# On packing chromatic number of subcubic outerplanar graphs

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**Abstract**

Although it has recently been proved that the packing chromatic number is unbounded on the class of subcubic graphs, there exists subclasses in which the packing chromatic number is finite (and small). These subclasses include subcubic trees, base-3 Sierpiski graphs and hexagonal lattices. In this paper we are interested in the packing chromatic number of subcubic outerplanar graphs. We provide asymptotic bounds depending on structural properties of the outerplanar graphs and determine sharper bounds for some classes of subcubic outerplanar graphs.

**Keywords:** packing colouring, packing chromatic number, outerplanar graphs, subcubic graphs.

**AMS Subject Classification: 05C12, 05C15, 05C70**

## 1 Introduction

Throughout this paper, we consider undirected simple graphs only, and for definitions and notations not defined here we refer to [2].

Let $G$ be a graph and $c$ a vertex $k$-colouring of $G$, i.e., a mapping $c : V(G) \to \{1, 2, \ldots, k\}$. We say that $c$ is a *packing $k$-colouring* of $G$ if vertices coloured with the same colour $i$ have pairwise distance greater than $i$. The *packing chromatic number* of $G$, denoted by $\chi_\rho(G)$ is the smallest integer $k$ such that $G$ has a packing $k$-colouring; if there is no such integer $k$ then we set $\chi_\rho(G) = \infty$. For a class of graphs $\mathcal{C}$, we say that the packing chromatic number of $\mathcal{C}$ is finite if there exists a positive integer $k$ such that $\chi_\rho(G) \leq k$ for every graph $G \in \mathcal{C}$.

The concept of a packing colouring of a graph, introduced by Goddard et al. in [13] under the name broadcast colouring, is inspired by frequency planning in wireless systems, in which it emphasizes the fact that signals can have different powers, providing a model for the frequency assignment problem. The packing chromatic number of lattices has been studied by several authors: for the infinite square lattice $\mathbb{Z}^2$, Soukal and Holub in [19] proved that $\chi_\rho(\mathbb{Z}^2) \leq 17$,

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while Ekstein et al. in [7] showed that $12 \leq \chi_\rho(\mathbb{Z}^2)$. Recently, Martin et al. in [17] improve the bounds by showing that $13 \leq \chi_\rho(\mathbb{Z}^2) \leq 15$. For the infinite hexagonal grid $\mathscr{H}$, Fiala et al. in [10] showed that $\chi_\rho(\mathscr{H}) \leq 7$, Korže and Vesel in [16] proved that $\chi_\rho(\mathscr{H}) \geq 7$. Finbow and Rall in [11] proved that the infinite triangular grid $\mathscr{T}$ is not packing colourable, i.e., $\chi_\rho(\mathscr{T}) = \infty$. The packing chromatic number of the Cartesian product of some graphs was investigated in [3, 10, 15]. Also, the packing chromatic number has been studied fot further graph classes in [3, 4, 13, 14, 20]. The computational complexity has been also studied: determining whether a graph has packing chromatic number at most 4 is an NP-complete problem [13] and determining whether a tree has packing chromatic number at most $k$ (with a tree and $k$ on input) is also a NP-complete problem [9].

Sloper in [18] showed that the infinite complete ternary tree $T$ has $\chi_\rho(T) = \infty$ while any tree $T$ with $\Delta(T) \leq 3$ is packing 7-colourable, hence it is natural to ask if all graphs with maximum degree 3 (often so-called *subcubic graphs*) have finite packing chromatic number. This question was raised by Goddard et al. [13]. Recently, a second open question has been proposed about the packing chromatic number of $S(G)$, when $G$ is subcubic [6, 12] ($S(G)$ being the graph obtained from $G$ by subdividing each edge once). Very recently, Balogh, Kostochka and Liu [1] proved that, for any integer $k$, almost all cubic graphs of order $n$ and of girth at least $2k + 2$ have packing chromatic number greater than $k$, hence answering negatively the question of Goddard et al. However, explicit constructions known so far are only for subcubic graphs with packing chromatic number up to 14 [5, 12]. Some subclasses of subcubic graph were also under consideration, see e.g. [3, 4].

Outerplanar graphs form a class of structured graphs (containing the class of trees), which are generally easy to color. Our aim is to find some classes of subcubic outerplanar graphs, which have finite packing chromatic number. We define these classes by giving restrictions on their structure - number of faces of different types, or, equivalently, on their weak dual. Note that, when a graph is not connected, we can colour each component separately satisfying the distance constraints of a packing colouring and the resulting colouring is packing as well. Thus, throughout the rest of this paper, we will consider connected outerplanar graphs only.

The paper is organized as follows. Section 2 presents an upper bound for 2-connected subcubic outerplanar graphs without internal face, i.e. with the weak dual as a path. Then, in Section 3, we use results from Section 2 in order to determine asymptotic bounds for some larger classes of subcubic outerplanar graphs restricted by the number of (internal) faces. In Section 4, we improve bounds from Section 3 for some specific classes of subcubic outerplanar graphs with a specific structure. Finally, in the last section, we present lower bounds for the packing chromatic number of subcubic outerplanar graphs and give concluding remarks. Table 1 summarizes the main results of this paper.

### 1.1 Preliminaries

Let $G$ be a graph and $A \subset V(G)$. We denote $G - A$ the subgraph of $G$ after deletion of all vertices of $A$ from $G$ and all edges incident to some vertex of $A$ in $G$. We further denote $G[A]$ the subgraph of $G$ induced by $A$, or equivalently, $G[A] = G - (V(G) \setminus A)$. Specifically, for $x \in V(G)$,

| Condition on the subcubic outerplanar graph $G$ | $\ell$ | Section |
|---|---|---|
| $G$ is 2-connected with no internal face | 15 | 2 |
| $G$ is 2-connected with at most $k$ internal faces | $17 \times 6^{3k} - 2$ | 3 |
| $G$ is connected with at most $k'$ faces | $9 \times 6^{k'} - 2$ | 3 |
| $G$ is 2-connected with one internal face | 51 | 4 |
| $G$ is connected with no internal face and with the block graph a path | 305 | 4 |

Table 1: Classes of subcubic outerplanar graphs and values of $\ell$ for which every relevant graph $G$ satisfies $\chi_\rho(G) \leq \ell$.

$G - x$ denotes the subgraph of $G$ after deletion of $x$ and all edges incident to $x$ from $G$.

An *outerplanar graph* $G$ can be represented by a *boundary cycle* $C$ containing all vertices of $G$, with non-crossing chords dividing the interior of $C$ into *faces*. A face $F$ of $G$ is called an *internal face* if $F$ contains more than two chords of $G$, and an *end face* of $G$ if $F$ contains only one chord of $G$; note that all remaining edges of an end face belong to $C$.

The *weak dual* of $G$, denoted by $\mathcal{T}_G$, is the graph with the vertex set as the set of all faces of $G$ and the edge set $E(G) = \{F, F' | \, F$ and $F'$ have an edge in common $\}$. We denote by $u_F$ the vertex of $\mathcal{T}_G$ corresponding to the face $F$ of $G$ and sometimes we identify a face $F$ and the corresponding vertex $u_F$ of $\mathcal{T}_G$. It is well known that the weak dual of a connected outerplanar graph is a forest and of a 2-connected outerplanar graph is a tree. Note that an end face of an outerplanar graph $G$ corresponds to a leaf of $\mathcal{T}_G$ and that an internal face of $G$ corresponds to a vertex of degree at least 3 in $\mathcal{T}_G$. Obviously, every end face of a 2-connected outerplanar graph contains at least one vertex of degree 2.

For a graph $G$, the *block graph* of $G$, denoted by $\mathcal{B}_G$, is the graph where vertices of $\mathcal{B}_G$ represent all maximal 2-connected subgraphs of $G$ (usually called *blocks*) and two vertices of $\mathcal{B}_G$ are adjacent whenever the corresponding blocks share a cut vertex.

For any $G_1 \subset G$, let $N(G_1) = \{u \in V(G) | \; uv \in E(G)$ for some $v \in V(G_1)\}$ be the *neighbourhood* of $G_1$ in $G$. Specifically, if $G_1 = \{v\}$, let $N(v)$ denote the neighbourhood of $v$ in $G$. For $X, Y \subseteq V(G)$, a *shortest (X,Y)-path* is a shortest path in $G$ between some vertex of $X$ and some vertex of $Y$. If $X$ contains a vertex $u$ only, then we write $u$ instead of $\{u\}$. Let $d_G(u, v)$ denote the *distance between two vertices* $u$ and $v$ in $G$, i.e., the length of a shortest $(u, v)$-path. Analogously, $d_G(X, Y)$ denote the *distance between* $X$ *and* $Y$, i.e., the length of a shortest $(X, Y)$-path in $G$.

In our proofs we will use the following statement presented by Goddard et al. in [13].

**Proposition A [13].** *Let $k$ be a positive integer. Then*

*i) Every cycle has packing chromatic number at most 4;*

*ii) There is a packing colouring of $P_\infty$ with colours $\{k, k+1, \ldots, 3k+2\}$;*

*iii) If $k \geq 34$, then there is a packing colouring of $P_\infty$ with colours $\{k, k+1, \ldots, 3k-1\}$.*

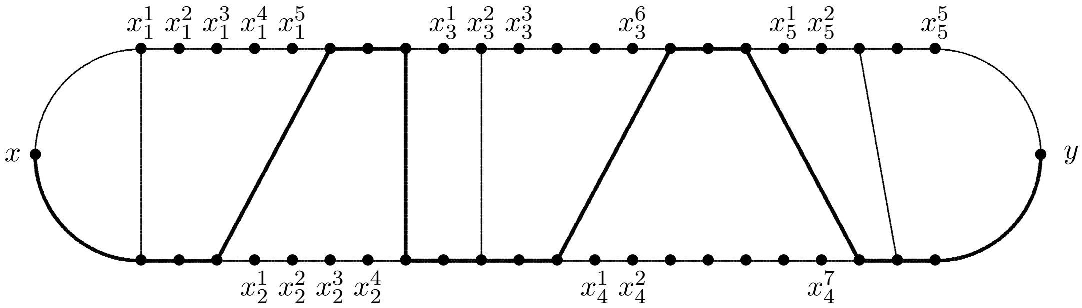


Figure 1: Structure of $G$ in Lemma 2.

## 2 2-connected subcubic outerplanar graphs with the weak dual a path

The following observation will be used in order to construct some useful shortest path in 2-connected subcubic outerplanar graphs. Moreover, it gives a description of 2-connected outerplanar graphs with the weak dual of these graphs.

**Observation 1.** *Let $G$ be a 2-connected outerplanar graph that is not a cycle. Then $G$ contains at least two end faces. Moreover, if $G$ contains no internal face, then $G$ has exactly two end faces.*

**Proof.** Considering the weak dual, $\mathcal{T}_G$ is a tree by connectedness of $G$. Since every nontrivial tree has at least two leaves and each leaf of $\mathcal{T}_G$ corresponds to some end face of $G$, each 2-connected outerplanar graph that is not a cycle contains at least two end faces. In particular, if $G$ has no internal face, then $\mathcal{T}_G$ is a path (the converse also holds), implying that $G$ has exactly two end faces. ■

We begin this section with the following lemma. This lemma will be used in Sections 3 and 4.

**Lemma 2.** *Let $G$ be a 2-connected outerplanar graph with no internal face and with $\Delta(G) \leq 3$. Let $x, y$ be a pair of vertices of degree 2 in $G$ such that $x$ belong to one of the end faces of $G$ and $y$ to the other one, and let $P$ be a shortest $x, y$-path in $G$. Then $G - P$ can be packing coloured with colours $\{1, 2, 3, 4\}$.*

**Proof.** Let $C$ denote the boundary cycle of $G$. If there is no chord in $G$, then $G$ is a cycle and, by Proposition A.i) , $\chi_\rho(G) \leq 4$.

Thus we may assume that $C$ contains some chords in $G$. Note that $C - P$ is not necessarily connected, but each component of $C - P$ is a path. Let $D_i$, $i = 1, \ldots, k$, denote the components of $C - P$ in an ordering from $x$ to $y$, and let $l_i$ denote the length of $D_i$. We further denote the vertices of each $D_i$ by $x_i^1, x_i^2, \ldots, x_i^{l_i}$ in an ordering starting from a vertex of $D_i$ which is closest to $x$ in $D_i$. The described structure is shown in Fig. 1, where the thick $x, y$-path depicts $P$.

We colour each component $D_i$ of $C - P$ with a pattern $1, 2, 1, 3$ starting from $x_i^1$ ($i = 1, 2, \ldots, k$), i.e., for each odd $j$, $x_i^j$ is coloured with colour 1, for each $j$ divisible by 4, $x_i^j$ obtains colour 3, and, for each even $j$ not divisible by 4, we colour vertex $x_i^j$ with colour 2, $j = 1, 2, \ldots, l_i$. We denote by $\chi$ the defined colouring. Since $P$ is a shortest path and $\Delta(G) \leq 3$, there is no collision between any pair of vertices coloured with colour 1 or 2, respectively. Suppose to the contrary that there is a pair of clashing vertices $a$ and $b$ coloured with colour 1. Clearly $a$ and $b$ belong to the same component $D_i$ of $C - P$ and $ab \in E(G) \setminus E(C)$ by the definition of $\chi$. But then we get a contradiction with the fact that $D_i$ is a path. Now suppose that there is a pair of clashing vertices $a$ and $b$ coloured with colour 2, i.e., $d_G(a, b) \leq 2$. Again, $a$ and $b$ must belong to the same component $D_i$ of $C - P$, otherwise $d_G(a, b) > 2$ and we obtain a contradiction. Analogously as for colour 1 we can show that $ab \notin E(G)$. Thus $a$ and $b$ must have a common neighbour $c$ in $G$. From the definition of $\chi$, $c \notin V(D_i)$, and $c \notin P$ since $\Delta(G) \leq 3$, a contradiction with the existence of $c$.

Therefore the only possible collision in the defined colouring $\chi$ could be between vertices coloured with colour 3. Analogously as for colours 1 and 2, any pair of clashing vertices $a$ and $b$ cannot be at distance one or two apart. Therefore any such collision happens for $a$ and $b$ with $d_G(a, b) = 3$. We will check and modify collisions in the defined colouring $\chi$ of the components $D_1, D_2, \ldots, D_k$ of $C - P$ one-by-one starting from $D_1$ and from the vertex $x_i^1$ in each $D_i$. Note that, in each step of the modification process, we check the modified colouring, not the original one. The following possible collisions can occur:

*Case 1:* $a$ and $b$ belong to different components $D_i$ and $D_j$ of $C - P$, $i, j \in \{1, 2, \ldots, k\}$. Since $d_G(a, b) = 3$, (up to a symmetry) $a = x_i^{l_i}$, $b$ belongs to the chord of $D_j \cup P$ which is closest to $a$, and $j = i+1$, otherwise we get $d_G(a, b) > 3$. Then we can modify the colouring $\chi$ of $D_j$ by recolouring vertices $x_j^s$, $s = 3, \ldots, l_j - 1$, with $\chi(x_j^s) := \chi(x_j^{s+1})$ and we set $\chi(x_j^{l_j}) \in \{1, 2, 3\}$ depending on the continuation of the pattern $1, 2, 1, 3$ in $D_j$.

*Case 2:* $a$ and $b$ belong to the same component $D_i$. Since $d_G(a, b) = 3$, $a$ and $b$ must belong to consecutive chords of $P \cup D_i$ and there is no vertex between these chords on $P$. We call such a pair of vertices coloured with colour 3 a *critical pair*. Consider a critical pair $a$ and $b$ such that $a = x_i^m$, $b = x_i^n$, $m < n$, and that there is no critical pair $a'$ and $a$ with $a' = x_i^o$, $o < m$. Then we modify the colouring of the vertices of $D_i$ starting at vertex $a$ by $\underline{3}, 1, 4, 1, \underline{2}, 1, 3, 1, 2, \ldots$ instead of $\underline{3}, 1, 2, 1, \underline{3}, 1, 2, 1, 3 \ldots$, i.e., we recolour the vertex $x_i^{m+2}$ with colour 4 and we switch colours 2 and 3 of the vertices $x_i^j$ for even $j > m+2$. Note that the underlined colours represent the critical pair $a$ and $b$. It is easy to verify that vertices coloured with colour 4 are mutually at distance more than 4 apart, implying that there is no collision between any pair of vertices coloured with colour 4.

After these modifications we obtain a colouring of all the vertices of $G - P$ with colours $\{1, 2, 3, 4\}$ satisfying the distance constraints of a packing colouring.

■

**Theorem 3.** *If $G$ is a 2-connected subcubic outerplanar graph with no internal face, then* $\chi_\rho(G) \leq 15$.

**Proof.** Let $x, y$ be any pair of vertices of degree 2 in $G$ belonging to distinct end faces of $G$. Let $P$ be a shortest $x, y$-path in $G$. By Lemma 2, the vertices of $V(G) \setminus V(P)$ can be coloured with colours from $\{1, 2, 3, 4\}$. Then the colouring can be completed in a packing 15-packing colouring of $G$ by colouring the vertices along the path $P$ starting at $x$ and using a packing colouring of the infinite path (since $P$ is a shortest path in $G$, then the distance between any pair of vertices of $P$ is the same on $P$ and on $G$). For this, we repeat the pattern $Q$ with colours from $\{5, \ldots, 15\}$ of length 36 along the vertices of $P$ starting at $x$:

$$5, 6, 7, 9, 13, 12, 5, 8, 6, 10, 7, 11, 5, 9, 14, 6, 8, 15, 5, 7, 13, 10, 6, 11, 5, 8, 9, 7, 12, 6, 5, 14, 10, 15, 8, 11$$

It is easy to check that any two colours $i$ in this repeating sequence are separated by at least $i$ integers. ■

Note that the previous pattern was found by a computer search.

# 3 Asymptotic results for subcubic outerplanar graphs

The main goal of this paper is to study the finiteness of the packing chromatic number of subcubic outerplanar graphs, i.e, we ask whether the packing chromatic number of an outerplanar graph with maximum degree at most 3 depends on the order of the graph or not. In this section we prove that, for any 2-connected outerplanar graph with a fixed number of internal faces and for any connected outerplanar graph with a fixed number of faces, the packing chromatic number does not depend on the order of the graph.

We begin this section by proving the following useful lemma that will also be used in Section 4. We recall that the weak dual of a 2-connected outerplanar graph is a tree and that $u_F$ is the vertex of the weak dual corresponding to the face $F$.

**Lemma 4.** *There exists a packing colouring of $P_\infty^+$ with colours $\{5, \ldots, 15\}$ such that the first vertex $x$ along the path is at distance at least $\lceil (i-5)/2 \rceil$ of any vertex of colour $i$.*

**Proof.** By considering the pattern from the proof of Theorem 3 starting at $x$, we can easily check that the first six vertices of $P_\infty^+$ satisfy the property. Since the colours used in the pattern from the proof of Theorem 3 are bounded by 15, the other vertices (other than the first six vertices) satisfy the property as well. ■

For positive integers $i$, $j$ and $k$, let $r_{i,j}^k \in \mathbb{Z}$ such that $r_{i,j}^k \equiv i - j \pmod{k}$ with minimum absolute value. The value $|r_{i,j}^k|$ corresponds to the distance between two vertices $i$ and $j$ in a cycle $C_k$ with vertex set $\{0, \ldots, k-1\}$ (the vertices are enumerated along the cycle). A subset of vertices $A$ of a graph $G$ is a *cycle-distance-preserved* set if there exists an ordering $v_A^0, \ldots, v_A^{|A|-1}$ of the vertices of $A$ satisfying $d_G(v_A^j, v_A^{j'}) \geq |r_{j,j'}^{|A|}|$, for integers $0 \leq j < j' \leq |A| - 1$.

**Lemma 5.** *For any positive integers $k$ and $n > 2$, there exists a packing coloring of the cycle $C_n$ with colours from $\{k, \ldots, 6k+4\}$.*

**Proof.** Let $C$ be a cycle of length $n$. First, if $n \leq 5k+5$, then we can colour each vertex of $C$ with a different colour from $\{k, \ldots, 6k+4\}$.

Second, if $5k+5 < n \leq 6k+4$, then we colour $3k$ consecutive vertices of $C$ with colours $k, \ldots, 3k-1, k, \ldots, 2k-1$, and colour the remaining $n-3k \leq 3k+4$ vertices of $C$ with mutually distinct colours from $\{3k, \ldots 6k+4\}$.

Third, suppose $n > 6k+4$. By Proposition A.ii), we can colour $P_\infty$ with colours from $\{k, \ldots, 3k+2\}$. Let $P'$ be any subpath of $C_n$ on $3k+2$ vertices. Since the distance between the two ends of $C_n - P'$ is at least $3k+3$ in $C_n - P'$ and exactly $3k+3$ in $C$, we can colour the vertices of $C - P'$ with the colours $\{k, \ldots, 3k+2\}$ (using Proposition A.ii) ) and the vertices of $P'$ with mutually distinct colours from $\{3k+3, \ldots, 6k+4\}$. ■

A subset of vertices $A$ of a graph $G$ is *decomposable* into $r$ cycle-distance-preserved sets if there exist $r$ sets of vertices $A_1$, $A_2$, $\ldots$, $A_r$, such that $A_1 \cup \ldots \cup A_r = A$ and for each integer $i$, $A_i$ is a cycle-distance-preserved set. The following lemma will be useful in order to prove Theorems 7 and 8.

**Lemma 6.** *Let $G$ be graph and let $A \subseteq V(G)$ be a subset decomposable into $r$ cycle-distance-preserved sets. The vertices of $A$ can be packing-coloured with colours $\{k, \ldots, 6^r(k+1)-2\}$, for any positive integer $k$.*

**Proof.** We proceed by induction on $r$. For $r=1$, since $A$ is a cycle-distance-preserved set, by Lemma 5, we can colour the vertices of $A$ with colours $\{k, \ldots, 6k+4\}$. Now suppose that a subset $A \subset V(G)$ is decomposable into $r+1$ cycle-distance-preserved sets. Using induction hypothesis we can colour the vertices of $A_1$, $\ldots$, $A_r$ with colours $\{k, \ldots, 6^r(k+1)-2\}$. For the vertices of $A_{r+1}$, by Lemma 5, we can use colours $\{6^r(k+1)-1, \ldots, k'\}$, where $k' = 6(6^r(k+1)-1)+4 = 6^{r+1}(k+1)-2$. Note that we do not need to change colours of the vertices from $\cup_{i=1}^{r}(A_i \cap A_{r+1})$ (in the case it is not empty). ■

The following theorem is one of our main results. It can be used in order to prove that some subcubic outerplanar graphs have finite packing chromatic number.

**Theorem 7.** *If $G$ is a 2-connected outerplanar graph with $\Delta(G) \leq 3$ and with $r$ internal faces, then $\chi_\rho(G) \leq 17 \times 6^{3r} - 2$.*

**Proof.** Let $F_1, \ldots, F_r$ denote the $r$ distinct internal faces of $G$, and $u_{F_1}, \ldots u_{F_r}$ the corresponding vertices of $\mathcal{T}_G$ (note that each $u_{F_i}$ has degree at least 3 in $\mathcal{T}_G$). Then, removing the vertices $u_{F_1}, \ldots, u_{F_r}$ from $\mathcal{T}_G$, we obtain a union of disjoint paths. For two vertices $u_{F_i}$ and $u_{F_j}$, $1 \leq i < j \leq r$, the path with one end vertex adjacent to $u_{F_i}$ and the other one to $u_{F_j}$ is denoted by $U_{i,j}$. For any $u_{F_i}$, $i = 1, \ldots, r$, the paths with one end vertex of degree 1 a the other one adjacent to $u_{F_i}$, are denoted by $U_i^1, \ldots, U_i^{\ell_i}$, where $\ell_i$ is the number of such paths for $u_{F_i}$. Note that some of the paths $U_{i,j}$,$U_i^j$ may be trivial or empty.

For any $i$, $1 \leq i \leq r$, let $B_i = N(F_i) \setminus \bigcup_{i=1}^{r} V(F_i)$, and let $B = \bigcup_{i=1}^{r}(F_i \cup B_i)$. Let $i$ and $j$ be integers such that $1 \leq i \leq r$, $1 \leq j \leq \ell_i$. Consider an end face $\hat{F}_i^j$ in $G$ corresponding to an end vertex of $V(U_i^j)$ of degree one in $\mathcal{T}_G$. We denote by $y_i^j$ a vertex of $\hat{F}_i^j$ of degree 2 (note that such a vertex always exists) and by $P_i^j$ a shortest $(B_i, y_i^j)$-path in $G$. Let $(p_i^j)_1, (p_i^j)_2, \dots$ denote vertices of $P_i^j$ in an ordering starting from the vertex of $B_i$.

Choose any vertex $z$ of $\mathcal{T}_G$ and let $\overrightarrow{\mathcal{T}_G}$ be the digraph obtained from $\mathcal{T}_G$ by replacing each edge $uv \in E(\mathcal{T}_G)$ satisfying $d_{\mathcal{T}_G}(u,z) < d_{\mathcal{T}_G}(v,z)$ with an arc from $u$ to $v$.

Now we consider the paths $U_{i,j}$ in $\mathcal{T}_G$. Let $i, j$ be positive integers such that $1 \leq i < j \leq r$ and $U_{i,j}$ is defined and has length at least one. Let $P_{i,j}$ be a shortest $(B_i, B_j)$-path and $p_{i,j}^1, p_{i,j}^2 \dots, p_{i,j}^{l_{i,j}}$ its vertices in an ordering starting from the vertex of $B_i$, if $d_{\mathcal{T}_G}(u_{F_i}, z) < d_{\mathcal{T}_G}(u_{F_j}, z)$, or from the vertex of $B_j$ otherwise (by $l_{i,j}$ we mean the order of $P_{i,j}$).

Let $P = \left( \bigcup_{1 \leq i \leq r} \bigcup_{1 \leq j \leq \ell_i} V(P_i^j) \right) \bigcup \left( \bigcup_{1 \leq i < j \leq k} V(P_{i,j}) \right)$.

*Step 1:* Colouring the vertices of $V(G) \setminus (B \cup P)$ with colours $\{1,2,3,4\}$.

We colour the vertices of $V(G) \setminus (B \cup P)$ by colouring each connected component (one by one) of $G - (B \cup P)$ in the same way as in the proof of Lemma 2, i.e., we use the pattern 1, 2, 1, 3. Note that the distance between any two vertices from $V(G) \setminus (B \cup P)$ in two different connected components of $G - B$ is at least 5. Moreover, we proceed as in the proof of Lemma 2 to avoid clashing vertices of colour 3, i.e., we use colour 4.

*Step 2:* Colouring vertices of $P$.

Let $i$, $j$, $i'$ and $j'$ be integers such that $U_{i,j}$ and $U_{i'}^{j'}$ are defined.

For the vertices of $P_{i'}^{j'}$, we use the pattern of Theorem 3 and Lemma 4 starting at the vertex $(p_{i'}^{k'})_3$. For the vertices of $P_{i,j}$, we use the pattern of Theorem 3 and Lemma 4, starting at the vertex $p_{i,j}^3$ and finishing at the vertex $p_{i,j}^{k_{i,j}-3}$. Note that every vertex of $V(\overrightarrow{\mathcal{T}_G})$ has in-degree at most one. This property, along with Lemma 4, ensure us that a vertex coloured with colour $a$ in $P_{i,j}$, $a \in \{5, \dots, 15\}$, is at distance at least $a+1$ from any other vertex coloured by $a$ in $P_{\bar{i},\bar{j}}$, for $1 \leq \bar{i} < \bar{j} \leq r$.

*Step 3:* Colouring the remaining vertices of $G$.

Let $w_{i,j}$ be a vertex among $\{p_{i,j}^2, p_{i,j}^{l_{i,j}-2}\}$ at distance 2 from a vertex of $V(F_i)$ (when $U_{i,j}$ is defined). Let $D_i$ be the set $\{(p_i^{j'})_2 | \ 1 \leq j' \leq \ell_i\} \cup \{w_{i,j} | \ U_{i,j}$ is defined, $1 \leq j \leq k\}$. Since the sets $V(F_i)$, $B_i$ and $D_i$ , $1 \leq i \leq r$, are cycle-distance-preserved sets, the set $\bigcup_{i=1}^r V(F_i) \bigcup_{i=1}^r B_i \bigcup_{i=1}^r D_i$ is decomposable into $3r$ cycle-distance-preserved sets. Hence, using Lemma 6, the remaining uncoloured vertices can be coloured with colours $\{16, \dots, 17 \times 6^{3r} - 2\}$.

■

Sloper in [18] defined an *expandable broadcast-colouring* of a complete binary tree $T$ as a colouring $c$ of $V(T)$ with colours $1, 2, \dots, 7$ such that:

(*i*) $\forall u, v \in V(T)$ $c(u) = c(v) \Rightarrow d_T(u,v) > c(u)$,

(*ii*) the root $x$ of $T$ has colour 1,

(*iii*) all vertices at even distance from $x$ have colour 1,

(*iv*) every vertex of colour 1 has at least one child of colour 2 or 3,

(*v*) $c(u) = 6, c(v) = 7 \Rightarrow d_T(u,v) \geq 5$,

(*vi*) $c(u) \in \{4,5,6,7\} \Rightarrow u$'s children each have children coloured with 2 and 3.

The following statement is true for a more general class of graphs than in Theorem 7 since it gives an upper bound for all connected outerplanar graphs (not necessarily 2-connected). However, since the parameter is the number of faces, the bound is weaker than the bound in Theorem 7.

**Theorem 8.** *If $G$ is a connected outerplanar graph with $r$ faces and $\Delta(G) \leq 3$, then $\chi_\rho(G) \leq 9 \times 6^r - 2$.*

**Proof.** Let $F_1, \dots, F_r$ denote all $r$ bounded faces of $G$. The graph $G - \bigcup_{i=1}^{r} V(F_i)$ consists of components $O_1, O_2, \dots, O_s$ such that each $O_j$ $(j = 1, \dots, s)$ is a tree. And, since $G$ is subcubic, each $O_j$ is subcubic as well. In the weak dual $\mathcal{T}_G$ of $G$, choose arbitrary vertex $z$ and let $F_z$ denote a face corresponding to $z$ in $G$. We colour the vertices of $G$ in two steps.

*Step 1:* Colouring the vertices of $O$ with colours $\{1, \dots, 7\}$.

Consider each component $O_i$ of $O$ separately $(i = 1, \dots, s)$ and let $z_i$ denote a vertex of $O_i$ closest to $F_z$. Then we use the expandable broadcast-colouring to colour vertices of $O_i$ with colours $1, 2, \dots, 7$. Note that $z_i$ has colour 1, the neighbour(s) of $z_i$ in $O_i$ has (have) colour 2 (and 3), vertices of $O_i$ at distance 2 from $z_i$ are coloured with colour 1 and vertices of $O_i$ at distance 3 have colours 4 and 5. Obviously, since $G$ is subcubic, $z_i$ is at distance at least 3 from any vertex of any $O_j \neq O_i$. Hence the above defined colouring satisfies the distance constraints of a packing colouring.

*Step 2:* Colouring the remaining vertices of $G$.

The sets $V(F_1), \dots, V(F_r)$ are cycle-distance-preserved sets. Hence, by Lemma 6, the remaining uncoloured vertices can be coloured with colours $\{8, \dots, 9 \times 6^r - 2\}$.

■

# 4 Some 2-connected outerplanar graphs with finite packing chromatic number

In this section we consider some special classes of subcubic outerplanar graphs for which we can decrease the upper bound on the packing chromatic number given in Theorem 7.

**Theorem 9.** *If $G$ is a 2-connected subcubic outerplanar graph with exactly one internal face, then $\chi_\rho(G) \leq 51$.*

**Proof.** Suppose $G$ is a 2-connected outerplanar graph with exactly one internal face and with $\Delta(G) \leq 3$. Let $C$ denote the boundary cycle of $G$ and $F$ the internal face of $G$. Let $C' = \{v_0, \dots, v_{N-1}\}$ denote the set of vertices which belong to $F$, with $v_i$ adjacent to $v_{i+1}$, for $0 \leq i < N$. When $N$ is odd, we suppose that $v_{N-1}$ is a vertex with $d_G(v_{N-1}) = 2$. Such a vertex exists since the number of vertices of degree 3 in $C'$ is even. By removing the edges of $C \cap F$ from $G$, and by removing the isolated vertices from the resulting graph, we obtain a graph $G'$ which is a disjoint union of 2-connected outerplanar graphs having no internal face.

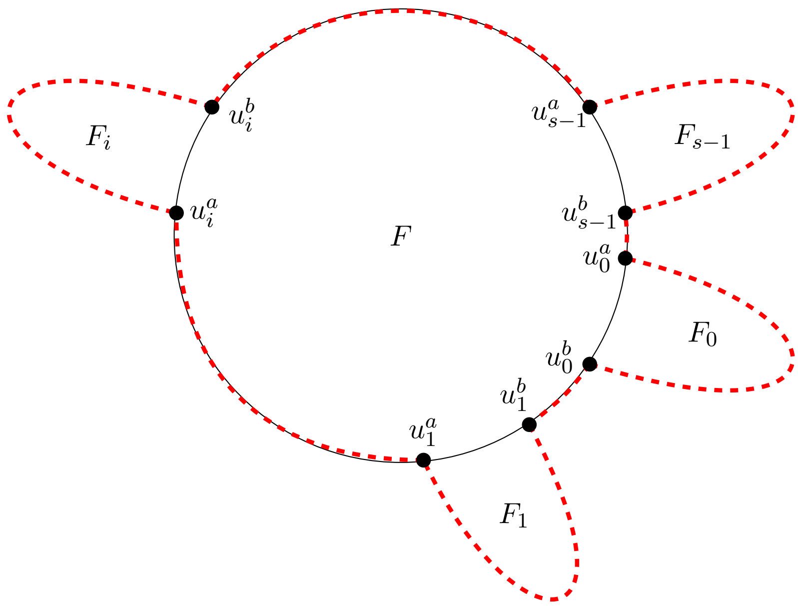


Figure 2: A 2-connected outerplanar graph with one internal face and its different subgraphs ($C$ is represented by a dashed line).

Let $F_0$, …, $F_{s-1}$ denote the 2-connected components of $G'$, enumerated in the clockwise order along the cycle $C$ in $G$ (for details, see Fig. 2). Note that, since any $F_i$ contains no internal face, each $F_i$ has exactly two end faces or $F_i$ is a cycle. Let $i$ be an integer with $0 \leq i < s$, and let $u_i^a$ and $u_i^b$ denote the two adjacent vertices of degree 3 in $G$ which belong to $V(F_i) \cap C'$, as it is depicted in Fig. 2. Let $y_i$ be a vertex of degree 2 in the end face of $F_i$ which does not contain $u_i^a$ (for $F_i$ a cycle we denote by $y_i$ a vertex of $F_i$ at maximum distance from $F$ in $G$). Let $x_i \in \{u_i^a, u_i^b\}$ denote a vertex at minimal distance from $y_i$. Finally, let $P_i$ be a shortest $(x_i, y_i)$-path in $G$. We further denote the vertices of each $P_i$ by $x_i, p_i^1, p_i^2, \ldots, y_i$ in an ordering starting from $x_i$. Let $D_i^1$, …, $D_i^{k_i}$ denote the connected components of $F_i - P_i$ with $D_i^1$ containing a vertex among $u_i^a$ and $u_i^b$ and with $D_i^k$ being at larger distance than $D_i^{k-1}$ from $x_i$, $2 \leq k \leq k_i$.

The proof will be organized as follows. First, we will colour the vertices of $C'$. Second, we will colour the vertices of $\cup_{0 \leq i < s} F_i - P_i$ with colour 1, 2 and 3. Note that the obtained colouring does not necessarily satisfy the distance constraints of a packing colouring of $G$. Third, we will modify colouring of some vertices of $F_i - P_i$ ($i = 0, \ldots, s-1$) to save colour 1 for some vertices of the paths $P_i$ and to prevent colisions in colour 2. Fourth, we will recolour some vertices of $F_0, \ldots, F_{s-1}$ with colour 4 in order to satysfy the distance constraints of a packing colouring. Finally, we will colour vertices of the paths $\cup_{0 \leq i < s} P_i \setminus \{x_i\}$.

*Step 1:* Colouring the vertices of $C'$ with colours 1, 2, $29, 30, \ldots, 45$.

For integers $j$, $j'$, let $r_{j,j'}$ be an integer such that $r_{j,j'} \equiv j - j' \pmod{N}$ and $-\lfloor N/2 \rfloor \leq r_{j,j'} \leq \lfloor N/2 \rfloor$. Note that $d_G(v_j, v_{j'}) = |r_{j,j'}|$. We begin with a partitioning of $C'$ into five

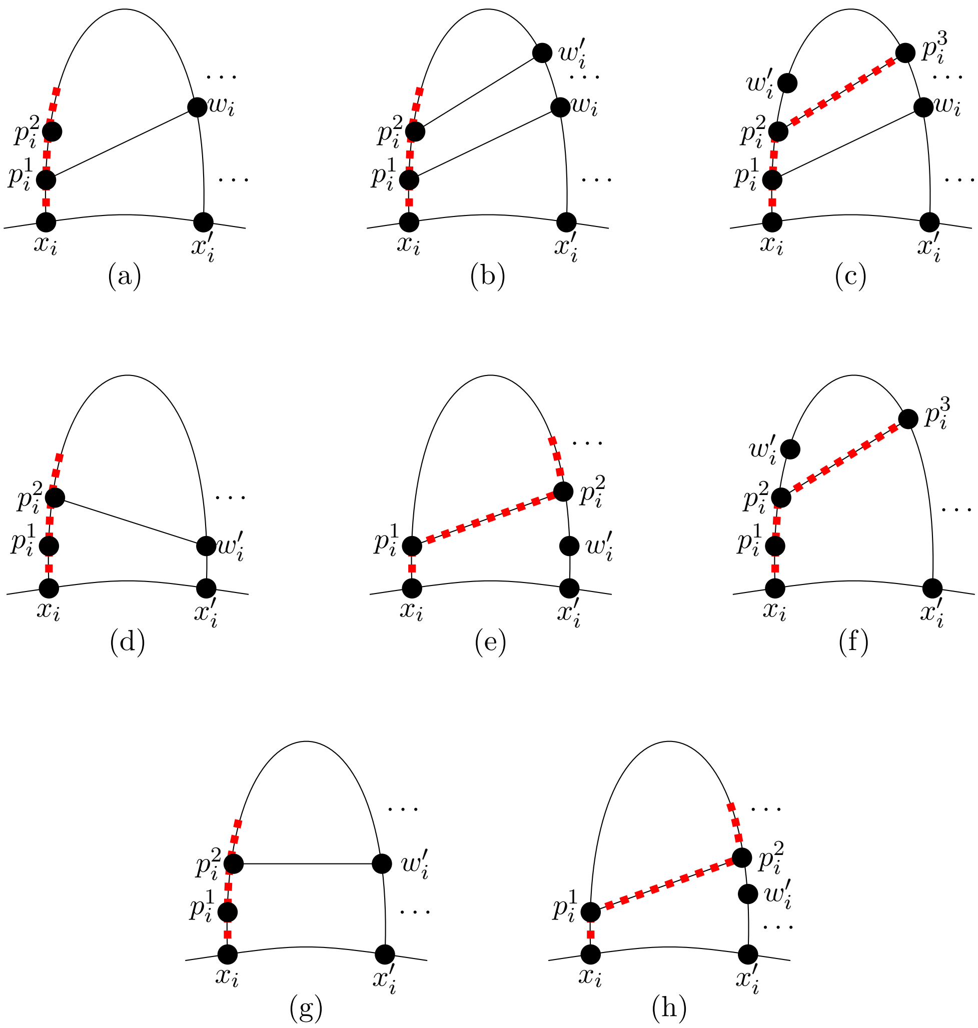


Figure 3: Eight configurations in step 3 ($P_i$ is represented by a dashed line).

subsets: $C_1' = \{v_j | \ j \equiv 0 \pmod 2, \ 0 \le j < N\}$, $C_2' = \{v_j | \ j \equiv 1 \pmod 4, \ 0 \le j < N\}$, $C_3' = \{v_j | \ j \equiv 3 \pmod{12}, \ 0 \le j < N\}$, $C_4' = \{v_j | \ j \equiv 7 \pmod{12}, \ 0 \le j < N\}$ and $C_5' = \{v_j | \ j \equiv 11 \pmod{12}, \ 0 \le j < N\}$. Let $m_k$ denote the vertex with largest index in $C_k'$, for $k \in \{1, 2, 3, 4, 5\}$. We use the following patterns to colour the vertices of $C'$.

1. if $|C_1'| \equiv 0 \pmod 2$ (or $|C_1'| \equiv 1 \pmod 2$), then we colour all vertices of $C_1'$ (or $C_1' \setminus \{m_1\}$, respectively) with colour 1;
2. we colour all vertices of $C_2' \setminus \{m_2\}$ with colour 2;
3. if $|C_3'| \equiv 0 \pmod 4$ (or $|C_3'| \equiv 1 \pmod 4$, respectively), then we use the pattern $29, 30, 35, 36, \ 29, 30, 35, 36, \dots, 29, 30, 35, 36$ to colour the vertices of $C_3'$ (or $C_3' \setminus \{m_3\}$, respectively);

   if $|C_3'| \equiv 2 \pmod 4$ (or $|C_3'| \equiv 3 \pmod 4$), then we use the pattern $29, 30, 35, 36, \ 29, 30, 35, 36, \dots, 29, 30, 35, \ 29, 30, 36$ to colour the vertices of $C_3'$ (or $C_3' \setminus \{m_3\}$, respectively);

4. if $|C'_4| \equiv 0 \pmod 4$ (or $|C'_4| \equiv 1 \pmod 4$), then we use the pattern $31, 32, 37, 38, 31, 32, 37, 38, \ldots, 31, 32, 37, 38$ to colour the vertices of $C'_4$ (or $C'_4 \setminus \{m_4\}$, respectively); if $|C'_4| \equiv 2 \pmod 4$ (or $|C'_4| \equiv 3 \pmod 4$), then we use the pattern $31, 32, 37, 38, 31, 32, 37, 38, \ldots, 31, 32, 37, 31, 32, 38$ to colour the vertices of $C'_4$ (or $C'_4 \setminus \{m_4\}$, respectively);
5. if $|C'_5| \equiv 0 \pmod 4$ (or $|C'_5| \equiv 1 \pmod 4$), then we use the pattern $33, 34, 39, 40, 33, 34, 39, 40, \ldots, 33, 34, 39, 40$ to colour the vertices of $C'_5$ (or $C'_5 \setminus \{m_5\}$, respectively); if $|C'_5| \equiv 2 \pmod 4$ (or $|C'_5| \equiv 3 \pmod 4$), then we use the pattern $33, 34, 39, 40, 33, 34, 39, 40, \ldots, 33, 34, 39, 33, 34, 40$ to colour the vertices of $C'_5$ (or $C'_5 \setminus \{m_5\}$, respectively);
6. when it is necessary, we use the colours 41, 42, 43, 44, 45 to colour the vertices of $\{m_k | \ 1 \leq k \leq 5\}$.

One can check that, for any pair of vertices $u, v$ of $C'$ with the same colour $k$, $d_G(u, v) > k$. For example, in the pattern $29, 30, 35, 36$ of length four, two vertices with the same colour are at distance at least 48, since the pattern has length four and we colour vertices with the same remainder modulo 12. The same goes for the pattern $29, 30, 35, 29, 30, 36$ of length six. Note that, for every pair of vertices $(u_i^a, u_i^b)$, at least one of them is coloured with 1.

*Step 2:* Colouring the vertices of $F_i - P_i$ with colours 1, 2 and 3, for every $i = 0, \ldots, s-1$.

Let $x'_i$ be the vertex among $u_i^a$ and $u_i^b$ different from $x_i$. Let $l_i$ be the order of $D_i^1$ and let $x'_i, x_i^1, \ldots, x_i^{l_i-1}$ be the vertices of $D_i^1$ in an ordering starting from $x'_i$. If $x'_i$ is coloured with colour 1, then we use the pattern $3, 1, 2, 1$ to colour the vertices $x_i^1, \ldots, x_i^{l_i-1}$. If $x'_i$ is not coloured with colour 1, then we use the pattern $1, 3, 1, 2$ to colour the vertices $x_i^1, \ldots, x_i^{l_i-1}$. Analogously as in the proof of Lemma 2, we colour vertices of $D_i^j$ ($j = 2, 3, \ldots, k_i$) using the pattern $1, 2, 1, 3$ starting from the vertex of $D_i^j$ at shortest distance from $C'$. At this step we do not change the colouring in order to avoid clashing vertices of colour 3.

*Step 3:* Recolouring some vertices in $F_i$, for every $i = 0, \ldots, s-1$.

In this step we deal with possible collisions in colour 2 between vertices of $F$ and vertices of $D_i^1$ at distance 2 from $F$. We also change colours of neighbours of $p_i^2$ coloured with 1 since, in Step 5, we will colour $p_i^2$ with 1 for reducing the number of colours used for the whole graph $G$.

Since we used the patterns $1, 3, 1, 2$ and $3, 1, 2, 1$ to colour the vertices of $V(D_i^1) \setminus \{x_i\}$, no vertex at distance 2 from $x'_i$ has colour 2. For any $i = 0, \ldots, s-1$, let $w_i$ denote the vertex of $F_i - P_i$ at distance 2 from $x_i$.

Case i) *$w_i$ has colour 2.* First suppose that $p_i^2$ has no neighbour of colour 1 (see Fig. 3(a)) or $p_i^2$ has a neighbour $w'_i$ with colour 1 in $D_i^1$ (see Fig. 3(b)). In both possibilities we recolour the vertices $x_i^j = w_i, x_i^{j+1}, x_i^{l_i-1}$ of $D_i^1$ with $\underline{4}, 2, 1, 3, 1, 2, 1 \ldots$ instead of $\underline{2}, 1, 3, 1, 2, \ldots$. Note that the underlined colours belong to the vertex $w_i$.

Now we assume that $p_i^2$ has a neighbor $w'_i$ of colour 1 which does not belong to $D_i^1$. Thus $p_i^2 p_i^3$ is a chord (see Fig. 3(c)). We recolour the vertices $x_i^j = w_i, x_i^{j+1}, \ldots, x_i^{l_i-1}$ of $D_i^1$ with $\underline{4}, 1, 2, 1, 3, \ldots$ instead of $\underline{2}, 1, 3, 1, 2, \ldots$ (the underlined colours belong to $w_i$), and the vertices of $D_i^2$ with pattern $\underline{2}, 1, 3, 1, \ldots$ instead of $\underline{1}, 2, 1, 3, \ldots$ (the underlined colours belong to $w_i'$). Note that if $w_i'$ was defined (had colour 1), the colour 1 was changed.

Case ii) $w_i$ *does not have colour* 2. If $p_i^2$ has no neighbour $w_i'$ of colour 1, then we do not modify the colouring of $D_i^1$ in this step. Suppose that $p_i^2$ has a neighbor $w_i'$ of colour 1. Suppose that $w_i'$ is a neighbor of $x_i'$ (see Fig. 3(d,e)). Clearly $x_i'$ does not have colour 1 and $x_i$ has colour 1 (by Step 1). Then we modify the path $P_i$ by replacing vertex $x_i$ with $x_i'$ and $p_i^1$ with $w_i'$ and recolour the modified path $P_i$ with pattern $3, 1, 2, 1, 3, \ldots$. Now suppose that $w_i'$ is at distance at least 2 from $x_i'$ and that $p_i^2 w_i'$ and $p_i^1 p_i^2$ are not chords (see Fig. 3(f)). We recolour vertices of $D_2^i$ with pattern $\underline{2}, 1, 3, 1, \ldots$ instead of $\underline{1}, 2, 1, 3, \ldots$. If $w_i'$ is at distance at least 2 from $x_i'$ and $p_i^2 w_i'$ is a chord (see Fig. 3(g)), we recolour vertices $x_i^j = w_i', x_i^{j+1}, \ldots x_i^{l_i-1}$ of $D_1^i$ with $\underline{4}, 1, 2, 1, 3, \ldots$ instead of $\underline{1}, 2, 1, 3, \ldots$ or $\underline{1}, 3, 1, 2, \ldots$. Finally, if $w_i'$ is at distance at least 2 from $x_i'$ and $p_i^1 p_i^2$ is a chord (see Fig. 3(h)), then we change the colour of $w_i'$ to 4. Note again that, in each possibility, the underlined colours belong to $w_i'$.

*Step 4:* Avoid colisions in colouring of vertices of $F_i - P_i$ for $i = 0, \ldots, s-1$.

Now we check and modify (analogously as in the proof of Lemma 2) the defined colouring of $F_i - P_i$ to avoid colisions between pairs of vertice with the same colour. Obviously, there is no colision between vertices coloured with colour 1 or 2. Hence the only possible colision is in colour 3. Let $a$ and $b$ be a pair of clashing vertices in colour 3. If $a$ and $b$ belong to the same component $D_i^k$ of $F_i - P_i$, $k, \in \{1, 2, \ldots, k_i\}$, then we proceed as in Case 2 of the proof of Lemma 2. Thus we may assume that $a$ and $b$ belong to different components $D_i^k$ and $D_i^{k'}$ of $F_i - P_i$, $k, k' \in \{1, 2, \ldots, k_i\}$. If we changed the colouring of $D_i^2$ in Step 3, then we recolour the vertices of $D_i^2$ starting from $w_i'$ (also defined in Step 3) with pattern $2, 3, 1, 2, 1, 3, 1, \ldots$ instead of $2, 1, 3, 1, 2, 1, \ldots$. Then we proceed as in Case 1 of the proof of Lemma 2.

Now we have to check that the vertices coloured with colour 4 in Step 3 are pairwise at distance at least 5, and that the vertices coloured with colour 4 in Step 3 are pairwise at distance at least 5 from the added vertices of colour 4 in Step 4.

Since we have used the patterns $1, 3, 1, 2$ and $3, 1, 2, 1$ to colour the vertices of $V(D_i^1) \setminus \{x_i\}$, the vertices $w_i$ and $w_i'$ are both at distance at least 2 from each of $x_i$, $x_i'$. Thus, the vertices coloured with colour 4 in Step 3 are at mutual distance at least 5.

Let $a$ be a vertex of colour 4 from Step 3 (one of $w_i$, $w_i'$ denoted in Step 3). Suppose that $b$ is a vertex of colour 4 not in $D_i^1$. The minimal distance between any vertex of $D_i^1$ and any vertex of $D_i^2$ is at least 3. Moreover, because we have proceeded as in the proof of Lemma 2, $b$ is at distance at least 2 from a vertex at minimal distance from $a$. Hence, $d(a, b) \geq 5$.

Now suppose $b$ is a vertex of colour 4 in $D_i^1$. Since, in every case, $a$ is at distance at least 3 from another vertex of colour 3 in $D_i^1$, we obtain that $d(a, b) \geq 5$.

*Step 5:* Colouring the vertices of $P_i \setminus \{x_i\}$ with colours 5 to 28 and 46 to 51, for every $i = 0, \ldots, s-1$.

We start with colouring of the vertices $p_i^2$ by 1 for each $i = 0, \ldots s-1$. Since we have changed the colours of the eventual neighbours of $p_i^2$ of colour 1 in Step 3, there are no possible collisions.

For the vertices of $P_i$, we use the pattern given in the proof of Theorem 3 beginning at the fourth vertex of $P_i$, i.e., the vertex $p_i^3$. By the proof of Lemma 4, we know that such a colouring satisfies the distance constraints of a packing colouring.

Let $B = \{p_i^1 |\ 0 \leq i < s\}$. We colour the vertices of $B$ with colours 16 to 28 and (if necessary) 46 to 51. For integers $j$, $j'$, let $r_{j,j'}$ be an integer such that $r_{j,j'} \equiv j - j' \pmod{s}$ and $-\lfloor s/2 \rfloor \leq r_{j,j'} \leq \lfloor s/2 \rfloor$. Note that the vertices $p_j^1$ and $p_{j'}^1$ are at distance $2|r_{j,j'}| + 1$. We begin by a partitioning of $B$ into three subsets $B_1$, $B_2$ and $B_3$, with $B_k = \{p_i^1 |\ i \equiv k \pmod{3},\ 0 \leq i < s\}$, $k = 0, 1, 2$. Let $m_k$ ($m'_k$) denote the vertex with the largest (second largest, respectively) index in $B_k$, for $k \in \{1, 2, 3\}$. We use the following patterns to colour the vertices of $B$.

1. For vertices of $B_1$, we use the pattern $16, 17, 18, 16, 17, 18, \ldots, 16, 17, 18$. If $|B_1| \equiv 1 \pmod{3}$ (or $|B_1| \equiv 2 \pmod{3}$), then we erase colours of $m_1$ (or of $m_1, m'_1$, respectively).
2. For vertices of $B_2$, we use the pattern
$$19, 20, 21, 25, 26, 19, 20, 21, 25, 26, \ldots, 19, 20, 21, 25, 26$$
when $|B_2| \equiv 0, 1, 2 \pmod{5}$, or the pattern
$$19, 20, 21, 25, 26, 19, 20, 21, 25, 26, \ldots, 19, 20, 21, 25, 26, 19, 20, 21, 25, 19, 20, 21, 26$$
when $|B_2| \equiv 3, 4 \pmod{5}$. Then, for $|B_2| \equiv 1, 4 \pmod{5}$ we erase colour of $m_2$, and for $|B_2| \equiv 2 \pmod{5}$ we erase colours of $m_2$ and $m'_2$.
3. For vertices of $B_3$, we use the pattern
$$22, 23, 24, 27, 28, 22, 23, 24, 27, 28, \ldots, 22, 23, 24, 27, 28$$
when $|B_3| \equiv 0, 1, 2 \pmod{5}$, or the pattern
$$27, 28, 22, 23, 24, 27, 28, \ldots, 22, 23, 24, 27, 28, 22, 23, 24, 27, 22, 23, 24, 28$$
when $|B_3| \equiv 3, 4 \pmod{5}$. Then, for $|B_3| \equiv 1, 4 \pmod{5}$ we erase colour of $m_3$, and for $|B_3| \equiv 2 \pmod{5}$ we erase colours of $m_3$ and $m'_3$.
4. When it is necessary, we use the colours 46, 47, 48, 49, 50, 51 to colour the vertices of $\{m_k, m'_k |\ 1 \leq k \leq 3\}$.

For checking that the defined colouring satisfies the distance constraints of a packing colouring, we recall that any two consecutive vertices in each $B_k$ ($k = 1, 2, 3$) are pairwise at distance at least 7, implying that vertices having the same colour are pairwise at distance at least 19 in $B_1$ and at distance at least 31 in $B_2$ and in $B_3$ (except some vertices of colours from 19 to 24 that can be at distance 25 apart). ■

Note that in some cases (depending on the size of $B$ and $C'$) we can be decrease the upper bound 51 of Theorem 9. For example, if the internal face $C'$ has length $4k$ ($k \in \mathbb{N}$) and if the number of 2-connected components of $G - C'$ is $15r$ $r \in \mathbb{N}$, then we use only 40 colours instead of 51.

The following statement will be used in the proof of Theorem 11.

**Proposition 10.** *There is a packing colouring of even vertices of $P_\infty$ with colours $\{k, k+1, \ldots, 2k-1\}$.*

**Proof.** For the colouring of the vertices of the path $P_\infty$ we use pattern $1, k, 1, k+1, 1, k+2, \ldots, 1, 2k-1$ and after deleting colour 1 we get the required colouring. Note that the distance between any pair of vertices coloured with the same colour in two consecutive copies of this pattern is $2k$. ■

The last class of outerplanar graphs we consider in this paper is a class of not necessarily 2-connected graphs.

**Theorem 11.** *Let $G$ be a connected outerplanar graph with no internal face and with $\Delta(G) \leq 3$, such that the block graph of $G$ is a path. Then $\chi_\rho(G) \leq 305$.*

**Proof.** Let $G$ be a graph, $B_G$ the block graph of $G$ and let $B_1, \ldots, B_k$ denote the blocks of $G$ such that $B_i, B_{i+1}$ are consecutive in $B_G$, $i = 1, \ldots, k-1$ (i.e., they are connected by a path which intersects no other block of $G$). Let $C_i$ denote the boundary cycle of $B_i$, $i = 1, \ldots, k$. Since $G$ contains no internal face, each $B_i$ contains no internal face as well, implying that every $B_i$ which is not a cycle contains exactly two end faces. Let $x_1$ denote any vertex of degree two in one end face of $B_1$, $x_k$ any vertex of degree two in one end face of $B_k$ and let $P$ denote a shortest $x_1, x_k$-path in $G$. Among all possible choices of the vertices $x_1, x_k$ we choose $x'_1$ and $x'_k$ such that the path $P$ is shortest possible. Let $P_i = P \cap B_i$, $i = 1, \ldots, k$. Obviously $P$ goes through all the blocks of $G$, hence $P_i$ is nonempty for each $i = 1, \ldots, k$, every $P_i$ is a path since the block graph of $G$ is a path, and each $P_i$ is shortest in $G$ since $P$ is shortest in $G$. In an orientation of $P$ from $x_1$ to $x_k$, we denote by $z_i$ the first vertex of $P$ in $B_i$ and by $z'_i$ the neighbour of $z_i$ in $B_i$ which does not belong to $P$. Note that there must be exactly one such vertex $z'_i$ in each $B_i$ since $P$ is shortest possible and, clearly, each $z_i$ must have degree three. For every block which is not a cycle, we further denote by $x_i$ any vertex of degree two in one end face of $B_i$, $i = 2, \ldots, k-1$ and by $y_i$ any vertex of degree two in the end face of $B_i$ which does not contain vertex $x_i$, $i = 1, \ldots, k$. Among all possible choices of the vertices $x_i, y_i$ $(i = 1, \ldots, k)$ we choose $x'_i, y'_i$ in such a way that $x'_i, P$-path and $y'_i, P$-path, respectively, is shortest possible. For each $i = 1, \ldots, k$, let $Q^y_i$ be a shortest $y'_i, P$-path in $G$, and analogously, for each $i = 2, \ldots, k-1$, let $Q^x_i$, be a shortest $x'_i, P$-path in $G$. Note that some of the paths $Q^x_i, Q^y_i$ may be trivial or empty (e.g., in the case when $B_i$ is a cycle). The structure of the graph $G$ is depicted in Fig. 4 (the thick path represents the path $P$).

Consider each block $B_i$ separately. Note that, for each $i = 1, \ldots, k$, the graph $G_i = G[B_i - (V(P_i) \cup V(Q^x_i) \cup V(Q^y_i))]$ consists of path components. Thus, analogously as in the proof of Lemma 2, we can colour vertices of $G_i$ with colours $1, 2, 3, 4$ using the periodic pattern $1, 2, 1, 3$ and modifications introduced in the proof of Lemma 2, starting at the end face of $B_i$ containing

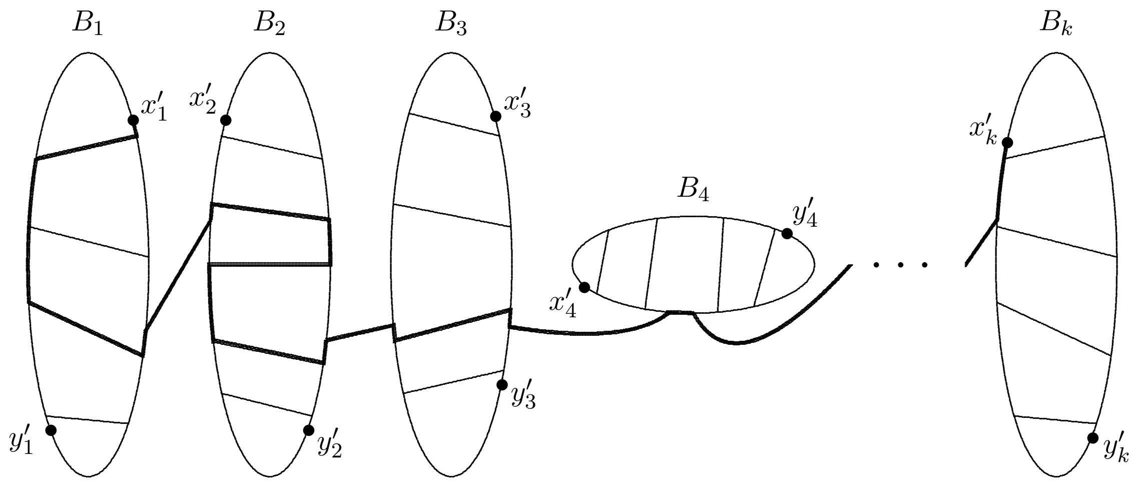


Figure 4: Structure of of blocks of the graph $G$ in Theorem 11.

vertex $x_i$. Note that, if $B_i$ is a cycle, $\chi_\rho(B_i) \leq 4$. Moreover, we can colour the vertices of $G_i$ in such a way that the vertex $z'_i$ gets colour 1 (if not so, then we can interchange roles of $x_i$ and $y_i$ for colouring of $V(G_i)$, i.e., we start such a colouring from $y_i$ instead of $x_i$).

Now we check that the defined colouring meets the conditions of a packing colouring. First, there is no colision in colours 1 and 2, since $\text{dist}_G(a, b) \geq 3$ for any $a \in V(G_i)$ and $b \in V(G_j)$, $i < j$. From the modifications described in the proof of Lemma 2, it follows that no end vertex of any path component of any $G_i$ is coloured with colour 4 since colour 4 was used for a vertex between vertices of a critical pair belonging to one path component. This implies that the distance between two vertices coloured with 4 which belong to different block $B_i, B_j$ is at least 5, hence there is no colision in colour 4. For colour 3, since no $z'_i$ is coloured with colour 3, there is no colision in colour 3 as well. And since there is no edge in $G$ connecting the path components of the blocks $B_i$, the defined colouring meets the distance constraints of a packing colouring of $\bigcup\limits_{i=1}^{k} [B_i - (V(P_i) \cup V(Q_i^x) \cup V(Q_i^y))]$.

Now we colour the paths $Q_i^x, Q_i^y$, $i = 1, 2, \ldots, k$. Rename these paths $Q_i^x, Q_i^y$ with $Q_j$, $j = 1, \ldots, 2k-2$ as they leave the path $P$ starting from $B_1$. Note that for $Q_i^x$ or $Q_i^y$ empty we set the corresponding $Q_j$ also empty. The distance from $P$ to any vertex of $Q_j$ in $G$ is the same as on $Q_j$, hence each $Q_j$ is a shortest path in $G$ between $P$ and the relevant vertex $x_i$ or $y_i$, respectively. Thus, by Lemma 4, we can colour the vertices of each path $Q_{2i-1}$ $(i = 1, 2, \ldots, k-1)$ with the same pattern $5, 6, \ldots$ using colours $5, 6, \ldots, 15$, starting at the vertex of $Q_{2i-1}$ at distance two from $P$. Analogously we can colour the vertices of the paths $Q_{2i}$ $(i = 1, 2, \ldots, k-1)$ using the same pattern $5, 6, \ldots$ using colours $5, 6, \ldots, 15$, starting at the vertex of $Q_{2i}$ at distance three from $P$. Then the distance between vertices on distinct paths $Q_m$, $Q_n$ coloured with colour 5 is at least $3 + 1 + 2$, the distance between vertices on distinct paths $Q_m$, $Q_n$ coloured with colour 6 is at least $4 + 1 + 3$, etc. Therefore the defined colouring of the paths $Q_j$, $j = 1, \ldots, 2k-2$ satisfies the distance contraints of a packing colouring.

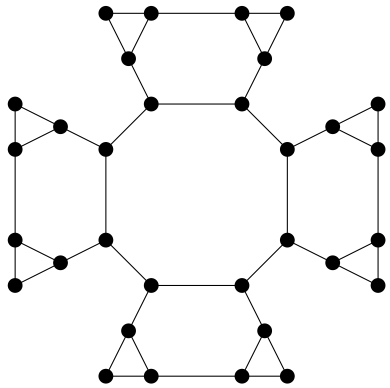

Figure 5: An outerplanar subcubic graph with packing chromatic number 7.

Now we colour the remaining vertices of $G$. We start with colouring of the path $P$ with a pattern using colours $16, 17, \dots, 50$ by Proposition A.ii). Then we colour all uncoloured vertices of $Q_j$ ($j = 1, \dots, 2k+2$) at distance one from $P$ with colours $51, 52, \dots, 152$ by Proposition A.iii). For the remaining vertices of $Q_{2i}$ at distance two from $P$, the distance between any such vertices on $Q_{2m}$ and $Q_{2n}$ ($m, n \in \{1, 2, \dots, k-1\}$, $m \neq n$) is at least $2|m-n|+4$. Hence we can colour these vertices with colours $153, \dots, 305$ by Proposition 10. ■

## 5 Concluding remarks

In the previous sections, we have determined some classes of outerplanar graphs with finite packing chromatic number. As for lower bounds, we are (only) able to state the following:

**Proposition 12.** *There exists* 2*-connected subcubic outerplanar graphs* $G$ *without internal faces and with packing chromatic number* 5*.*

**Proof.** It has been proven in [13] that $\chi_\rho(G) = 5$ for $G = P_n \square P_2$ and $n \geq 6$. ■

**Proposition 13.** *There exists a* 2*-connected subcubic outerplanar graph with packing chromatic number* 7*.*

**Proof.** We have checked, by computer, that the graph $G$ illustrated in Figure 5 has packing chromatic number 7 by verifying that every proper colouring of $G$ with 6 colours is not a packing colouring and by determining a packing colouring of $G$ with 7 colours.

■

Brešar et al. [3] have proven that for any finite graph $G$, the graph $G \boxtimes P_\infty$ has finite packing chromatic number. The degree of $G \boxtimes P_\infty$ can be arbitrary large. This property illustrates the fact that the degree of a graph is not the only parameter to consider in order to have finite packing chromatic number. Maybe the fact that the weak dual is a path (and is not any tree) helps to bound the packing chromatic number. It remains an open question to determine if the packing chromatic number of subcubic outerplanar graphs is finite or not.

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## References

[1] J. Balogh, A. Kostochka, and X. Liu: Packing chromatic number of subcubic graphs. *arXiv*:1703.09873, 2017.

[2] J. A. Bondy and U. S. R. Murty: *Graph Theory.* Springer, 2008.

[3] B. Brešar, S. Klavžar, and D. F. Rall: On the packing chromatic number of Cartesian products, hexagonal lattice, and trees. *Discrete Appl. Math.* **155**(17):2303–2311, 2007.

[4] B. Brešar, S. Klavžar, and D. F. Rall: Packing Chromatic Number of Base-3 Sierpiski Graphs. *Graphs and Combin.* **32**(4):1315–1327, 2016.

[5] B. Brešar, S. Klavžar, D. F. Rall, and K. Wash: Packing chromatic number under local changes in a graph. *Discrete Math.* **340**:1110–1115, 2017.

[6] B. Brešar, S. Klavžar, D. F. Rall, and K. Wash: Packing chromatic number, $(1, 1, 2, 2)$-colorings, and characterizing the Petersen graph. *Aequat. Math.* **91**:169-184, 2017.

[7] J. Ekstein, J. Fiala, P. Holub, and B. Lidický: The packing chromatic number of the square lattice is at least 12. *arXiv*:1003.2291v1, 2010.

[8] J. Fiala and P. A. Golovach: Complexity of the packing coloring problem for trees. *Discrete Appl. Math.* **158**:771–778, 2010.

[9] J. Fiala and P. A. Golovach: Complexity of the packing coloring problem for trees. *Discrete Appl. Math.*, **158**(7):771–778, 2010.

[10] J. Fiala, S. Klavžar, and B. Lidický: The packing chromatic number of infinite product graphs. *European J. Combin.* **30**(5):1101–1113, 2009.

[11] A.S. Finbow and D.F. Rall: On the packing chromatic number of some lattices. *Discrete Appl. Math.* **158**:1224-1228, 2010.

[12] N. Gastineau and O. Togni: $S$-Packing Colorings of Cubic Graphs. *Discrete Math.* **339**:2461–2470, 2016.

[13] W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris, and D. F. Rall: Broadcast chromatic numbers of graphs. *Ars Comb.* **86**:33-49, 2008.

[14] J. Ekstein, P. Holub, and O. Togni: The packing colouring of distance graph $D(k, t)$. $D$iscrete Appl. Math. **1**67:100–106, 2014.

[15] Y. Jacobs, E. Jonck, and E. Joubert: A lower bound for the packing chromatic number of the Cartesin product of cycles. $C$entr. Eur. J. Math. **1**1(7):1344–1357, 2013.

[16] D. Korže and A. Vesel: On the packing chromatic number of square and hexagonal lattice. *Ars Mathematica Contemporanea* **7**:13–22, 2014.

[17] B. Martin, F. Raimondi, T. Chen, and J. Martin: The packing chromatic number of the infinite square lattice is between 13 and 15. *Discrete Appl. Math.* **225**: 136–142, 2017.

[18] C. Sloper: Broadcast-coloring of trees. *Reports in Informatics* **233**:1-11, 2002.

[19] R. Soukal and P. Holub: A Note on Packing Chromatic Number of the Square Lattice. *Electron. J. of Combinatorics* **17**:447–468, 2010.

[20] O. Togni: On packing Colorings of Distance Graphs. *Discrete Appl. Math.* **167**:280–289, 2014.