Structures of opposition induced by relations:the Boolean and the gradual cases

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The Boolean and the gradual cases

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Abstract The paper shows that a cube of opposition, a structure that generalizes the square of opposition invented in ancient logic, can be generated from the composition of a binary relation with a subset, by the effect of set complementation on the subset, on the relation, or on the result of the composition. Since the composition of relations is encountered in many areas, the structure of opposition exhibited by the cube of opposition has a universal flavor. In particular, it applies to information processing-oriented settings such as rough set theory, possibility theory, or formal concept analysis. We then discuss how this structure extends to a fuzzy relation and a fuzzy subset, and the graded cube of opposition thus obtained provides an organized view of the different existing compositions of fuzzy relations. The paper concludes by pointing out areas of research where the cube of opposition, or its graded version are of interest.

Keywords Square of opposition · Composition of relations · Fuzzy relation · Multiple-valued logic

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1 Introduction

The square of opposition is a structure that exhibits various forms of opposition between four logical statements, and which was originally introduced by Ancient Greek logicians, in relation to the study of syllogisms. This structure is induced by the interplay of an inside and an outside involutive negation operating on the statements. This structure was revisited by medieval logicians before being finally somewhat forgotten with the advent of modern logic more concerned with the formalization of deduction than with issues in knowledge representation related to the use of negation. The square of opposition was tentatively revived in the middle of the XX^{th} century when it was discovered that it can be completed into a hexagon including three squares of opposition, and that the resulting hexagonal structure plays an important role in the organization of concepts [9]. More recently, under the impulse of Jean-Yves Béziau [3, 4], there has been an increasing surge of interest for the square of opposition and its extensions for applications in philosophical and mathematical logic, linguistics, psychology, semiotics, and artificial intelligence [5–7].

A cube of opposition can be obtained by duplicating the square of opposition by means of a third negation. This cube is not mentioned very often, and it was rediscovered several times under different guises. It apparently appears for the first time under the form of a cube described by Reichenbach in his modern study of syllogisms [38], where he points out that Aristotle himself mentioned the negative oppositions on the back facet of the cube, but did not incorporate them in his theory. The use of the cube for syllogistic reasoning has been recently revisited in [16]. The statements associated to the back facet produced by the third negation already appear in [26], and in [15] (where lower case counterparts of the traditional names of the vertices of the square of opposition are introduced for the vertices of the back facet). The same oppositions received a different geometrical representation with an octagon in [28] and more recently in [27]. A similar cube also arises in the study of dual pairs (such as necessity / possibility, universal / existential quantifiers, conjunction / disjunction) in [17].

The cube of opposition was found again in [23], where it was noticed that Piaget's group [36] of transformations of a compound statement is at work inside the cube. The presence of the cube of opposition structure was then noticed in different formal settings of interest dealing with the processing of information, such as formal concept analysis, possibility theory, abstract argumentation, or rough set theory [1, 13, 23].

Also recently, Murinová and Novák [32, 33] have proposed a graded extension of the square of opposition based on Łukasiewicz triangular norm and co-norm, for quantified fuzzy statements. Moreover, they have introduced intermediary layers in the original square of opposition between the top edge linking universally quantified statements, and the bottom edge linking existentially quantified statements, for accommodating soft quantifiers such as "most", or "a few", which then correspond to the vertices of these intermediary layers. Independently of a particular knowledge representation need, suggestions for systematically extending the logical links that should hold between the vertices of a square, of a hexagon, or of a cube of opposition have been provided in [24]. This is of interest when the vertices may be graded, as for instance in possibility theory. This raises the question of extending the generation process of the cube in the fuzzy (graded) case [43].

The paper is organized in the following way. The next section, Section 2, starts with a refresher on the square of opposition insisting on the logical links between the vertices and the dependencies between these links. Section 3 shows how the square of opposition, and then a cube and hexagons, are naturally induced from a relation. Section 4 discusses the graded extensions of these structures on the basis of the links that should exist between the vertices. Section 5 studies the graded cube of opposition induced by a fuzzy relation, which offers a structured organization of the existing fuzzy relational compositions [2, 42]. Sections 2 and 3 are a revised and extended version of the first half of [13], while Section 4 elaborates from [24], and Section 5 is fully new.

2 The square of opposition: a refresher

Let us start with a refresher on the Aristotelian square of opposition [34]. The traditional square involves four logically related statements exhibiting universal or existential quantifications: a statement **A** of the form "every x is p" is negated by the statement **O** "some x is not p", while a statement like **E** "no x is p" is clearly in more general opposition to the first statement (**A**). These three statements, together with the negation of the last one, namely **I** "some x is p", give birth to the Aristotelian square of opposition in terms of quantifiers **A** : $\forall x\ p(x)$, **E** : $\forall x\ \neg p(x)$, **I** : $\exists x\ p(x)$, **O** : $\exists x\ \neg p(x)$. This square, pictured in Fig. 1, is usually denoted by the letters **A**, **I** (affirmative half) and **E**, **O** (negative half). The names of the vertices come from a traditional Latin reading: **A**ff**I**rmo, n**E**g**O**).

As can be seen, different relations hold between the vertices. We take them as a basis for a formal definition of the square of opposition. Namely,

Definition 1 In a classical square of opposition **AIEO**, the following relations are supposed to hold:

(a) **A** and **O** are the negation of each other, as well as **E** and **I**;
(b) **A** entails **I**, and **E** entails **O**;
(c) **A** and **E** cannot be true together, but may be false together;
(d) **I** and **O** cannot be false together, but may be true together.

Note that we assume that there are some x for avoiding existential import problems in Fig. 1, and having the condition (b) satisfied. Another well-known instance of this square is in terms of the *necessary* ($\Box$) and *possible* ($\Diamond$) modalities, with the following reading **A** : $\Box p$, **E** : $\Box\neg p$, **I** : $\Diamond p$, **O** : $\Diamond\neg p$, where $\Diamond p =_{def} \neg\Box\neg p$ (with $p \neq \bot, \top$). Then the entailment from **A** to **I** is nothing but the axiom (D) in modal logic, namely $\Box p \to \Diamond p$.

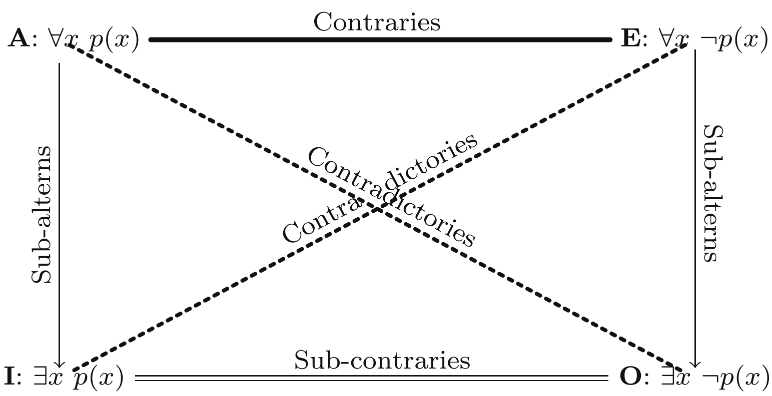

Fig. 1 Square of opposition

We can describe the links that hold between the vertices of the square using propositional logic, as in, e.g. [31, 41], where **A**, **I**, **E**, and **O** are now associated with Boolean variables, i.e., **A**, **I**, **E**, and **O** are the truth values of statements. Then, the following counterparts of the four above conditions can be easily checked.

Proposition 1 *In a square of opposition **AIEO**, the following holds:*

(a) *The diagonal link between **A** and **O**, which represents the symmetrical relation of contradiction, is an exclusive or, namely* $\neg(\boldsymbol{A} \equiv \boldsymbol{O})$*, or if we prefer* $\boldsymbol{A} \equiv \neg\boldsymbol{O}$*. Similarly,* $\boldsymbol{E} \equiv \neg\boldsymbol{I}$*.*

(b) *The vertical arrows represent implication relations* $\boldsymbol{A} \to \boldsymbol{I}$ *and* $\boldsymbol{E} \to \boldsymbol{O}$*.*

(c) *The link between **A** and **E** which represents the symmetrical relation of contrariety, corresponds to mutual exclusion, namely* $\neg\boldsymbol{A} \vee \neg\boldsymbol{E}$ *should hold.*

(d) *The link between **I** and **O** which represents the symmetrical relation of subcontrariety, is a disjunction, namely* $\boldsymbol{I} \vee \boldsymbol{O}$ *holds.*

Let us notice that these conditions are not independent. Indeed, at least the following dependency links hold in any square of opposition.

Proposition 2 *Let **AIEO** be a square of opposition and (a)–(d) the conditions listed in Proposition 1. Then,*

(dep 1) *(a)(b)⇒(c):* $\neg\boldsymbol{A} \vee \neg\boldsymbol{E}$ *is a consequence of* $\boldsymbol{A} \equiv \neg\boldsymbol{O}$ *and* $\boldsymbol{E} \to \boldsymbol{O}$ *(or of* $\boldsymbol{E} \equiv \neg\boldsymbol{I}$ *and* $\boldsymbol{A} \to \boldsymbol{I}$*) in the square. But,* $\neg\boldsymbol{A} \vee \neg\boldsymbol{E}$ *and* $\boldsymbol{E} \to O$ *(resp.* $\boldsymbol{A} \to \boldsymbol{I}$*) are not enough for entailing* $\boldsymbol{A} \equiv \neg\boldsymbol{O}$ *(resp.* $\boldsymbol{E} \equiv \neg\boldsymbol{I}$*).*

(dep 2) *(a)(b)⇒(d):* $\boldsymbol{I} \vee \boldsymbol{O}$ *is a consequence of* $\boldsymbol{A} \equiv \neg\boldsymbol{O}$ *and* $\boldsymbol{A} \to \boldsymbol{I}$ *(or of* $\boldsymbol{E} \equiv \neg\boldsymbol{I}$ *and* $\boldsymbol{E} \to \boldsymbol{O}$*). But,* $\boldsymbol{I} \vee \boldsymbol{O}$ *and* $\boldsymbol{A} \to \boldsymbol{I}$ *(resp.* $\boldsymbol{E} \to \boldsymbol{O}$*) are not enough for entailing* $\boldsymbol{A} \equiv \neg\boldsymbol{O}$ *(resp.* $\boldsymbol{E} \equiv \neg\boldsymbol{I}$*).*

(dep 3) *(a)(c)⇒(b)(d) & (a)(d)⇒(b)(c):* $\boldsymbol{A} \equiv \neg\boldsymbol{O}$*,* $\boldsymbol{E} \equiv \neg\boldsymbol{I}$*, together with* $\neg\boldsymbol{A} \vee \neg\boldsymbol{E}$ *entail* $\boldsymbol{A} \to \boldsymbol{I}$*,* $\boldsymbol{E} \to \boldsymbol{O}$ *and* $\boldsymbol{I} \vee \boldsymbol{O}$*. Similarly,* $\boldsymbol{A} \equiv \neg\boldsymbol{O}$*,* $\boldsymbol{E} \equiv \neg\boldsymbol{I}$*, together with* $\boldsymbol{I} \vee \boldsymbol{O}$ *entail* $\boldsymbol{A} \to \boldsymbol{I}$*,* $\boldsymbol{E} \to \boldsymbol{O}$ *and* $\neg\boldsymbol{A} \vee \neg\boldsymbol{E}$*.*

Let us summarize the situation. One may consider that the involutive nature of negation expressed by (a) is crucial. Moreover, note that the formulas encoding (b), (c), and (d) together do not entail the formulas associated with (a). Besides, on the one hand conditions (a) and (b) entail conditions (c) and (d), while on the other hand conditions (a) and (c) entail conditions (b) and (d) (or in a dual manner, conditions (a) and (d) entail conditions (b) and (c)). This leaves us with three options for defining a formal square of opposition with independent conditions: i) either we can regard (a) and (b), or ii) (a) and (c), or iii) (a) and (d), as the basic requirements.

3 Structure of opposition induced from a relation and a subset

In this section we describe a very simple process that generates the basic structures of opposition: square, cube, and hexagons. It involves a binary relation and a subset that are composed together.

3.1 A relation-based square of opposition

Let us now consider a binary relation R on a Cartesian product $X \times Y$ (one may have $Y = X$). We assume $R \neq \emptyset$. Let xR denote the set $\{y \in Y \mid (x, y) \in R\}$. We write xRy when $(x, y) \in R$ holds, and $\neg(xRy)$ when $(x, y) \notin R$. Let R^t denote the transposed relation, defined by $xR^t y$ if and only if yRx, and yR^t will be equivalently denoted as $Ry = \{x \in X \mid (x, y) \in R\}$.

Moreover, we assume that $\forall x,\ xR \neq \emptyset$, which means that the relation R is *serial*, namely $\forall x, \exists y$ such that xRy; this is also referred to in the following as the *X-normalization condition*. In the same way R^t is also supposed to be serial, i.e., $\forall y,\ Ry \neq \emptyset$ (Y-normalization). We further assume that the complementary relation $\overline{R}$ ($x\overline{R}y$ iff $\neg(xRy)$), and its transpose are also serial, i.e., $\forall x,\ xR \neq Y$ and $\forall y,\ Ry \neq X$. These conditions enforce a non trivial relation between X and Y. In the following, set complementations will be denoted by means of overbars.

Let S be a subset of Y. We assume $S \neq \emptyset$ and $S \neq Y$. The relation R and the subset S, also considering its complement $\overline{S}$, give birth to the two following subsets of X, namely the (left) images of S and $\overline{S}$ by R

$$R(S) = \{x \in X \mid \exists s \in S, xRs\} = \{x \in X \mid S \cap xR \neq \emptyset\} = \bigcup_{s \in S} Rs \tag{1}$$

$$R(\overline{S}) = \{x \in X \mid \exists s \in \overline{S}, xRs\} = \bigcup\nolimits_{s \in \overline{S}} Rs$$

and their complements

$$\overline{R(S)} = \{x \in X \mid \forall s \in S, \neg(xRs)\} = \overline{\bigcup\nolimits_{s \in S} Rs} = \bigcap\nolimits_{s \in S} \overline{Rs} = \bigcap\nolimits_{s \in S} \overline{R}s$$

$$\overline{R(\overline{S})} = \{x \in X \mid \forall s \in \overline{S}, \neg(xRs)\} = \{x \in X \mid xR \subseteq S\} = \overline{\bigcup_{s \in \overline{S}} Rs} = \bigcap_{s \in \overline{S}} \overline{R}s \tag{2}$$

The four subsets thus defined can be nicely organized into a square of opposition, see Fig. 2. Indeed, it can be checked that the set counterparts of the relations (a)–(d) of Proposition 1

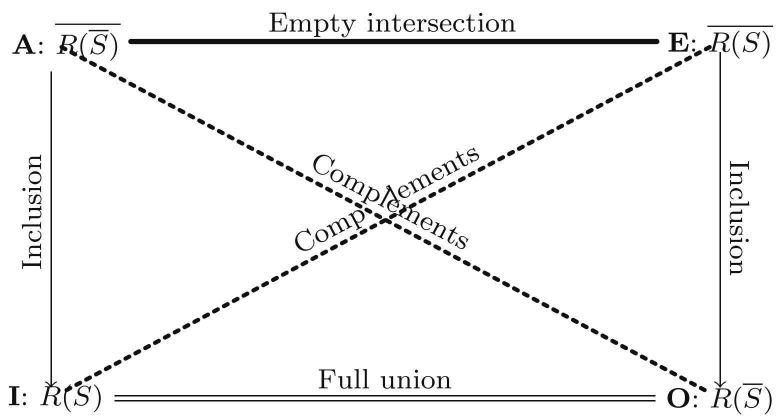

Fig. 2 Square of opposition induced by a relation R and a subset S

existing between the logical statements of the traditional square of opposition still hold here. Namely,

(a) $\overline{R(\overline{S})}$ and $R(\overline{S})$ are complements of each other, as are $\overline{R(S)}$ and $R(S)$;
they correspond to the diagonals of the square;

(b) $\overline{R(\overline{S})} \subseteq R(S)$, and $\overline{R(S)} \subseteq R(\overline{S})$,
thanks to the *X-normalization condition* $\forall x,\ xR \neq \emptyset$. These inclusions are represented by vertical arrows in Fig. 2;

(c) $\overline{R(\overline{S})} \cap \overline{R(S)} = \emptyset$;
this empty intersection corresponds to the thick line in Fig. 2,
and one may have $\overline{R(\overline{S})} \cup \overline{R(S)} \neq Y$;

(d) $R(S) \cup R(\overline{S}) = X$;
this full union corresponds to the double thin line in Fig. 2,
and one may have $R(S) \cap R(\overline{S}) \neq \emptyset$.

Conditions (c)-(d) hold also thanks to the X-normalization of R.

Note that this fits with a regular modal logic reading of this square where R is viewed as an accessibility relation defined on $X \times X$, and S as the set of models of a proposition p. Indeed, $\Box p$ (resp. $\Diamond p$) is true in world x means that p is true at *every* (resp. at *some*) possible world accessible from x; this corresponds to $\overline{R(\overline{S})}$ (resp. $R(S)$) which is the set of worlds where $\Box p$ (resp. $\Diamond p$) is true. It is easy to check axiom K is valid.

Proposition 3 *The relational square satisfies axiom K:* $(\Box(p \to q) \to (\Box p \to \Box q))$.

Proof $\Box(p \to q) \to (\Box p \to \Box q)$ writes in set theoretic terms

$$\overline{R(\overline{S \to T})} \to (\overline{R(\overline{S})} \to \overline{R(\overline{T})}) = Y$$

where $S \to T$ stands for $\overline{S} \cup T$. Developing, it comes

$$R(S \cap \overline{T}) \cup R(\overline{S}) \cup \overline{R(\overline{T})} = Y$$

It means $R(\overline{T}) \subseteq R(S \cap \overline{T}) \cup R(\overline{S})$, but note that $R(S \cap \overline{T}) \cup R(\overline{S}) = R((S \cap \overline{T}) \cup \overline{S}) = R(\overline{T} \cup \overline{S})$, so that it holds indeed. $\Box$

Moreover, the entailment from **A** to **I** is the axiom (D) of modal logic which is known to require serial accessibility relations [10]. So the logical framework of the relational square of opposition seems to be that of the logic KD.

3.2 The relational cube

Let us also consider the complementary relation $\overline{R}$, namely $x\overline{R}y$ if and only if $\neg(xRy)$. We further assume that $\overline{R} \neq \emptyset$ (i.e., $R \neq X \times Y$). Moreover we have also assumed the X-normalization of $\overline{R}$, i.e., $\forall x, \exists y\ \neg(xRy)$. In the same way as previously, we get four other subsets of X from $\overline{R}$. Namely,

$$\overline{R}(\overline{S}) = \{x \in X \mid \exists s \in \overline{S}, \neg(xRs)\} = \{x \in X \mid S \cup xR \neq X\} = \bigcup_{s \in \overline{S}} \overline{R}s \tag{3}$$

$$\overline{R}(S) = \{x \in X \mid \exists s \in S, \neg(xRs)\} = \bigcup\nolimits_{s \in S} \overline{R}s$$

and their complements

$$\overline{\overline{R}(\overline{S})} = \{x \in X \mid \forall s \notin S, xRs\} = \overline{\bigcup\nolimits_{s\in\overline{S}} \overline{R}s} = \bigcap\nolimits_{s\in\overline{S}} \overline{\overline{R}s} = \bigcap\nolimits_{s\in\overline{S}} Rs$$

$$\overline{\overline{R}(S)} = \{x \in X \mid \forall s \in S, xRs\} = \{x \in X \mid S \subseteq xR\} = \overline{\bigcup_{s\in S} \overline{R}s} = \bigcap_{s\in S} Rs \tag{4}$$

The eight subsets involving R and its complement can be organized into a cube of opposition [12, 13] (see Fig. 3). Similar cubes have been recently shown to extend the traditional square of opposition in terms of quantifiers [23], or in the particular setting of abstract argumentation (for the complement of the attack relation) [1]. As can be seen, the front facet of the cube in Fig. 3 is nothing but the square in Fig. 2, and the back facet is a similar square induced by $\overline{R}$. Its vertices are denoted by lower case letters **a**, **i**, **e**, and **o** [15, 23]. Neither the top and bottom facets, nor the side facets are squares of opposition in the above sense since condition (a) is violated: Diagonals do not link complements in these squares.

More precisely, in top and bottom squares, diagonals change R into $\overline{R}$ and vice versa; there is no counterpart of condition (b), in the top square condition (c) holds for the pairs of subsets associated with vertices **A**-**E** and with vertices **a**-**e**, while condition (d) fails (top square), and conversely in the bottom square, condition (d) holds for the pairs of subsets associated with vertices **I**-**O** and with vertices **i**-**o** while condition (c) fails. Moreover, the top and bottom facets exhibit other empty intersection relationships and full union relationships respectively. Indeed in the top facet, e.g. $\overline{\overline{R}(S)} \cap \overline{R(S)} = \emptyset$, since $\overline{\overline{R}(S)} = \{x \in X \mid S \subseteq xR\}$ and $\overline{R(S)} = \{x \in X \mid S \cap xR = \emptyset\}$. Similarly in the bottom facet, e.g. $\overline{R}(S) \cup R(S) = X$, since $\overline{R}(S) = \{x \in X | \exists s \in S, \neg(xRs)\}$ and $R(S) = \{x \in X | \exists s \in S, xRs\}$. This is pictured in Fig. 4 (in order not to overload Fig. 3).

For side facets, condition (b) clearly holds, while both conditions (c)-(d) fail. In side facets, vertices linked by diagonals are exchanged by changing R into $\overline{R}$ (and vice versa) and by applying the overall complementation. These diagonals express set inclusions: $\overline{R(S)} = \{x \in X | \forall s \in S, \neg(xRs)\} \subseteq \{x \in X | \exists s \in S, \neg(xRs)\} = \overline{R}(S)$. In the same way, we have $\overline{\overline{R}(\overline{S})} \subseteq R(\overline{S})$, $\overline{\overline{R}(S)} \subseteq R(S)$, and $\overline{R(\overline{S})} \subseteq \overline{R}(\overline{S})$, as pictured in Fig. 5 for the left side. Mind that while diagonals in front and back facets express complementations, they express inclusions in side facets, empty intersections in top facet, and full union in bottom facets.

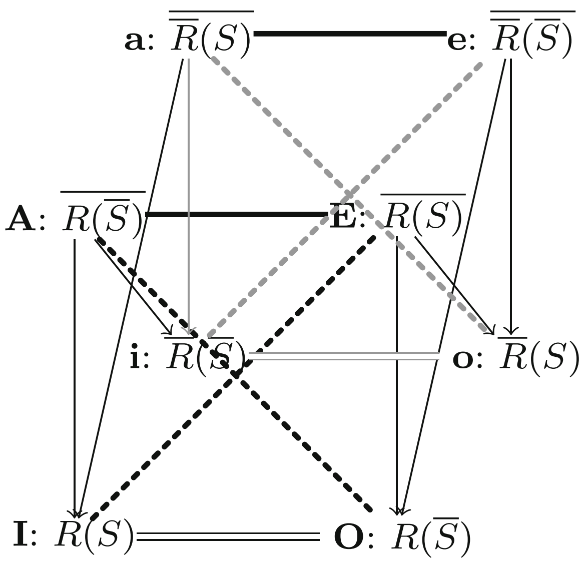

Fig. 3 Cube of opposition induced by a relation R and a subset S

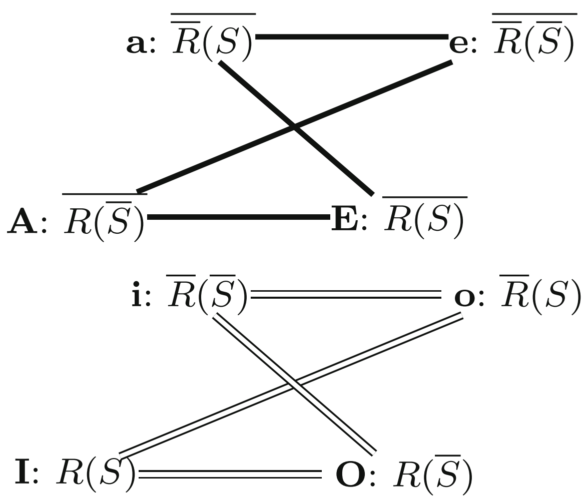

Fig. 4 Top and bottom facets of the cube of opposition

As can be seen, the conditions associated with **A**, **a**, **I**, and **i** can be written respectively $xR \cap \overline{S} = \emptyset$, $\overline{xR} \cap S = \emptyset$, $xR \cap S \neq \emptyset$, $\overline{xR} \cap \overline{S} \neq \emptyset$, while **O**, **o**, **E**, and **e** are respectively associated with their negation. Note that although the vertices of these side squares present some forms of structural opposition, e.g. **A** and **a**, these squares strongly differ from the original square of opposition. Moreover, the conditions associated to the vertices involve all possible ways of comparing the subsets xR and S possibly using their complements. Indeed the conditions $xR \subseteq S$, $S \cap xR = \emptyset$, $S \subseteq xR$, and $S \cup xR = X$ express the four possible inclusion relations of xR wrt S or $\overline{S}$ and define distinct subsets of X. Thus, the 4 subsets $\overline{R(\overline{S})}$, $\overline{R(S)}$, $\overline{\overline{R}(S)}$, and $\overline{\overline{R}(\overline{S})}$ (or their complements $R(\overline{S})$, $R(S)$, $\overline{R}(S)$, and $\overline{R}(\overline{S})$) constitute distinct pieces of information in the sense that one cannot be deduced from the others [18].

In the cube of opposition, three negations are at work, the usual outside one, and two inside ones respectively applying to the relation and to the subset - this gives birth to the eight vertices of the cube - while in the front and back squares (but also in the top and bottom squares) only two negations are at work. Besides, it is obvious that a similar cube can be built for the transpose relation R^t and a subset $T \subseteq X$, then inducing eight other remarkable subsets, now, in Y. This leads us to assume the X-normalizations of R^t and $\overline{R^t}$ $(= \overline{R}^t)$, which is nothing but the Y-normalizations of R and $\overline{R}$ ($\forall y, \exists t, tRy$, and $\forall y, \exists t', \neg(t'Ry)$), as already announced. In other words, we have to assume the seriality of R, R^t, $\overline{R}$, and $\overline{R}^t$.

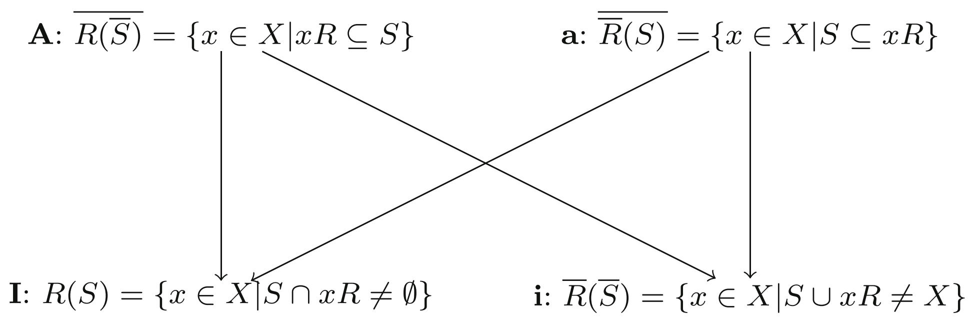

Fig. 5 Left side facet of the cube of opposition induced by a relation R and a subset S

As a summary, we can now state the definition of a cube of opposition.

Definition 2 In a cube of opposition **AIEOaieo**, the following relations are supposed to hold:

Front and back facets:

(a) **A** and **O** are the negation of each other, as well as **E** and **I**, **a** and **o**, and **e** and **i**;
(b) **A** entails **I**, **E** entails **O**, **a** entails **i**, and **e** entails **o**;
(c) **A** and **E** cannot be true together, but may be false together; the same for **a** and **e**;
(d) **I** and **O** cannot be false together, but may be true together; the same for **i** and **o**.

Side facets:

(*e*) **A** entails **i**;
(*f*) **a** entails **I**;
(*g*) **e** entails **O**;
(*h*) **E** entails **o**.

Top and bottom facets :

(*i*) **a** and **E** cannot be true together, but may be false together;
(*j*) the same for **A** and **e**;
(*k*) **i** and **O** cannot be false together, but may be true together;
(*l*) the same for **I** and **o**.

The 8 subsets corresponding to the vertices of the cube of opposition can receive remarkable interpretations in the settings of possibility theory (assuming no intermediary degree between fully possible and impossible) [23], formal concept analysis [13, 23], and rough set theory [11, 13]. The front facet of the cube corresponds to the core of rough set theory, while formal concept analysis operators are on the left-hand side face, and the set functions of possibility theory are associated with the eight vertices of the cube.

It is also interesting to complete the different squares corresponding to the facets of the cube into hexagons, as we are going to see.

3.3 From squares to hexagons

As proposed and advocated by Blanché [8, 9], it is always possible to complete a classical square of opposition into a hexagon:

Definition 3 Given a square of opposition **AEIO**, a *hexagon of opposition* is made of the four square vertices plus the bottom vertex $\mathbf{Y} =_{def} \mathbf{I} \wedge \mathbf{O}$, and the top vertex $\mathbf{U} =_{def} \mathbf{A} \vee \mathbf{E}$.

The hexagon fully exhibits the logical relations inside a structure of opposition generated by the three mutually exclusive situations **A**, **E**, and **Y** (**Y** is the conjunction of the negation of **A** and the negation of **E**), where two vertices linked by a diagonal are contradictories, **A** and **E** entail **U**, while **Y** entails both **I** and **O**. Moreover it can be checked that $\mathbf{I} = \mathbf{A} \vee \mathbf{Y}$ and $\mathbf{O} = \mathbf{E} \vee \mathbf{Y}$. Conversely, three mutually exclusive situations playing the roles of **A**, **E**, and **Y** always give birth to a hexagon [23], which is made of three squares of opposition: **AEOI**, **AYOU**, and **EYIU**, as in Fig. 6. The interest of this hexagonal construct has been

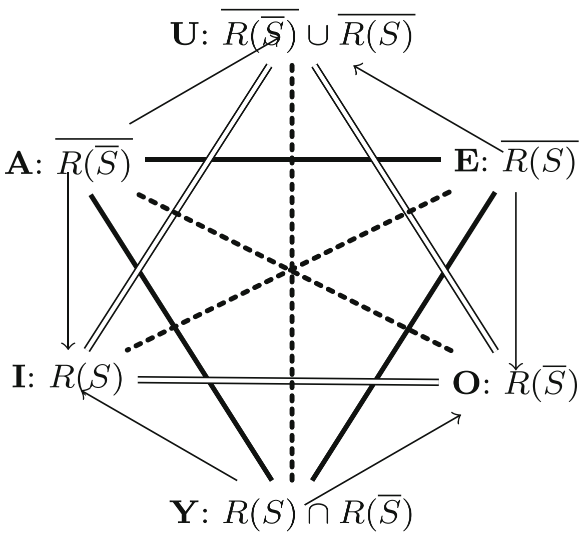

Fig. 6 Hexagon associated with the front facet of the cube

rediscovered and advocated again by Béziau [3] in recent years in particular for solving delicate questions in paraconsistent logic modeling.

Applying this idea to the front facet of the cube of opposition induced by a relation and a subset, we obtain the hexagon of Fig. 6, associated with the tri-partition $\{\overline{R(\overline{S})}, \overline{R(S)}, R(S) \cap R(\overline{S})\}$. Note that indeed $R(\overline{S}) = \overline{R(S)} \cup (R(S) \cap R(\overline{S}))$ (since $R(\overline{S}) \supseteq \overline{R(S)}$). Similarly, $R(S) = \overline{R(\overline{S})} \cup (R(S) \cap R(\overline{S}))$. In Fig. 6, arrows ($\rightarrow$) indicate set inclusions ($\subseteq$). A similar hexagon is associated with the back facet, changing R into $\overline{R}$.

Another type of hexagon can be associated with side facets. The one corresponding to the left-hand side facet is pictured in Fig. 7. Now, not only the arrows of the sides of the hexagon correspond to set inclusions, but also the diagonals (oriented downwards). Indeed $\overline{R(\overline{S})} \subseteq \overline{R}(\overline{S})$ and $\overline{\overline{R}(S)} \subseteq R(S)$. Moreover, since $\overline{R(S)} \subseteq \overline{R}(S)$ and using the inclusions corresponding to the vertical edges of the cube, we get

$$\overline{R(\overline{S})} \cup \overline{\overline{R}(S)} \subseteq R(S) \cap \overline{R}(\overline{S}),$$

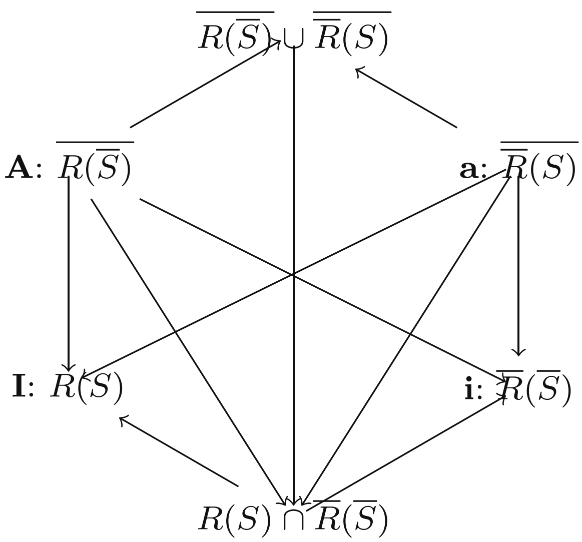

Fig. 7 Hexagon induced by the left-hand side square

and for the right-hand side square, by De Morgan duality, we have

$$\overline{R(S)} \cup \overline{\overline{R}(\overline{S})} \subseteq R(\overline{S}) \cap \overline{R}(S).$$

One may wonder if one can build useful hexagons from the bottom and top squares of the cube. It is less clear, as shown by the following discussion. Indeed, if we consider the four subsets involved in the bottom square, namely $R(S)$, $R(\overline{S})$, $\overline{R}(S)$ and $\overline{R}(\overline{S})$ (the top square has their complements as vertices), they are weakly related through $R(S) \cup R(\overline{S}) = X$, $\overline{R}(S) \cup \overline{R}(\overline{S}) = X$, $R(S) \cup \overline{R}(S) = X$ and $R(\overline{S}) \cup \overline{R}(\overline{S}) = X$. Still $R(S) \cup \overline{R}(\overline{S})$ or $\overline{R}(S) \cup R(\overline{S})$ (or their complements in the top square, $\overline{R(S)} \cap \overline{\overline{R}(\overline{S})}$ or $\overline{\overline{R}(S)} \cap \overline{R(\overline{S})}$) are compound subsets that may be of interest for some particular understanding of relation R. Note that in general, $R(S) \cup \overline{R}(\overline{S}) \neq X$ and $\overline{R}(S) \cup R(\overline{S}) \neq X$. Note also that similar combinations changing $\cap$ into $\cup$ and vice versa, namely $\overline{\overline{R}(S)} \cup \overline{R(\overline{S})}$ and $R(S) \cap \overline{R}(\overline{S})$ already appear in the hexagons associated with the side facets of the cube, while $R(S) \cap R(\overline{S})$ and $\overline{R}(S) \cap \overline{R}(\overline{S})$ are the **Y**-vertices of the hexagons associated with the front and back facets of the cube of opposition.

Besides, it would be also possible to complete the top facet into yet another type of hexagon by taking the complements of $\overline{\overline{R}(S)} \cup \overline{R(\overline{S})}$ or of $\overline{R(S)} \cup \overline{\overline{R}(\overline{S})}$, which clearly have empty intersections with the subsets attached to vertices **A** and **a** and to vertices **E** and **e** respectively. However, the intersection of the two resulting subsets, namely $\overline{R}(S) \cap R(\overline{S})$ and $R(S) \cap \overline{R}(\overline{S})$ is not necessarily empty. A dual construct could be proposed for the bottom facet.

3.4 Boolean closure of the cube

In search of the hexagons arising from the different facets of the cube, several combinations of the vertices have been considered, obtaining new subsets of the universe from R and S. One may push this procedure further, and study the Boolean closure of the cube, that is all the non-trivial (i.e., different from the empty set and the universe) subsets that can be obtained from the starting eight ones using standard set operations. We notice that the hexagon is the Boolean closure of the square.

As we may expect, the Boolean closure of the cube is more complicated to compute than the Boolean closure of the square. At least one may wonder if it contains other cubes of the same kind. Indeed, one may consider the 3 potential cubes made by pairing the 3 squares of the hexagon of the front facet with the 3 corresponding squares of the hexagon of the back facet. However, the 2 additional potential cubes thus obtained do not make cubes of opposition in the above sense, since the expected relations on the side facets do not hold; for instance, **A** does not entail **u**, where $\mathbf{u} = \mathbf{a} \vee \mathbf{e}$.

4 Gradual structure of opposition

We have examined the abstract logical structure underlying the classical square of opposition and the derived structures. The statements associated to the vertices were supposed to be true or false, and the membership in the counterpart in terms of relation and subset was all-or-nothing. We now discuss how these structures could be generalized when truth, or membership, become graded. This is of interest since various settings, for which structures of opposition are relevant, are naturally graded, such as possibility theory.

4.1 The graded square of opposition

Let us now associate a degree in [0, 1] to each vertex **A**, **E**, **O**, and **I** of the square of opposition, namely α, ϵ, o, and ι respectively. Let n be an involutive negation operator (i.e., $n(1) = 0$; $n(n(s)) = s$), and $*$ a symmetrical conjunction (i.e., $s * t = t * s$; $0 * s = 0$; $1 * s = s$). Let $\oplus$ denote the De Morgan dual of $*$, namely, $s \oplus t = n(n(s) * n(t))$, $\Rightarrow$ the strong implication, i.e., $s \Rightarrow t = n(s * n(t)) = n(s) \oplus t$, associated with $*$. Note that $s \Rightarrow t = 1$ is equivalent to $s * n(t) = 0$.

Definition 4 A *graded square of opposition* $\alpha, \epsilon, o, \iota$ is a square where each corner is associated to a degree in [0, 1] and constraints (a)–(d) are rendered as follows.

(a) **A** and **O** are contradictories, i.e., the negation of each other, as well as **E** and **I**, can be encoded by

$$\alpha = n(o) \qquad \text{and} \qquad \epsilon = n(\iota)$$

(b) **A** and **I** are sub-alterns, so **A** entails **I**. Also **E** and **O** are sub-alterns and **E** entails **O**. This translates into $\alpha \Rightarrow \iota = 1$ and $\epsilon \Rightarrow o = 1$, i.e.,

$$\alpha * n(\iota) = 0 \qquad \text{and} \qquad \epsilon * n(o) = 0;$$

(c) **A** and **E** are contraries, hence they cannot be true together, but may be false together. It can be encoded by

$$\alpha * \epsilon = 0 \text{ or equivalently } n(\alpha * \epsilon) = 1;$$

(d) **I** and **O** are sub-contraries, hence they cannot be false together, but may be true together. It can be encoded by

$$n(\iota) * n(o) = 0 \text{ or equivalently } n(n(\iota) * n(o)) = 1, \text{ i.e., } \iota \oplus o = 1.$$

It is straightforward to check that we still have for any involutive negation and any symmetrical conjunction that conditions (a) and (b) entail conditions (c) and (d) (i.e., dep 1 and dep 2 in Proposition 2), conditions (a) and (c) entail conditions (b) and (d), and conditions (a) and (d) entail conditions (b) and (c) (i.e., dep 3).

Let us consider the various algebraic structures that can support the square of opposition.

Kleene systems Taking min for the conjunction $*$, we get the constraints

(a) $\alpha = n(o)$; $\epsilon = n(\iota)$;
(b) $\min(\alpha, n(\iota)) = 0$ and $\min(\epsilon, n(o)) = 0$;
(c) $\min(\alpha, \epsilon) = 0$;
(d) $\max(\iota, o) = 1$.

Then using $n(s) = 1 - s$ together with $* = \min$, we get a Kleene system leading to a square of opposition. An instantiation of such a graded square of opposition is provided by the main set functions in possibility theory applied to classical events [20].

Example 1 Given a (finite) set Ω, and a possibility distribution $\pi : \Omega \to [0, 1]$, assumed to be normalized (i.e., $\exists \omega \in \Omega, \pi(\omega) = 1$), let us denote by $\Pi(A) = \max_{\omega \in A} \pi(\omega)$ the possibility degree of a proposition with set of models A, and by $N(A) = 1 - \Pi(\overline{A})$ the conjugate necessity degree. We can instantiate the gradual square of opposition by letting $\alpha = N(A)$, $\epsilon = N(\overline{A})$, $\iota = \Pi(A)$, $o = \Pi(\overline{A})$. Then (a) is nothing but the duality between possibility and necessity degrees. Conditions (b), (c) and (d) hold as well, since respectively we have in possibility theory that $\Pi(A) < 1$ entails $N(A) = 0$ (or equivalently $N(A) > 0$ entails $\Pi(A) = 1$), $\min(N(A), N(\overline{A})) = 0$, and $\max(\Pi(A), \Pi(\overline{A})) = 1$.

Łukasiewicz systems If we take Łukasiewicz triangular norm [29, 40], namely, $\max(0, \cdot + \cdot - 1)$ for the conjunction $*$, the conditions write

(a) $\alpha = n(o); \epsilon = n(\iota)$;
(b) $n(\iota) \leq 1 - \alpha$ and $n(o) \leq 1 - \epsilon$;
(c) $\alpha + \epsilon \leq 1$;
(d) $n(\iota) + n(o) \leq 1$.

Choosing again $n(s) = 1 - s$ simplifies the above conditions into

(a) $\alpha = 1 - o; \epsilon = 1 - \iota$;
(b) $\alpha \leq \iota$ and $\epsilon \leq o$;
(c) $\alpha + \epsilon \leq 1$;
(d) $\iota + o \geq 1$.

Such a graded extension of the square of opposition is also a remarkable instance of another point of view we may think of for extending the square of opposition with grades. Namely, starting only from a symmetrical conjunction $*$, we may both build the implication $\to$ and the negation ν by residuation. Namely, $s \to t = \max\{x \in [0, 1] \mid s * x \leq t\}$, and $\nu(s) = s \to 0$. Taking $* = \min$, it yields Gödel implication $s \to t = 1$ if $s \leq t$ and $s \to t = t$ otherwise. Unfortunately in this case, the negation obtained, namely $\nu(s) = 1$ if $s = 0$, and $\nu(s) = 0$ if $s > 0$, is no longer many-valued, nor involutive, and then the symmetries of the square of opposition are completely lost, that is conditions (a)–(d) do not hold. However, starting with $* = \max(0, \cdot + \cdot - 1)$, we obtain Łukasiewicz implication $s \to t = \min(1, 1 - s + t)$, and then $\nu(s) = 1 - s$ as a negation, which is involutive. This again corresponds to the above system of constraints.

Note that the latter system of constraints is weaker than the Kleene system, since we have the following entailments

(b) $\min(\alpha, 1 - \iota) = 0$ entails $\alpha \leq \iota$ and $\min(\epsilon, n(o)) = 0$ entails $\epsilon \leq o$;
(c) $\min(\alpha, \epsilon) = 0$ entails $\alpha + \epsilon \leq 1$;
(d) $\max(\iota, o) = 1$ entails $\iota + o \geq 1$.

An instantiation of the above weaker system is provided by possibility theory where possibility and necessity measures are extended to fuzzy events, as explained below.

Example 2 (Example 1 continued) The possibility of a fuzzy event A [44], represented by its membership function $\mu_A : \Omega \to [0, 1]$, is defined by $\Pi(A) = \max_{\omega \in \Omega} \min(\mu_A(\omega), \pi(\omega))$, and its necessity by $N(A) = 1 - \Pi(\overline{A})$. Letting $\alpha = N(A)$, $\epsilon = N(\overline{A})$, $\iota = \Pi(A)$, $o = \Pi(\overline{A})$ again, (a) always expresses the duality between possibility and necessity degrees, while condition (b), (c) and (d) still hold in a weaker way with respect to the case of classical events, since we have for fuzzy events $N(A) \leq \Pi(A)$, $N(A) + N(\overline{A}) \leq 1$, and $\Pi(A) + \Pi(\overline{A}) \geq 1$ respectively.

More generally, we have the following result for nilpotent triangular norms [29, 40] (i.e., triangular norms $*$ that are continuous and such that for each x in the open interval (0, 1) there is a natural number n such that $x * \cdots * x$ (n times) equals 0). $* = \max(0, \cdot + \cdot - 1)$ is the prototype of nilpotent triangular norms.

Proposition 4 *Any system $(*, n)$ with $*$ a nilpotent triangular norm and $n(x) = 1 - x$ that satisfies conditions (a) $\alpha = n(o)$, $\epsilon = n(\iota)$, (b) $\alpha \leq \iota$ and $\epsilon \leq o$, (c) $\alpha * \epsilon = 0$, (d) $n(\iota) * n(o) = 0$ satisfies the dependencies dep 1, dep 2, and dep 3.*

Zadeh systems Note that the preservation of the whole structure of dependencies in the square of opposition relies on the use of the strong implication associated with the conjunction, for expressing conditions (b). Instead, we may prefer to choose the implication independently from $*$, and use for (b) weaker conditions such as the ones obtained with residuated implications, namely $\alpha \leq \iota$ and $\epsilon \leq o$. These weaker (b)-conditions are of interest for dealing for instance with fuzzy sets whose inclusion is defined pointwisely by such inequalities. As an illustration, take $n(s) = 1 - s$, $* = \min$, together with the conditions $\alpha \leq \iota$ and $\epsilon \leq o$, we obtain the following extension of the square of opposition, which may be of interest, but where the entailment of (c) and (d) from (a) and (b) no longer holds:

(a) $\alpha = 1 - o$; $\epsilon = 1 - \iota$;
(b) $\alpha \leq \iota$ and $\epsilon \leq o$;
(c) $\min(\alpha, \epsilon) = 0$;
(d) $\max(\iota, o) = 1$;

Combining squares In all the previous examples, we always considered triangular norms as conjunctions, even if associativity, and increasingness played no role. A situation where these two properties are required is when considering the idea of combining squares of opposition as recently proposed in [37].

Proposition 5 *Given two squares of opposition **A**, **I**, **E**, **O** and **A**′, **I**′, **E**′, **O**′, it can be checked that $\boldsymbol{A} \wedge \boldsymbol{A}'$, $\boldsymbol{I} \vee \boldsymbol{I}'$, $\boldsymbol{E} \wedge \boldsymbol{E}'$, $\boldsymbol{O} \vee \boldsymbol{O}'$ make another square of opposition satisfying the four characteristic conditions. This fact extends to the graded case using the same conjunction $*$ and disjunction $\oplus$ for the combination of the squares.*

Proof With obvious notations, the following can be easily checked:

(a) $\alpha = n(o)$ and $\alpha' = n(o')$ entail $\alpha * \alpha' = n(o \oplus o')$. Similarly, $\epsilon = n(\iota)$ and $\epsilon' = n(\iota')$ entail $\epsilon * \epsilon' = n(\iota \oplus \iota')$.
(b) $\alpha * n(\iota) = 0$ and $\alpha' * n(\iota') = 0$ entail $\alpha * \alpha' * n(\iota) * n(\iota') = 0$ using commutativity and associativity, i.e., $(\alpha * \alpha') * n(\iota \oplus \iota') = 0$. Similarly, $\epsilon * n(o) = 0$ and $\epsilon' * n(o') = 0$ entail $(\epsilon * \epsilon') * n(o \oplus o) = 0$.
(c) $\alpha * \epsilon = 0$ and $\alpha' * \epsilon' = 0$ entail $(\alpha * \alpha') * (\epsilon * \epsilon') = 0$ (using associativity).
(d) $n(\iota) * n(o) = 0$ and $n(\iota') * n(o') = 0$ entail $n(\iota \oplus \iota') * n(o \oplus o') = 0$.

□

The combination of the squares then remains associative thanks to the associativity of triangular norms. Besides, note that in case we use the weak form of (b), namely $\alpha \leq \iota$ and $\epsilon \leq o$, then $\alpha \leq \iota$ and $\alpha' \leq \iota'$ entail, from the increasingness of $*$, that $\alpha * \alpha' \leq \iota * \iota'$ (and we have a similar entailment for the second condition). An example where this combination of squares make sense is again exemplified using possibility theory.

Example 3 (Examples 1 and 2 continued) If two squares of opposition correspond respectively to possibility and necessity functions $\Pi_i, N_i, i = 1, 2$ based on two possibility distributions π_1 and π_2, then the two corresponding squares of opposition can be combined into one square by setting $\alpha = \min(N_1(A), N_2(A))$, $\epsilon = \min(N_1(\overline{A}), N_2(\overline{A}))$, $\iota = \max(\Pi_1(A), \Pi_2(A))$, $o = \max(\Pi_1(\overline{A}), \Pi_2(\overline{A}))$, which yields a Kleene square of opposition for non fuzzy events. The reason is that $\max(\Pi_1(A), \Pi_2(A)) = \max_{\omega \in A} \max(\pi_1(\omega), \pi_2(\omega))$ is the possibility function induced by the possibility distribution $\max(\pi_1, \pi_2)$ [21]. For fuzzy events, only the weak form of (b) will be preserved.

4.2 Graded cube of opposition

As for the square of opposition, graded extensions can be proposed for the cube. Let us suppose that the front and back of the cube are square of opposition and α, ι, ϵ, o, and α', ι', ϵ', o' be the grades associated to vertices **A**, **I**, **E**, **O** and **a**, **i**, **e**, **o**. Then, we can define a graded cube of opposition in the following way.

Definition 5 Given an involutive negation n, a symmetrical conjunction $*$, and using the strong implication associated with $*$, the constraints associated to the front facet and the back facet of the cube yield

(a) $\alpha = n(o)$, $\epsilon = n(\iota)$ and $\alpha' = n(o')$ and $\epsilon' = n(\iota')$;
(b) $\alpha * n(\iota) = 0$, $\epsilon * n(o) = 0$ and $\alpha' * n(\iota') = 0$, $\epsilon' * n(o') = 0$;
(c) $\alpha * \epsilon = 0$ and $\alpha' * \epsilon' = 0$;
(d) $n(\iota) * n(o) = 0$ and $n(\iota') * n(o') = 0$.

Then, the constraints associated with the side facets (e)–(h) and with the top and bottom facets (i)–(l) read as:

(e)	$\alpha * n(\iota') = 0$	(i)	$\alpha' * \epsilon = 0$
(f)	$\alpha' * n(\iota) = 0$	(j)	$\alpha * \epsilon' = 0$
(g)	$\epsilon' * n(o) = 0$	(k)	$n(\iota') * n(o) = 0$
(h)	$\epsilon * n(o') = 0.$	(l)	$n(\iota) * n(o') = 0.$

Then, it can be straightforwardly checked that given conditions (a)-(b)-(c)-(d), conditions (e)-(f)-(g)-(h) are equivalent to conditions (i)-(j)-(k)-(l). Moreover, one may think of using the weak view of implication, which amounts to writing for the front and the back facets (b') in place of (b):

(b') $\alpha \leq \iota$, $\epsilon \leq o$ and $\alpha' \leq \iota'$, $\epsilon' \leq o'$;

and for the side facets:

(e') $\alpha \leq \iota'$;
(f') $\alpha' \leq \iota$;
(g') $\epsilon' \leq o$;
(h') $\epsilon \leq o'$.

In the general case, the top (equiv., bottom) and side constraints cannot be derived from the conditions on the square neither one from the other. For instance, if we consider the drastic triangular norm as the conjunction $*$, namely $s * t = s$ if $t = 1$, $s * t = t$ if $s = 1$, and $s * t = 0$ otherwise, it is easy to find values that satisfy the top / bottom constraints and not the side ones. Conversely, using $* = \min$, we easily get that side facets can be satisfied without satisfying the top / bottom ones.

If we put more constraints on the conjunction, we have the following result.

Proposition 6 *If $*$ is a left-continuous triangular norm and its residuated implication $\to_*$ has contrapositive symmetry (that is, $x \to_* y = n(y) \to_* n(x)$), then the top and bottom conditions are equivalent to the side ones.*

Proof From $\alpha' * \epsilon = 0$, we get $\alpha' * n(\iota) = 0$ which, given the hypothesis, holds iff $\alpha' \leq \iota$ (see [25]). The other conditions are obtained in a similar way. □

This result should not come as a surprise since the prototypical case of residuated implication having contrapositive symmetry is Łukasiewicz implication, which is also a strong implication with respect to $* = \max(0, \cdot + \cdot - 1)$.

Weaker results can be proved for other triangular norms. For instance,

Proposition 7 *In case of $* = \min$ we have that top / bottom conditions imply the (e')-(f')-(g')-(h') version of side ones.*

Proof Condition $\min(\alpha', \epsilon) = 0$ implies $\alpha' = 0$ or $\epsilon = 0$ and consequently $\alpha' \leq \iota = n(\epsilon)$. On the other hand, α' can be less or equal to ι with $\alpha \neq 0$ and $\iota \neq 1$. □

The converse of the above proposition does not generally hold.

Such a graded cube can receive different instantiations. One would be in terms of fuzzy relations as discussed in the last section of this paper. Another is in terms of (graded) possibility theory [22], as briefly indicated below.

Example 4 (Examples 1, 2 and 3 continued) Assuming that the normalized possibility distribution $\pi : \Omega \to [0, 1]$, is also such that $1 - \pi$ is normalized (i.e., $\exists \omega \in \Omega, \pi(\omega) = 0$), let us denote by $\Delta(A) = \min_{\omega \in A} \pi(\omega)$ the strong / guaranteed possibility degree of a proposition with set of models A, and by $\nabla(A) = 1 - \Delta(\overline{A})$ the conjugate degree. We can instantiate the gradual square of opposition by letting $\alpha' = \Delta(A)$, $\epsilon' = \Delta(\overline{A})$, $\iota' = \nabla(A)$, $o' = \nabla(\overline{A})$. Thanks to the duality between Δ and ∇, and normalization of $1 - \pi$, it can be checked that $\alpha', \epsilon', \iota', o'$ makes a square of opposition for $* = \min$, and $n(s) = 1 - s$. Since $\Delta(A) \leq \Pi(A)$ and $\nabla(A) \leq N(A)$, the constraints of the side facets hold under the form (e')-(f')-(g')-(h').

4.3 The gradual hexagon

Now, let us investigate the graded counterpart of the hexagon. We will denote as ν and γ the degrees of truth associated to U and Y (since they both collapse into the letter upsilon in Greek). Then, we should have

$$\nu = \alpha \oplus \epsilon \qquad \text{and} \qquad \gamma = \iota * o \tag{5}$$

If we want to preserve **AYOU** as a square of opposition, we should have that

(a) $\nu = n(\epsilon)$. This is true once condition (5) is satisfied. Indeed, $\nu = \alpha \oplus \epsilon = n(\iota) \oplus n(o) = n(\iota * o) = n(\gamma)$.
(b) A entails U and Y entails O, that is $\alpha \leq \nu$ and $\gamma \leq o$[1]. Using again condition (5), this means $\alpha \leq \alpha \oplus \epsilon$ and $\iota * a \leq o$, which is true for any choice of monotonic conjunction $*$ and disjunction $\oplus$, and in particular for all triangular norms and triangular conorms.
(c) $\alpha * \gamma = 0$. This condition, generally, does not follow from the previous ones, so that we should impose it.
(d) $n(o) * n(\nu) = 0$. This condition is equivalent to the previous one.

Similarly, in the case of the square **EYIU**, we should impose that Y entails I, E entails U and $\epsilon * \gamma = 0$ (or equivalently, $n(i) * n(\nu) = 0$).

Proposition 8 *In the case of nilpotent triangular norms such as* $* = \max(0, \cdot + \cdot - 1)$, *condition* (5) *alone ensures that all the three squares **AEOI**, **AYOU**, and **EYIU** satisfy conditions (a)–(d) as well as the set of dependencies among them.*

Proof $\alpha * \gamma = 0$ leads to the constraint $n(\alpha * n(\nu)) = 1$, which is equivalent to $n(\alpha * n(\alpha \oplus \epsilon)) = 1$, i.e., we have, $\alpha * (n(\alpha) * n(\epsilon)) = 0$. Assuming that $*$ is a triangular norm, and applying associativity we get $(\alpha * n(\alpha)) * n(\epsilon) = 0$. Since it should hold for any ϵ, it leads to require that $\alpha * n(\alpha) = 0$ for any α, which amounts to saying that $*$ is a nilpotent triangular norm, typically $*$ is the Łukasiewicz triangular norm $s * t = \max(0, s + t - 1)$. Then, we already know, by Proposition 4 that with a nilpotent triangular norm and the four constraints (a)-(b)-(c)-(d), all the dependencies are satisfied for the three squares **AEOI**, **AYOU**, and **EYIU**. We thus get a perfect graded extension of the hexagon. □

Summarizing,

– in order to have a hexagonal structure with three squares of opposition, it is sufficient to satisfy condition (5), condition (c) (for the three squares) and the fact that $*$ and $\oplus$ are dual triangular norms;
– condition (5) and a nilpotent triangular norm (with its dual co-norm) are sufficient to satisfy condition (a)-(d) for all the three squares as well as their dependencies.

Conversely, given a fuzzy partition in the sense of Ruspini [39], we can generate a hexagon.

Proposition 9 *Let* α, ϵ *and* γ *be three numbers such that* $\alpha + \epsilon + \gamma = 1$ *(a fuzzy partition). If we consider the three other numbers* $\iota = n(\epsilon)$, $o = n(\alpha)$, $\nu = n(\gamma)$, *where* $n = 1 - (\cdot)$ *is the standard involutive negation and* $*$ *the Łukasiewicz triangular norm, we obtain a hexagon of opposition.*

Proof It is clear that $\alpha * \epsilon = \alpha * \gamma = \gamma * \epsilon = 0$, since $\alpha + \epsilon \leq 1$ and so on. Moreover, we have $\alpha \to \iota = \min(1, 1 - \alpha + 1 - \epsilon) = 1$ (since $\alpha + \epsilon \leq 1$); we also have $\alpha \to \nu =$

[1]In the following, we use the weak form of the entailment. Indeed if we want to keep the strong form, Y entails O, for instance, would translate into $\iota * o * n(o) = 0$, which would require that $*$ is such that $s * n(s) = 0$, which would lead to take $*$ as a nilpotent triangular norm.

$\min(1, 1-\alpha+1-\gamma) = 1$, and $\gamma \to \iota = \min(1, 1-\gamma+1-\epsilon) = 1$. It can be checked that all the other hexagonal constraints are satisfied as well. □

This result could be generalized to any nilpotent triangular norm using a partition of the form $\varphi(\alpha) + \varphi(\epsilon) + \varphi(\gamma) = 1$ where φ is the additive generator of the triangular norm.

Note also that the constraint $\alpha + \epsilon + \gamma = 1$ can be weakened into $\alpha + \epsilon + \gamma \leq 1$ without any harm for the triple square-of-opposition structure. This should not come as a surprise since the mutual exclusiveness of the three situations **A**, **E**, **Y** is the key condition for having the hexagon in the non-graded case, while there is no covering condition. We notice that these two different hexagons are named, respectively, *strong* and *weak* in [35]. The context there is different, since based on modal logic, but the interpretation is similar. Indeed, a weak hexagon is such that the disjunction of the three vertices is not necessarily true whereas it is true in a strong hexagon and in both cases the conjunction of any two (out of three) vertices is false.

Finally, let us consider the hexagon built on the side of the cube (see Fig. 7). With respect to the constraints holding on the cube, we should add the ones concerning **U** and **Y**: $\alpha \leq \gamma$, $\alpha \leq \nu$, $\alpha' \leq \gamma$, $\alpha' \leq \nu$, $\gamma \leq \iota$, $\gamma \leq \iota'$, $\nu \leq \gamma$. From this list, we notice that it must hold that $\alpha \leq 0.5$. This is due to the fact that $\alpha \leq \nu$ and $\alpha \leq n(\nu) = \gamma$. The same holds for α'. Then, since it should hold that $\nu \leq \gamma = n(\nu)$, we get that $\nu \leq 0.5$ and $\gamma \geq 0.5$ and consequently $\iota, \iota' \geq 0.5$. That is, the upper part of the hexagon is made of all values less or equal to 0.5 and the lower part of all values greater or equal to 0.5.

5 Structures of opposition and the composition of two relations

In this section, we consider, as basic bricks to build our structures of opposition, two relations and the different possibilities to combine them. This is a well-known topic both in the Boolean and fuzzy settings [2, 14, 30]. We will discuss how these approaches fit into our framework.

5.1 The Boolean case

Let us suppose that $R \subseteq (X \times Y)$ and $S \subseteq (Y \times Z)$, we want to build new relations on $X \times Z$ by a composition of R, S. In order to build a square of opposition, and in particular to guarantee that the entailments from top to bottom of the square hold, we have to take care of the normalization conditions. That is, R, S and their transpose should be serial and consequently xR, Ry, yS and Sz cannot be empty. Assuming this condition we have that the four corners of the square read as follows.

$$
\begin{array}{ll}
\textbf{(I)} & R \circ S = \{(x,z) | \exists y : xRy \text{ and } ySz\} = \{(x,z) | xR \cap Sz \neq \emptyset\} \\
\textbf{(E)} & \overline{R \circ S} = \{(x,z) | \forall y : xRy \Rightarrow \neg ySz\} = \{(x,z) | xR \subseteq \overline{Sz}\} \\
\textbf{(A)} & \overline{R \circ \overline{S}} = \{(x,z) | \forall y : xRy \Rightarrow ySz\} = \{(x,z) | xR \subseteq Sz\} \\
\textbf{(O)} & R \circ \overline{S} = \{(x,z) | \exists y : xRy \text{ and } \neg ySz\} = \{(x,z) | xR \cap \overline{Sz} \neq \emptyset\}
\end{array}
$$

We notice that (**A**) is the Bandler and Kohout [30] subproduct, (**I**) is the basic composition of relations, and clearly (**O**) and (**E**) are their negations. To be more precise, (**A**) is the De Baets and Kerre [14] version of the subproduct due to the presence of the normalization conditions, which are not considered by Bandler and Kohout.

Example 5 Let us consider the example 2.1 "medical diagnosis" taken from [14]. So, R is a relation between a set of patients X and symptoms Y and S a relation between symptoms and illnesses Z. Then, we have the following interpretation: $x(\mathbf{I})z$ iff the patient x has at least one symptom associated to the illness z; $x(\mathbf{A})z$ iff all the symptoms of x are symptoms of the illness z. Now, the negations of these two relations can be interpreted as: $x(\mathbf{E})z$ iff x has none of the symptoms of z; $x(\mathbf{O})z$ iff x has some symptom not related to the illness z.

Similarly, and with the normalization conditions applied to the negation of R and S, the back square of the cube is made by the four corners

(**o**) $\overline{R} \circ S = \{(x, z) | \exists y : \neg xRy \text{ and } ySz\} = \{(x, z) | \overline{xR} \cap Sz \neq \emptyset\}$

(**a**) $\overline{\overline{R} \circ S} = \{(x, z) | \forall y : xRy \Leftarrow ySz\} = \{(x, z) | Sz \subseteq xR\}$

(**e**) $\overline{\overline{R} \circ \overline{S}} = \{(x, z) | \forall y : \neg xRy \Rightarrow ySz\} = \{(x, z) | \forall y : xRy \text{ or } ySz\} = \{(x, z) | xR \cup Sz = Y\}$

(**i**) $\overline{R} \circ \overline{S} = \{(x, z) | \exists y : \neg xRy \text{ and } \neg ySz\} = \{(x, z) | \overline{xR} \cap \overline{Sz} \neq \emptyset\}$

In this case, we have that (**a**) is the De Baets and Kerre reading of Bandler and Kohout superproduct and (**o**) its negation. On the other hand, (**e**) and its negation (**i**) seem to be new ways to compose relations. The corner (**e**) expresses the idea that x and z are in relation if they are in connection (through R and S) to the rest of the world.

Example 6 (Example 5 continued) Coming back to the example of medical diagnosis, we have the following interpretations: $x(\mathbf{a})z$ iff the patient x has all the symptoms of the illness z (and perhaps some more); $x(\mathbf{o})y$ iff there exists a symptom of the illness z that the patient x does not have; $x(\mathbf{e})z$ iff the symptoms possessed by x plus the symptoms characterizing z cover the whole universe of symptoms; finally $x(\mathbf{i})z$ is the negation of this last situation: there exists a symptom not possessed by x and not characterizing z.

Now, let us turn our attention to the hexagons. The one built on the front face adds the two corners to the traditional square:

(**U**) $\overline{R \circ \overline{S}} \cup \overline{R \circ S} = \{(x, z) | xR \subseteq Sz \quad \text{or} \quad xR \cap Sz = \emptyset\}$

(**Y**) $R \circ \overline{S} \cap R \circ S = \{(x, z) | xR \cap \overline{Sz} \neq \emptyset \quad \text{and} \quad xR \cap Sz \neq \emptyset\}$

That is, (**U**) represents the case where x, z are in relation if z is connected to all or none of the elements connected to x. On the other hand, (**Y**) implies that there exists at least one element related to x and to z and another one related to x and not to z.

Example 7 (Examples 5 and 6 continued) In terms of Example 5, (**U**) represents the situation where the symptoms of a patient x are all or none of the symptoms of an illness z. The corner (**Y**) represents the case of a patient with some symptoms of an illness z and some symptoms not from z.

If we consider the hexagons, build on the left side (see Fig. 7), the new corners introduce new operators, which however do not seem to have an interesting interpretation with respect to the composition of relations. They are the union of (**A**) and (**a**), i.e., the union of super and sub-product, and the intersection of (**I**) (the standard composition) and (**i**). On the other hand, in this case one of the possible hexagons built on the top deserves some interest, since the conjunction of (**a**) and (**A**) defines the square-product, the fourth and last operation introduced by Bandler and Kohout. This particular hexagon is completed by the new operator (**e**) $\cup$ (**E**).

5.2 The gradual case

By giving a reading of the composition of relations in terms of characteristic functions, De Baets and Kerre [14] give two different generalizations of Bandler and Kohout operations. We start from them in order to give a generalization of the cube to the gradual case.

Let $*$ be a triangular norm, R and S fuzzy relations, $\rightarrow$ a fuzzy implication. Then, the standard composition of relations, corresponding to the corner (**I**) is:

$$(R \circ S)(x, z) = \sup_{y \in Y} R(x, y) * S(y, z)$$

By its negation we obtain (**E**), using $n(s) = 1 - s$:

$$(\overline{R \circ S})(x, z) = 1 - \sup_{y \in Y} R(x, y) * S(y, z) = \inf_{y \in Y} (1 - R(x, y)) \oplus (1 - S(y, z))$$

The corner (**A**) corresponds to the fuzzified subproduct. The two (different) definitions given by De Baets and Kerre are:

$$(\mathbf{A}) \qquad \min \left\{ \inf_{y \in Y} R(x, y) \rightarrow S(y, z), \sup_{y \in Y}(R(x, y) * S(y, z)) \right\}$$

$$(\mathbf{A}') \qquad \min \left\{ \inf_{y \in Y} R(x, y) \rightarrow S(y, z), \sup_{y \in Y} R(x, y), \sup_{y \in Y} S(y, z) \right\}$$

Assuming that R and S are serial means that the two last terms in (**A′**) are equal to 1. Moreover as the sup–$*$ composition preserves seriality, the second term in (**A**) is also 1. So, for serial fuzzy relations the extra terms in (**A**) and (**A′**) can be deleted, (**A**)=(**A′**) and reads $\inf_{y \in Y} R(x, y) \rightarrow S(y, z)$.

Under the seriality assumption, (**A**) does reduce to $\overline{R \circ \overline{S}}$ in the crisp case since then $\overline{R \circ \overline{S}} \subseteq R \circ S$. It is also true for (**A′**) since $\sup_{y \in Y}(R(x, y) * S(y, z)) \leq \sup_{y \in Y} \min(R(x, y), S(y, z)) \leq \min(\sup_{y \in Y} R(x, y), \sup_{y \in Y} S(y, z))$. The negations of the two above expressions correspond to the corner (**O′**):

$$(\mathbf{O}) \qquad \max \left\{ \sup_{y \in Y}(1 - (R(x, y) \rightarrow S(y, z))), \inf_{y \in Y} (1 - R(x, y)) \oplus (1 - S(y, z)) \right\}$$

$$(\mathbf{O}') \qquad \max \left\{ \sup_{y \in Y}(1 - (R(x, y) \rightarrow S(y, z))), \inf_{y \in Y} 1 - R(x, y), \inf_{y \in Y} 1 - S(y, z) \right\}$$

Again, if the fuzzy relations are serial, the second term in (**O**) and the second and third terms in (**O′**) can be dropped. Now, let us see if the requirements to get a gradual square of opposition hold. We assume that we use the weak entailment since as already said, it fits with fuzzy set inclusion. By definition we have that (**A**) and (**O**), (**A′**) and (**O′**), (**E**) and (**I**) are complements. Then, it is clear that (**A**) is less than or equal to (**I**) and (**E**) is less than or equal to (**O**). On the other hand, it may happen that (**A′**) is greater than (**I**) and similarly (**E**) is greater than (**O′**). For instance, if for all $y \in Y$ we have: $R(x, y) = \frac{1}{3}$, $S(y, z) = \frac{1}{2}$ and we consider the product triangular norm and its residuum, we get that the degree of (**A′**) is $\frac{1}{3}$ and the degree of (**I**) is $\frac{1}{6}$. Finally, the constraint $\alpha * \epsilon = 0$ requires that $k * (1 - k) = 0$

(where $k = \sup_{y\in Y} R(x, y) * S(y, z)$). Again, this holds if $*$ is a nilpotent triangular norm (see Proposition 4).

As a conclusion, (**A**), (**I**), (**E**) and (**O**) form a square of opposition once $*$ is a nilpotent triangular norm.

Instead of computing (**E**) and (**O**) as negations of (**I**) and (**A**), we can obtain them directly as a fuzzification of the corresponding Boolean formulae. In the case of (**E**) we can proceed as for (**A**) by substituting $S(y, z)$ with its negation $1 - S(y, z)$. So we get two possible definitions

$$(\mathbf{E}\mathrm{b}) \qquad \min\left\{\inf_{y\in Y} R(x, y) \to 1 - S(y, z), \sup_{y\in Y} R(x, y), \sup_{y\in Y} 1 - S(y, z)\right\}$$

$$(\mathbf{E}\mathrm{b}') \qquad \min\left\{\inf_{y\in Y} R(x, y) \to 1 - S(y, z), \sup_{y\in Y}(R(x, y) * 1 - S(y, z))\right\}$$

However, with these definitions we immediately lose the possibility to define a square of opposition since (**I**), the standard $*$-based composition of fuzzy relations, and (**E**) are not complement of each other.

We can proceed similarly for the back square. So, (**a**) is the fuzzification of the super-product and (**o**) its negation:

$$(\mathbf{a}) \qquad \min\left\{\inf_{y\in Y} S(y, z) \to R(x, y), \sup_{y\in Y}(R(x, y) * S(y, z))\right\}$$

$$(\mathbf{o}) \qquad \max\left\{\sup_{y\in Y} 1 - (S(y, z) \to R(x, y)), \inf_{y\in Y}(1 - R(x, y)) \oplus (1 - S(y, z))\right\}$$

Then, (**i**) the standard composition of $\overline{R}$ and $\overline{S}$, and (**e**) its negation:

$$(\mathbf{i}) \quad \sup_{y\in Y}[1 - (R(x, y) \oplus S(x, y))]$$

$$(\mathbf{e}) \quad \inf_{y\in Y}[R(x, y) \oplus S(x, y)]$$

Thus, we have a fuzzy cube of opposition built by the corners (**A**), (**I**), (**E**), (**O**) in the front face and (**a**), (**i**), (**e**), (**o**) in the back face, by considering a nilpotent triangular norm as conjunction.

Now, if we desire to build the hexagons on this cube, we must define the degrees associated to the two new corners (**U**) $=$ (**A**) $\sqcup$ (**E**) and (**Y**) $=$ (**I**) $\sqcap$ (**O**), where $\sqcup$ and $\sqcap$ are, respectively a union and an intersection operation. If we consider as intersection the nilpotent triangular norm used to define the cube and its dual triangular conorm as union, we get a perfect fuzzification of the hexagon as explained in Section 4.3.

Also in this case, we can build a hexagon on the top and obtain the fuzzified version of the square product between the two relations R and S. In order to exactly match De Baets and Kerre definition, it is sufficient to take as conjunction of (a) and (A) the min operator.

6 Conclusion

For a long time, one has been interested in particular instantiations of the square and hexagon of opposition, and these structures have been found fruitful to lay bare all operators of interest in an exhaustive and structured way in these instantiations. In this paper, we have looked at the structures associated to the square, cube and hexagon for themselves, showing that they can be generated by a simple device based on the composition of a relation and a subset. The fact that relations and subsets are encountered everywhere makes the universality of these structures even clearer. Moreover, their study naturally suggests the question of generalizing these structures when the statements corresponding to the vertices are associated with degrees. We have shown how this extension enables us to retrieve the different ways of composing fuzzy relations. These graded structures apply to a variety of settings useful in knowledge representation, including modal logic [19], possibility theory [24], rough set theory [13], formal concept analysis [23], abstract argumentation [1], and analogical proportions [31]. It should contribute to bridge the gaps between these settings, and may induce new developments in each of them. These are topics for further research.

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