A Lattice Formulation of the F 4 Completion Procedure

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A lattice formulation of the noncommutative F_4 procedure

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Abstract

We introduce a new procedure for constructing noncommutative Gröbner bases using a lattice formulation of completion. This leads to a lattice description of the noncommutative F_4 procedure. Our procedure is based on the lattice structure of reduction operators which provides a lattice description of the confluence property. We relate reduction operators to noncommutative Gröbner bases, we show the Diamond Lemma for reduction operators and we deduce the lattice interpretation of the F_4 procedure. Finally, we illustrate our procedure with a complete example.

Keywords: lattice structure, noncommutative F_4 procedure, reduction operators.

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1 Introduction

The objective of the paper is to introduce a new procedure for constructing noncommutative Gröbner bases which turns out to be a lattice formulation of the noncommutative F_4 procedure. This formulation is based on a description of the completion procedure using linear algebra techniques and is motivated by the development of effective methods in homological algebra using such techniques [1, 2, 9, 13, 14, 18].

The F_4 procedure is an improvement of the Buchberger's one where several S-polynomials are reduced into normal forms simultaneously. Improvements and optimisations of Buchberger's

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procedure were first introduced in the context of polynomial ideals, where selections strategies [4, 5, 12] and criteria for avoiding useless critical pairs [6, 7, 11, 15, 16] were investigated. The F_4 completion procedure was also introduced for polynomial ideals [10], it is adapted to the noncommutative case [19] and an implementation of this adaptation can be found in the system MAGMA.

Our lattice formulation of F_4 uses the approach due to Bergman [3] who described reduction systems over noncommutative algebras by *reduction operators*. The latter admit a lattice structure inducing lattice formulations of confluence and completion that we present now.

Lattice formulations of confluence and completion. A reduction operator relative to a well-ordered set $(G,\ <)$ is an idempotent linear endomorphism T of the $\mathbb{K}$-vector space $\mathbb{K}G$ spanned by G such that for every $g \notin \operatorname{im}(T)$, $T(g)$ is a linear combination of elements of G strictly smaller than g. We denote by $\mathbf{RO}(G,\ <)$ the set of reduction operators relative to $(G,\ <)$.

From [8, Proposition 2.1.14], the kernel map induces a bijection between $\mathbf{RO}(G,\ <)$ and subspaces of $\mathbb{K}G$, so that $\mathbf{RO}(G,\ <)$ admits a lattice structure defined in terms of kernels:

i. $T_1 \preceq T_2$ if $\ker(T_2) \subseteq \ker(T_1)$,

ii. $T_1 \wedge T_2 = \ker^{-1}(\ker(T_1) + \ker(T_2))$,

iii. $T_1 \vee T_2 = \ker^{-1}(\ker(T_1) \cap \ker(T_2))$.

Given a subset F of $\mathbf{RO}(G,\ <)$, we denote by $\wedge F$ the lower-bound of F, that is the reduction operator whose kernel is the sum of kernels of elements of F. We get the following lattice formulation of confluence: F is said to be *confluent* if the image of $\wedge F$ is equal to the intersection of images of elements of F. Recall from [8, Corollary 2.3.9] that F is confluent if and only if the reduction relation on $\mathbb{K}G$ defined by $v \longrightarrow T(v)$ for every $T \in F$ and every $v \notin \operatorname{im}(T)$ is confluent. Moreover, recall from [8, Theorem 3.2.6] that the completion of F is done by the operator $C^F = (\wedge F) \vee \left(\vee \overline{F}\right)$, where $\overline{F}$ is a subset of $\mathbf{RO}(G,\ <)$ defined from F and $\vee\overline{F}$ is the upper-bound of $\overline{F}$, that is $F \cup \{C^F\}$ is a confluent subset of $\mathbf{RO}(G,\ <)$.

In Section 3, the operator C^F is used to reduce simultaneously several S-polynomials into normal forms using a triangular process such as the F_4 procedure does. For that, we introduce *presentations by operators* which relate reduction operators to noncommutative Gröbner bases.

Reduction operators and presentations of algebras. A presentation by operator of an associative $\mathbf{A}$ is a triple $(X,\ <,\ S)$, where X is a set, $<$ is a monomial order on the set of noncommutative monomials X^* and S is a reduction operator relative to $(X^*,\ <)$ such that $\mathbf{A}$ is isomorphic to the quotient of the free algebra over X by the two-sided ideal spanned by $\ker(S)$.

In order to describe all the reductions induced by S we consider the "extensions" of S, that is the operators which applied to a monomial $w_1w_2w_3$ gives $w_1S(w_2)w_3$. The presentation $(X,\ <,\ S)$ is said to be *confluent* if the set of extensions of S is a confluent subset of $\mathbf{RO}(G,\ <)$. From [8, Proposition 3.3.10], the presentation $(X,\ <,\ S)$ is confluent if and only if the set of elements $w - S(w)$ with $w \notin \operatorname{im}(S)$ is a noncommutative Gröbner basis of $I(\ker(S))$. This link between reduction operators and noncommutative Gröbner bases enables us to show the Diamond Lemma in terms of reduction operators in Proposition 2.2.8.

Our procedure for constructing confluent presentations by operators, and thus noncommutative Gröbner bases, is given in Section 3.1. At the step number d of the procedure, we reduce the S-polynomials of the current presentation $\left(X,\ <,\ S^d\right)$ into normal forms using a set of reduction operators F_d. The operator at the step $d+1$ is $S^{d+1} = S^d \wedge C^{F_d}$. Denoting by $\overline{S}$ the lower-bound of all the operators S^d, the triple $\left(X,\ <,\ \overline{S}\right)$ is called the *completed presentation* of $\mathbf{A}$. The main result of the paper is Theorem 3.2.5 which asserts that a completed presentation is confluent. In Section 3.3, we show how to implement our procedure with a complete example as an illustration.

Organisation of the paper

Section 2.1 is a recollection of results from [8]: we recall the definitions and properties of reduction operators, their confluence and completion used in the sequel. In Section 2.2, we define presentations by operators, the confluence property of such presentations, we formulate and we show the Diamond Lemma for reduction operators. In Section 3.1, we write our completion procedure and define completed presentations. In Section 3.2, we show that a completed presentation is confluent. In Section 3.3, we illustrate our completion procedure with a complete example based on the computation of lattice operations of reduction operators.

2 Reduction operators

2.1 Lattice structure of reduction operators

Throughout the paper, $\mathbb{K}$ denotes a commutative field. Given a set G, we denote by $\mathbb{K}G$ the vector space spanned by G. Given a well-order $<$ on G, the leading generator of a nonzero element $v \in \mathbb{K}G$ is written $\lg(v)$. We extend the order $<$ on G into a partial order on $\mathbb{K}G$ in the following way: we have $u < v$ if $u = 0$ and $v \neq 0$ or if $\lg(u) < \lg(v)$.

Definition 2.1.1. A *reduction operator relative to* $(G,\ <)$ is an idempotent endomorphism T of $\mathbb{K}G$ such that for every $g \in G$, we have $T(g) \leq g$. We denote by $\mathbf{RO}(G,\ <)$ the set of reduction operators relative to $(G,\ <)$. Given $T \in \mathbf{RO}(G,\ <)$, a generator $g \in G$ is said to be a *T-normal form* or *T-reducible* according to $T(g) = g$ or $T(g) \neq g$, respectively. We denote by $\mathrm{nf}(T)$ the set of T-normal forms and by $\mathrm{red}(T)$ the set of T-reducible generators.

Lattice structure, confluence and completion. Recall from [8, Proposition 2.1.14] that the restriction of the kernel map $T \longmapsto \ker(T)$ to $\mathbf{RO}(G,\ <)$ is a bijection. Using the inverse $\ker^{-1}$, the set $\mathbf{RO}(G,\ <)$ admits a lattice structure for the operations

i. $T_1 \preceq T_2$ if $\ker(T_2) \subseteq \ker(T_1)$,

ii. $T_1 \wedge T_2 = \ker^{-1}(\ker(T_1) + \ker(T_2))$,

iii. $T_1 \vee T_2 = \ker^{-1}(\ker(T_1) \cap \ker(T_2))$.

Recall from [8, Lemma 2.1.18] that we have the following implication

$$T_1 \preceq T_2 \Longrightarrow \mathrm{nf}(T_1) \subseteq \mathrm{nf}(T_2)\ ^{\mathrm{a}}. \tag{1}$$

[a] In [8], the notation $\mathrm{red}(T)$ stands for *reduced generators* and correspond to $\mathrm{nf}(T)$ in the present paper. The notation $\mathrm{red}(T)$ of the present paper corresponds to $\mathrm{nred}(T)$ of [8] which means *nonreduced generators.*

Given a nonempty subset F of $\mathbf{RO}(G, <)$, we denote by $\mathrm{nf}(F)$ and $\wedge F$ the set of normal forms for each $T \in F$ and the lower-bound of F, respectively. From (1), $\mathrm{nf}(\wedge F)$ is included in $\mathrm{nf}(T)$ for every $T \in F$, so that $\mathrm{nf}(\wedge F)$ is included in $\mathrm{nf}(F)$. We write

$$\mathrm{obs}(F) = \mathrm{nf}(F) \setminus \mathrm{nf}(\wedge F). \tag{2}$$

The set F is said to be *confluent* if $\mathrm{obs}(F)$ is the empty set. In Section 3.2, we use two characterisations of the confluence property in terms of reduction operators. First, recall from [8, Theorem 2.2.5] that F is confluent if and only if it has the *Church-Rosser property*, that is for every $v \in \mathbb{K}G$, there exist $T_1, \cdots, T_r \in F$ such that $(\wedge F)(v) = (T_r \circ \cdots \circ T_1)(v)$. Moreover, from [8, Proposition 2.2.12], F is confluent if and only if it is *locally confluent*, that is for every $v \in \mathbb{K}G$ and for every $(T, T') \in F \times F$, there exist $v' \in \mathbb{K}G$ and $T_1, \cdots, T_r, T'_1, \cdots, T'_k \in F$ such that $v' = (T_r \circ \cdots \circ T_1)(T(v))$ and $v' = (T'_k \circ \cdots \circ T'_1)(T'(v))$. Finally, we recall how a set of reduction operators is completed into a confluent one.

Definition 2.1.2. A *complement of* F is an element C of $\mathbf{RO}(G, <)$ such that

i. $(\wedge F) \wedge C = \wedge F$,

ii. $\mathrm{obs}(F) \subseteq \mathrm{red}(C)$.

The *F-complement* is the operator $C^F = (\wedge F) \vee \left(\vee \overline{F}\right)$, where $\vee \overline{F}$ is equal to $\ker^{-1}(\mathbb{K}\mathrm{nf}(F))$.

Recall from [8, Proposition 3.2.2] that a reduction operator C satisfying $(\wedge F) \wedge C = \wedge F$ is a complement of F if and only if $F \cup \{C\}$ is confluent. Recall from [8, Theorem 3.2.6] that the F-complement is a complement of F.

2.2 Presentations by operators

In this section, we relate the confluence property for reduction operators to noncommutative Gröbner bases and we prove the Diamond Lemma for reduction operators.

Given a set X, we denote by X^* the set of noncommutative monomials over X and we identify the free algebra over X with $\mathbb{K}X^*$, equipped with the multiplication induced by concatenation of monomials. A *monomial order* over X^* is a well-founded total strict order $<$ on X^* such that the following conditions are fulfilled:

i. $1 < w$ for every monomial w different from 1,

ii. for every $w_1, w_2, w, w' \in X^*$ such that $w < w'$, we have $w_1 w w_2 < w_1 w' w_2$.

For any $f \in \mathbb{K}X^*$, the leading monomial of f is written $\mathrm{lm}(f)$ instead of $\mathrm{lg}(f)$.

Definition 2.2.1. A *presentation by operator* of an associative algebra $\mathbf{A}$ is a triple $(X, <, S)$ where

i. X is a set and $<$ is a monomial order on X^*,

ii. S is a reduction operator relative to $(X^*, <)$ such that $\mathbf{A}$ is isomorphic to $\mathbb{K}X^*/I(\ker(S))$, where $I(\ker(S))$ is the two-sided ideal spanned by $\ker(S)$.

We fix an algebra $\mathbf{A}$ together with a presentation by operator $(X,\ <,\ S)$ of $\mathbf{A}$. For every integer n, we denote by $X^{(n)}$ and $X^{(\leq n)}$ the set of monomials of length n and of length smaller or equal to n, respectively. For every integers n and m such that $(n,\ m)$ is different from $(0,\ 0)$, we consider the reduction operator

$$S_{n,m} \ = \ \mathrm{Id}_{\mathbb{K}X^{(\leq n+m-1)}} \ \oplus \ \left(\mathrm{Id}_{\mathbb{K}X^{(n)}} \otimes S \otimes \mathrm{Id}_{\mathbb{K}X^{(m)}}\right).$$

Explicitly, for every $w \in X^*$, $S_{n,m}(w)$ is defined by: if the length of w is strictly smaller than $n+m$, then $S_{n,m}(w) \ = \ w$, else we let $w = w_1w_2w_3$ where w_1 and w_3 have length n and m, respectively and we have $S_{n,m}(w) = w_1S(w_2)w_3$. We also let $S_{0,0} = S$.

Definition 2.2.2. The set of all the operators $S_{n,m}$ with $(n,\ m) \ \in \ \mathbb{N}^2$, is called the *reduction family* of $(X,\ <,\ S)$. The presentation $(X,\ <,\ S)$ is said to be *confluent* if its reduction family is a confluent subset of $\mathbf{RO}\,(X^*,\ <)$.

Recall from [8, Proposition 3.3.10] that $(X,\ <,\ S)$ is confluent if and only if the set of elements $w \ - \ S(w)$ with $w \ \in \ \mathrm{red}\,(S)$ is a noncommutative Gröbner basis of $I\,(\ker(S))$, that is $\mathrm{red}\,(S)$ spans leading monomials of I as a monomial ideal.

Example 2.2.3. Let $X = \{x,\ y,\ z\}$ and let $<$ be the deg-lex order induced by $x \ < \ y \ < \ z$. Consider the algebra presented by $(X,\ <,\ S)$ where S is defined on the basis X^* by $S(yz) \ = \ x$, $S(zx) \ = \ xy$ and $S(w) \ = \ w$ for every monomial w different from yz and zx. We have

$$\begin{aligned} yxy - xx \ &= \ (yxy - yzx) \ - \ (xx - yzx) \\ &= \ (yS(zx) - yzx) \ - \ (S(yz)x - yzx) \\ &= \ A \ + \ B \end{aligned}$$

where $A \ = \ (S_{1,0} - \mathrm{Id}_{\mathbb{K}X^*})\,(yzx)$ and $B \ = \ (\mathrm{Id}_{\mathbb{K}X^*} - S_{0,1})\,(yzx)$. Hence, $yxy \ - \ xx$ belongs to $\ker\,(\wedge F)$ where F is the reduction family of the presentation, so that yxy is $\wedge F$-reducible. Moreover, yxy belongs to $\mathrm{nf}\,(F)$, so that yxy belongs to $\mathrm{obs}\,(F)$ and F is not confluent. Thus, $(X,\ <,\ S)$ is not a confluent presentation of $\mathbf{A}$.

In Section 3.1 we formulate our procedure for constructing confluent presentations by operators using *critical branchings* that we introduce in Definition 2.2.4. These branchings are analogous to *ambiguities* for Gröbner bases. An ambiguity with respect to $<$ of a subset R of $\mathbb{K}X^*$ is a tuple $b = (w_1,\ w_2,\ w_3,\ f,\ g)$ where $w_1,\ w_2,\ w_3$ are monomials such that $w_2 \ \neq \ 1$, $f,\ g$ belong to R and one of the following two conditions is fulfilled:

1. $w_1w_2 \ = \ \mathrm{lm}\,(f)$ and $w_2w_3 \ = \ \mathrm{lm}\,(g)$.

2. $w_1w_2w_3 \ = \ \mathrm{lm}\,(f)$ and $w_2 \ = \ \mathrm{lm}\,(g)$.

The S-polynomial of b is written $\mathrm{sp}\,(b)$, that is $\mathrm{sp}\,(b) \ = \ fw_3 \ - \ w_1g$ or $\mathrm{sp}\,(b) \ = \ f \ - \ w_1gw_3$ according to b is of the form 1 or 2, respectively. The ambiguity b is said to be *solvable relative to* $<$ if there exists a decomposition

$$\mathrm{sp}\,(b) \ = \ \sum_{i=1}^{n} \lambda_i w_i f_i w'_i, \tag{3}$$

where, for every $i \in \{1, \cdots, n\}$, λ_i is a non-zero scalar, $w_i,\ w'_i$ are monomials and f_i is an element of R such that $w_i\mathrm{lm}\,(f_i)\,w'_i \ < \ w_1w_2w_3$. The Diamond Lemma [3, Theorem 1.2] asserts that R is a noncommutative Gröbner basis of $I(R)$ if and only if every critical branching of R with respect to $<$ is solvable relative to $<$.

Our purpose is to formulate and to prove the Diammond Lemma for reduction operators. Until the end of the section, we fix some notations: $\mathbf{A}$ is an associative algebra and $(X,\ <,\ S)$ is a presentation by operator of $\mathbf{A}$. For every pair of integers $(n,\ m)$, we consider the operator $S_{n,m}$ defined such as the beginning of the section. We denote by R the set of elements $w - S(w)$ with $w \in \operatorname{red}(S)$.

Definition 2.2.4. A *critical branching* of $(X,\ <,\ S)$ is a triple $b = (w,\ (n,\ m),\ (n',\ m'))$ where w is a monomial and $(n,\ m)$ and $(n',\ m')$ are couples of integers such that

i. w belongs to $\operatorname{red}(S_{n,m}) \cap \operatorname{red}\left(S_{n',m'}\right)$,

ii. $n = 0$ or $n' = 0$,

iii. $m = 0$ or $m' = 0$,

iv. $n + n' + m + m'$ is strictly smaller than the length of w.

The *S-polynomial* of b is $\operatorname{SP}(b) = S_{n,m}(w) - S_{n',m'}(w)$ and the *source* of b is the monomial w.

Remark 2.2.5. The roles of $(n,\ m)$ and $(n',\ m')$ being symmetric, we do not distinguish $(w,\ (n,\ m),\ (n',\ m'))$ and $(w,\ (n',\ m'),\ (n,\ m))$.

Definition 2.2.6. Let $w \in X^*$ and let $f \in \mathbb{K}X^*$. We say that f admits a $(S,\ w)$*-type decomposition* if it admits a decomposition

$$f = \sum_{i=1}^{n} \lambda_i w_i^1 \left(w_i - S(w_i)\right) w_i^2,$$

where, for every $i \in \{1, \cdots, n\}$, λ_i is a non-zero scalar, w_i^1, w_i^2 and w_i are monomials such that w_i belongs to $\operatorname{red}(S)$ and $w_i^1 w_i w_i^2 < w$.

Lemma 2.2.7. *There is a one-to-one correspondence $b \longmapsto \tilde{b}$ between critical branchings of $(X,\ <,\ S)$ and ambiguities of R with respect to $<$. Moreover, a critical branching b of source w admits a (S, w)-type decomposition if and only if $\tilde{b}$ is solvable relative to $<$.*

Proof. Let us show the first part of the lemma. Let $b = (w,\ (n,\ m),\ (n',\ m'))$ be a critical branching of $(X,\ <,\ S)$. In order to define $\tilde{b}$, we distinguish four cases depending on the values of n and m:

Case 1: $(n,\ m) = (0,\ 0)$. We write $w = w_1 w_2 w_3$, where the lengths of w_1 and w_3 are equal to n' and m', respectively. By definition of a critical branching, w and w_2 belong to $\operatorname{red}(S)$ and we let $\tilde{b} = \Big(w_1,\ w_2,\ w_3,\ w - S(w),\ w_1\left(w_2 - S(w_2)\right) w_3\Big)$. By definition of a critical branching, $n + n' + m + m' = n' + m'$ is strictly smaller than the length of w. In particular, w_2 is not the empty word, so that the tuple $\tilde{b}$ is an ambiguity of R with respect to $<$ of the form 2.

Case 2: $n = 0$ and $m \neq 0$. By definition of a critical branching, $m' = 0$. If n' is also equal to 0, we have $(n',\ m') = (0,\ 0)$, so that we exchange the roles of $(n,\ m)$ and $(n',\ m')$ and we recover the first case. If $n' \neq 0$, we write $w = w_1w_2w_3$, where the lengths of w_1 and w_3 are equal to n' and m, respectively. In particular, b being a critical branching, the monomials w_1w_2 and w_2w_3 belong to $\mathrm{red}\,(S)$ and w_2 is different from 1. Hence, $\tilde{b} = \Big(w_1,\ w_2,\ w_3,\ w_1w_2 - S(w_1w_2),\ w_2w_3 - S(w_2w_3)\Big)$, is an ambiguity of R with respect to $<$.

Case 3: $n \neq 0$ and $m = 0$. By definition of a critical branching, n' is equal to 0. Exchanging the roles of $(n,\ m)$ and $(n',\ m')$, we recover the second case.

Case 4: $n \neq 0$ and $m \neq 0$. By definition of a critical branching, the pair $(n',\ m')$ is equal to $(0,\ 0)$. Exchanging the roles of $(n,\ m)$ and $(n',\ m')$, we recover the first case.

We have a well-defined map $b \longmapsto \tilde{b}$ between critical branchings of $(X,\ <,\ S)$ and ambiguities of R with respect to $<$. Now, we define the inverse map $\tilde{b} \longmapsto b$. Let $\tilde{b} = (w_1,\ w_2,\ w_3,\ f,\ g)$ be an ambiguity of R with respect to $<$ and let $w = w_1w_2w_3$.

- If $\tilde{b}$ is an ambiguity of the form 1, let n and m' be the lengths of w_1 and w_3, respectively. The word w_2 being non-empty, $n + m'$ is strictly smaller than the length of w, so that $b = (w,\ (n,\ 0),\ (0,\ m'))$ is a critical branching of $(X,\ <,\ S)$.
- If $\tilde{b}$ is of the form 2, let n and m be the lengths of n and m, respectively. Then, $b = (w,\ (n,\ m),\ (0,\ 0))$ is a critical branching of $(X,\ <,\ S)$.

Such defined, the two composites of $b \longmapsto \tilde{b}$ and $\tilde{b} \longmapsto b$ are identities.

Let us show the second part of the lemma. Given a critical branching b, $\mathrm{sp}\,(b)$ and $\mathrm{sp}\left(\tilde{b}\right)$ are equal. Letting w the source of w, a $(S,\ w)$-type decomposition of $\mathrm{sp}\,(b)$ is precisely a decomposition of the from (3). That shows the second part of the lemma.

□

The Diamond Lemma for reduction operators is formulated as follows:

Proposition 2.2.8. *The presentation $(X,\ <,\ S)$ is confluent if and only if for every critical branching b of source w, $\mathrm{SP}(b)$ admits a $(S,\ w)$-type decomposition.*

Proof. The two-sided ideal $I(R)$ spanned by R is equal to $I\,(\ker(S))$. Hence, from [8, Proposition 3.3.10], $(X,\ <,\ S)$ is confluent if and only if R is a noncommutative Gröbner basis of $I(R)$. From the Diamond Lemma, the presentation $(X,\ <,\ S)$ is confluent if and only if every ambiguity of R with respect to $<$ is solvable relative to $<$. Thus, from Lemma 2.2.7, $(X,\ <,\ S)$ is confluent if and only if for every critical branching b of source w the S-polynomial $\mathrm{sp}\,(b)$ admits a $(S,\ w)$-type decomposition. □

Example 2.2.9. Considering the presentation of Example 2.2.3, we have one critical branching $b_1 = (yzx,\ (1,\ 0),\ (0,\ 1))$ and we have $\mathrm{sp}\,(b_1) = yxy - xx$. This S-polynomial does not admit a $(S,\ yzx)$-type decomposition so that we recover that the presentation is not confluent.

3 Completion procedure

In Section 3.1, we formulate our procedure for constructing confluent presentations by operators and we show the correctness of this procedure in Section 3.2. Throughout Section 3, we fix the following notations:

i. $\mathbf{A}$ is an algebra and $(X,\ <,\ S)$ is a presentation by operator of $\mathbf{A}$.

ii. Given a reduction operator $T \in \mathbf{RO}(X^*,\ <)$ and a pair of integers $(n,\ m)$, the operator $T_{n,m}$ is defined such as the beginning of Section 2.2.

iii. For every $f \in \mathbb{K}X^*$, we write $T(f) = \ker^{-1}(\mathbb{K}f)$. Explicitly, $(T(f))(\mathrm{lm}(f))$ is equal to $\mathrm{lm}(f) - 1/\mathrm{lc}(f)\, f$ and all other monomial is a normal form for $T(f)$. Moreover, we write $\mathrm{supp}(f)$ the *support* of f, that is the set monomials occurring in the decomposition of f with a nonzero coefficient.

iv. Given a subset $E \subseteq \mathbb{K}X^*$, we write $\mathrm{lm}(E)$ the set of leading monomials of elements of E.

3.1 Formulation

Our procedure requires a function called `normalisation` with inputs a finite set $E \subset \mathbb{K}X^*$ and a reduction operator $U \in \mathbf{RO}(X^*,\ <)$ and with output a finite set of reduction operators. Then, `normalisation`$(E,\ U)$ is defined as follows:

1. Let $M = \left(\bigcup_{f\in E} \mathrm{supp}(f)\right) \setminus \mathrm{lm}(E)$ and $F = \{T(f) \mid f \in E\}$.
2. `while` $\exists\ w_1ww_2 \in M$ such that $w \in \mathrm{red}(U)$,
   - **i.** we add $T(w_1(w - U(w))w_2)$ to F,
   - **ii.** we remove w_1ww_2 from M,
   - **iii.** we add $\mathrm{supp}(w_1U(w)w_2)$ to M.
3. `normalisation`$(E,\ U)$ is the set F obtained when the loop `while` is over.

The loop `while` is terminating beacause E is finite and $<$ is a monomial order.

We formulate our completion procedure. We assume that the presentation $(X,\ <,\ S)$ is *finite*, that is X is finite and $\ker(S)$ is finite-dimensional. In particular, the set of critical branchings of $(X,\ <,\ S)$ is finite.

Algorithm 1 Completion procedure

Initialisation:

- $d := 0$,
- $S^d := S$,
- $Q_d := \emptyset$ and $P_d := \left\{\text{critical branchings of } \left(X, <, S^d\right)\right\}$,
- $E_d := \left\{w - {S^d}_{n,m}(w) \mid (w, (n, m), (n', m')) \in P_d\right\}$.

while $Q_d \neq P_d$ **do**

$F_d := \texttt{normalisation}(E_d, S^d)$;

$S^{d+1} := S^d \wedge C^{F_d}$;

$Q_{d+1} := P_d$;

$d = d + 1$;

$P_d := \left\{\text{critical branchings of } \left(X, <, S^d\right)\right\}$;

$E_d := \left\{w - {S^d}_{n,m}(w) \mid (w, (n, m), (n', m')) \in P_d \setminus Q_d\right\}$;

end while

This first and the last instruction of the loop **while** make sense because we have the following:

Lemma 3.1.1. *Let d be an integer.*

1. *The kernels of S^d and C^{F_d} are finite-dimensional.*
2. *The set Q_d is included in P_d.*

Proof. We show Point 1 by induction on d. The kernel of $S^0 = S$ is finite-dimensional by hypotheses. Let $d \in \mathbb{N}$ and assume that the kernel of S^d is finite-dimensional. Let $M_d = \bigcup_{f \in E_d} \operatorname{supp}(f)$ be the union of words appearing in E_d. The elements of F_d are only acting on M_d, so that we have the inclusion

$$\ker\left(C^{F_d}\right) \subset \mathbb{K}M_d. \tag{4}$$

The kernel of S^d being finite-dimensional by induction hypothesis, the set of critical branchings of $\left(X, <, S^d\right)$ is finite. Hence, E_d and M_d are finite sets, so that $\ker\left(C^{F_d}\right)$ is finite-dimensional from (4). Moreover, by definition of $\wedge$, $\ker\left(S^{d+1}\right)$ is equal to $\ker\left(S^d\right) + \ker\left(C^{F_d}\right)$, so that $\ker\left(S^{d+1}\right)$ is finite-dimensional.

Let us show Point 2. By construction, Q_d is equal to P_{d-1}, that is Q_d is the set of critical branchings of $\left(X, <, S^{d-1}\right)$. Let $(w, (n, m), (n', m'))$ be such a critical branching, so that we have

$$w \in \operatorname{red}\left(\left(S^{d-1}\right)_{n,m}\right) \cap \operatorname{red}\left(\left(S^{d-1}\right)_{n',m'}\right). \tag{5}$$

Moreover, by construction, we have $S^d \preceq S^{d-1}$. Hence, from implication (1) (see page 3), we have

$$\operatorname{red}\left(S^{d-1}\right) \subset \operatorname{red}\left(S^d\right). \tag{6}$$

From (5) and (6), w belongs to $\operatorname{red}\left({S^d}_{n,m}\right) \cap \operatorname{red}\left({S^d}_{n',m'}\right)$, so that $(w,\ (n,\ m),\ (n',\ m'))$ is a critical branching of $\left(X,\ <,\ S^{d+1}\right)$, that is it belongs to P_d. Thus, Q_d is included in P_d. □

Remark 3.1.2. Our procedure requires to compute lower-bound of reduction operators relative to $(X^*,\ <)$. In Section 3.3, we give the implementation of $\ker^{-1}$ for totally ordered finite sets, so that it cannot be used for a set of monomials. However, from Lemma 3.1.1, the kernels of S^d and C^{F_d} are finite-dimensional, so that these two operators can be computed by restrictions over finite-dimensional subspaces of $\mathbb{K}X^*$. We illustrate how works such computations in Section 3.3.

Our procedure has no reason to terminate since there exist finitely presented algebras with no finite Gröbner basis [17, Section 1.3]. If the procedure terminates after d iterations of the loop **while**, we let $S^n = S^d$ for every integer $n \geq d$, so that the sequence $\left(S^d\right)_{d\in\mathbb{N}}$ is well-defined if the procedure terminates or not. We let

$$\overline{S} = \bigwedge_{d\in\mathbb{N}} S^d.$$

Definition 3.1.3. The triple $\left(X,\ <,\ \overline{S}\right)$ is called the *completed presentation* of $(X,\ <,\ S)$.

The purpose of the next section is to show that the completed presentation of $(X,\ <,\ S)$ is a confluent presentation of $\mathbf{A}$, that is our procedure computes a noncommutative Gröbner basis.

3.2 Soundness

In this section, we say reduction operator instead of reduction operator relative to $(X^*,\ <)$.

Lemma 3.2.1. *Let $w \in X^*$ and let T and T' be two reduction operators such that $T' \preceq T$.*

1. *Let $(n,\ m)$ be a pair of integers such that w is $T_{n,m}$-reducible. Then, $\left(T_{n,m} - T'_{n,m}\right)(w)$ admits a $(T',\ w)$-type decomposition.*
2. *Let $f \in \mathbb{K}X^*$ admitting a $(T,\ w)$-type decomposition. Then, f admits a $(T',\ w)$-type decomposition.*

Proof. Let us show Point 1. We let $w = w^{(n)}w'w^{(m)}$, where $w^{(n)}$ and $w^{(m)}$ have length n and m, respectively. Let

$$T(w') = \sum_{i=1}^{k} \lambda_i w_i, \tag{7}$$

be the decomposition of $T(w')$ with respect to the basis X^*. By hypotheses, T' is smaller than T, that is $\ker(T) \subseteq \ker(T')$, so that $T' \circ T$ is equal to T'. Hence, we have

$$\begin{aligned}\left(T_{n,m} - T'_{n,m}\right)(w) &= w^{(n)}\left(T(w') - T'(w')\right)w^{(m)} \\ &= w^{(n)}\left(T(w') - T'\left(T(w')\right)\right)w^{(m)}.\end{aligned}$$

From (7), we obtain

$$\left(T_{n,m} - T'_{n,m}\right)(w) = \sum_{i=1}^{k} \lambda_i w^{(n)} \left(w_i - T'(w_i)\right) w^{(m)}. \tag{8}$$

By hypotheses, w is $T_{n,m}$-reducible, so that w' is T-reducible and each w_i is strictly smaller than w' for $<$. The strict order $<$ being monomial, each $w^{(n)}w_iw^{(m)}$ is strictly smaller than $w^{(n)}w'w^{(m)} = w$, so that (8) is a $(T',\ w)$-type decomposition of $\left(T_{n,m} - T'_{n,m}\right)(w)$.

Let us show Point 2. Let

$$f = \sum_{i=1}^{n} \lambda_i w_i^1 \left(w_i - T(w_i)\right) w_i^2, \tag{9}$$

be a $(T,\ w)$-type decomposition of f. Letting

$$A = \sum_{i=1}^{n} \lambda_i w_i^1 \left(w_i - T'(w_i)\right) w_i^2 \quad \text{and} \quad B = \sum_{i=1}^{n} \lambda_i w_i^1 \left(T(w_i) - T'(w_i)\right) w_i^2,$$

f is equal to $A - B$. The decomposition (9) being $(T,\ w)$-type, each $w'_i = w_i^1 w_i w_i^2$ is strictly smaller than w, so that A is $(T',\ w)$-type. For every $i \in \{1, \cdots, n\}$, let n_i and m_i be the lengths of w_i^1 and w_i^2, respectively, so that we have $B = \sum_{i=1}^{n} \lambda_i \left(T_{n_i,m_i} - T'_{n_i,m_i}\right)(w'_i)$. Each w_i being T-reducible, each w'_i is T_{n_i,m_i}-reducible. Hence, from Point 1 of the lemma, each $\left(T_{n_i,m_i} - T'_{n_i,m_i}\right)(w'_i)$ admits a $(T',\ w'_i)$-type decomposition, so that it admits a $(T',\ w)$-type decomposition since w'_i is strictly smaller than w. Hence, B admits a $(T',\ w)$-type decomposition, so that f also admits such a decomposition.

□

Notation. For every integer d, let F_d be the reduction family of $\left(X,\ <,\ S^d\right)$, that is F_d is equal to $\left\{ \left(S^d\right)_{n,m} \mid (n,\ m) \in \mathbb{N}^2 \right\}$.

Lemma 3.2.2. *Let d be an integer, let $(w,\ (n,\ m),\ (n',\ m')) \in P_d \setminus Q_d$ and let f be the S-polynomial of $(w,\ (n,\ m), (n',\ m'))$.*

1. *$(\wedge F_d)(f)$ is equal to 0.*
2. *f admits a $\left(S^{d+1},\ w\right)$-type decomposition.*

Proof. Let us show Point 1. The two elements $w - \left(S^d\right)_{n,m}(w)$ and $w - \left(S^d\right)_{n',m'}(w)$ belong to E_d by construction of the latter. Hence, by definition of the function `normalisation`, the operators $T_1 = T\left(w - \left(S^d\right)_{n,m}(w)\right)$ and $T_2 = T\left(w - \left(S^d\right)_{n',m'}(w)\right)$ belong to F_d, so that $f = (w - {S^d}_{n,m}(w)) - (w - {S^d}_{n',m'}(w))$ belongs to the kernel of $T_1 \wedge T_2$. The latter is included in the kernel of $\wedge F_d$, which shows Point 1.

Let us show Point 2. The operator C^{F_d} being a complement of F_d, we have

$$\wedge \left(F_d \cup \left\{C^{F_d}\right\}\right) = \wedge F_d, \tag{10}$$

and $F_d \cup \left\{C^{F_d}\right\}$ is confluent (see the paragraph after Definition 2.1.2), that is it has the Church-Rosser property (see the paragraph before Definition 2.1.2). Hence, from Point 1 of the lemma and Relation (10), there exist $T_1, \cdots, T_r \in F_d \cup \left\{C^{F_d}\right\}$ such that

$$\left(T_r \circ \cdots \circ T_1\right)(f) = 0. \tag{11}$$

We let $f_1 = (\mathrm{Id}_{\mathbb{K}X^*} - T_1)(f)$ and for every $k \in \{2, \cdots, r\}$, $f_k = (\mathrm{Id}_{\mathbb{K}X^*} - T_k)(T_{k-1} \circ \cdots \circ T_1(f))$. From (11), we have

$$f = \sum_{k=1}^{r} f_k. \tag{12}$$

The tuple $(w, (n, m), (n', m'))$ being a critical branching of $(X, <, S^d)$, w belongs to $\mathrm{red}\left(\left(S^d\right)_{n,m}\right) \cap \mathrm{red}\left(\left(S^d\right)_{n',m'}\right)$, so that the leading monomial of f is strictly smaller than w. Moreover, each T_i is either of the form $T\left(w_1(w_2 - S^d(w_2))w_3\right)$, or is equal to C^{F_d}. Hence, each f_i admits a $\left(S^d, w\right)$-type decomposition or a $\left(C^{F_d}, w\right)$-type decomposition. The reduction operators S^d and C^{F_d} being smaller than S^{d+1}, each f_i admits a $\left(S^{d+1}, w\right)$-type decomposition from Point 2 of Lemma 3.2.1, so that f admits a $\left(S^{d+1}, w\right)$-type decomposition from (12).

□

Proposition 3.2.3. *Let d be an integer. For every $(w, (n, m), (n', m')) \in Q_d$, the S-polynomial $\left(S^d\right)_{n,m}(w) - \left(S^d\right)_{n',m'}(w)$ admits a $\left(S^d, w\right)$-type decomposition.*

Proof. We show the proposition by induction on d. The set Q_0 being empty, Proposition 3.2.3 holds for $d = 0$. Assume that for every $(w, (n, m), (n', m')) \in Q_d$, ${S^d}_{n,m}(w) - {S^d}_{n',m'}(w)$ admits a (S_d, w)-type decomposition. We let

$$\begin{aligned} A &= \left(S^d\right)_{n',m'}(w) - \left(S^{d+1}\right)_{n',m'}(w) \quad , \\ B &= \left(S^d\right)_{n,m}(w) - \left(S^{d+1}\right)_{n,m}(w), \\ C &= \left(S^d\right)_{n,m}(w) - \left(S^d\right)_{n',m'}(w). \end{aligned}$$

We have

$$\left(S^{d+1}\right)_{n,m}(w) - \left(S^{d+1}\right)_{n',m'}(w) = A - B + C.$$

By construction, S^{d+1} is smaller than S^d. Moreover, $(w, (n, m), (n', m'))$ being a critical branching, w belongs to $\mathrm{red}\left(\left(S^d\right)_{n,m}\right) \cap \mathrm{red}\left(\left(S^d\right)_{n',m'}\right)$. Hence, from Point 1 of Lemma 3.2.1, A and B admit a $\left(S^{d+1}, w\right)$-type decomposition. It remains to show that C admits a $\left(S^{d+1}, w\right)$-type decomposition. By construction, Q_{d+1} is equal to P_d, so that it contains Q_d from Point 2 of Lemma 3.1.1. If $(w, (n, m), (n', m'))$ does not belong to Q_d, C admits a $\left(S^{d+1}, w\right)$-type decomposition from Point 2 of Lemma 3.2.2. If $(w , (n, m), (n', m'))$ belongs to Q_d, C admits a $\left(S^d, w\right)$-type decomposition by induction hypothesis. Hence, from Point 2 of Lemma 3.2.1, C admits a $\left(S^{d+1}, w\right)$-type decomposition.

□

Recall that the lower-bound of the operators S^d is written $\overline{S}$. The last lemma we need to prove Theorem 3.2.5 is

Lemma 3.2.4. *1. The sequence $(I_d)_{d\in\mathbb{N}}$ of ideals spanned by $\ker\left(S^d\right)$ is constant.*

2. $\mathrm{Red}\left(\overline{S}\right)$ is equal to $\bigcup_{d\in\mathbb{N}} \mathrm{Red}\left(S^d\right)$.

Proof. Let us show Point 1. By definition of the function `normalisation`, the kernel of each element of F_d is included in I_d. In particular, $\ker(\wedge F_d) = \sum_{T \in F_d} \ker(T)$ is also included in I_d. Moreover, C^{F_d} being a complement of F_d, it is smaller than $\wedge F_d$, that is its kernel is included in the one of $\wedge F_d$. In particular, $\ker\left(C^{F_d}\right)$ is included in I_d, so that $\ker\left(S^{d+1}\right)$, which by definition is equal to $\ker\left(S^d\right) + \ker\left(C^{F_d}\right)$, is also included in I_d. Hence, the sequence $(I_d)_{d\in\mathbb{N}}$ is not increasing. Moreover, the sequence $\left(S^d\right)_{d\in\mathbb{N}}$ is not increasing by construction, which means that $\left(\ker\left(S^d\right)\right)_{d\in\mathbb{N}}$ is not decreasing. Hence, $(I_d)_{d\in\mathbb{N}}$ constant.

Let us show Point 2. The equality we want to prove means that the set $F = \left\{S^d \mid d \in \mathbb{N}\right\}$ is confluent. From Newman's Lemma (see the paragraph before Definition 2.1.2) in terms of reduction operators, it is sufficient to show that F is locally confluent. Let $f \in \mathbb{K}X^*$ and let d and d' be two integers which we assume to satisfy $d \geq d'$. In particular, we have $S^{d'} \preceq S^d$, so that $S^d \circ S^{d'}$ is equal to $S^{d'}$. Hence, $\left(S^d \circ S^{d'}\right)(f)$ and $S^d(f)$ are equal, so that F is locally confluent.

□

Theorem 3.2.5. *Let* $\mathbf{A}$ *be an algebra and let* $(X, <, S)$ *be a presentation by operator of* $\mathbf{A}$. *The completed presentation of* $(X, <, S)$ *is a confluent presentation of* $\mathbf{A}$.

Proof. Let $\overline{S}$ be the lower-bound of the operators S^d.

First, we show that $\left(X, <, \overline{S}\right)$ is a presentation of $\mathbf{A}$. From Point 1 of Lemma 3.2.4, the ideal spanned by the kernels of the operators S^d is equal to the ideal I spanned by the kernel of $S^0 = S$. In particular, the ideal spanned by $\ker\left(\overline{S}\right) = \sum_{d\in\mathbb{N}} \ker\left(S^d\right)$ is equal to I. Hence, $(X, <, S)$ being a presentation of $\mathbf{A}$, $\left(X, <, \overline{S}\right)$ is also a presentation of $\mathbf{A}$.

Let us show that this presentation is confluent. From the Diamond Lemma, it is sufficient to show that for each critical branching $b = (w, (n, m), (n', m'))$ of $\left(X, <, \overline{S}\right)$, the S-polynomial $\mathrm{sp}(b)$ admits a $\left(\overline{S}, w\right)$-type decomposition. From Point 2 of Lemma 3.2.4, there exist integers d and d' such that $w \in \mathrm{red}\left(\left(S^d\right)_{n,m}\right) \cap \mathrm{red}\left(\left(S^{d'}_{n',m'}\right)\right)$. Without lost of generalities, we may assume that d is greater or equal to d', so that b is a critical branching of $\left(X, <, S^d\right)$, that is it belongs to $P_d = Q_{d+1}$. We let

$$
\begin{aligned}
A_d &= \left(S^{d+1}\right)_{n',m'}(w) - \overline{S}_{n',m'}(w),\\
B_d &= \left(S^{d+1}\right)_{n,m}(w) - \overline{S}_{n,m}(w),\\
C_d &= \left(S^{d+1}\right)_{n,m}(w) - \left(S^{d+1}\right)_{n',m'}(w).
\end{aligned}
$$

We have

$$
\mathrm{sp}(b) = A_d - B_d + C_d. \tag{13}
$$

From Proposition 3.2.3, b being an element of Q_{d+1}, C_d admits a $\left(S^{d+1}, w\right)$-type decomposition, so that it admits a $\left(\overline{S}, w\right)$-type decomposition from Point 2 of Lemma 3.2.1. Moreover, S^{d+1} being smaller than S^d, w belongs to $\mathrm{red}\left(\left(S^{d+1}\right)_{n,m}\right) \cap \mathrm{red}\left(\left(S^{d+1}\right)_{n',m'}\right)$. The operator $\overline{S}$ being smaller than S^{d+1}, A_d and B_d also admit a $\left(\overline{S}, w\right)$-type decomposition from Point 1 of Lemma 3.2.1. Hence, from (13), $\mathrm{sp}(b)$ admits a $\left(\overline{S}, w\right)$-type decomposition.

□

Example 3.2.6. In Section 3.3, we compute the completed presentation of Example 2.2.3. It is given by the operator defined by $\overline{S}(yz) = x$, $\overline{S}(zx) = xy$, $\overline{S}(yxy) = xx$, $\overline{S}(yxx) = xxz$, $\overline{S}(yxxx) = xxxy$ and $\overline{S}(w) = w$ for all other monomial w.

3.3 Example

In this section, we compute the completed presentation of Example 3.2.6. Before that, we show how to use Gaussian elimination to compute lattice operations and completion for reduction operators relative to totally ordered finite sets. We use the SageMath software, written in Python.

Lattice operations and completion. Let $(G, <)$ be a totally ordered finite set. The set G being finite, the Gaussian elimination provides a unique basis $\mathscr{B}$ of any suspace $V \subseteq \mathbb{K}G$ such that for every $e \in \mathscr{B}$, $\mathrm{lc}\,(e)$ is equal to 1 and, given two different elements e and e' of $\mathscr{B}$, $\mathrm{lg}\,(e')$ does not belong to the decomposition of e. The operator $T = \ker^{-1}(V)$ satisfies $T(\mathrm{lg}\,(e)) = \mathrm{lg}\,(e) - e$ for every $e \in \mathscr{B}$ and $T(g) = g$ if g is not a leading generator of $\mathscr{B}$. Moreover, we represent the subspaces of $\mathbb{K}G$ by lists of generating vectors and for any list of vectors `L`, let `reducedBasis(L)` be the basis of $\mathbb{K}$`L` obtained by Gaussian elimination.

First, we define the function `operator` which takes as input a list of vectors `L` and returns $\ker^{-1}(\mathbb{K}\mathtt{L})$. We deduce the functions which compute the lattice operations of $\mathbf{RO}\,(G, <)$.

```
def operator(G):
    L=reducedBasis(G)
    n=len(L[0])
    V=VectorSpace(QQ,n)
    v=V.zero()
    G=(lg(L[0])-1)*[v]+[L[0]]
    k=len(L)
    for i in [1..k-1]:
        G=G+(lg(L[i])-lg(L[i-1])-1)*[v]+[L[i]]
    G=G+(n-lg(L[k-1]))*[v]
    return identity_matrix(QQ,n)-matrix(G).transpose()

def lowerBound(T_1,T_2):
    V_1,V_2=kernel(T_1.transpose()),kernel(T_2.transpose())
    G_1,G_2=basis(V_1),basis(V_2)
    L_1,L_2=reducedBasis(G_1),reducedBasis(G_2)
    G=L_1+L_2
    L=reducedBasis(G)
    return operator(L)

def upperBound(T_1,T_2):
    V_1,V_2=kernel(T_1.transpose()),kernel(T_2.transpose())
    V=V_1.intersection(V_2)
    G=basis(V)
    L=reducedBasis(G)
    return operator(L)
```

By definition of the F-complement, we need an intermediate function with input a reduction operator T and output $\ker^{-1}(\mathbb{K}\mathrm{nf}\,(T))$. We define this function before defining the one of the

F-complement.

```
def tilde(T):
    n,L=T.nrows(),[]
    for i in [0..n-1]:
        j,k=i,n-i-1
        if T[i,i]==1: L=L+[vector(j*[0]+[1]+k*[0])]
    return operator(L)

def complement(L):
    n,C,T=len(L),L[0],tilde(L[0])
    for i in [1..n-1]: C=lowerBound(C,L[i])
    for j in [1..n-1]: T=upperBound(T,tilde(L[j]))
    return lowerBound(C,T)
```

Example. Now, we use our implementation to compute the completed presentation of Example 2.2.3: we consider the algebra $\mathbf{A}$ presented by $(X,\ <,\ S)$ where $X = \{x,\ y,\ z\}$, $<$ is the deg-lex order induced by $x\ <\ y\ <\ z$ and $S(yz)\ =\ x$, $S(zx)\ =\ xy$ and $S(w)\ =\ w$ for every monomial w different from yz and zx.

Recall that S^d denotes the operator of the presentation at the beginning of step d of the procedure, P_d is the set of critical branchings of $\left(X,\ <,\ S^d\right)$, $Q_d\ =\ P_{d-1}$, $E_d\ =\ \left\{w\ -\ {S^d}_{n,m}(w)\ \mid\ (w,\ (n,\ m),\ (n',\ m'))\ \in\ P_d \setminus Q_d\right\}$ and $F_d\ =\ \mathtt{normalisation}(E_d,\ S^d)$. Moreover, we represent reduction operators by matrices. For that, we use that the operators appearing in the procedure act nontrivially on finite-dimensional subspaces of $\mathbb{K}X^*$ spanned by an ordered set of monomials $w_1\ <\ w_2\ <\ \cdots\ <\ w_n$.

At the first step, we have $d\ =\ 0$. The presentation $\left(X,\ <,\ S^0\right)$ has one critical branching $b_1\ =\ (yzx,\ (1,\ 0),\ (0,\ 1))$ and we have $P_0\ =\ \{b_1\}$ and $E_0\ =\ \left\{yzx - xx,\ yzx - yxy\right\}$. We have $F_0\ =\ \left\{T_1,\ T_2\right\}$ where the matrices of the restrictions of T_1 and T_2 to the subspace spanned by $xx\ <\ yxy\ <\ yzx$ are

$$T_1\ =\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ \text{ and }\ T_2\ =\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

that is $T_1(yzx)\ =\ xx$ and $T_2(yzx)\ =\ yxy$. The matrice of $C^{F_0}\ =\ \text{complement}\,([T_1,\ T_2])$ restricted to $\mathbb{K}\{xx,\ yxy,\ yzx\}$ is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

The operator $S^1\ =\ S \wedge C^{F_0}$ can be computed by restriction to the subspace spanned by $x\ <\ xx\ <\ xy\ <\ yz\ <\ zx\ <\ yxy$ and the matrices of the restrictions of S^0 and C^{F_0} to this subspace are

$$S^0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C^{F_0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We obtain that S^1 is the operator defined by $S^1(yz) = x$, $S^1(zx) = xy$, $S^1(yxy) = xx$ and $S^1(w) = w$ for every monomial w different from yz, zx and yxy.

The presentation $\left(X, <, S^1\right)$ has two new critical branchings b_2 and b_3 equal to $(yxyz, (2, 0), (0, 1))$ and $(yxyxy, (2, 0), (0, 2))$, respectively. We have $P_1 = \{b_1, b_2, b_3\}$, $P_1 \setminus Q_1 = \{b_2, b_3\}$ and $E_1 = \left\{yxyz - xxz, yxyz - yxx, yxyxy - xxxy, yxyxy - yxxx\right\}$. Moreover, $F_1 = \texttt{normalisation}\left(E_1, S^1\right)$ is equal to

$$\left\{\begin{matrix} T_3 = T\left(yxyz - xxz\right), & T_4 = T\left(yxyz - yxx\right) \\ T_5 = T\left(yxyxy - xxxy\right), & T_6 = T\left(yxyxy - yxxx\right) \end{matrix}\right\},$$

where $T(f) = \ker^{-1}(\mathbb{K}f)$. The restriction of C^{F_1} to $\mathbb{K}\{xxz, yxx, xxxy, yxxx, yxyz, yxyxy\}$ is

$$C^{F_1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and we obtain that $S^2 = S^1 \wedge C^{F_1}$ is defined by $S^2(yz) = x$, $S^2(zx) = xy$, $S^2(yxy) = xx$, $S^2(yxx) = xxz$, $S^2(yxxx) = xxxy$ and all other monomial is a normal form for S^2.

The computation of the operator C^{F_2} gives the identity operator of size 11, which corresponds to the monomials $x^4 < x^3y < x^2zx < yx^3 < x^5 < x^3yz < yx^3z < yxyx^2 < x^3yxyx < yx^4y < yxyx^3$. Hence, no new critical branching is created at this step and the procedure stops.

References

[1] David J. Anick. On the homology of associative algebras. *Trans. Amer. Math. Soc.*, 296(2):641–659, 1986.

[2] Roland Berger. Confluence and Koszulity. *J. Algebra*, 201(1):243–283, 1998.

[3] George M. Bergman. The diamond lemma for ring theory. *Adv. in Math.*, 29(2):178–218, 1978.

[4] Anna M. Bigatti, Massimo Caboara, and Lorenzo Robbiano. Computing inhomogeneous Gröbner bases. *J. Symbolic Comput.*, 46(5):498–510, 2011.

[5] Miguel A. Borges-Trenard, Mijail Borges-Quintana, and Teo Mora. Computing Gröbner bases by FGLM techniques in a non-commutative setting. *J. Symbolic Comput.*, 30(4):429–449, 2000.

[6] Bruno Buchberger. A criterion for detecting unnecessary reductions in the construction of Gröbner-bases. In *Symbolic and algebraic computation (EUROSAM '79, Internat. Sympos., Marseille, 1979)*, volume 72 of *Lecture Notes in Comput. Sci.*, pages 3–21. Springer, Berlin-New York, 1979.

[7] Massimo Caboara, Martin Kreuzer, and Lorenzo Robbiano. Efficiently computing minimal sets of critical pairs. *J. Symbolic Comput.*, 38(4):1169–1190, 2004.

[8] Cyrille Chenavier. Reduction operators and completion of rewriting systems. *J. Symbolic Comput.*, 84:57–83, 2018.

[9] Sergio Chouhy and Andrea Solotar. Projective resolutions of associative algebras and ambiguities. *J. Algebra*, 432:22–61, 2015.

[10] Jean-Charles Faugère. A new efficient algorithm for computing Gröbner bases (F_4). *J. Pure Appl. Algebra*, 139(1-3):61–88, 1999. Effective methods in algebraic geometry (Saint-Malo, 1998).

[11] Jean-Charles Faugère. A new efficient algorithm for computing Gröbner bases without reduction to zero (F_5). In *Proceedings of the 2002 International Symposium on Symbolic and Algebraic Computation*, pages 75–83 (electronic). ACM, New York, 2002.

[12] Jean-Charles Faugère, Patrizia Gianni, Daniel Lazard, and Teo Mora. Efficient computation of zero-dimensional Gröbner bases by change of ordering. *J. Symbolic Comput.*, 16(4):329–344, 1993.

[13] Yves Guiraud, Eric Hoffbeck, and Philippe Malbos. Convergent presentations and polygraphic resolutions of associative algebras. 65 pages, December 2017.

[14] Yuji Kobayashi. Gröbner bases of associative algebras and the Hochschild cohomology. *Trans. Amer. Math. Soc.*, 357(3):1095–1124 (electronic), 2005.

[15] Martin Kreuzer and Lorenzo Robbiano. *Computational commutative algebra. 2*. Springer-Verlag, Berlin, 2005.

[16] Martin Kreuzer and Lorenzo Robbiano. *Computational commutative algebra 1.* Springer-Verlag, Berlin, 2008. Corrected reprint of the 2000 original.

[17] Teo Mora. An introduction to commutative and noncommutative Gröbner bases. *Theoret. Comput. Sci.*, 134(1):131–173, 1994. Second International Colloquium on Words, Languages and Combinatorics (Kyoto, 1992).

[18] Stewart B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.*, 152:39–60, 1970.

[19] Xingqiang Xiu. *Non-commutative Gröbner bases and applications.* PhD thesis, Universität Passau, 2012.