}.$$

Using the asymptotic (10) with $p = 1$, we deduce that there exists a constant c_1 such that for any $n \geqslant 1$ we have $\mathrm{Cat}(n) \leqslant c_1 4^n (n+1)^{-3/2}$. We can thus use Lemma 1 with $\beta = 3/2$ to deduce that there exists a constant $c_2 > 0$ such that for every $n \geqslant 1$, the probability that two of the trees $\tau_1^{(n)}, \dots, \tau_p^{(n)}$ have size larger than $a > 0$ is less than $c_2 a^{-1/2}$. Hence, for large n's the right-hand side of (11) becomes

$$\varepsilon_{n,a} + \sum_{i=1}^p \left(\sum_{0\leqslant n_1,\dots,\widehat{n_i},\dots,n_p\leqslant a} \frac{\mathrm{Cat}\left(n - \sum_{j\neq i} n_j\right)}{\mathrm{Cat}(n,p)} E[f_i(T_{n-\sum_{j\neq i} n_j})] \prod_{j\neq i} \mathrm{Cat}(n_j) E[f_j(T_{n_j})] \right),$$

where $\varepsilon_{n,a} \leqslant c_3 a^{-1/2}$ uniformly in $n \geqslant 1$, for some finite constant c_3 depending on the functions f_i. Moreover, (8) implies that $E[f(T_n)] \to E[f(T_\infty)]$ for any bounded continuous

functional for $\mathrm{d}_{\mathrm{tree}}$ as $n \to \infty$. So, using once more the asymptotic (10), we can let $n \to \infty$ followed by $a \to \infty$ in the last display and obtain

$$E\left[\prod_{i=1}^{p} f_i(\tau_i^{(n)})\right] \quad \xrightarrow[n\to\infty]{} \quad \sum_{i=1}^{p} \frac{1}{p} E[f_i(T_\infty)] \prod_{j\neq i} \sum_{n_j=0}^{\infty} \frac{\mathrm{Cat}(n_j)}{2\cdot 4^{n_j}} E[f_j(T_{n_j})]. \tag{12}$$

The sum over indices n_j is $E[f_j(T)]$, where T is a critical geometric Galton–Watson tree, which proves the desired result. □

2.3 Quadrangulations with a boundary from treed bridges

2.3.1 Finite bijection

The bijection presented in this section is taken from [11] with minor adaptations. This is a one-to-one correspondence between, on the one hand, the set $\mathcal{Q}^{\bullet}_{n,p}$ of all rooted and pointed quadrangulations with a boundary of perimeter $2p$ and size n and, on the other hand, the set of all labeled treed bridges of length $2p$ and size n. We only present the mapping from labeled treed bridges to quadrangulations, the reverse direction can be found in [11].

Let $\mathbf{b} = (\mathrm{b}_p; \theta_1, \dots, \theta_p)$ be a labeled treed bridge of size n and perimeter $2p$. We consider a representation $\mathcal{E}$ of this treed bridge in the plane. Recall that the labeling of the vertices of this representation is denoted by $\ell_{\mathbf{b}}$. Let $\mathscr{C}$ be the set of corners of the infinite component of $\mathbb{R}^2 \backslash \mathcal{E}$ that are also incident to vertices belonging to the grafted trees, that is, we erase the corners coming from angular sectors around the vertices of the cycle that are not down-steps of the bridge (see Fig. 7). This set inherits a clockwise cyclic order.

We now associate a quadrangulation with $\mathbf{b}$ by the following device: we start by putting an extra vertex denoted by ρ in the infinite component of $\mathbb{R}^2 \setminus \mathcal{E}$. Then for each corner $c \in \mathscr{C}$, we draw an edge between c and the first corner $c' \in \mathscr{C}$ for the clockwise order such that the label $\ell_{\mathbf{b}}(c')$ of c' equals $\ell_{\mathbf{b}}(c) - 1$: this corner is called *the successor* of c. If there is no such corner (this happens only if the corner c has minimal label) then we draw an edge between c and ρ. This construction can be done in such a way that the edges are non-crossing. After erasing the representation $\mathcal{E}$ of $\mathbf{b}$ we obtain a quadrangulation q of size n with a boundary, whose vertex set is the union of the vertices of (the embeddings of) θ_i for $i \in \{1, \dots, p\}$ plus the extra vertex ρ.

Note that there is a one-to-one order-preserving correspondence between the edges of the cycle of $\mathcal{E}$ and the edges of the external face of q, see Fig. 7 where the external face is inside the drawing. The distinguished oriented edge $\vec{e}$ of q is the edge that corresponds to the first step of the bridge oriented in such a way that the external face is on the right-hand side of $\vec{e}$. We denote the rooted quadrangulation $(q, \vec{e})$ pointed at ρ by $\Phi(\mathbf{b})$. Furthermore, using the identification of the vertices of the map $\Phi(\mathbf{b})$ with $\cup_i \theta_i \cup \{\rho\}$, for every $u \in \cup_i \theta_i$ we have

$$\ell_{\mathbf{b}}(x) - \ell_{\mathbf{b}}(\rho) \quad = \quad \mathrm{d}_{\mathrm{gr}}(x, \rho), \tag{13}$$

where d_{gr} is the graph distance in $\Phi(\mathbf{b})$.

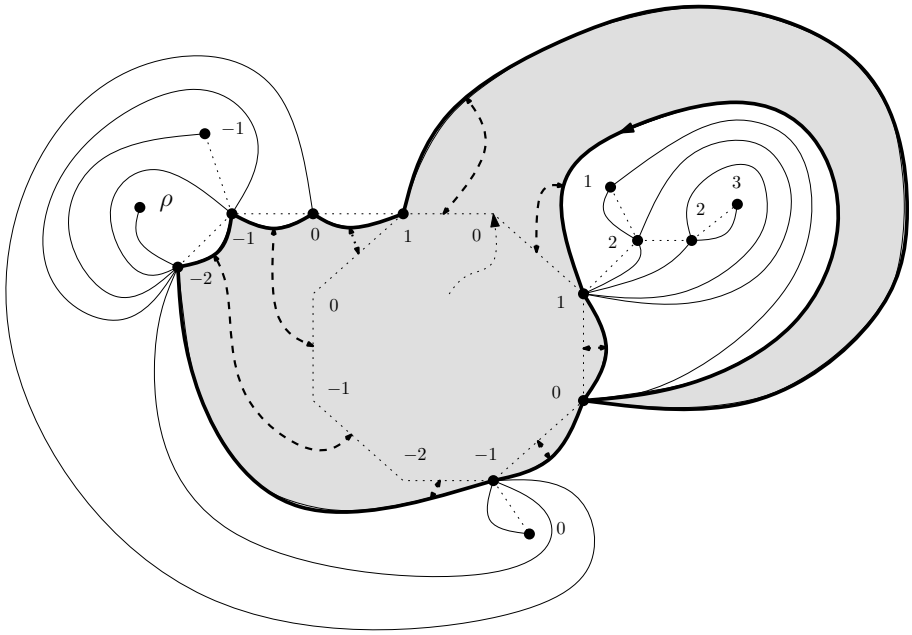


Figure 7: Construction of the rooted and pointed quadrangulation with a boundary associated with a treed bridge. The treed bridge is represented in dotted lines, the quadrangulation in solid lines, its boundary in thick lines and finally the correspondence between the embedded cycle and the boundary of the map is highlighted with dashed lines.

2.3.2 Extended construction

We can extend the preceding mapping Φ to the case when the treed bridge $\mathbf{b}$ is infinite but still belongs to $\mathscr{B}_p$. The extension is very similar to that of [14]. Almost all the steps of the construction of Φ remain the same, the only exception being that every corner attached to a tree in the infinite component of the embedding will find a successor, that is, there is no need to add an extra vertex ρ. The extended mapping, which we still denote Φ, associates with every infinite labeled treed bridge in $\mathscr{B}_p$ an infinite rooted quadrangulation q with a boundary of perimeter $2p$ (this quadrangulation is not pointed anymore) whose vertex set is the union of the vertices of the trees of the bridge. The correspondence between the cycle of the representation and the boundary of the map is still preserved. The shifted labels lose their interpretation (13) (see (15) in Theorem 2). However, since any neighboring vertices in the quadrangulation q have labels that differ in absolute value by exactly 1, we deduce that for any vertices u, v in the resulting quadrangulation we have (with the identification of the vertices of the quadrangulation with those of the trees of $\mathbf{b}$)

$$\mathrm{d}_{\mathrm{gr}}(u,v) \quad \geqslant \quad |\ell_{\mathbf{b}}(u) - \ell_{\mathbf{b}}(v)|. \tag{14}$$

Proposition 2. *The extended Schaeffer mapping $\Phi : \mathscr{B}_p \longrightarrow \mathcal{Q}$ is continuous with respect to the metrics $\mathrm{d}_{\mathrm{bridge}}$ and $\mathrm{d}_{\mathrm{map}}$.*

Proof. The proof is similar to that of [14, Proposition 2.5] and is left to the reader. □

3 The UIPQ with a general boundary

3.1 Construction

The following theorem in an extension of the main result of [14] to the case of quadrangulations with a general boundary. Recall that $Q_{n,p}$ is uniformly distributed over $\mathcal{Q}_{n,p}$.

Theorem 2. *(i) For any $p \in \{1, 2, 3, \ldots\}$ we have the following convergence in distribution for $\mathrm{d}_{\mathrm{map}}$*

$$Q_{n,p} \xrightarrow[n\to\infty]{(d)} Q_{\infty,p},$$

where $Q_{\infty,p}$ is called the uniform infinite planar quadrangulation with a boundary of perimeter $2p$. *If $\mathbf{B}_{\infty,p}$ is a uniform infinite labeled treed bridge of length $2p$ then $Q_{\infty,p} = \Phi(\mathbf{B}_{\infty,p})$ in distribution.*

(ii) If $Q_{\infty,p} = \Phi(\mathbf{B}_{\infty,p})$ then, with the identification of the vertices of $Q_{\infty,p}$ with those of the trees of $\mathbf{B}_{\infty,p}$, we have for any $u, v \in Q_{\infty,p}$

$$\lim_{z\to\infty} \big(\mathrm{d}_{\mathrm{gr}}(u,z) - \mathrm{d}_{\mathrm{gr}}(v,z)\big) \quad = \quad \ell_{\mathbf{B}_{\infty,p}}(u) - \ell_{\mathbf{B}_{\infty,p}}(v), \tag{15}$$

where $z \to \infty$ means that the graph distance between z and the origin tends to infinity.

Proof of the first part of Theorem 2. An application of Euler's formula shows that every quadrangulation with a boundary of perimeter $2p$ and size n has exactly $n+p+1$ vertices. We deduce from the preceding section that after forgetting the distinguished point, the rooted quadrangulation $\Phi(\mathbf{B}_{n,p})$ is uniform over $\mathcal{Q}_{n,p}$. The first part of the theorem then follows from Proposition 1 and Proposition 2.
For the second part of the theorem, notice that the case $p = 1$ is proved in [14]. The general case will follow from this case using a surgical operation that we present in Section 4 below. □

Remark 3. In the construction $Q_{\infty,p} = \Phi(\mathbf{B}_{\infty,p})$, since the origin of the root edge of Q_∞ automatically has label 0, the formula (15) can be used to recover $\ell_{\mathbf{B}_{\infty,p}}$ as a measurable function of $Q_{\infty,p}$. Using an extension of the reversed construction "Φ^{-1}" (see [11]) it can be proved following the lines of [14] that the treed bridge $\mathbf{B}_{\infty,p}$ itself can be recovered as a measurable function of $Q_{\infty,p}$. We leave details to the interested reader.

Remark 4. Recall that the UIPQ with a simple boundary was defined from the standard UIPQ in Section 1.3. It is also possible to define the UIPQ with a simple boundary $\widetilde{Q}_{\infty,p}$ as the variable $Q_{\infty,p}$ conditioned on having a simple boundary. However, thanks to the asymptotics (4) and (5) the probability that the boundary of $Q_{\infty,p}$ is simple is easily seen to be

$$P(\partial Q_{\infty,p} \text{ is simple}) \quad = \quad \frac{\widetilde{C}_p}{C_p} \quad \underset{p\to\infty}{\sim} \quad \sqrt{3}\left(\frac{9}{16}\right)^p.$$

This exponential decay is not useful if we want to derive properties of $\widetilde{Q}_{\infty,p}$ from properties of $Q_{\infty,p}$ for large p's. For that purpose we develop in Section 4 another link between these objects based on a pruning procedure.

3.2 Aperture of the UIPQ with a general boundary

Recall that if q is a quadrangulation with a boundary, the *aperture* of q is the maximal graph distance between any two points of the boundary

$$\mathrm{aper}(q) \quad = \quad \max\{\mathrm{d}_{\mathrm{gr}}(u,v) : u,v \in \partial q\}.$$

The main result of this section provides bounds on the aperture of $Q_{\infty,p}$ for large p's. It is based on the construction $Q_{\infty,p} = \Phi(\mathbf{B}_{\infty,p})$ of Theorem 2 and on properties of some specific geodesics in $Q_{\infty,p}$.

Theorem 3. *The aperture of $Q_{\infty,p}$ is exponentially concentrated around the order of magnitude $\sqrt{p}$. More precisely, there exist $c_1, c_2 > 0$ such that for all $\lambda > 0$ and every $p \in \{1,2,3,\ldots\}$ we have*

$$P\big(\mathrm{aper}(Q_{\infty,p}) \notin [\lambda^{-1}\sqrt{p}, \lambda\sqrt{p}]\big) \quad \leqslant \quad c_1 \exp(-c_2\lambda^{2/3}).$$

Fix $p \in \{1,2,3,\ldots\}$. To simplify notation, we write $\mathbf{B} = (B_p; \Theta_1, \ldots, \Theta_p) \in \mathscr{B}_p$ for the uniform infinite labeled treed bridge of length $2p$ and assume that $Q_{\infty,p} = \Phi(\mathbf{B})$. We will always use a representation $\mathcal{E}$ of $\mathbf{B}$ and identify the trees $\Theta_1, \ldots, \Theta_p$ and the bridge B_p with their embeddings in the representation, see Fig. 6. Recall that the shifted labels of the trees are denoted by $\ell_{\mathbf{B}}$. We denote by $\mathscr{C}$ the set of corners of $\mathcal{E}$ that are associated with some vertex of $\cup_i \Theta_i$. If $c, c' \in \mathscr{C}$ we denote by $[c, c']$ the set of corners of $\mathscr{C}$ that are in-between c and c' for the clockwise order. For $1 \leqslant i \leqslant p$, the set of corners attached to the tree Θ_i is $[c_{L,i}, c_{R,i}]$ where $c_{L,i}$ and $c_{R,i}$ respectively denote the left and right most corner of the root vertex of Θ_i in $\mathcal{E}$, see Fig. 8.

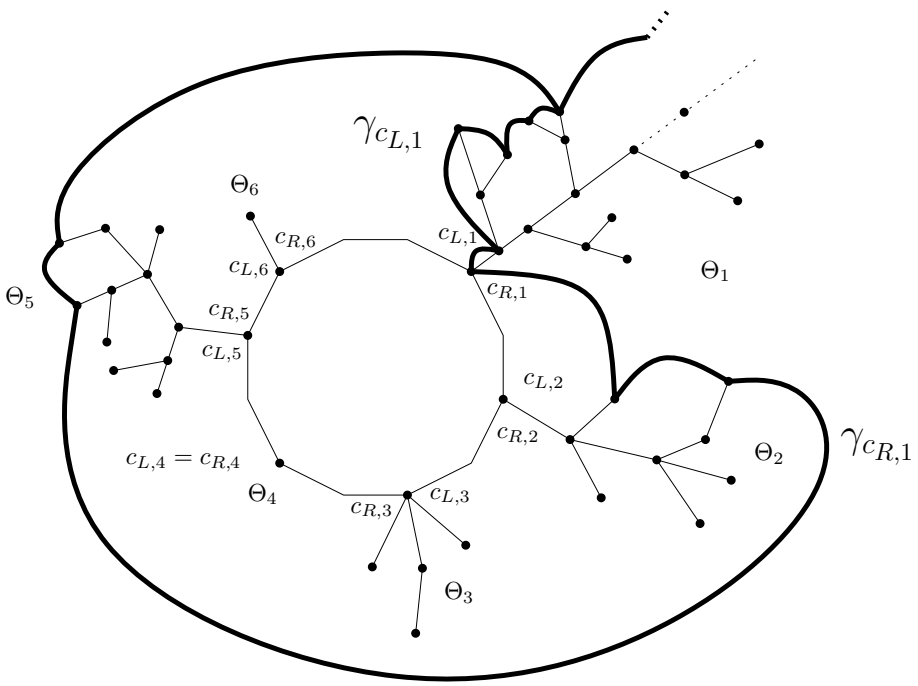


Figure 8: Left and right most corners of the trees $\Theta_1, \ldots, \Theta_6$ and the geodesics $\gamma_{c_{L,1}}$ and $\gamma_{c_{R,1}}$.

Simple geodesic. We recall the notion of simple (or maximal) geodesic, see [14, Definition 3.5]. Let $c \in \mathscr{C}$. We can construct a path in the quadrangulation $\Phi(\mathbf{B})$ by starting with the corner c and following iteratively its successors. This path is called *the simple geodesic starting from* c and is denoted by γ_c, see Fig. 8 for examples. It is easy to see, thanks to (14), that this path is actually a geodesic in the quadrangulation. If $\mathbf{B}$ were finite, this path would eventually end at ρ. In general, if $c, c' \in \mathscr{C}$, then γ_c and $\gamma_{c'}$ merge at a corner c'' of label

$$\ell_{\mathbf{B}}(c'') \quad = \quad \max\left\{ \min_{[c,c']} \ell_{\mathbf{B}}, \min_{[c',c]} \ell_{\mathbf{B}} \right\} - 1. \tag{16}$$

Proof of Theorem 3. We will suppose, without loss of generality, that Θ_1 is the uniform infinite labeled tree so that $\Theta_2, \dots, \Theta_p$ are uniformly labeled critical geometric Galton–Watson trees. Let us start with a preliminary observation.

Warm-up. Imagine that we construct the two simple geodesics $\gamma_{c_{L,1}}$ and $\gamma_{c_{R,1}}$ starting from the extreme corners of the root of Θ_1 and denote by C the cycle they form until their meeting point, see Fig. 8. We let $M = \max\{\ell_{\mathbf{B}}(u) : u \in \cup_{i=2}^{p} \Theta_i \text{ or } u = \mathrm{root}(\Theta_1)\}$ and $m = \min\{\ell_{\mathbf{B}}(u) : u \in \cup_{i=2}^{p} \Theta_i \text{ or } u = \mathrm{root}(\Theta_1)\}$. By the remark made on simple geodesics, one sees that the length of C is less than $2(M - m + 1)$ and that every vertex in the external face of $Q_{\infty,p}$ is linked to C by a simple geodesic of length less than $M - m + 1$. We deduce that

$$\mathrm{aper}(Q_{\infty,p}) \quad \leqslant \quad 3(M - m + 1). \tag{17}$$

Although M and m are typically of order $\sqrt{p}$, yet it is possible that with a probability of order p^{-2}, a specific tree, say Θ_2, has a height larger than p^2 and thus contains labels of order $\pm p$. If that happens then $M - m$ becomes of order p and not $\sqrt{p}$ anymore. Thus the exponential concentration presented in Theorem 3 cannot follow directly from (17). The idea is to modify the cycle C in order to bypass the large trees among $\Theta_2, \dots, \Theta_p$.

Bridge. Since $Q_{\infty,p}$ is constructed from $\mathbf{B}$, we know that the edges of its boundary are in correspondence with the edges of the cycle of $\mathcal{E}$, in particular the $\ell_{\mathbf{B}}$-labeling of the vertices of $\partial Q_{\infty,p}$ corresponds to the values $(X_0, \dots, X_{2p-1})$ of the bridge B_p. From the lower bound (14) we deduce that if $\Delta_p = \max X_i - \min X_i$ we have

$$\mathrm{aper}(Q_{\infty,p}) \quad \geqslant \quad \Delta_p. \tag{18}$$

Since B_p is a uniform bridge with $2p$ steps, classical results (which easily follow from the arguments of [18]) show that there exist positive constants c_1 and c_2 such that for all $p \geqslant 0$ we have

$$P\big(\Delta_p \notin [\lambda^{-1}\sqrt{p}, \lambda\sqrt{p}]\big) \quad \leqslant \quad c_1 \exp(-c_2\lambda^2). \tag{19}$$

Shortcut. As we said, the strategy is to build a simple path surrounding the external face in a very similar fashion as in the warmup but that shortcuts large trees. Let us be more precise. For every $i \in \{1, \dots, p\}$, declare the tree Θ_i "good" if the maximal displacement of the labels in Θ_i is in absolute value less than $\sqrt{p}$, that is if

$$\sup_{u,v \in \Theta_i} |\ell_{\mathbf{B}}(u) - \ell_{\mathbf{B}}(v)| \leqslant \sqrt{p}.$$

Call Θ_i "bad" otherwise. For any $i \in \{2, \dots, p\}$, the probability that Θ_i is bad is less than $c_3 p^{-1}$, for some $c_3 > 0$ (see e.g. [14, Lemma 3.9]). Hence, if K is the number of bad trees among $\{\Theta_2, \dots, \Theta_p\}$, then for any $1 \leqslant k \leqslant p$ we have

$$P(K \geqslant k) \quad \leqslant \quad \binom{p}{k} \left(\frac{c_3}{p}\right)^k \quad \leqslant \quad \frac{c_3^k}{k!}. \tag{20}$$

We now construct the path shortcutting these trees. Recall that for $1 \leqslant i \leqslant p$, we denote $c_{L,i}, c_{R,i}$ respectively the left-most and right-most corners of the root vertex of Θ_i in $\mathcal{E}$. We start our journey with $\gamma_{c_{R,1}}$ and move along it. As soon as $\gamma_{c_{R,1}}$ meets a bad tree Θ_i, we proceed as follows. From $c_{L,i}$ we start the simple geodesic $\gamma_{c_{L,i}}$. We know that it requires less than $\sqrt{p} + \Delta_p + 1$ steps for $\gamma_{c_{L,i}}$ to merge with $\gamma_{c_{R,1}}$ (which happens in Θ_i), then we bypass Θ_i by going backwards along $\gamma_{c_{L,i}}$ to reach the root of Θ_i and finally continue the process with $\gamma_{c_{R,i}}$. Since Θ_1 is obviously bad we also shortcut it. See Fig. 9 below.

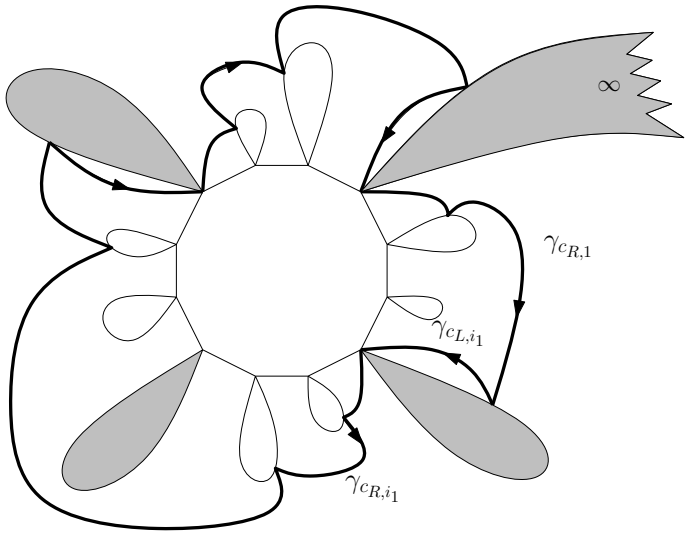


Figure 9: An illustration of the proof, the bad trees are represented in gray and the cycle constructed out of simple geodesics shortcutting them by a fat black line.

At the end of the process we get a simple cycle denoted by $\mathsf{C}_{\text{short}}$, which surrounds the external face of $Q_{\infty,p}$ and whose length is at most

$$\text{Length}(\mathsf{C}_{\text{short}}) \quad \leqslant \quad 2(K+1)(\sqrt{p} + \Delta_p + 1). \tag{21}$$

Furthermore, similarly as in the warm-up part, it is easy to see that every vertex of the external face of $Q_{\infty,p}$ can be connected to $\mathsf{C}_{\text{short}}$ via a simple geodesic of length less than $\sqrt{p} + \Delta_p + 1$, thus the aperture of $Q_{\infty,p}$ is less than $(K+3) \cdot (\sqrt{p} + \Delta_p + 1)$, which together with (18) gives

$$\Delta_p \quad \leqslant \quad \text{aper}(Q_{\infty,p}) \quad \leqslant \quad (K+3)(\sqrt{p} + \Delta_p + 1). \tag{22}$$

Now, for $\lambda > 1$ and $p \geqslant 1$, noticing that we have $12\lambda \geqslant (\lambda^{2/3} + 3)(2 + \lambda^{1/3})$ we get from the previous display that

$$\begin{aligned} P(\text{aper}(Q_{\infty,p}) \leqslant \lambda^{-1}\sqrt{p}) \quad &\leqslant \quad P(\Delta_p \leqslant \lambda^{-1}\sqrt{p}) \\ P(\text{aper}(Q_{\infty,p}) \geqslant 12\lambda\sqrt{p}) \quad &\leqslant \quad P(\Delta_p \geqslant \lambda^{1/3}\sqrt{p}) + P(K \geqslant \lambda^{2/3}). \end{aligned}$$

We use (19) and (20) to see that the probabilities of the right hand side are bounded by $c_4 \exp(-c_5 \lambda^{2/3})$ for some constants $c_4, c_5 > 0$, which completes the proof of the theorem. □

4 Pruning

4.1 Pruning of $Q_{\infty,p}$

Recall from Section 1.2 that we can decompose a quadrangulation with a general boundary into the irreducible component containing the root edge on which quadrangulations with a general boundary are attached. We now aim at a decomposition with respect to a "big" irreducible component, which is not necessarily the one designated by the root edge. We call this operation *pruning*.

Let q be a rooted quadrangulation with a boundary. Suppose that there is a unique irreducible component in q with maximal size. We call this irreducible component the *core* of q and denote it by $\mathrm{Core}(q)$. The core is either rooted at the original root edge of the map if it lies on the boundary of the core, or on the oriented edge preceding the component carrying the root edge (in that case this component is not reduced to the vertex map). Attached to this core we find quadrangulations with a general boundary denoted by $\mathrm{Part}_0(q), \mathrm{Part}_1(q), \ldots, \mathrm{Part}_{2r-1}(q)$, ordered starting from the root of the core, where $2r$ is the perimeter of the core. Note that some of these components can be reduced to the vertex map †. The components attached to the core are rooted at their last oriented edge visited during a clockwise contour of the external face (keeping the external face on the right). See Fig. 10 below.

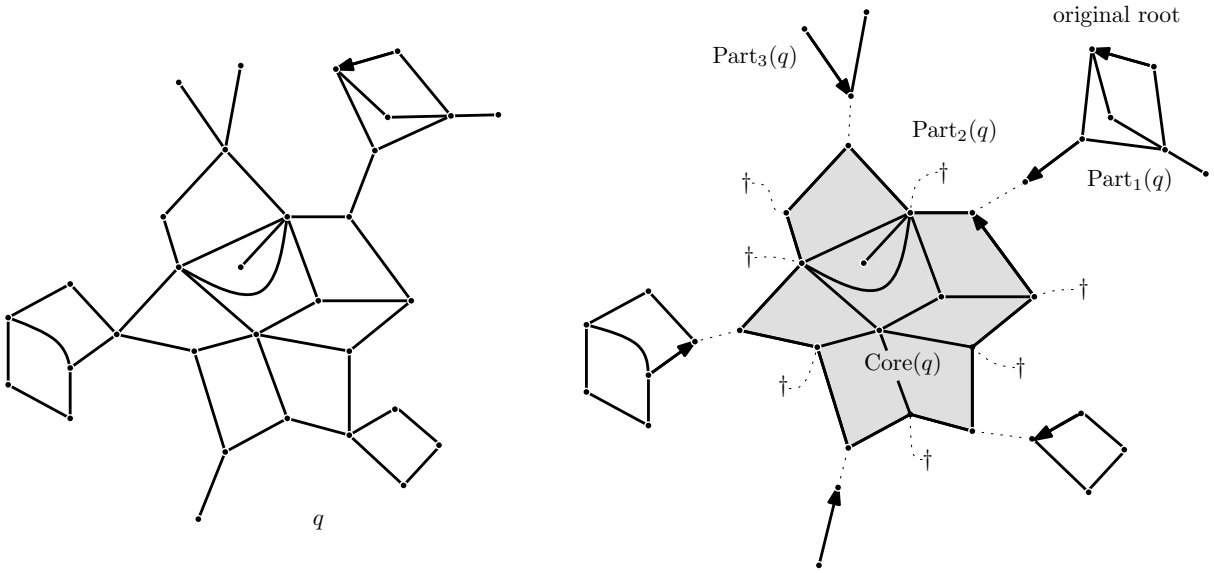


Figure 10: Illustration of the pruning.

The quadrangulation q can be recovered from $\mathrm{Core}(q), \mathrm{Part}_0(q), \mathrm{Part}_1(q), \ldots, \mathrm{Part}_{2r-1}(q)$ if we are given a number $\mathrm{R}(q) \in \{1, \ldots, |\partial \mathrm{Part}_0(q)| + 1\}$ to specify the location of the original root edge of the initial quadrangulation: on the first, second, ..., $|\partial \mathrm{Part}_0(q)|$-th oriented edge of $\mathrm{Part}_0(q)$ if $\mathrm{R}(q) \in \{1, \ldots, |\partial \mathrm{Part}_0(q)|\}$, or on the core just before $\mathrm{Part}_0(q)$ if $\mathrm{R}(q) = |\partial \mathrm{Part}_0(q)| + 1$.

If there are several irreducible components with maximal size, we set $\mathrm{Core}(q)$ and all the components to be equal to † and $\mathrm{R}(q) = 0$ by convention. The pruning is still possible when dealing with a rooted quadrangulation with a boundary that contains a unique infinite irreducible component, which is automatically the core.

Let $\mathcal{B}_p$ be the set of (possibly infinite) quadrangulations with a boundary of a fixed perimeter $2p$ and at most one infinite irreducible component. We claim that every $q \in \mathcal{B}_p$ is a continuity point for the functions $\mathrm{Core}, \mathrm{Part}_i$ and R with respect to $\mathrm{d}_{\mathrm{map}}$ (here we make a slight abuse of notation: since the set of indices i for which Part_i is defined depends on q, this means that for q' close enough to q this set of indices will be the same). We only sketch the proof, leaving details to the reader.

Let q_n converge to q in $\mathcal{B}_p$. If q is finite then there is nothing to prove since we have $q_n = q$ for all n large enough. So assume that q has a unique infinite irreducible component. Let $D = \max\{\mathrm{diam}(\mathrm{Part}_i(q)) : 0 \leqslant i \leqslant 2r-1\}$ and fix $r' > D + p + 1$. Fix n large enough so that $B_{r'}(q_n) = B_{r'}(q)$. Note that $B_{r'}(q)$ contains all the edges of the boundary of q and q_n, as well as the vertices of the irreducible components of these two quadrangulations that are at distance less than $D+1$ from the boundary. Hence knowing this ball is enough to deduce that q_n admits a unique irreducible component of maximal size (necessarily contained in $\mathrm{Core}(q)$), which is thus $\mathrm{Core}(q_n)$, and that the smaller irreducible components of q_n are the same as those of q. Moreover, $\mathrm{Core}(q_n)$ and $\mathrm{Core}(q)$ must coincide within the ball of radius $r'-p$ centered at their root vertices, which implies that $\mathrm{d}_{\mathrm{map}}(\mathrm{Core}(q_n), \mathrm{Core}(q)) \to 0$ as $n \to \infty$ since r' was arbitrary. For the same reason, the components of q_n attached to its core by a cut vertex are the same as those of q, and connect through the same cut vertices, and $\mathrm{R}(q_n)$ coincides with $\mathrm{R}(q)$, entailing the claimed continuity property.

On the other hand, if q has at least two irreducible components of maximal size, then it is necessarily finite by definition of $\mathcal{B}_p$, and thus $q_n = q$ for every n large enough.

Proposition 3. *For every $p \in \{1,2,3,\ldots\}$, $Q_{\infty,p}$ is in $\mathcal{B}_p$ almost surely.*

Proof. We suppose that $Q_{\infty,p} = \Phi(\mathbf{B})$ as in the proof of Theorem 3. Recall that in the warmup part of the proof of Theorem 3 we built a cycle C using the two simple geodesics starting from the extreme left and right corners of the infinite tree in $\mathbf{B}$. Since this cycle is a simple path it must be contained in a single irreducible component of $Q_{\infty,p}$. Furthermore, it is easy to see that only finitely many vertices are in-between the external face of $Q_{\infty,p}$ and C, see Fig. 8. This shows that the irreducible component containing C is the unique infinite component of $Q_{\infty,p}$. □

In the following theorem, Z_i are i.i.d. random variables distributed according to

$$P(Z=k) = \frac{\sum_{n=0}^{\infty} 12^{-n} 8^{-k} q_{n,k}}{W_c(8^{-1})}, \qquad k \in \{0,1,2,\ldots\}.$$

Theorem 4 (Pruning with fixed perimeter)**.** *For every $p \geqslant r \geqslant 1$, conditionally on the event $\{|\partial\mathrm{Core}(Q_{\infty,p})| = 2r\}$ of probability*

$$P(|\partial\mathrm{Core}(Q_{\infty,p})| = 2r) \quad = \quad \frac{r^{-1}(9/2)^{-r}\widetilde{C}_r}{p^{-1}8^{-p}C_p} P(Z_1 + \ldots + Z_{2r} = p - r),$$

the core of $Q_{\infty,p}$ is distributed as a UIPQ with a simple boundary of perimeter $2r$.

Proof. Recall that $Q_{n,p}$ (resp. $\widetilde{Q}_{n,p}$) denotes a random variable, uniformly distributed over $\mathcal{Q}_{n,p}$ (resp. $\widetilde{\mathcal{Q}}_{n,p}$). We also denote by U_k a uniform variable over $\{1,2,\ldots,k\}$. Fix $p \geqslant 1$ and $r \geqslant 1$, and let $g, f_1, \ldots, f_{2r}$ be positive bounded continuous functions for the distance $\mathrm{d}_{\mathrm{map}}$ and $e : \mathbb{R} \to \mathbb{R}_+$ be a bounded positive continuous function with $e(0) = 0$. As an immediate consequence of Proposition 3 and part (i) in Theorem 2, we deduce that $Q_{n,p}$

has a unique irreducible component with maximal size with a probability tending to 1 as $n \to \infty$. Thus we have, by the continuity properties of Core, Part_i and R on the set $\mathcal{B}_p$ that supports the law of $Q_{\infty,p}$ by Proposition 3,

$$E\left[g\big(\mathrm{Core}(Q_{n,p})\big)e(\mathrm{R}(Q_{n,p}))\prod_{i=0}^{2r-1} f_i\big(\mathrm{Part}_i(Q_{n,p})\big)\mathbf{1}_{|\partial\mathrm{Core}(Q_{n,p})|=2r}\right]$$
$$\xrightarrow[n\to\infty]{} E\left[g\big(\mathrm{Core}(Q_{\infty,p})\big)e(\mathrm{R}(Q_{\infty,p}))\prod_{i=0}^{2r-1} f_i\big(\mathrm{Part}_i(Q_{\infty,p})\big)\mathbf{1}_{|\partial\mathrm{Core}(Q_{\infty,p})|=2r}\right]. \quad (23)$$

Moreover, the pruning decomposition leads to

$$\begin{aligned} &E\left[g\big(\mathrm{Core}(Q_{n,p})\big)e(\mathrm{R}(Q_{n,p}))\prod_{i=0}^{2r-1} f_i\big(\mathrm{Part}_i(Q_{n,p})\big)\mathbf{1}_{|\partial\mathrm{Core}(Q_{n,p})|=2r}\right] \\ &= \frac{1}{q_{n,p}}\sum_{n\geqslant m\geqslant 0}\widetilde{q}_{m,r}E\big[g(\widetilde{Q}_{m,r})\big] \\ &\quad\times\sum_{\substack{p_0+\ldots+p_{2r-1}=p-r\\ m>n_0,\ldots,n_{2r-1}\geqslant 0\\ n_0+\ldots+n_{2r-1}=n-m}}(2p_0+1)E\left[e(U_{2p_0+1})\right]\prod_{i=0}^{2r-1} q_{n_i,p_i}E\big[f_i(Q_{n_i,p_i})\big]. \end{aligned} \quad (24)$$

Similarly as in the proof of Proposition 1, one can use the asymptotics (4), (5) and Lemma 1 with $\beta = 5/2$ to deduce that the probability that one of the components $\mathrm{Part}_0(Q_{n,p}), \mathrm{Part}_1(Q_{n,p}), \ldots$ has a size larger than $a > 0$ is bounded above by $ca^{-3/2}$ for some constant $c > 0$ uniformly in n. So we can let $n \to \infty$ followed by $a \to \infty$ in the formula (24) and obtain by (7), (23) and asymptotics (4) and (5) that

$$\begin{aligned} &E\left[g\big(\mathrm{Core}(Q_{\infty,p})\big)e(\mathrm{R}(Q_{\infty,p}))\prod_{i=0}^{2r-1} f_i\big(\mathrm{Part}_i(Q_{\infty,p})\big)\mathbf{1}_{|\partial\mathrm{Core}(Q_{\infty,p})|=2r}\right] \\ &= \frac{\widetilde{C}_r}{C_p}E\big[g(\widetilde{Q}_{\infty,r})\big]\sum_{p_1+\ldots+p_{2r}=p-r}(2p_1+1)E\left[e(U_{2p_1+1})\right] \\ &\quad\times\prod_{i=1}^{2r}\sum_{n_i=0}^{\infty}12^{-n_i}q_{n_i,p_i}E\big[f_i(Q_{n_i,p_i})\big]. \end{aligned} \quad (25)$$

The last expression is the fundamental "pruning formula". It can be used to derive the distribution of the core and the components of $Q_{\infty,p}$ for a fixed p. Let us proceed. Fix $p \geqslant r \geqslant 1$, so that (25) specializes when $e = f_1 = \ldots = f_{2r} = 1$ to

$$\begin{aligned} E[g(\mathrm{Core}(Q_{\infty,p}))\mathbf{1}_{|\partial\mathrm{Core}(Q_{\infty,p})|=2r}] &= \frac{\widetilde{C}_r}{C_p}E[g(\widetilde{Q}_{\infty,r})]\sum_{p_1+\ldots+p_{2r}=p-r}(2p_1+1)\prod_{i=1}^{2r}\sum_{n_i=0}^{\infty}12^{-n_i}q_{n_i,p_i} \\ &= \frac{p}{r}\frac{\widetilde{C}_r}{C_p}E[g(\widetilde{Q}_{\infty,r})]\sum_{p_1+\ldots+p_{2r}=p-r}\prod_{i=1}^{2r}\sum_{n_i=0}^{\infty}12^{-n_i}q_{n_i,p_i}, \end{aligned}$$

where we got rid of the term $(2p_1+1)$ by an obvious symmetry argument, to the cost of adding the prefactor $(2r)^{-1}\sum_{i=1}^{2r}(2p_i+1) = p/r$. Recalling the definition of the bivariate

function $W(g,z)$ and $W_c(z) = W(12^{-1}, z)$, we can further re-write the last expression as

$$\begin{aligned}\frac{p}{r}\frac{\widetilde{C}_r}{C_p}E[g(\widetilde{Q}_{\infty,r})]8^{p-r}W_c(8^{-1})^{2r} \sum_{p_1+\ldots+p_{2r}=p-r}\prod_{i=1}^{2r}\frac{\sum_{n_i=0}^{\infty}12^{-n_i}8^{-p_i}q_{n_i,p_i}}{W_c(8^{-1})} \\ = \frac{p}{r}\frac{(9/2)^{-r}\widetilde{C}_r}{8^{-p}C_p}E[g(\widetilde{Q}_{\infty,r})] \sum_{p_1+\ldots+p_{2r}=p-r}\prod_{i=1}^{2r}\frac{\sum_{n_i=0}^{\infty}12^{-n_i}8^{-p_i}q_{n_i,p_i}}{W_c(8^{-1})}\,,\end{aligned}$$

where we used the fact that $W_c(8^{-1}) = 4/3$, see (3). We interpret the last sum as $P(Z_1+Z_2+\ldots+Z_{2r} = p-r)$, where $Z_1, \ldots, Z_{2r}$ are as in the statement of the theorem. □

4.2 Proof of Theorem 1

As a first application of Theorem 4, let us now prove Theorem 1. To this end, we first make some preliminary observations. By definition of $W(g,z)$, the generating function of Z_1 is given by

$$\begin{aligned}E[s^{Z_1}] &= \frac{W_c(s8^{-1})}{W_c(8^{-1})} = \frac{(1-\sqrt{1-s})(2s-1+\sqrt{1-s})}{s^2} \\ &\underset{s\uparrow 1}{=} s + 2(1-s)^{3/2} + o((1-s)^{3/2})\,.\end{aligned}$$

By standard results on stable domains of attraction [9], this expression entails that the random variable Z_1 is in the domain of attraction of a stable random variable with exponent $3/2$. More precisely, since moreover $E[Z_1] = 1$ by differentiating the previous expression, it holds that

$$\frac{Z_1+\ldots+Z_n-n}{n^{2/3}} \xrightarrow[n\to\infty]{(d)} \mathcal{Z}'\,,$$

where the Laplace transform of $\mathcal{Z}'$ is given by $E[\exp(-\lambda\mathcal{Z}')] = \exp(2\lambda^{3/2})$ for every $\lambda \geqslant 0$. More precisely, if h denotes the density of the law of $\mathcal{Z}'$, then the Gnedenko-Kolmogorov local limit theorem for lattice variables (see [16, Theorem 4.2.1]) entails that

$$\sup_{k\in\mathbb{Z}}\left|n^{2/3}P(Z_1+Z_2+\ldots+Z_n = n+k) - h\Big(\frac{k}{n^{2/3}}\Big)\right| \xrightarrow[n\to\infty]{} 0\,. \tag{26}$$

Now, for a given $r \geqslant 1$, set $p = 3r$ in the display of Theorem 4. By using (26) with $n = 2r$ and $k = 0$, we obtain $P(Z_1+\ldots+Z_{2r} = 2r) \sim h(0)(2r)^{-2/3}$ as $r\to\infty$. On the other hand, the asymptotic behavior for $C_p, \widetilde{C}_r$ entails that (still when $p = 3r$)

$$\frac{r^{-1}(9/2)^{-r}\widetilde{C}_r}{p^{-1}8^{-p}C_p} \xrightarrow[r\to\infty]{} 3\,. \tag{27}$$

From this and Theorem 4, we conclude that

$$P(|\partial\mathrm{Core}(Q_{\infty,3r})| = 2r) \quad\sim\quad Ar^{-2/3}\,,$$

where $A = 3\cdot 2^{-2/3}\cdot h(0) \in (0,\infty)$. Using Theorem 4, we deduce that there exists $A' \in (0,\infty)$ such that for any non-negative measurable function g,

$$E[g(\widetilde{Q}_{\infty,r})] = E\big[g(\mathrm{Core}(Q_{\infty,3r}))\,\big|\,|\partial\mathrm{Core}(Q_{\infty,3r})| = 2r\big] \leqslant A'r^{2/3}E[g(\mathrm{Core}(Q_{\infty,3r}))]\,.$$

From this, Theorem 3, and the obvious fact that $\mathrm{aper}(\mathrm{Core}(Q_{\infty,3r})) \leqslant \mathrm{aper}(Q_{\infty,3r})$, we conclude that

$$\begin{aligned} P(\mathrm{aper}(\widetilde{Q}_{\infty,r}) \geqslant \lambda\sqrt{r}) &\leqslant A' r^{2/3} P(\mathrm{aper}(\mathrm{Core}(Q_{\infty,3r})) \geqslant \lambda\sqrt{r}) \\ &\leqslant A' r^{2/3} P(\mathrm{aper}(Q_{\infty,3r}) \geqslant \lambda\sqrt{r}) \\ &\leqslant A' c_1 r^{2/3} \exp(-3^{-1/3} c_2 \lambda^{2/3}) \,. \end{aligned}$$

This yields Theorem 1.

Remark 5. Since the aperture decreases through the pruning, we can only provide an upper bound on the aperture of quadrangulations with a simple boundary using our method. Secondly, although Theorem 3 entails that $(p^{-1/2}\mathrm{aper}(Q_{\infty,p}))_{p\geqslant 0}$ is tight, our results do not imply a similar property for $(p^{-1/2}\mathrm{aper}(\widetilde{Q}_{\infty,p}))_{p\geqslant 0}$ due to the additional factor $p^{2/3}$. However, we believe that $(p^{-1/2}\mathrm{aper}(Q_{\infty,p}))_{p\geqslant 0}$ and $((3p)^{-1/2}\mathrm{aper}(\widetilde{Q}_{\infty,p}))_{p\geqslant 0}$ converge in distribution to the same non-degenerate random variable.

4.3 Asymptotics of the perimeters

As a second application of Theorem 4, we will see that as $p \to \infty$ the core of $Q_{\infty,p}$ has a perimeter which is roughly a third of the original quadrangulation. This supports [11, Section 5] where the authors proved that quadrangulations with a simple boundary of perimeter p have the same large scale structure as quadrangulations with a general boundary of perimeter $3p$.

Proposition 4. *We have the following convergence in probability*

$$\frac{|\partial \mathrm{Core}(Q_{\infty,p})|}{2p} \xrightarrow[p\to\infty]{(P)} \frac{1}{3}.$$

More precisely, it holds that

$$\frac{|\partial \mathrm{Core}(Q_{\infty,p})| - 2p/3}{p^{2/3}} \xrightarrow[p\to\infty]{(d)} \mathcal{Z}\,,$$

where $\mathcal{Z}$ is a spectrally negative stable random variable with exponent $3/2$, with Laplace transform given by

$$E[\exp(\lambda\mathcal{Z})] = \exp\left(\left(\frac{2}{3}\right)^{5/2}\lambda^{3/2}\right), \qquad \lambda \geqslant 0\,.$$

Proof. Let $p \geqslant 1$ and $x \in \mathbb{R}$, and let $r = \lfloor p/3 \rfloor + \lfloor xp^{2/3} \rfloor$. We again use the local limit theorem (26) by specializing it to $n = 2r$ and $k = p - 3r$, and use the asymptotic equivalents for $C_p, \widetilde{C}_r$ with the same limit as in (27). Together with Theorem 4 this implies

$$p^{2/3} P\Big(\frac{1}{2}|\partial \mathrm{Core}(Q_{\infty,p})| - \lfloor p/3 \rfloor = \lfloor xp^{2/3} \rfloor\Big) \xrightarrow[p\to\infty]{} \frac{3^{5/3}}{2^{2/3}} h\Big(-\frac{3^{5/3}}{2^{2/3}}x\Big)\,.$$

By Scheffé's lemma and elementary computations using the Laplace transform of h, this implies the second claim in the statement. The first claim on convergence in probability is a simple consequence of the latter. □

4.4 Randomizing the perimeters

In this section, we argue that (25) gives a particularly nice probabilistic interpretation of the pruning operation, to the cost of randomizing the perimeters of the maps under consideration. Let us introduce some notation. Let $C(z) = \sum C_p z^p$ be the generating function of the C_p's (with $C_0 = 0$) and set $\vec{W}_c(z) := W_c(z) + 2z\partial_z W_c(z)$. Notice that $\sum_{n\geqslant 0} q_{n,p}(2p+1)12^{-n} = [z^p](W_c(z)+2z\partial_z W_c(z)) = [z^p]\vec{W}_c(z)$. From the exact expressions of $C_p(.)$ (4) and (6) we get that

$$\begin{aligned} C(z) &= \frac{2z}{\sqrt{\pi}(1-8z)^{3/2}}, \\ \vec{W}_c(z) &= \frac{1-4z-\sqrt{1-8z}}{8z^2}. \end{aligned} \tag{28}$$

For $0 \leqslant z \leqslant 1/8$, we denote by $Q_{f,z}$ (resp. $\vec{Q}_{f,z}$) a random finite quadrangulation with a general boundary such that the size of $Q_{f,z}$ equals n and its perimeter $2p$ with probability $12^{-n}z^p W_c(z)^{-1}$ (resp. $(2p+1)12^{-n}z^p\vec{W}_c(z)^{-1}$). In particular, conditionally on n and p, both $Q_{f,z}$ and $\vec{Q}_{f,z}$ are uniform over $\mathcal{Q}_{n,p}$, and the r.v. $\vec{Q}_{f,z}$ has the law of $Q_{f,z}$ biased by its perimeter plus 1. Since $W_c(1/8)$ and $\vec{W}_c(1/8)$ are finite, both $Q_{f,z}$ and $\vec{Q}_{f,z}$ make sense for $z = 1/8$. We call these random quadrangulations "free critical Boltzmann (extra rooted) quadrangulations with parameter z". Finally, for $0 < z < 1/8$, let $\mathcal{P}_z$ and $\widetilde{\mathcal{P}}_z$ be random variables distributed according to

$$P(\mathcal{P}_z = p) = \frac{z^p C_p}{C(z)}, \qquad \text{and} \qquad P(\widetilde{\mathcal{P}}_z = r) = \widetilde{C}_r\big(zW_c^2(z)\big)^r \frac{\vec{W}_c(z)}{C(z)W_c(z)}. \tag{29}$$

We recall that $C_0 = \widetilde{C}_0 = 0$. The lines leading to (25) (or a direct calculation) show that $\widetilde{C}(zW_c^2(z))\vec{W}_c(z) = C(z)W_c(z)$ for every $0 \leqslant z < 1/8$ (where $\widetilde{C}$ is the generating function of the $\widetilde{C}_r$'s) so that $\widetilde{\mathcal{P}}_z$ is well defined.

In the remaining of this work if $\mathcal{P}$ is an integer-valued random variable, we denote by $Q_{\infty,\mathcal{P}}$ a random variable such that conditionally on $\{\mathcal{P} = p\}$, $Q_{\infty,\mathcal{P}}$ is distributed as $Q_{\infty,p}$ (and similarly for $\widetilde{Q}_{\infty,\mathcal{P}}$).

Theorem 5 (Pruning with random perimeter)**.** *For every $z \in (0, 1/8)$, we have the following equality in distribution*

$$\mathrm{Core}\big(Q_{\infty,\mathcal{P}_z}\big) \overset{(d)}{=} \widetilde{Q}_{\infty,\widetilde{\mathcal{P}}_z}.$$

Furthermore, conditionally on $|\partial\mathrm{Core}(Q_{\infty,\mathcal{P}_z})|$, the core and the components of $Q_{\infty,\mathcal{P}_z}$ are independent, the latter being distributed as follows: the first component is distributed according to $\vec{Q}_{f,z}$ (the location $\mathrm{R}(Q_{\infty,\mathcal{P}_z})$ of the root being uniform over $\{1, 2, \ldots, |\partial\mathrm{Part}_0(Q_{\infty,\mathcal{P}_z})|+1\}$), and all the other $|\partial\mathrm{Core}(Q_{\infty,\mathcal{P}_z})| - 1$ components are distributed according to $Q_{f,z}$.

Proof. Fix $z \in (0, 1/8)$. We multiply both members of (25) by $z^p C_p$, sum over all $p \geqslant 1$

and then divide by $C(z)$ to deduce that

$$E\left[g\big(\mathrm{Core}(Q_{\infty,\mathcal{P}_z})\big)e(\mathrm{R}(Q_{\infty,\mathcal{P}_z}))\prod_{i=0}^{2r-1}f_i\big(\mathrm{Part}_i(Q_{\infty,\mathcal{P}_z})\big)\mathbf{1}_{|\partial\mathrm{Core}(Q_{\infty,\mathcal{P}_z})|=2r}\right]$$
$$= \frac{\vec{W}_c(z)}{C(z)W_c(z)}\widetilde{C}_r\big(zW_c^2(z)\big)^r\left(E\big[g(\widetilde{Q}_{\infty,r})\big]E\Big[f_1(\vec{Q}_{f,z})E\Big[e\big(U_{|\partial\vec{Q}_{f,z}|+1}\big)\Big]\Big]\prod_{i=2}^{2r}E\big[f_i(Q_{f,z})\big]\right)$$
$$= P(\widetilde{\mathcal{P}}_z=r)\left(E\Big[g(\widetilde{Q}_{\infty,r})\Big]E\Big[f_1(\vec{Q}_{f,z})E[e(U_{|\partial\vec{Q}_{f,z}|+1})]\Big]\prod_{i=2}^{2r}E\big[f_i(Q_{f,z})\big]\right).$$

This implies the result. □

4.5 Interpretation of the labels when $Q_{\infty,p}=\Phi(\mathbf{B}_{\infty,p})$

In this section we finally prove how the pruning can be used to deduce the second part of Theorem 2 from the results of [14]. Let us recall the setting. Fix $p\in\{1,2,3,\dots\}$ and let $\mathbf{B}$ be a uniform infinite labeled treed bridge of length $2p$. We consider $Q_{\infty,p}=\Phi(\mathbf{B})$. We aim at showing that, with the identification of the vertices of $Q_{\infty,p}$ with those of $\mathbf{B}$, a.s. for every $u,v\in Q_{\infty,p}$ we have

$$\lim_{z\to\infty}\big(\mathrm{d}_{\mathrm{gr}}(u,z)-\mathrm{d}_{\mathrm{gr}}(v,z)\big) \;=\; \ell_{\mathbf{B}}(u)-\ell_{\mathbf{B}}(v).$$

Proof of the second part of Theorem 2. In order to prove the last display, it only suffices to prove the property, denoted by $(*)$, that the left-hand side does have an a.s. limit as $z\to\infty$. Indeed, assuming $(*)$, we can adapt the end of the proof of [14, Lemma 3.13]: for $u,v\in\mathbf{B}$ we consider two simple geodesics γ_u and γ_v starting from any corners associated with u and v in $\mathbf{B}$ (these are thus proper geodesics in the terminology of [14]). By (16), these two paths eventually merge with a difference of steps equal to $|\ell_{\mathbf{B}}(u)-\ell_{\mathbf{B}}(v)|$. Since γ_u and γ_v are geodesics, for any z on these paths after the merging point we have

$$\mathrm{d}_{\mathrm{gr}}(u,z)-\mathrm{d}_{\mathrm{gr}}(v,z)=\ell_{\mathbf{B}}(u)-\ell_{\mathbf{B}}(v),$$

so sending $z\to\infty$ along $\gamma_u\cap\gamma_v$ implies the claim by using property $(*)$.
It thus suffices to prove that $(*)$ holds for $Q_{\infty,p}$. In fact, it is easy to see that we can restrict our attention to those u,v that belong to the Core of $Q_{\infty,p}$. Indeed, if u is inside a finite irreducible component C and z is inside $\mathrm{Core}(Q_{\infty,p})$ then if $u_\circ$ denotes the vertex common to $\mathrm{Core}(Q_{\infty,p})$ and C then we have

$$\mathrm{d}_{\mathrm{gr}}(u,z)=\mathrm{d}_{\mathrm{gr}}(u,u_\circ)+\mathrm{d}_{\mathrm{gr}}(u_\circ,z).$$

Thanks to Theorem 4, for any $1\leqslant r\leqslant p$, conditionally on $\{\partial\mathrm{Core}(Q_{\infty,p})=2r\}$ we have $\mathrm{Core}(Q_{\infty,p})=\widetilde{Q}_{\infty,r}$ in distribution. We are thus reduced to prove that $\widetilde{Q}_{\infty,r}$ satisfies $(*)$ for all $r\geqslant 1$. To show this, we will make some plastic surgery with $\widetilde{Q}_{\infty,p}$ in order to come back to the setup of [14], which deals with the full-plane UIPQ.

More precisely, let us consider the quadrangulation $Q'_{\infty,r}$ obtained after filling the external face of $\widetilde{Q}_{\infty,r}$ with a quadrangulation with a simple boundary of perimeter $2r$ made of $2r+1$ "layers" of quadrangles such that the last layer is connected to the root edge, see Fig. 11.

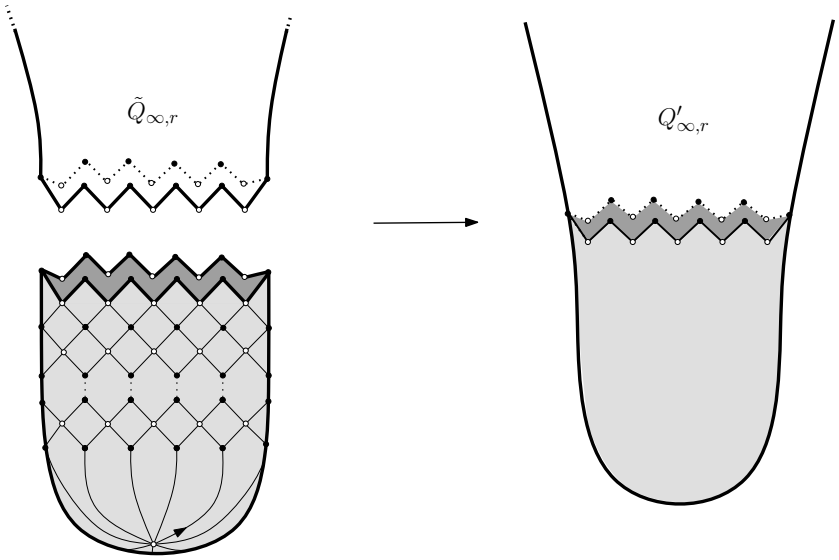


Figure 11: The filling operation.

This operation is reversible, that is, given $Q'_{\infty,r}$ and the number r, we can recover $\widetilde{Q}_{\infty,r}$. Most importantly, the filling has been done in such a way that for any $u, v \in \widetilde{Q}_{\infty,r}$ we have

$$\mathrm{d}_{\mathrm{gr}}^{Q'_{\infty,r}}(u,v) \quad = \quad \mathrm{d}_{\mathrm{gr}}^{\widetilde{Q}_{\infty,r}}(u,v). \tag{30}$$

Indeed it is easy to see that for any $u, v \in \widetilde{Q}_{\infty,r}$ we can find a geodesic path between u and v that does not enter the grafted region. Furthermore, the law of $Q'_{\infty,r}$ is absolutely continuous with respect to the law of the UIPQ (this can be seen, e.g. using the spatial Markov property of the UIPQ, see [6]). Since the UIPQ satisfies property $(*)$ a.s. (see [14]) we deduce that the same holds for $Q'_{\infty,r}$. By (30), the same also holds for $\widetilde{Q}_{\infty,r}$. □

5 Open boundary, open questions

5.1 UIPQ of the half-plane

In this section we let $p \to \infty$ and define the UIPQ with an infinite general and simple boundary of infinite perimeter. We then extend the pruning procedure to these infinite quadrangulations. The proofs are only sketched or left to the reader.

5.1.1 General boundary

We start by introducing the limit of the uniform infinite labeled treed bridges as $p \to \infty$.

Let $(X_n)_{n\in\mathbb{Z}}$ be a two-sided simple random walk with $X_0 = 0$ and having uniform increments in $\{+1, -1\}$. Independently of $(X_n)_{n\in\mathbb{Z}}$, let $(\Theta_i)_{i\in\mathbb{Z}}$ be a sequence of independent uniformly labeled geometric critical Galton–Watson trees (all finite). The object $\mathbf{B}_{\infty,\infty} = ((X_n)_{n\in\mathbb{Z}}, (\Theta_i)_{i\in\mathbb{Z}})$ is *the uniform infinite labeled treed bridge of infinite length*. It obviously appears as a limit of the uniform infinite labeled treed bridge $\mathbf{B}_{\infty,p} = (B_p; \Theta_{1,p}, \dots, \Theta_{p,p})$ of length $2p$ as $p \to \infty$ in the following sense: if $B_p = (X_0^{(p)}, \dots, X_{2p-1}^{(p)})$, then for any

$m \geqslant 1$ we have

$$(X^{(p)}_{[i]})_{-m\leqslant i\leqslant m} \xrightarrow[p\to\infty]{(d)} (X_i)_{-m\leqslant i\leqslant m},$$

when $[i]$ stands for the representative of i modulo $2p$ that belongs to $\{0,\dots,2p-1\}$. Furthermore the trees grafted on the down-steps $[i]$ such that $-m \leqslant i \leqslant m$ are asymptotically i.i.d. critical geometric Galton–Watson trees since the probability that one of these trees is the infinite one tends to 0 as $p \to \infty$.

We can associate with the infinite treed bridge $\mathbf{B}_{\infty,\infty}$ a representation by grafting the trees Θ_i to the down-steps of the walk (X_n) and shifting their labels, see Fig. 12. We then (once again) extend the Schaeffer mapping Φ to this object in a straightforward manner and define a random infinite quadrangulation with an infinite perimeter denoted by $\Phi(\mathbf{B}_{\infty,\infty})$, see Fig. 12.

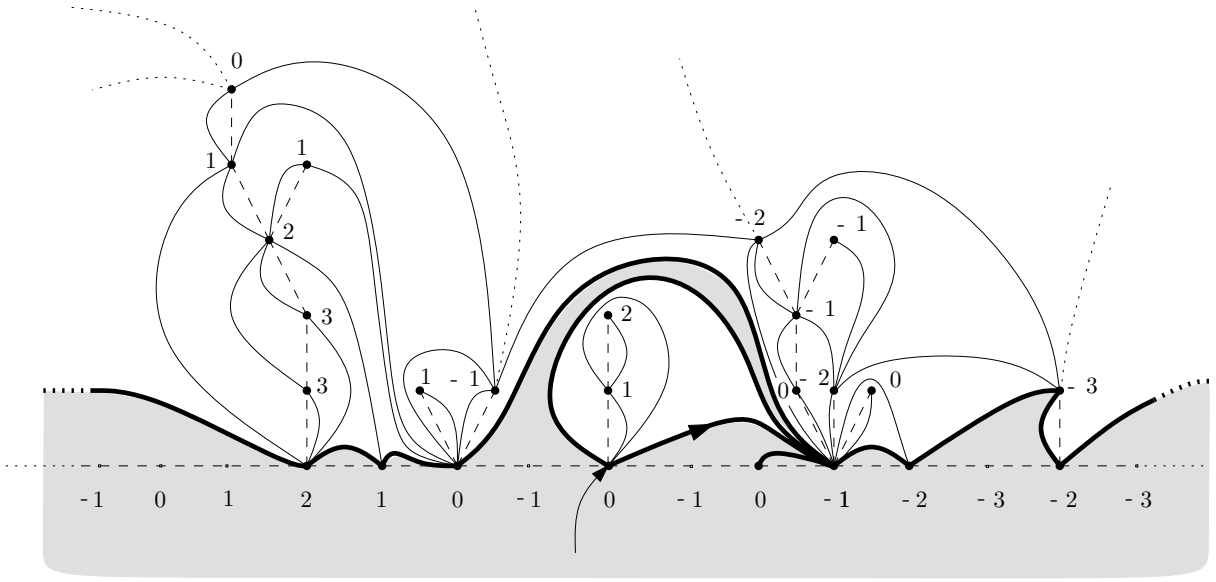


Figure 12: Second extension of Φ. The infinite labeled treed bridge is represented with dashed lines, the map with solid lines and its boundary with thick lines.

Proposition 5. *We have the following convergence*

$$Q_{\infty,p} \xrightarrow[p\to\infty]{(d)} Q_{\infty,\infty},$$

in distribution for $\mathrm{d}_{\mathrm{map}}$*, where* $Q_{\infty,\infty}$ *is a random infinite rooted quadrangulation with an infinite boundary that can be constructed from the uniform infinite treed bridge of infinite perimeter via the extended Schaeffer mapping, that is,* $Q_{\infty,\infty} = \Phi(\mathbf{B}_{\infty,\infty})$ *in distribution.*

This is essentially an adaptation of the first part of Theorem 2 (and Proposition 2). The proof is left to the interested reader.

5.1.2 Simple boundary.

Recall the notation used in Theorem 5. Using the exact expression of $C(.)$ we deduce that for $z \in (0, 1/8)$ we have

$$E[\mathcal{P}_z] \quad = \quad \frac{zC'(z)}{C(z)} \quad = \quad \frac{1+4z}{1-8z}.$$

Thus the average perimeter of $Q_{\infty,\mathcal{P}_z}$ is asymptotically equivalent to $3(1-8z)^{-1}$ as $z \to 1/8$ and it is easy to see from the singularity analysis of $C(.)$ that $\mathcal{P}_z \to \infty$ in distribution as $z \to 1/8$. Using Proposition 4 (or by a direct analysis) we see that $\widetilde{\mathcal{P}}_z \to \infty$ as well as $z \to 1/8$. It follows from the last theorem that $Q_{\infty,\mathcal{P}_z} \to Q_{\infty,\infty}$ as $z \to 1/8$. Using Theorem 5 we would like to deduce that $\widetilde{Q}_{\infty,\widetilde{\mathcal{P}}_z} \overset{(d)}{=} \mathrm{Core}(Q_{\infty,\mathcal{P}_z})$ converges in distribution as $z \to 1/8$.

We have to be a bit careful at this point, because Core is not a continuous mapping on the full set of infinite quadrangulations with boundaries. However, suppose that $q_n, n \geqslant 1$ are quadrangulations with boundaries respectively of perimeter $2p_n$ for some sequence $p_n, n \geqslant 1$, and assume that q_n has a well-defined, infinite core. Recall that $\mathrm{Part}_i(q_n), 0 \leqslant i \leqslant 2r_n - 1$ denote the components of q_n pending from the core, which are finite quadrangulations with boundaries, and that the extra number $\mathrm{R}(q_n)$ allows one to recover the root edge of q_n. By convention, for $i \in \mathbb{Z}$, let $\mathrm{Part}_i(q_n)$ be equal to $\mathrm{Part}_{[i]}(q_n)$, where $[i]$ is the unique integer of the form $i+2kr_n$ with $k \in \mathbb{Z}$ that lies in $\{0, 1, \ldots, 2r_n-1\}$. We assume that as $n \to \infty$,

- q_n converges to a limit q for the distance $\mathrm{d}_{\mathrm{map}}$,
- r_n converges to ∞,
- for every $i \in \mathbb{Z}$, $\mathrm{Part}_i(q_n)$ converges to a finite quadrangulation with a boundary $q_{(i)}$.

(Note that the third point is not implied by the first point.) Under these hypotheses, it is elementary to see that q also has a well-defined, infinite core, which is a quadrangulation with an infinite simple boundary, and that $\mathrm{d}_{\mathrm{map}}(\mathrm{Core}(q_n), \mathrm{Core}(q)) \to 0$ as $n \to \infty$. Moreover, if we label the boundary vertices by $\mathbb{Z}$ in the natural way, the vertices with labels 0 and 1 being the origin and end vertices of the root edge, then for $i \in \mathbb{Z}$, the map $q_{(i)}, i \in \mathbb{Z}$ is the component of q grafted by its root to the i-th vertex of the boundary of $\mathrm{Core}(q)$. Finally, it holds that $\mathrm{R}(q_n) \to \mathrm{R}(q)$.

We apply this to the convergence of $Q_{\infty,\mathcal{P}_z}$ to $Q_{\infty,\infty}$. Indeed, recall that $Q_{\infty,\mathcal{P}_z}$ has a well-defined infinite core, and that the pending components are independent (and independent of the core) with same distribution as $Q_{f,z}$ (or $\vec{Q}_{f,z}$ for the root component). Now as $z \to 1/8$, these maps converge in distribution to random maps distributed as $Q_{f,1/8}$ (or $\vec{Q}_{f,1/8}$), which are a.s. finite. By Skorokhod's representation theorem, we may assume that these convergences, as well as the convergence of $Q_{\infty,\mathcal{P}_z} \to Q_{\infty,\infty}$ and $|\partial\mathrm{Core}(Q_{\infty,\mathcal{P}_z})| \to \infty$, hold in the a.s. sense (the last convergence comes from the fact that the perimeters under consideration have same distribution as $2\widetilde{\mathcal{P}}_z$). By applying the observation of the preceding paragraph, we conclude that $Q_{\infty,\infty}$ has a well-defined core, which is the limit of $\mathrm{Core}(Q_{\infty,\mathcal{P}_z}) \overset{(d)}{=} \widetilde{Q}_{\infty,\widetilde{\mathcal{P}}_z}$. This core is a quadrangulation with an infinite simple boundary, and the pending components are independent with same law as $Q_{f,1/8}$ (or $\vec{Q}_{f,1/8}$ for the root component), and independent of the core. We have obtained the following proposition.

Proposition 6. *It holds that*

$$\widetilde{Q}_{\infty,\widetilde{\mathcal{P}}_z} \xrightarrow[z\to 1/8]{(d)} \widetilde{Q}_{\infty,\infty}\,,$$

where the limit is a random quadrangulation with an infinite simple boundary. Moreover, the core and the components of $Q_{\infty,\infty}$ are all independent, the core being distributed as $\widetilde{Q}_{\infty,\infty}$, the first component as $\vec{Q}_{f,1/8}$, and the other components as $Q_{f,1/8}$.

One can wonder whether the maps $\widetilde{Q}_{\infty,p}$ converge in distribution to the limit $\widetilde{Q}_{\infty,\infty}$. This convergence is not implied by our approach, but is nonetheless true and can be obtained by adapting techniques of [1] (where the analog of $\widetilde{Q}_{\infty,\infty}$ has been defined in the triangular case). One of the advantages of working with objects such as $\widetilde{Q}_{\infty,\infty}$ is the very simple form that takes the spatial Markov property, see [1, 3].

5.2 Comments, questions

Extending the techniques of this paper, it is possible to show that the graph distance between the origin and the kth vertex on the boundary of $Q_{\infty,\infty}$ is of order $\sqrt{|k|}$ for $k \in \mathbb{Z}$ and to translate this result to the simple boundary case via the pruning procedure extended in Theorem 5. We present here a couple of open questions related to the models $Q_{\infty,\infty}$ and $\widetilde{Q}_{\infty,\infty}$.

Open Question 1. *In the construction $Q_{\infty,\infty} = \Phi(\mathbf{B}_{\infty,\infty})$, is it the case that the $\ell_{\mathbf{B}_{\infty,\infty}}$-labels have the same interpretation as in Theorem 2, that is, for every $u, v \in Q_{\infty,\infty}$ we have*

$$\lim_{z\to\infty} \big(\mathrm{d}_{\mathrm{gr}}(z,u) - \mathrm{d}_{\mathrm{gr}}(z,v)\big) \;=\; \ell_{\mathbf{B}_{\infty,\infty}}(u) - \ell_{\mathbf{B}_{\infty,\infty}}(v) \quad ?$$

We next move to surgical considerations. Consider two independent copies of $\widetilde{Q}_{\infty,\infty}$ and glue them together along the boundary with coinciding roots to form a rooted quadrangulation of the plane denoted by $\mathfrak{Q}_\infty$ together with an two-sided infinite self-avoiding path. Also, consider the "closing" operation that consists in zipping the boundary of $\widetilde{Q}_{\infty,\infty}$ to get an infinite rooted quadrangulation $\mathscr{Q}_\infty$ with an infinite self-avoiding path on it.

Open Question 2. *Are the laws of $\mathfrak{Q}_\infty$ and $\mathscr{Q}_\infty$ absolutely continuous or singular with respect to the law of Q_∞? Study the self-avoiding walks obtained on them: in particular, are they diffusive?*

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