Further results on sampled-data stabilization of time-delay systems

Mattia Mattioni, Salvatore Monaco, Dorothée Normand-Cyrot

Further results on sampled-data stabilization of time-delay systems ⋆

Mattia Mattioni *,** **Salvatore Monaco** * **Dorothée Normand-Cyrot** **

* *Dipartimento di Ingegneria Informatica, Automatica e Gestionale* A. Ruberti *(La Sapienza University of Rome); Via Ariosto 25, 00185 Rome, Italy (e-mail: {mattia.mattioni,salvatore.monaco}@uniroma1.it).*
** *Laboratoire de Signaux et Systèmes (L2S, CNRS-CentraleSupelèc); 3, Rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: {mattioni,cyrot}@l2s.centralesupelec.fr).*

Abstract: The paper deals with the stabilization of nonlinear sampled-data systems affected by non-entire input-delays. The approach combines input-Lyapunov matching, prediction and Immersion and Invariance concepts. The result is also detailed for linear dynamics and illustrated on a benchmark example.

Keywords: Delay systems, Asymptotic stabilization, Digital implementation

1. INTRODUCTION

In the late 50s, the work by Smith (1959) inspired a wide research community activity on time-delay systems pursuing many different approaches with their own benefits (e.g., Michiels and Niculescu (2014), Mazenc et al. (2014), Fridman (2014) and references therein).
More recent developments concern nonlinear systems where the design is carried out via reduction or prediction-based methodologies (Mazenc and Bliman (2006), Krstic (2009), Krstic (2010), Califano et al. (2011), Bekiaris-Liberis and Krstic (2013), Karafyllis et al. (2016)). Nevertheless, a lot of questions still remain unanswered, mainly due to the fact the retarded system is intrinsically infinite dimensional.
From the late 90s, an increasing interest has been addressed to systems under sampling where measures are periodically sampled (over intervals of length $\delta \in \mathbb{R}^+$) and the control is possibly piecewise constant over the sampling period. Huge attention is devoted to analyze the performances of sample-and-hold strategies in presence of delays (Karafyllis and Krstic (2012), Mazenc and Normand-Cyrot (2013), Mazenc et al. (2013), Pepe (2014), Monaco et al. (2016)). Significant improvements have been made basically exploiting the fact that the sampled-data delayed system is finite dimensional and admits a finite (extended) hybrid state-space representation whenever the delay is affecting the input or the measurements. Whenever a digital stabilizing controller for the delay-free system is computable, compensation of the delay can be pursued by implementing the former feedback either via prediction over a finite time interval (Karafyllis and Krstic (2012)) or through discrete-time prediction mappings (Monaco et al. (2012)). The consequent design procedures are based on the assumption that the sampling period can be chosen as directly proportional to the delay-length (i.e., $\tau = N\delta$ for some integer $N \in \mathbb{N}$). Moreover, the so-defined controllers strictly depend on the computability of the predicted state, so restricting their applicability to integrable dynamics or strict feedforward forms.
Along these lines, our contribution firstly concerned the design of sampled-data controllers for the delay-free system ensuring the same Lyapunov performances as an ideal continuous-time stabilizing feedback. This is achieved by designing a digital feedback ensuring matching of the behavior of the continuous-time Lyapunov function along the continuous-time (ideally) controlled trajectories (Input-Lyapunov Matching - ILM - Monaco et al. (2011), Tanasa et al. (2016)). Starting from such a delay-free design, discrete-time predictor-based solutions can easily be defined. Our second contribution relies on the improvement of the prediction-based sampled-data with respect to prediction errors. This is attained via the concept of Immersion and Invariance (I&I, Astolfi et al. (2008)) which is shown to be adequately shaped to deal with input-delayed dynamics under sampling (Monaco et al. (2016)).
In this paper, we extend these ideas to the case of *non-entire* delays in the sense that $\tau = N\delta + \sigma$ with $\sigma \in]0, \delta[$. An equivalent sampled-data dynamics can be still associated to this class of delayed dynamics (over a state space of dimension $N+1$) according to a double step sampling procedure. Exploiting the consequent cascade structure, a two step prediction-based design is proposed. First, the non-entire part of the delay is compensated via a digital stabilizing state feedback (parametrized by δ and σ) ensuring ILM, at the time instants $t = k\delta + \sigma$ ($k \geq 0$), of the evolution of the continuous-time Lyapunov along the delay-free continuous-time trajectories (under an ideal continuous-time feedback). In this sense, we extend the concept of ILM to non-entire input-delayed dynamics under sampling. Then, the actual control law is obtained through usual discrete-time N-step ahead prediction. The resulting predictor-based feedback is further modified according to I&I to improve robustness with respect to prediction errors. For computational facilities, approximate solutions guaranteeing local stabilizing properties are discussed.

The problem is stated in Section 2. In Section 3, the sampled ILM design is developed for the delayed dynamics when $\tau = \sigma$

⋆ This work was partially supported by a CNRS-ST2I International Scientific Project - PICS - for cooperation between France and Italy. Mattia Mattioni thanks the *Université Franco-Italienne/Università Italo-Francese* (UFI/UIF) for supporting his mobility from France to Italy within his PhD program.

and a Lyapunov function is defined. In Section 4, the main results are settled. In Section 5, the LTI case is discussed as a case study while simulations are reported for the van der Pol oscillator in Section 6. Section 7 concludes the paper.

Notations and definitions: All the functions and vector fields defining the dynamics are assumed smooth over the respective definition spaces. M_U (resp. M_U^I) denotes the space of measurable, locally bounded and smooth functions $u : \mathbb{R} \to U$ ($u : I \to U$, $I \subset \mathbb{R}$) with $U \subseteq \mathbb{R}$. $\mathscr{U}_\delta \subseteq M_U$ denotes the set of piecewise constant functions over time intervals of length $\delta \in]0, T^*[$, a finite time interval; i.e. $\mathscr{U}_\delta = \{u \in M_U \text{ s.t. } u(t) = u_k, \forall t \in [k\delta, (k+1)\delta[; k \geq 0\}$. Given a vector field f, L_f denotes the Lie derivative operator, $L_f = \sum_{i=1}^n f_i(\cdot)\frac{\partial}{\partial x_i}$. e^{L_f}(or e^f, when no confusion arises) denotes the associated Lie series operator, $e^f := 1 + \sum_{i\geq 1} \frac{L_f^i}{i!}$. A function $R(x,\delta) = O(\delta^p)$ is said of order $\delta^p; p \geq 1$ if whenever it is defined it can be written as $R(x,\delta) = \delta^{p-1}\tilde{R}(x,\delta)$ and there exist a function $\theta \in \mathscr{K}_\infty$ and $\delta^* > 0$ s. t. $\forall \delta \leq \delta^*$, $|\tilde{R}(x,\delta)| \leq \theta(\delta)$. We denote by the same $\circ$ the composition of functions and operators.

2. PROBLEM SETTLEMENT

In this paper, we consider nonlinear input-delayed dynamics over $\mathbb{R}^n$ of the form

$$\dot{x}(t) = f(x(t), u(t-\tau)) \tag{1}$$

with equilibrium x_* (i.e., $f(x_*, 0) = 0$), $u \in M_U^{[-\tau,\infty)}$ and known delay $\tau \geq 0$. When no delay is affecting the input (i.e., $\tau = 0$), we are referring to (1) as the *delay free dynamics*; namely,

$$\dot{x}(t) = f(x(t), u(t)). \tag{2}$$

The following standing assumptions are set.

A. The delay free dynamics is smoothly stabilizable; i.e., there exists a smooth feedback $u(t) = \gamma(x)$ with $\gamma(x_*) = 0$ and a proper [1] Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $\mathrm{L}_{f(\cdot,\gamma)}V(x) < 0$ with $\frac{\mathrm{L}_{f(\cdot,u)}V(x)}{\partial u}\big|_{u=\gamma(x)} \neq 0$ for all $x \in \mathbb{R}^n / \{x_*\}$.
B. The system (2) is forward complete [2];
C. $u \in \mathscr{U}_\delta$ and measures are available only at the sampling instants $t = k\delta$ $(k \geq 0)$ where δ denotes the sampling period;
D. $\tau = N\delta + \sigma; \sigma \in]0, \delta[$ for a suitable integer $N \in \mathbb{N}$.

In this context, we address the problem of stabilizing the retarded system (1) under non-entire delay via sampled-data predictor-based feedback.

2.1 The sampled-data extended state pace representation

Under Assumptions **C** and **D**, (1) can be described as an extended *equivalent hybrid model* over $\mathbb{R}^n \times \mathbb{R}^{N+1}$ provided by

$$\dot{x}(t) = f(x(t), v_k^1); \quad t \in [k\delta, k\delta + \sigma[\tag{3a}$$
$$\dot{x}(t) = f(x(t), v_k^2); \quad t \in [k\delta + \sigma, (k+1)\delta[\tag{3b}$$
$$v_{k+1}^1 = v_k^2, \quad \ldots \quad, v_{k+1}^{N+1} = u_k \tag{3c}$$

with $v_k^i = u((k-N-2+i)\delta) := u_{k-N-2+i}$ for $i = 1, \cdots, N+1$. By integrating (3) over time-intervals of length δ and initial condition $x_k := x(k\delta)$, one defines the *sampled-data equivalent dynamics* to (1) over $\mathbb{R}^n \times \mathbb{R}^{N+1}$ as

$$x_{k+1} = F^\delta(\sigma, x_k, v_k^1, v_k^2) = e^{\sigma f(\cdot, v_k^1)} \circ e^{(\delta-\sigma) f(\cdot, v_k^2)} x\big|_{x_k} \tag{4a}$$
$$v_{k+1}^1 = v_k^2 \quad , \ldots, \quad v_{k+1}^{N+1} = u_k. \tag{4b}$$

The sampled dynamics (4a) is parameterized by both δ and σ. In most cases, a closed-form solution for (4a) does not exist and each flow is described by its exponential series expansion. For computational purposes, approximations are actually used in practice and defined as truncations at finite order $p \in \mathbb{N}$ (in powers of δ) of the exponential series expansion defining (4a).

Remark 2.1. When $\tau = 0$ in (1) (i.e., $\sigma = 0$ and $N = 0$ in (3)), (4a) recovers the sampled-data equivalent model of the delay-free dynamics (2) that is provided by the mapping

$$x_{k+1} = F^\delta(x_k, u_k) := e^{\delta f(\cdot, u_k)} x\big|_{x_k} \tag{5}$$

with $u_k = v_k^2$.

For the sake of the dissertation, we are rewriting (4) in the cascade form below

$$x_{k+1}^e = F_e^\delta(\sigma, x_k^e, v_k^2) \tag{6a}$$
$$v_{k+1}^2 = v_k^3, \quad \ldots, \quad v_{k+1}^{N+1} = u_k \tag{6b}$$

to underline the $x^e = (x^\top, v^1)^\top$-dynamics over $\mathbb{R}^{n+1}$. Whenever $N = 0$, $v_k^2 = u_k$ and (6) reduces to the x^e-dynamics.

Given the cascade structure of equations (6), the design is backstepping-like in the sense that we first design a *fictitious-feedback* on v^2 for stabilizing the x_e-subsystem and then the effective control u for the whole cascade. Under the standing assumptions **A** to **D**, we prove:

(1) the existence of a smooth fictitious feedback $v_k^2 = K^\delta(\sigma, x_k^e)$ that guarantees Global Asymptotical Stability (GAS) of the equilibrium $(x_*^\top, 0)^\top$ of (6a) (Theorem 3.1);
(2) the existence of a predictor-based control $u_k = K^\delta(\sigma, x_k^{ep})$ with suitably defined discrete-time predictor state x_k^{ep} yielding GAS of the equilibrium $(x_*^\top, 0, \mathbf{0}_N^\top)^\top$ of the extended (6), (Theorem 4.2) and, equivalently, S-GAS [3] of (1);
(3) I&I stabilizability of the sampled-data dynamics (6) so providing a modified predictor-based controller ensuring S-GAS of the closed-loop equilibrium of (1) (Proposition 4.1) with intrinsic robustness improvement with respect to prediction errors; stability properties under approximate solutions are established as well (Theorem 4.1).

3. SAMPLED STABILIZATION OF THE X^E-DYNAMICS

Consider the x^e-dynamics (6a) with equilibrium $(x_*, 0)$. We aim at defining a *fictitious feedback* $v_k^2 = K^\delta(\sigma, x_k, v_k^1)$ (with $K^\delta(\sigma, x, v^1) : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+1} \to \mathbb{R}$) ensuring GAS of the closed-loop equilibrium of (6a). For this purpose, we extend the concept of ILM to retarded systems of the form (1) whenever Assumption **A** holds. In doing so, we underline that the x-dynamics in (6a) rewrites as affected by distributed delay; namely, it takes the form $x_{k+1} = F^\delta(\sigma, x_k, v_k^2, v_{k-1}^2)$.
The following result is instrumental and generalizes to (1) the notion of ILM (Monaco et al. (2011)).

[1] $V : \mathbb{R}^n \to \mathbb{R}$ is *proper* if $\forall\, r > 0$, $V^{-1}([0,r]) = \{x \in \mathbb{R}^n\ V(x) \geq r\}$ is compact.
[2] Assuming the delay free dynamics forward complete ensures that the delayed one (1) is complete too (Karafyllis and Krstic (2012))

[3] By S-GAS of the equilibrium of (1) we mean that the sampled-data control ensures stability of the equilibrium x_* of (1) at the time instants $t = k\delta + \sigma$, $k \geq 0$.

Lemma 3.1. Consider the system (1) under Assumptions **A** to **C**. Then, when $\tau = 0$, there exists a smooth mapping $\gamma^\delta(\cdot) : \mathbb{R}^n \to \mathbb{R}$ in the form

$$\gamma^\delta(x) = \gamma(x) + \sum_{i\geq 1} \frac{\delta^i}{(i+1)!}\gamma_i(x)$$

solution of the ILM equality

$$V(F^\delta(x_k, \gamma^\delta(x_k))) - V(x_k) = \int_{k\delta}^{(k+1)\delta} \mathrm{L}_{f(\cdot,\gamma(\cdot))}V(x(s))\mathrm{d}s \quad (7)$$

for any constant value $x_k = x(k\delta)$ ($k \geq 0$). As a consequence, the feedback $u_k = \gamma^\delta(x_k)$ makes the closed-loop equilibrium of (2) (resp. (5) GAS at the sampling instants $t = k\delta$, for any $k \geq 0$ (resp. GAS).

Theorem 3.1. Consider the system (1) under Assumptions **A** to **D** and let the mapping $\gamma^\delta(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be solution to (7). Introduce the predictor

$$x_k^p := x(k\delta + \sigma) = F^\sigma(x_k, v_k^1) \quad (8)$$

with $F^\sigma(x, v^1) = e^{\sigma f(\cdot, v^1)}x\big|_{x_k}$, evolving according to

$$x_{k+1}^p = F^\delta(x_k^p, v_k^2) \quad (9)$$

with $x_{k+1}^p := x((k+1)\delta + \sigma)$ and $x_0^p = F^\sigma(x_0, v_0^1)$. Then, setting

$$K^\delta(\sigma, x_k, v_k^1) := \gamma^\delta(F^\sigma(x_k, v_k^1)) = \gamma^\delta(x_k^p) \quad (10)$$

$$V^e(x, v^1) := V(F^\sigma(x, v^1)) = e^{\sigma f(\cdot, v^1)}V(x) \quad (11)$$

the *fictitious feedback* $v_k^2 := K^\delta(\sigma, x_k, v_k^1)$ ensures GAS of the closed-loop equilibrium of (6a) with predictor-dynamics

$$x_{k+1}^p = F^\delta(x_k^p, \gamma^\delta(x_k^p)) \quad (12)$$

with Lyapunov function $V^e(x, v)$ in (11).

Proof. From Lemma 3.1, we infer that when $\sigma = 0$, the fictitious feedback $v^2 = v^1 = \gamma^\delta(x)$ stabilizes (5) in the sampled-data sense. When $\sigma \in]0, \delta[$, one introduces the predictor variable (8) and computes $x_{k+1}^p = x(k\delta + \sigma + \delta)$ so getting

$$x_{k+1}^p = e^{\sigma f(\cdot, v_k^2)}x\big|_{x_{k+1}} \quad (13)$$

with, according to (6a),

$$x_{k+1} = e^{\sigma f(\cdot, v_k^1)} \circ e^{(\delta-\sigma) f(\cdot, v_k^2)}x\big|_{x_k} = e^{(\delta-\sigma) f(\cdot, v_k^2)}x\big|_{x_k^p}. \quad (14)$$

Substituting now (14) into (13), one obtains (9) that coincides with the delay-free (5) when setting $u_k = v^2$. Since $\gamma^\delta(\cdot)$ satisfies (7), one sets $v^2 = K^\delta(\sigma, x, v^1)$ as in (10) so that the closed-loop predictor sampled dynamics (9) has a GAS equilibrium in x_*^p with Lyapunov function $V(x^p)$. As a consequence, the resulting closed-loop dynamics (6a) satisfies

$$x_{k+1} = F^\delta(\sigma, x_k, v_k^1, \gamma^\delta(x_k^p)) = e^{\sigma f(\cdot, v_k^1)} \circ e^{(\delta-\sigma) f(\cdot, \gamma^\delta(x_k^p))}x\big|_{x_k}$$

$$v_{k+1}^1 = \gamma^\delta(x_k^p) \quad (15)$$

where the predictor $x_k^p = e^{\sigma f(\cdot, v_k^1)}x\big|_{x_k}$ is computable at any sampling instant $t = k\delta$. For proving closed-loop stability, we have to show that $V^e(x, v)$ in (11) is a Lyapunov function for (6a). For this purpose, it is sufficient to note that, by construction, the fictitious feedback $K^\delta(\sigma, x_k, v_k^1)$ satisfies

$$\Delta_k V^e(x, v^1) = \int_{k\delta+\sigma}^{(k+1)\delta+\sigma} \mathrm{L}_{f(\cdot,\gamma(\cdot))}V(x(s))\mathrm{d}s$$

with $\Delta_k V^e(x, v^1) = V^e(x_{k+1}, K^\delta(\sigma, x_k, v_k^1)) - V^e(x_k, v_k^1)$. Thus, one recovers the ILM equality (7) at the time instants $t = k\delta + \sigma$ with initial condition $x_0^p = x(\sigma) = e^{-\sigma f(\cdot, v_0^1)}(x_0)$. GAS of the equilibrium $(x_*, 0)$ of (6a) follows. ◁

Remark 3.1. $K^\delta(\sigma, x, v^1)$ is smoothly parameterized by δ and σ. By construction one verifies that $\lim_{\sigma\to 0} K^\delta(\sigma, x, v^1) = \gamma^\delta(x)$ and $\lim_{\sigma\to\delta} K^\delta(\sigma, x, v^1) = \gamma^\delta(F^\delta(x, v^1))$.

4. THE MAIN RESULT

4.1 The predictor based controller

Considering now the complete dynamics (6), we show that the predictor-based feedback $u_k = K^\delta(\sigma, x_{k+N}^p, v_{k+N}^1) = \gamma^\delta(x_k^{ep})$ with extended predicted state $x_k^{ep} := (x_{k+N}^{p\top}, v_{k+N}^1)^\top$ (with $x_{k+N}^p := x((k+N)\delta + \sigma))$ ensures GAS of the equilibrium of (6) at the time instants $t = k\delta + \sigma$ ($k \geq 0$). For, we exploit the fact that the study of the extended predictor can be addressed starting from the one of (6a) under virtual feedback $v_k^2 = K^\delta(\sigma, x_k^p, v_k^1)$.

Theorem 4.1. Consider (1) under Assumptions **A** to **D**. Introduce the predictor $x_k^{ep} := (\bar{x}_k^{p\top}, v_{k+N}^1)^\top$ with $v_{k+N}^1 = v_k^N$, $\bar{x}_k^p = x_{k+N}^p$ and, in detail,

$$\bar{x}_k^p = e^{\sigma f(\cdot, v^1)} \circ e^{\delta f(\cdot, v^2)} \circ \cdots \circ e^{\delta f(\cdot, v^{N+1})}x\big|_{x_k}$$

whose evolutions are described by the difference mapping

$$\bar{x}_{k+1}^p = F^\delta(\bar{x}_k^p, u_k) \quad (16a)$$

$$v_{k+N+1}^1 = u_k. \quad (16b)$$

Then, the predictor-based feedback

$$u_k = K^\delta(\sigma, \bar{x}_k^p, v_{k+N}^1) := K^\delta(\sigma, x_k^{ep}) \quad (17)$$

ensures GAS of the equilibrium of the closed loop dynamics (6) at the time instants $t = k\delta + \sigma$ ($k \geq 0$) with Lyapunov function $V^{ep}(x, v^1, \cdots, v^{N+1}) = V^e(x_{k+N}^p, v_{k+N}^1)$. S-GAS of the equilibrium of the retarded dynamics (1) follows.

Proof. Defining x_k^{ep} as stated, one verifies that the x^{ep}-dynamics recovers the x^e-ones in the sense that one gets

$$x_{k+1}^{ep} = F_e^\delta(0, x_k^{ep}, u_k)$$

with

$$F_e^\delta(0, \bar{x}_k^{ep}, u_k) = \left([F^\delta(\bar{x}_k^{ep}, u_k)]^\top \;\; u_k\right)^\top.$$

Accordingly, $\bar{x}_k^p = x_{k+N}^p$ and the predictor-based feedback is provided by

$$u_k = K^\delta(\sigma, x_{k+N}^p, v_{k+N}^1) := \gamma^\delta(x_k^{ep})$$

so recovering

$$u_k = e^{\delta f(\cdot, v_k^2)} \circ \cdots \circ e^{\delta f(\cdot, v_k^{N+1})}\gamma^\delta(x_k^p) = \gamma^\delta(x_{k+N}^p).$$

Following the lines of the proof of Proposition 3.1, the closed-loop predictor dynamics

$$\bar{x}_{k+1}^p = F^\delta(\bar{x}_k^p, \gamma^\delta(\bar{x}_k^p)) \quad (18)$$

$$v_{k+1}^N = \gamma^\delta(\bar{x}_k^p) \quad (19)$$

is GAS as it coincides with (12). GAS of (6) (and hence S-GAS of (1)) follows along the same lines of the proof of Proposition 3.1 by noting that after $N+1$ step the dynamics of (6) reduces to $x_{k+N+1}^e = F_e^\delta(\sigma, x_k^{ep}, \gamma^\delta(\sigma, \bar{x}_k^p))$ and, more in details,

$$x_{k+N+1} = F^\delta(\sigma, \bar{x}_k^p, v_{k+N}^1, \gamma^\delta(\bar{x}_k^p)) \quad (20)$$

$$v_{k+N+1}^1 = \gamma^\delta(\bar{x}_k^p). \quad (21)$$

or, in compact form as, $x_{k+N+1}^e = F_e^\delta(\sigma, \bar{x}_k^{pe}, \gamma^\delta(\bar{x}_k^p))$ so recovering (15) that is GAS. ◁

Remark 4.1. The compensating feedback $K^{\delta}(\sigma, x^{pe})$ in (17) is based on a N-step prediction of the feedback defined over the reduced dynamics (6a) in Proposition 3.1. As underlined in the proof of Theorem 4.2, the consequent feedback is equivalent to the one resulting from a prediction over $N\delta+\sigma$ of the delay-free feedback $\gamma^{\delta}(x_k)$ in Lemma 3.1 over the x-dynamics (4a).

4.2 The I&I feedback for $N > 0$

The proposed predictor-based control can be further modified according to a result in Monaco et al. (2016). The following Theorem follows from Theorem 3.1.

Proposition 4.1. Let (1) fulfil Assumptions **A** to **B** and (6) be its extended sampled-data equivalent model. Then, $\forall \delta \in]0, T^*[, \sigma \in]0, \delta[$, (6) is I&I stabilizable with *target dynamics*

$$x_{k+1} = F^{\delta}(\sigma, x_k, K^{\delta}(\sigma, x_k, v_k^1)), \quad v_{k+1}^1 = K^{\delta}(\sigma, x_k, v_k^1) \quad (22)$$

rewritten with $x^e = (x^\top, v^1)^\top$ as

$$x_{k+1}^e = \alpha^{\delta}(\sigma, x_k^e)$$

with $\alpha^{\delta}(\sigma, x_k^e) = ([F^{\delta}(\sigma, x_k^e, K^{\delta}(\sigma, x_k^e))]^\top,\ K^{\delta}(\sigma, x_k^e))$.

Proof. In order to prove the result one has to show that hypotheses **H1** to **H4** of Theorem 2.2 in Monaco et al. (2016) are verified. From Proposition 3.1 it is straightforward to conclude that the closed-loop equilibrium $(x_*^\top, 0)^\top$ of the target (22) is GAS so that **H1** is ensured. Then, it follows that the *immersion and invariance* condition is guaranteed by setting the immersion mapping $\pi^{\delta} : \mathbb{R}^{n+1} \times \mathbb{R}^N \to \mathbb{R}^n$ and on-the-manifold control $c^{\delta} : \mathbb{R}^{n+1} \to \mathbb{R}$ as

$$\pi^{\delta}(x_k^e) = (x_k^{e\top}, K^{\delta}(\sigma, x_k^e), \dots, K^{\delta}(\sigma, x_{k+N-1}^e))^\top \quad (23a)$$

$$c^{\delta}(x_k^e) = K^{\delta}(\sigma, x_{k+N}^e). \quad (23b)$$

where

$$K^{\delta}(\sigma, x_{k+i}^e) = ([(\alpha^{\delta}(\sigma, x_k^e))^i]^\top, v_{k+i}^1)^\top$$

$$(\alpha^{\delta}(\sigma, x_k^e))^i = e^{\sigma f(\cdot, v_k^1)} \circ e^{\delta f(\cdot, v_{k+1}^1)} \circ e^{\delta f(\cdot, v_{k+i}^1)} x\big|_{x_k}$$

for $i = 1, \dots, N$. As a consequence, **H3** is fulfilled by implicitly defining the manifold as $\mathscr{M}^{\delta} = \{(x^{e\top}, v^2, \dots v^{N+1})^\top \in \mathbb{R}^{n+1} \times \mathbb{R}^N \text{s. t. } \phi^{\delta}(x^e, v^2, \dots v^{N+1}) = \mathbf{0}_N\}$ with

$$\phi^{\delta}(x^e, v^2, \dots v^{N+1}) = (\phi_1^{\delta}(x^e, v^2), \dots, \phi_N^{\delta}(x^e, v^2, \dots v^{N+1})^\top$$

$$\phi_i^{\delta}(x^e, v^2, \dots v^{i+1}) = v_k^{i+1} - K^{\delta}(\sigma, (F_e^{\delta}(\sigma, x^e)^i)$$

$$(F_e^{\delta}(\sigma, x^e)^i = ([(F^{\delta}(\sigma, x_k, v_k^1))^i]^\top, v_k^{i+1})$$

$$(F^{\delta}(\sigma, x_k, v_k^1))^i = e^{\sigma f(\cdot, v_k^1)} \circ e^{\delta f(\cdot, v_k^2)} \circ e^{\delta f(\cdot, v_k^{i+1})} x\big|_{x_k}$$

for $i = 1, \dots, N$. Accordingly, one defines the off-the-manifold component $z = (z^1, \dots, z^N)$ by setting $z^i = \phi_i^{\delta}(x^e, v^2, \cdots, v^{i+1})$ with $z_0 = \phi^{\delta}(x_0^e, v_0^2, \cdots, v_0^{N+1})$ .Thus, stabilization in closed-loop of (6) is achieved by any feedback $u = \psi^{\delta}(x^e, v^2, \cdots v^{N+1}, z)$ such that $\psi^{\delta}(\pi^{\delta}(x^e), \mathbf{0}_N) = c^{\delta}(x^e)$ that is designed so to drive $z \to 0$ while guaranteeing boundedness of all the (x, v, z) trajectories. ◁

Before showing how to construct the I&I feedback, we first notice that the dynamics of the extended system $(x^{e\top}, v^2, \dots, v^N, z)$ coordinates is described by a cascade structure composed of (6) plus the z-dynamics provided by

$$z_{k+1}^1 = z_k^2, \quad \dots, \quad z_{k+1}^N = z_k^{N+1}$$

$$z_{k+1}^{N+1} = u_k - K^{\delta}(\sigma, (F_e^{\delta}(\sigma, x^e)^N).$$

or, in more compact form, as

$$z_{k+1} = \mathscr{A} z_k + \mathscr{B}(u_k - K^{\delta}(\sigma, (F_e^{\delta}(\sigma, x^e)^i)) \quad (24)$$

with

$$\mathscr{A} = \begin{pmatrix} \mathbf{0}_{(N-1)\times 1} & \mathrm{I}_{(N-1)\times(N-1)} \\ \mathbf{0}_{1\times(N-1)} & \mathbf{0}_{1\times(N-1)} \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} \mathbf{0}_{1\times(N-1)} \\ 1 \end{pmatrix}. \quad (25)$$

Accordingly, stabilization in closed-loop is achieved by a the feedback of the form

$$\psi^{\delta}(x^e, v^2, \cdots v^{N+1}, z) = K^{\delta}(\sigma, (F_e^{\delta}(\sigma, x^e)^N) + Lz_k$$

where L is chosen so that $\mathscr{A} + \mathscr{B}L$ is Schur. Unfortunately, in most cases the above feedback cannot be exactly computed. Hence, only approximate solutions can be implemented so loosing, in general, global stability properties in closed-loop. The following result is stated while the proof can be carried out along the lines of the one of Proposition 3.2 in Monaco et al. (2016).

Theorem 4.2. Let (1) verify Assumptions **A** to **D**. Then, sampled-data (local) asymptotic stabilization of the equilibrium x_* of the retarded dynamics (1) is achieved by any feedback

$$u_k = K^{\delta}(\sigma, (\alpha^{\delta}(\sigma, x_k^e))^N) + L^{\delta}(x_k^e) z_k \quad (26)$$

where $L^{\delta}(x_k^e)$ is chosen to achieve $\lim_{k\to\infty} z_k = 0$ with boundedness of the state trajectories of (6)-(24).

Remark 4.2. Writing the feedback as (26) underlines that the I&I feedback is composed of the predictor-based feedback plus a term that represent a feedback on the prediction error. As a matter of fact, when the manifold is reached (i.e., $z \equiv 0$), the I&I feedback (26) recovers the predictor-based one; namely, $K^{\delta}(\sigma, (\alpha^{\delta}(\sigma, x_k^e))^N) = \gamma^{\delta}(x_{k+N}^p)$ with $x_{k+N}^p = e^{\delta f(\cdot, v^2)} \circ \dots \circ e^{\delta f(\cdot, v^{N+1})} x\big|_{x_k^p}$ and $x_k^p = F^{\sigma}(x_k, v_k^1)$.

5. THE LINEAR CASE

Consider, as a case study, the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t-\tau) \quad (27)$$

where $x \in \mathbb{R}^n$ verifying assumptions **B** to **D** while Assumption **A** reformulates as follows:

LA. When $\tau = 0$ the couple (A, B) is stabilizable and the continuous-time feedback $u = Fx$ stabilizes in closed-loop with Lyapunov function $V(x) = x^\top Qx$, $Q > 0$ such that $(A+BF)^\top Q + Q(A+BF) < 0$ and QB is full rank.

In the LTI case, the sampled-data equivalent dynamics are exactly computable. Thus, we are now specifying the results in Theorems 4.1 and 4.2 to the present case. As well known, the LTI nature of (27) is preserved under sampling so that the extended sampled-data equivalent model gets the form

$$x_{k+1} = A^{\delta} x_k + A^{\delta-\sigma} B^{\sigma} v_k^1 + B^{\delta-\sigma} v_k^2 \quad (28a)$$

$$v_{k+1}^1 = v_k^2, \quad \dots, \quad v_{k+1}^{N+1} = u_k \quad (28b)$$

where $x_k = x(k\delta)$ for $k \geq 0$ and $A^{\delta} = e^{\delta A}$; $B^{\sigma} = \int_0^{\sigma} e^{sA} B \mathrm{d}s$. From the above definitions, it is straightforward to verify that $A^{\delta-\sigma} B^{\sigma} + B^{\delta-\sigma} = A^{\sigma} B^{\delta-\sigma} + B^{\sigma} = B^{\delta}$.

The predictor-based feedback

Corollary 5.1. Consider (27) under Assumptions **LA** and **B** to **D** and let F^{δ} be computed as the solution to the ILM equality

$$(A^{\delta} + B^{\delta} F^{\delta})^\top Q (A^{\delta} + B^{\delta} F^{\delta}) = e^{(A+BF)^\top \delta} Q e^{(A+BF)\delta}. \quad (29)$$

Then, the predictor-based feedback

$$u_k = F^\delta A^{N\delta} x_k^p + \sum_{i=0}^{N-1} F^\delta A^{i\delta} B^\delta v_k^{N+1-i} \tag{30}$$

with

$$x_k^p := x(k\delta+\sigma) = A^\sigma x_k + B^\sigma v_k^1 \tag{31}$$

and initial conditions (x_0, v_0, u_0) so that

$$u_0 = F^\delta A^{N\delta}(A^\sigma x_0 + B^\sigma v_0^1) + \sum_{i=0}^{N-1} F^\delta A^{i\delta} B^\delta v_0^{N+1-i}$$

asymptotically stabilizes (28) As a consequence, (30) asymptotically stabilizes (27) at the time instants $t = k\delta+\sigma$, $k \geq 0$.

Proof. From Lemma 3.1, one has that, by construction, when $\tau = 0$ the feedback $u_k = F^\delta x_k$ stabilizes the delay-free system in closed-loop. By straightforwardly applying Proposition 3.1, one sets $K^\delta(\sigma, x_k, v_k^1) = F^\delta(A^\sigma x_k + B^\sigma v_k^1)$ so that the fictitious feedback $v_k^2 = F^\delta x_k^p$ with x_k^p as in (31) makes

$$x_{k+1} = A^\delta x_k + A^{\delta-\sigma} B^\sigma v_k^1 + B^{\delta-\sigma} F^\delta x_k^p, \quad v_{k+1}^1 = F^\delta x_k^p \tag{32}$$

asymptotically stable in closed-loop with asymptotically stable predictor dynamics provided by

$$x_{k+1}^p = (A^\delta + B^\delta F^\delta) x_k^p \tag{33}$$

with $x_0^p = A^\sigma x_0 + B^\sigma v_0^1$. As a matter of fact, one obtains that, when $k \to \infty$, $x_k^p \to x_*$ and thus $x_k \to (A^{\delta-\sigma} + B^{\delta-\sigma} F^\delta) x_*$ and $v_k^1 \to F^\delta x_* = 0$ and so $x_k \to A^{\delta-\sigma} x_*$. Accordingly, Theorem 4.1 applies now by defining the real feedback as the prediction of the virtual feedback $K^\delta(\sigma, x_k, v^1) = F^\delta x_k^p$, N-step ahead over the x^p-dynamics (33); namely, one gets

$$u_k = K^\delta(\sigma, x_{k+N}^p, v_{k+N}^1) = F^\delta A^{N\delta} x_k^p + \sum_{i=0}^{N-1} F^\delta A^{i\delta} B^\delta v_k^{N+1-i}.$$

Because of the cascade structure, after $N+1$ steps the dynamics (28) in closed-loop recovers the feedback dynamics (32) with asymptotically stable equilibrium $(x_*^\top, 0)$; i.e., one gets

$$\begin{aligned} x_{k+N+1} &= A^\delta x_{k+N} + A^{\delta-\sigma} B^\sigma v_{k+N}^1 + B^{\delta-\sigma} F^\delta x_{k+N}^p \\ v_{k+N+1}^1 &= F^\delta x_{k+N}^p. \end{aligned}$$

$\lhd$

5.1 The I&I feedback for $N > 0$

Theorem 4.2 is specified below to (27).

Corollary 5.2. Consider (27) under Assumptions **LA** and **B** to **D** and let F^δ be computed as the solution to the ILM equality (29). Then, (28) is I&I stabilizable with target dynamics

$$\begin{aligned} x_{k+1} &= (A^\delta + B^{\delta-\sigma} F^\delta A^\sigma) x_k + (A^{\delta-\sigma} B^\sigma + B^{\delta-\sigma} F^\delta B^\sigma) v_k^1 \\ v_{k+1}^1 &= F^\delta A^\sigma x_k + F^\delta B^\sigma v_k^1. \end{aligned} \tag{34}$$

Thus, asymptotic stability of (28) is ensured by the feedback

$$u_k^I = L^\delta z_k + F^\delta (A^\delta + B^\delta F^\delta)^N x_k^p + \sum_{i=0}^{N-1} F^\delta A^{i\delta} B^\delta z_k^{N-i} \tag{35}$$

where x_k^p is as in (31), L^δ is chosen so that the matrix $\mathscr{A} + \mathscr{B} L^\delta$ (with $\mathscr{A}$ and $\mathscr{B}$ as in (25)) is Schur and $z = (z^1, \ldots, z^N)^\top$ with

$$z_k^i = v_k^{i+1} - F^\delta (A^\delta + B^\delta F^\delta)^{i-1} x_k^p \tag{36}$$

for $i = 1, \ldots, N$ and $z_0^i = v_0^{i+1} - F^\delta (A^\delta + B^\delta F^\delta)^{i-1} x_0^p$.

Proof. Again we have to show that hypotheses **H1** to **H4** of Theorem 2.2 in Monaco et al. (2016). First, we notice that the target dynamics (34) coincides with (32) so proving that **H1** is fulfilled. **H2** is guaranteed by setting

$$\pi^\delta(x^p) = (\Pi_1^{\delta\top}, \ldots, \Pi_N^\delta)^\top x^p, \quad c^\delta(x^p) = C^\delta x^p$$

with $\Pi_1^\delta = \mathrm{I}$, $\Pi_i^\delta = F^\delta (A^\delta + B^\delta F^\delta)^{i-1}$ $(i = 2, \ldots, N)$ and $C^\delta = F^\delta (A^\delta + B^\delta F^\delta)^N$. Accordingly, **H3** is verified by idefining the invariant set as $\mathscr{M}^\delta = \{(x^\top, v^1; v^2, \ldots, v^{N+1}) \in \mathbb{R}^{n+1} \times \mathbb{R}^N \text{ s.t. } v_k^{i+1} - F^\delta (A^\delta + B^\delta F^\delta)^{i-1} (A^\delta x + B^\delta v^1) = 0\}$ so deducing the off-the-set component z as in (36) with dynamics

$$z_{k+1} = \mathscr{A} z_k + \mathscr{B}\big[u_k - F^\delta (A^\delta + B^\delta F^\delta)^N x_k^p + \sum_{i=0}^{N-1} F^\delta A^{i\delta} B^\delta z_k^{N-i})\big].$$

Accordingly, the feedback u_k^I in (35) satisfies **H4** as in closed-loop one gets $z_{k+1} = (\mathscr{A} + \mathscr{B} L^\delta) z_k$ that is asymptotically stable so proving the result. $\lhd$

6. THE VAN DER POL EXAMPLE

Let the van der Pol oscillator dynamics be described by

$$\dot{x}_2 = x_1 - x_1^2 u, \quad \dot{x}_1 = u \tag{37}$$

and denote $x = (x_1, x_2)^\top$. The smooth continuous-time feedback $u = -3x_1 - \frac{x_1^3}{3} - x_2$ makes the closed-loop equilibrium GAS with Lyapunov function $V(x) = x_1^2 + \frac{x_1^4}{3} + x_1 x_2 + \frac{1}{2} x_2^2$. When $u \in \mathscr{U}_\delta$, by applying Lemma 3.1, the *approximate* sampled-data feedback in $u = \gamma^\delta(x) = -3x_1 - \frac{x_1^3}{3} - x_2 + \frac{\delta}{2}(x_1^3 + 8x_1 + 3x_2) + O(\delta^2)$ makes the closed-loop equilibrium S-GAS. Assume that a delay $\tau = \delta + \sigma$ is acting on the control in (37). One computes the extended hybrid sampled-data equivalent dynamics to (37) as

$$\begin{aligned} x_{2k+1} =& x_{2k} + \delta x_1 - v^1(\sigma(\sigma-\delta) - \frac{\sigma^2}{2}) - \frac{1}{3}(x_{1k} + \sigma v_k^1 \\ & - v_k^2(\sigma-\delta))^3 + \frac{1}{2} v_k^2 (\sigma-\delta)^2) + \frac{1}{3} x_{1k}^3 \\ x_{1k+1} =& x_{1k} + \sigma v_k^1 + (\delta-\sigma) v_k^2, \quad v_{k+1}^1 = v_k^2, \quad v^{2k+1} = u_k. \end{aligned}$$

The predictor-based feedback (Theorem 4.1) is provided by

$$\begin{aligned} u =& -(\frac{\delta^2}{6} + \frac{\delta}{2} + 1)(x_2 + \sigma x_1 + \frac{\delta^2}{2} v^2 + \frac{\sigma^2}{2} v^1 + \frac{x_1^3}{3} + \delta(x_1 + \\ & \sigma v^1)) - (\frac{\delta^2}{2} v^2 + x_1 + \sigma v^1)(\frac{3\delta}{2} + \frac{\delta^2}{6}(2\delta+3) + \frac{\delta^2}{2} + 3) \end{aligned}$$

while the I&I feedback gets the form provided in Theorem 4.2 with a static gain $L \in (-1, 1)$.

Simulations are performed for $N = 1$, different values of δ and σ. We compare the predictor-based (PB) and the I&I feedbacks (Theorems 4.1 and 4.2) with the sampled-data (SD) delay-free one. As one might expect, both the design strategies yield good performances when δ and σ are small (Figure 1). Though, since approximate solutions are applied, the I&I feedback yields improved performances with respect to the PB one (Figure 2). Furthermore, we compare the behavior of the Lyapunov function of the closed-loop PB system ($V^e(x, v)$, Theorem 4.1) with the ones of the continuous-time (CT) and sampled-data (SD) delay-free systems (Assumption **A** and Lemma 3.1). In this case, simulations confirm that matching of the CT behavior is achieved by the predictor-based feedback at the time instants $t = k\delta + \sigma$. Though, such a property is lost in Figure 2 as the sampling period increases basically due to the fact that only the approximated prediction-based feedback is implemented.

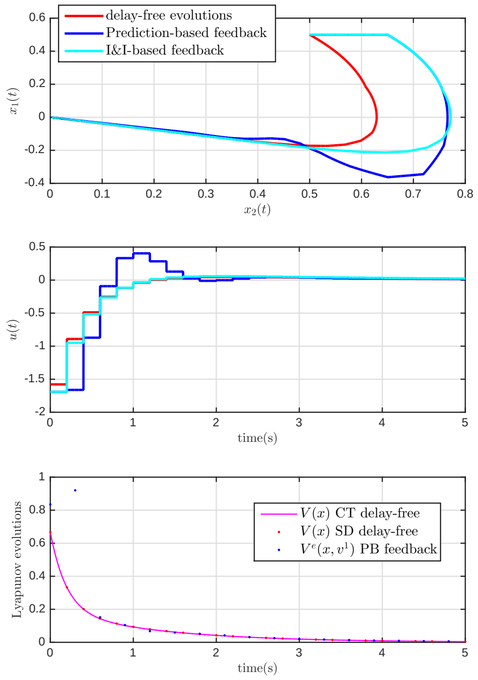

Fig. 1. $\delta = 0.2$ s, $N = 1$ and $\sigma = 0.1$ s

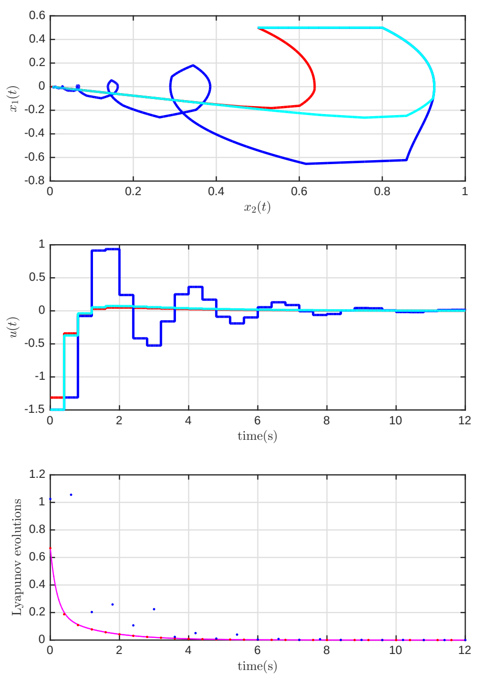

Fig. 2. $\delta = 0.4$ s, $N = 1$ and $\sigma = 0.2$ s

7. CONCLUSION

In this paper, we addressed the problem of stabilizing nonlinear dynamics affected by non-entire input-delays under sampling. The main problem relied on a distributed delay acting on the sampled-data mapping of the x-dyamics. Two digital stabilizing feedbacks with Lyapunov functions have been proposed based on the definition of a discrete-time predictor.

REFERENCES

Astolfi, A., Karagiannis, D., and Ortega, R. (2008). *Nonlinear and adaptive control with applications*. Springer Publishing Company.

Bekiaris-Liberis, N. and Krstic, M. (2013). *Nonlinear Control under Nonconstant Delays*. Advances in Design and Control. SIAM.

Califano, C., Marquez-Martinez, L.A., and Moog, C.H. (2011). Extended Lie brackets for nonlinear time-delay systems. *IEEE Transactions on Automatic Control*, 56(9), 2213–2218.

Fridman, E. (2014). *Introduction to Time-Delay Systems: Analysis and Control*. Systems & Control: Foundations & Applications. E. Fridman.

Karafyllis, I. and Krstic, M. (2012). Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold. *IEEE Transactions on Automatic Control*, 57(5), 1141–1154.

Karafyllis, I., Malisoff, M., Mazenc, F., and Pepe, P. (eds.) (2016). *Recent Results on Nonlinear Delay Control Systems*, volume 4 of *Advances in Delays and Dynamics*. Springer International Publishing.

Krstic, M. (2010). Input delay compensation for forward complete and strict-feedforward nonlinear systems. *IEEE Transactions on Automatic Control*, 55(2), 287–303.

Krstic, M. (2009). *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Birkhauser.

Mazenc, F. and Bliman, P. (2006). Backstepping design for time-delay nonlinear systems. *IEEE Transactions on Automatic Control*, 51(1), 149–154.

Mazenc, F., Malisoff, M., and Dinh, T.N. (2013). Robustness of nonlinear systems with respect to delay and sampling of the controls. *Automatica*, 49(6), 1925–1931.

Mazenc, F., Malisoff, M., and Niculescu, S.I. (2014). Reduction model approach for linear time-varying systems with delays. *IEEE Transactions on Automatic Control*, 59(8), 2068–2082.

Mazenc, F. and Normand-Cyrot, D. (2013). Reduction model approach for linear systems with sampled delayed inputs. *IEEE Transactions on Automatic Control*, 58(5), 1263 – 1268.

Michiels, W. and Niculescu, S. (2014). *Stability, control, and computation for time-delay systems. An eigenvalue based approach*. Advances in Design and Control. SIAM.

Monaco, S., Normand-Cyrot, D., and Mattioni, M. (2016). Sampled-data stabilization of nonlinear dynamics with input delays through immersion and invariance. *IEEE Transactions on Automatic Control*, PP(99), 1–1.

Monaco, S., Normand-Cyrot, D., and Tanasa, V. (2012). Digital stabilization of input delayed strict feedforward dynamics. In *Proc. 51st IEEE-CDC, Maui, Hawaii*, 7535–7540.

Monaco, S., Normand-Cyrot, D., and Tiefensee, F. (2011). Sampled-data stabilizing feedback; a pbc approach,. *IEEE Transactions on Automatic Control*, 907-912(4), 907–912.

Pepe, P. (2014). Stabilization in the sample-and-hold sense of nonlinear retarded systems. *SIAM Journal on Control and Optimization*, 52(5), 3053–3077.

Smith, O.J.M. (1959). A controller to overcome dead time. *Indian Scientist Association in Japan*, 2(6), 28–33.

Tanasa, V., Monaco, S., and Normand-Cyrot, D. (2016). Backstepping control under multi-rate sampling. *IEEE Transactions on Automatic Control*, 61(5), 1208 – 1222.