# A graphic approach to Euler's method

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**Abstract:** To solve differential equations and study transcendental curves appearing in problems of geometry, celestial mechanics, ballistics and physics, mathematicians have imagined numerous approaches since the 17th century. Alongside integration by quadratures and the series method, we can notably quote the polygonal method formalised by Euler in 1768. He directly used Leibniz's vision of curves as polygons made up of segments of infinitely tiny tangents. After an historical introduction and the study of an appropriate extract from the work by Euler on integral calculus, this chapter recounts a teaching experiment with 18 year olds, the aim of which was to introduce the notion of differential equations with support from the graphic version of the polygonal method. Through the purely geometric construction of integral curves formed from tiny segments of tangents, the students were able to make useful transfers between algebra and geometry and actively discover the first concepts of infinitesimal calculation.

**Keywords:** Tangent to a curve, Euler's polygonal method, Graphic method, Differential equations, Integral calculus, Exponential curve, Augustin-Louis Cauchy, Leonhard Euler, Gottfried Wilhelm Leibniz.

## The origins of Euler's method

Since 2001, Euler's method has played a significant place in the 16-18 year old curriculum for those studying the science subjects, both in mathematics and physics. This construction process, close to integrating differential equations, is generally performed numerically, the necessary calculations being carried out with the help of a programmable calculator or spreadsheet. In a previous paper (Tournès, 2007 pp. 263-285), I proposed that one could rely on a purely graphical version of the same method so that the notion of differential equations at high school level had greater meaning. To illustrate this in a concrete way, I am going to describe the teaching method I devised taking inspiration from history and that I was pleased to try out with my final year of upper secondary school students.

In the 17th century the initial problems that led to differential equations were from either geometry or physics. The geometrical problems were linked to the properties of tangents, curves, squaring the circle and rectification (the process of finding the length of a curve segment). Physics involved the swinging of a pendulum and research into isochronal curves, the paths of light rays in a medium with a variable refractive index, orthogonal trajectory problems and the vibration of a string fixed at both ends. These problems, sometimes anecdotal, often appeared as challenges between corresponding scientists. Closely linked to the invention of infinitesimal calculus by Newton and Leibniz, these led to the most simple differential equations and everyday cases of quadrature (numerical integration). For nearly a century the principal approach to these equations was algebraic in nature because mathematicians tried to express their solutions in finite form using traditional algebraic operations as well as the new methods of differentiation and integration.

From the start of the 18th century however, more systematic and more difficult problems arose which did not yield to the basic methods formulated by Newtonian mechanics. Mathematical physics provided numerous equations with partial derivatives which led, by separating the variables, to ordinary differential equations. The mechanics of points and solid bodies gave rise directly to such equations. At the heart of this proliferation there were two areas that played specific roles in perfecting these new methods of dealing with differential equations: celestial mechanics and projectiles. Certainly two-body problems and the trajectory of a cannonball in a vacuum that can be integrated by quadrature are different from three-body problems under the influence of gravity and taking air resistance into account when dealing with a projectile. Alongside integration by quadrature two other key routes were explored.

The first is writing the unknown functions as infinite series. This was started by Newton in 1671 and for a long time favoured by the English school of thought. In 1673 Leibniz began using infinite series closely followed by the Bernoulli brothers and other continental mathematicians. This was widely practised in mathematical physics and celestial mechanics sparking a considerable revolution in functions reaching far beyond Descartes' algebraic expressions. Little by little the explosion of infinity into algebraic calculation led to questioning the formal calculations and on to deep reflection about the notion of convergence.

The second method, the polygonal method is found in the early works of the founding fathers of infinitesimal calculus. It is linked to the Leibnizian concept of curves as polygons consisting of an infinity of infinitesimally tiny sides as elements of tangents. For example, in 1694 Leibniz constructed the paracentric isochrone (the curve traced out by a mass moving under gravity such that it distances itself from a fixed point at a constant speed) by means of a succession of segments of tangents as close as possible to the actual arc. On this occasion he writes (Leibniz, 1989, p. 304):

> Thus we will obtain a polygon […] replacing the unknown curve, that is to say *a Mechanical curve replacing a Geometric curve*, at the same time we clearly see that it is possible to make the Geometric curve pass through a given point, since such a curve is the *limit* where the convergent polygons definitely fade.

We can recognise the idea in this that Cauchy uses at a later date to demonstrate the famous theory of existence (Cauchy, 1981, p. 55) which, after a few tweaks, became the Cauchy-Lipschitz theorem. Between Leibniz and Cauchy it was Euler who formalised the polygonal method and from it created the numerical method. Since this really worked for its applications history has remembered it as Euler's method.

## A wordy mathematician, master of the pen

Leonhard Euler (1707-1783) could single-handedly embody the mathematics of the 18th century. Euler studied in his natal town of Basel, Switzerland where his father was the protestant pastor. In 1726 he was offered a post at the Academy of Science in St Petersburg to take over from Nicholas Bernoulli. A few years later, after his position in society was assured, he married Katharina Gsell. She was the daughter of a painter from St Petersburg and like him, of Swiss origin. They had thirteen children, of whom only five survived: Euler took pleasure in recounting that he had made some of his most important mathematical discoveries while holding a baby in his arms while the other children played around him. In 1741, on the invitation of Frederick the Great, Euler joined the Academy of Science in Berlin, where he was to stay until 1766. He then returned to St Petersburg for the last years of his life. Although he had become blind, he pursued his scientific activities without pause with the help of his sons and other members of the Academy.

Writing equally in Latin, German or French Euler maintained a regular correspondence with most of the continental mathematicians, finding himself at the crossroads of contemporary research. Gifted with an extraordinary creative power, he constructed an immense work which significantly enhanced progress in all areas of mathematics and physics. Begun in 1911, the publication of his complete works is still not finished in spite of the 76 volumes that have already appeared: 29 volumes on mathematics, 31 for mechanics and astronomy, 12 for physics and various works and 4 for correspondence.

In particular, Euler's work on differential equations is considerable. With great skill he explored the ideas launched by his predecessors and pushed the majority of them a great deal further. In his research on Riccati's equation $y' = a(x)y^2 + b(x)y + c(x)$, so important because its integration is equivalent to that of the second order linear equation $y'' = a(x)y' + b(x)y + c(x)$ omnipresent in mathematical physics, Euler had recourse to all methods imaginable: series, definite integrals depending on a parameter, continuous fractions, tractional motion etc. This determination cannot just be explained by mathematical reasons as Euler needed to integrate second order linear equations in many of his works on geometry and physics. From 1728 he met second order equations dealing with the movement of a pendulum in a resistant medium. In 1733 the calculation of the length of a quarter of an ellipse led him to a second order linear equation, then on to one of Riccati's equations. In 1736, following Daniel Bernoulli, he tackled the oscillations of a vertically hung chain, homogenous or not. Later, in 1764, he took an interest in the vibrations of a circular membrane. In this later research he deals with different second order differential equations which he does not know how to integrate exactly. We see him making use more and more frequently of series, and this use becomes systematic from 1750. This was how, on several occasions, he came across Bessel equations and their equivalents leading to the first general expression of Bessel functions.

## Euler's text

When equations cannot be integrated by quadratures and do not lend themselves readily to the series method, but for which a solution has to be found at all costs, at least approximately, for practical reasons, Euler resorted to the polygonal method. We find Euler's method initial appearance in the first volume of *Institutiones calculi integralis*, published in St Petersburg in 1768. However, Euler had already used this method on at least two occasions: in 1753 for his research on the trajectory of a body in a resistant medium and in 1759 to determine the perturbations of a planet or comet (Tournès, 1997, pp. 158-167). This work on ballistics and celestial mechanics shows that, for Euler, practice preceded theory: it is only after having rubbed shoulders at length with substantial applications that the great mathematician was able to perfect the simplified didactic text of 1768. We give below an English translation of the extract from the work which corresponds to what is actually taught in upper secondary. This will allow interested teachers to let their students discover Euler's method based on the original text. Here is the passage in question (Euler, 1768, pp. 424-425), which does not require commentary as it is so clear and instructive.

Problem 85

650. Whatever the differential equation might be, determine its complete integral in the most approximate way.

Solution

Let there be a differential equation between two variables $x$ and $y$. This equation will be in the form $\frac{dy}{dx} = V$ where $V$ is any function of $x$ and $y$. Moreover, when you are calculating a definite integral, you must interpret it in such a way that if you give $x$ a fixed value, for example $x = a$, the other variable should take on a given value, for example $y = b$. Let us first deal with finding the value of $y$ when $x$ is given a slightly different value to $a$. In other words, let us find $y$ when $x = a + \omega$. Now since $\omega$ is a small quantity, the value of $y$ remains close to $b$. That is why, if $x$ only varies from $a$ to $a + \omega$, it is possible to consider the quantity $V$ as constant in that interval. So, having said $x = a$ and $y = b$, it will follow that $V = A$ and for this slight change we will have $\frac{dy}{dx} = A$ and by integration $y = b + A(x - a)$, a constant having been added so that $y = b$ when $x = a$. Let us therefore assume that, when $x = a + \omega$, $y = b + A\omega$. In the same way we can allow ourselves to advance further from these last steps by means of more small steps until we finally reach values as far from the initial values as we wish. In order to show this more clearly, let us set them out in succession the following way:

| Variables | Successive values |
|---|---|
| $x$ | $a,\ a',\ a'',\ a''',\ a^{\text{iv}},\ \ldots\ 'x,\ x$ |
| $y$ | $b,\ b',\ b'',\ b''',\ b^{\text{iv}},\ \ldots\ 'y,\ y$ |
| $V$ | $A,\ A',\ A'',\ A''',\ A^{\text{iv}},\ \ldots\ 'V,\ V$ |

Obviously, from the first values $x = a$ and $y = b$, one can derive $V = A$, but then for the second values we will have $b' = b + A(a' - a)$, the difference $a' - a$ having been chosen as small as desired. From that, supposing $x = a'$ and $y = b'$, we will calculate $V = A'$ and so for the third values we will obtain $b'' = b' + A'(a'' - a')$, from which having established $x = a''$ and $y = b''$, it will follow that $V = A''$. So for the fourth values, we will have $b''' = b'' + A''(a''' - a'')$ and from that, supposing $x = a'''$ and $y = b'''$, we determine $V = A'''$ and we can advance towards values that are as far removed from the originals as we wish. Now the first series which illustrates the successive values of $x$ can be increasing or decreasing provided the change is by very small amounts.

Corollary 1

651. For each individual tiny interval, the calculation is done in the same way and thus the values which depend successively on each other are obtained. By this method for all the individual values given for $x$, the corresponding values can be determined.

Corollary 2

652. The smaller the values of $\omega$, the more precise are the values obtained for each interval. However, mistakes made in each interval, even if they are even smaller, increase to a greater number.

Corollary 3

653. Now in this calculation, errors derive from the fact that we consider at each interval the two values of $x$ and $y$ as constants and consequently the function $V$ is held to be a constant. It follows that the more the value of $V$ varies from one interval to the next, the greater the errors.

If Euler's treatise of 1768 has been recorded by history it is undoubtedly because for the first time the polygonal method was clearly set out for didactic purposes at the same time it took on the entirely numerical form which we know today. Before Euler, according to the ancient geometric view of analysis, one did not study functions but constructed curves. Therefore the polygonal method was initially a way to determine geometrically the new transcendental curves which appeared alongside infinitesimal calculus rather than a numerical process. This aspect, moreover, survived well after Euler at the heart of graphical construction practised by engineers up to the Second World War: numerous variations on and improvements for the polygonal method then came to light to calculate the integrals of differential equations graphically. (Tournès, 2003, pp. 458-468).

In the last year of upper secondary school, in order to introduce the notion of differential equations, I think it could be pertinent to return to the original geometric meaning of the Euler-Cauchy method: constructing a curve based on the knowledge of its tangents and carrying out this construction entirely by geometric means without any recourse to numerical calculation. This is what I demonstrated at the IREM in Réunion and what I am going to present here.

## Account of classroom activities

I set up this strategy in a final year science class covering two two-hour sessions. The trial took place at the Le Verger High School, at Sainte-Marie in Réunion. It was Mr Jean-Claude Lise's class and I am most grateful for his welcome and collaboration. The students had previously met Euler's methods with their mathematics and physics teachers in the traditional numeric form, carrying out the calculations on a spreadsheet.

**First session: where the students see the exponential in a new light**

The first session was dedicated to the construction of an exponential function, the keystone of the final year programme. I began with a brief historical outline of Euler: the main stages of his life in Basel, St Petersburg and Berlin; the immensity of his writings in mathematics and physics; some details on certain of his works which link with what is taught in high school. Then after having quickly run through the extract given earlier from the 1768 *Institutiones calculi integralis* and having made the link between the students' knowledge on differential equations, I told them that my objective was to get them to apply Euler's method in a different way, no longer numerically but purely graphically, by replacing all the calculations by geometric constructions with a ruler and compass. For that they first had to learn the basics of graphical construction, as they appear in the first pages of Descartes' *Géométrie* (Descartes, 1637, pp. 297-298). So I suggested to the class the following preparatory activity: "given one segment one unit in length and two segments of length $x$ and $y$, construct segments of length $x+y$, $x-y$, $x\times y$, $x/y$". Getting underway was laborious, the students having great difficulty in recalling Thalès' theorem and in applying it in context. They nevertheless managed the synthesis of Figure 7.1, completed by several of them on the interactive whiteboard.

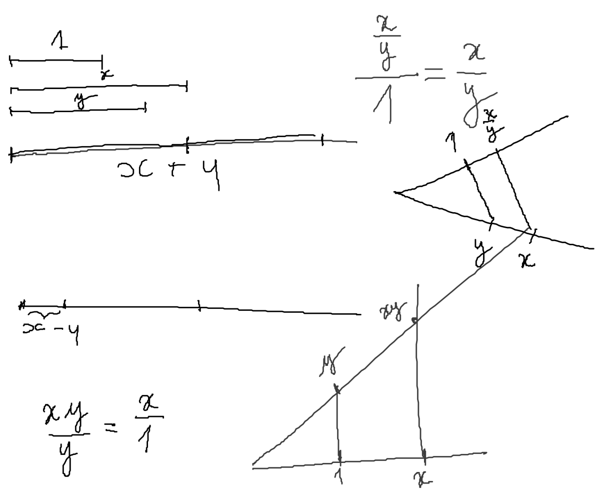


*Figure 7.1.* Basic constructions.

Then we moved on to the exponential function by replacing the differential equation $y' = y$ by the equation with finite differences $\Delta y = y\,\Delta x$. I first asked the class to explain the basic construction which would allow progress from a given point $(x, y)$ to the neighbouring point $(x+\Delta x, y+\Delta y)$ by drawing a small segment of the tangent. The students easily understood how to transform the original ordinate $y$ at the starting point into a slope for the required tangent (see Figure 7.2). To do this it was enough to extend by a unit to the left of the point $(x,0)$ and then to join the point $(x-1,0)$ to the point $(x, y)$; thus one obtains a segment of slope $y$ which now only needs to be extended.

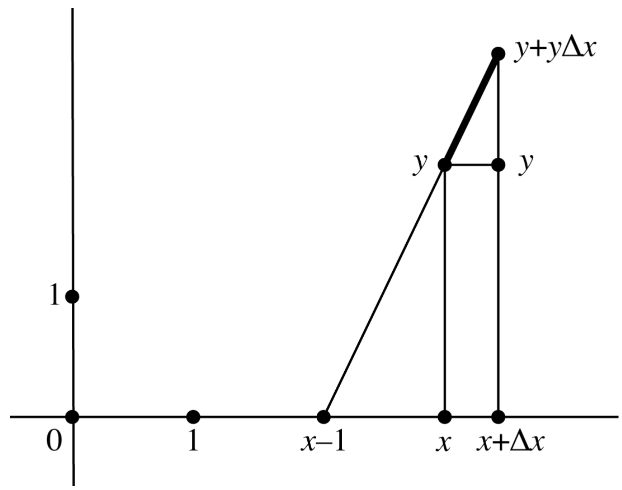


*Figure 7.2.* Basic construction of a tangent to the exponential.

Once this basic construction was understood they had to repeat it in their own way from the starting point $(0,1)$ to obtain a polygonal line approaching the graph of the exponential function. Figure 7.3 shows three quite different examples of students' work; on the third a confusion between the chosen unit (2 cm) and the process of the subdivision used for the construction (1 cm) can be seen, which means that the student treated the equations as $y' = 2y$.

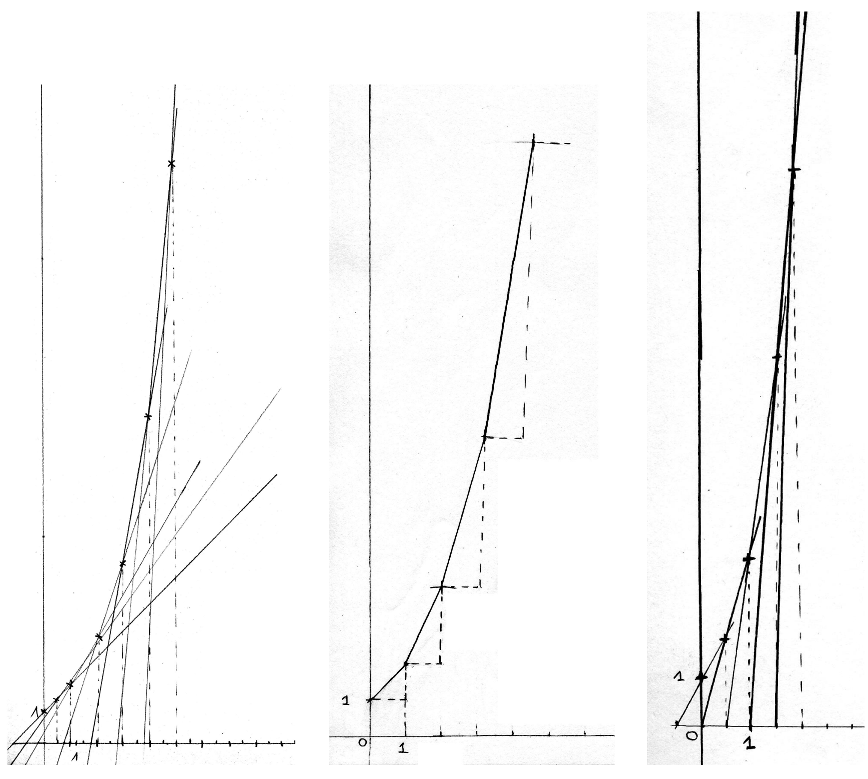

*Figure 7.3.* Three constructions of the exponential.

At this point I digressed so we could delve deeper into Euler's method and show its implicit variation. Explicitly one moves from a point $(x, y)$ to a neighbouring point

$(x + \Delta x,\, y + \Delta y)$ using the tangent at the starting point. In a symmetrical way, we can use the tangent at the end point, i.e. replace the differential equation $y' = y$ by the equation that uses finite differences $\Delta y = (y + \Delta y)\Delta x$. We speak of the implicit method because the difference $\Delta y$ is not directly given, but determined implicitly by the previous equation. In the case of the exponential this equation is easily solved and we get:

$$\Delta y = \frac{y\,\Delta x}{1 - \Delta x}.$$

I then asked the students to do this basic construction with finite differences and to repeat it to arrive at a second construction approached from the exponential (see Figure 7.4). Some quicker students neatly incorporated the two figures on one single figure.

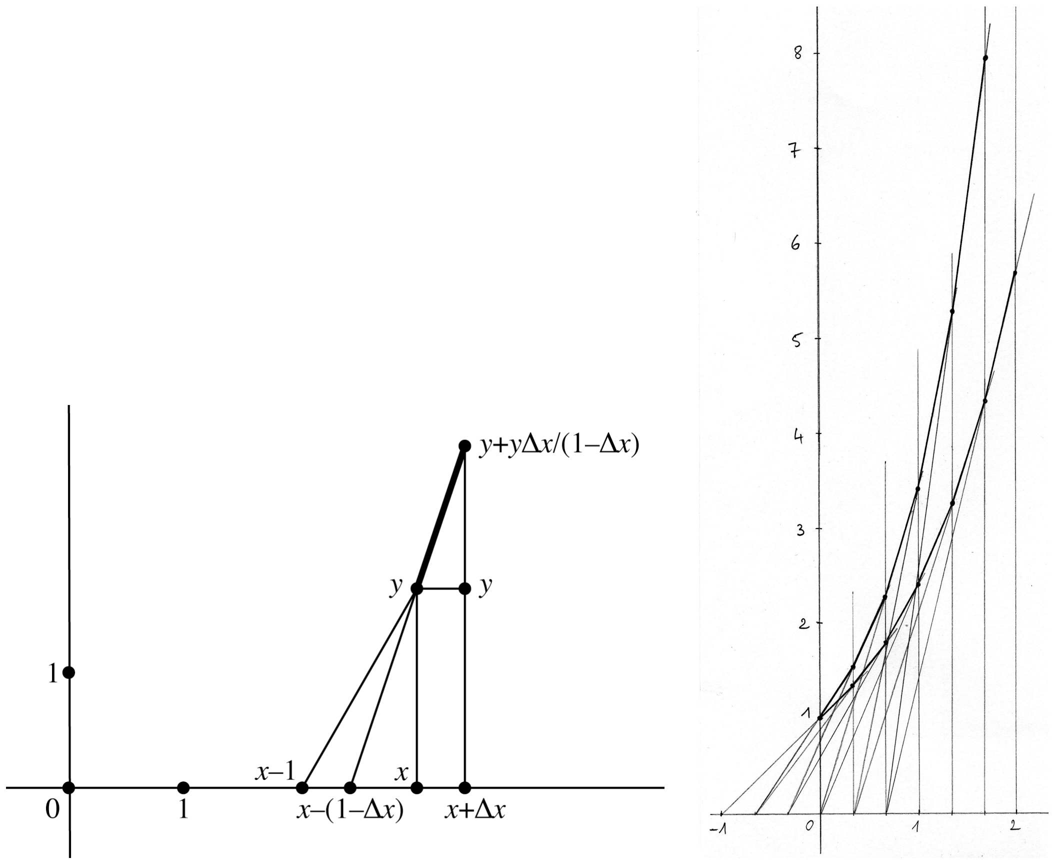


*Figure 7.4.* Constructing the exponential using Euler's implicit method.

The session ended with an analysis of this final figure: we tried to understand why the true curve defined by the differential equation, supposing it existed, had to be situated between the two polygonal lines provided by the explicit and implicit methods. We concluded by saying that a much better approximate polygonal line would be obtained by taking the average of the two $y$ values for each $x$ value.

**Second session: where the students dealt with two Baccalaureate topics in an unusual way**

During my second session with the students they had to deal with two topics from the course on Euler's method by practising the new techniques of graphical construction they had just discovered. One of these topics had been given by their teacher two weeks ago in a mock exam and I asked them to work on the other as a piece of homework in the week between my two sessions. In this way they had all the elements on hand to compare the numeric and graphic approaches of the two individual differential equations. Here is the start of the first topic we worked on:

We were to study the functions $f$ which could be derived in $[0,+\infty[$ subject to

$$(1)\qquad \begin{cases} \text{for all } x \in \left[0,+\infty\right[,\ f(x)f'(x)=1 \\ f(0)=1. \end{cases}$$

Part A

Let a function $f$ exist which will satisfy (1). Euler's method allows the construction of a series of points $(M_n)$ near the curve represented by $f$. The step $h=0.1$ is chosen.

Then the coordinates $(x_n, y_n)$ of the points $M_n$ obtained by this method satisfy:

$$\begin{cases} x_0 = 0 \\ y_0 = 1 \end{cases} \text{ and } \begin{cases} x_{n+1} = x_n + 0.1 \\ y_{n+1} = y_n + \dfrac{0.1}{y_n} \end{cases} \text{ for all } n \in \mathbf{N}.$$

Calculate the coordinates of the points $M_1, M_2, M_3, M_4, M_5$ (rounded to the nearest thousandth). […]

The problem continues by getting the students to check that the function $f(x)=\sqrt{2x+1}$ is the only solution and asking them to compare the values of $f(0.1), f(0.2), f(0.3), f(0.4), f(0.5)$ to those previously obtained by Euler's method.

I gave the students a challenge: graphically construct a polygonal line from $x=0$ to $x=0.5$ with step $h=0.1$, without doing any numerical calculation, then measure with a 20 cm ruler the values of the corresponding ordinates and compare them to those found by the numerical method. At this stage in the progress of the work I gave no more guidance and left the students to fend for themselves. The completion of the basic construction associated with the equation $\Delta y = \Delta x / y$ took most of them an extremely long time. Figure 7.5 illustrates a way of organising this construction, but the students using their own initiative found many other ways. Figure 7.6 shows four pieces of student work, all very different. Reaching such a result took some more than an hour of intense work. Often there were several false starts or careless errors. I came away convinced that if one allows the students time to get involved in what they are doing they achieve remarkable results.

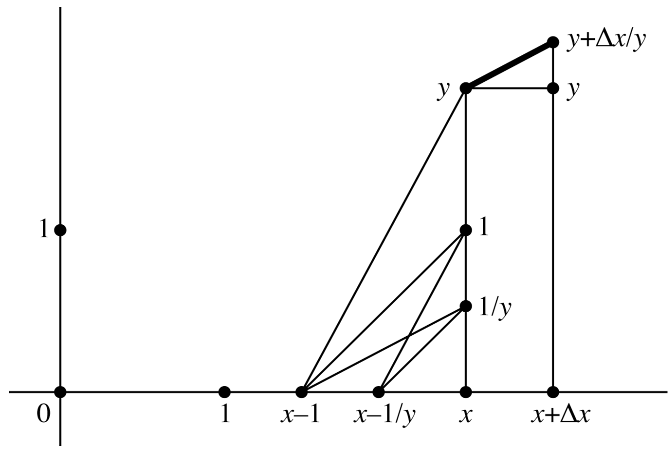


*Figure 7.5.* Basic construction of a tangent for the equation $y' = 1/y$.

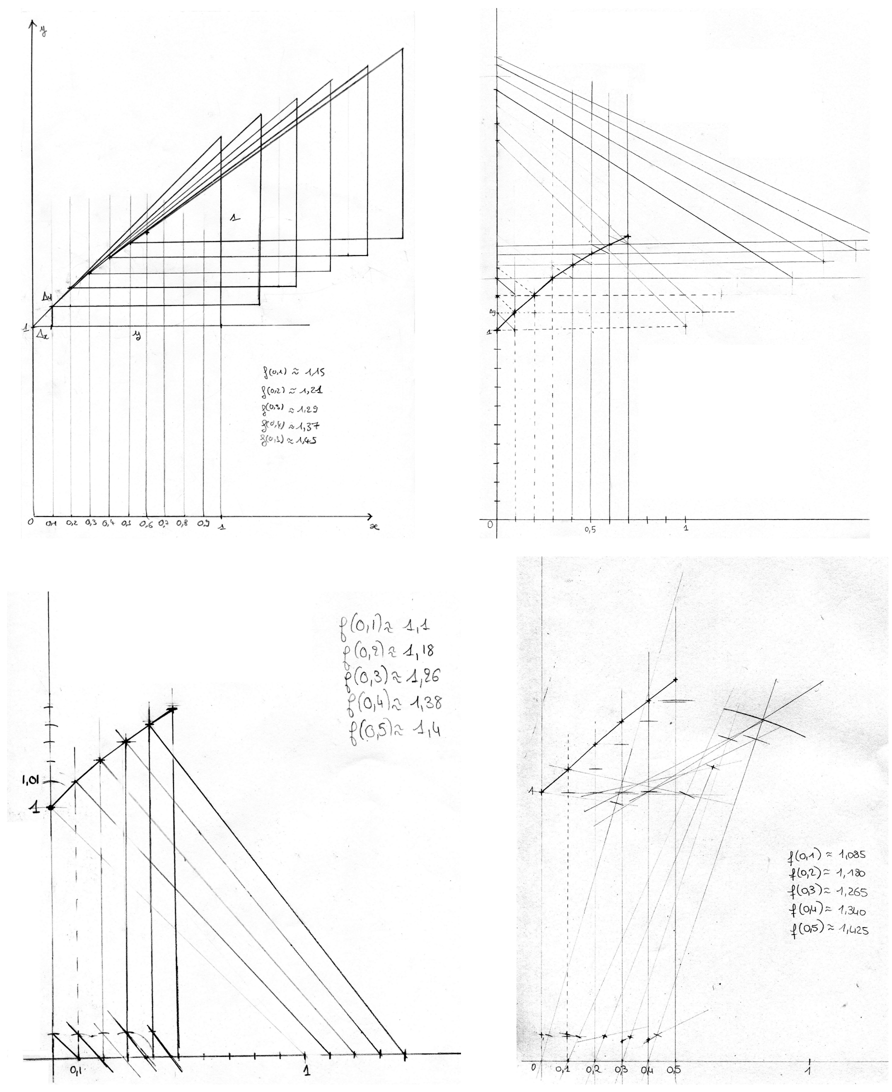


*Figure 7.6.* Four constructions of the equation $y' = 1/y$ with $y(0) = 1$.

The more advanced students could then turn their attention in a similar way to the second syllabus topic, following the same path as before. The start of the problem, reproduced below, deals with a differential equation using Euler's method. Here one is given the explicit function

$$f(x) = 2\left(\frac{e^{4x}-1}{e^{4x}+1}\right),$$

which allows it to be studied directly and bring attention to the asymptote $y = 2$.

> The plane is given an orthonormal basis $(0,\vec{i},\vec{j})$. We are interested in the functions $f$ derivable in $[0,+\infty[$ satisfying the conditions
>
> $$\begin{cases} (1):\ \text{for all real } x \text{ in } \left[0,+\infty\right[,\ f'(x) = 4-\left[f(x)\right]^2 \\ (2):\ f(0) = 0. \end{cases}$$
>
> We admit that there is a unique function $f$ satisfying (1) and (2) simultaneously.
>
> The two parts can be dealt with independently. The annex will be completed and submitted with the work at the end of the text.
>
> Part B Follow-up study
>
> To obtain a representative approximation of the curve $f$ we use Euler's method with a step length of 0.2. Thus we obtain a succession of points marked $(M_n)$, an abscissa $x_n$ and ordinate $y_n$ such that
>
> $$\begin{cases} x_0 = 0 \ \text{for all natural numbers } n,\ x_{n+1} = x_n + 0.2 \\ y_0 = 0 \ \text{for all for all natural numbers } n,\ y_{n+1} = -0,2y_n^2 + y_n + 0.8. \end{cases}$$
>
> a.i. The coordinates of the first few points are shown in the table below. Complete the table. Give your answers to the nearest $10^{-4}$.
>
> Annex: Part B
>
> | $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
> |---|---|---|---|---|---|---|---|---|
> | $x_n$ | 0 | 0.2 | 0.4 | | | | | |
> | $y_n$ | 0 | 0.8000 | 1.4720 | | | | | |
>
> […]

Now well versed in graphical construction, the students are tasked with finding a basic construction using the equation with finite differences $\Delta y = (4 - y^2)\Delta x$ (Figure 7.7) and with carefully constructing an approximate integral curve (see Figure 7.8), which will allow them to compare the diagram with the numerical values in the table.

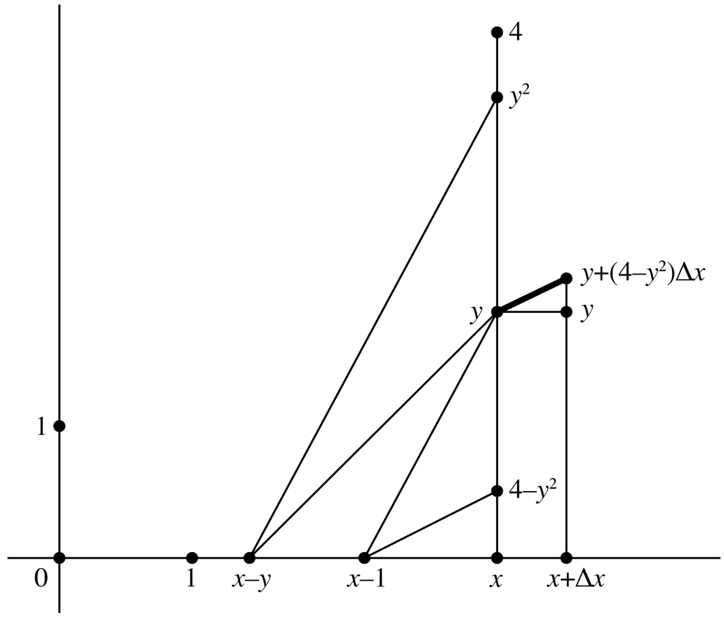


*Figure 7.7*. Basic construction of a tangent for the equation $y' = 4 - y^2$.

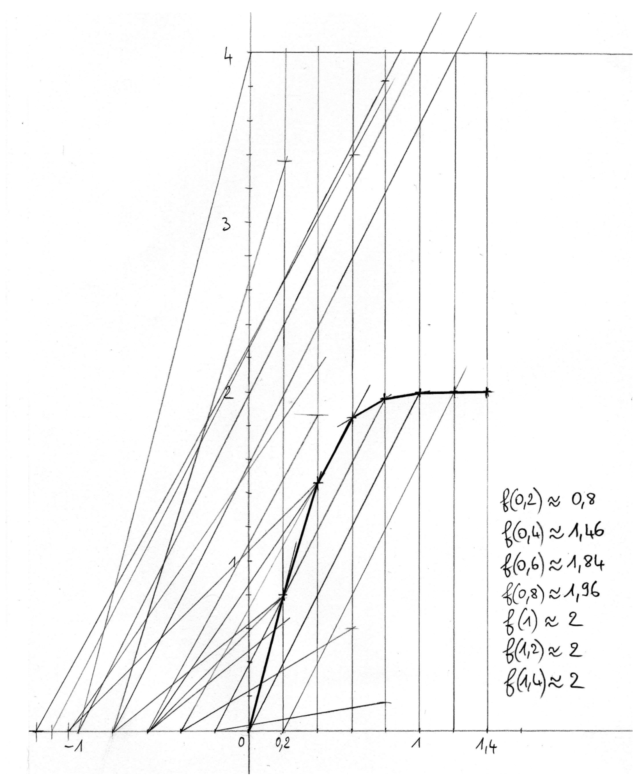


*Figure 7.8*. A construction of the equation $y' = 4 - y^2$ with $y(0) = 0$.

At the end of the trial I am convinced that these practical tasks inspired by the history of the polygonal method allow a revision of basic geometric knowledge learnt in secondary school (ages 11-16). They also create the opportunity for fruitful interaction between algebra and geometry as well as offering a gentle introduction to analysis. They lead to acquiring a kinaesthetic feel of the tangent describing the curve from the differential equation. The old expression of ‘the inverse problem of tangents’ takes on its full meaning here: the students experience this problem by physically drawing the tangent and following its movement step by step. After these very telling graphic investigations it should be easier for them to move from the small to the infinitely small; from the discrete to the continuous; and to imagine the ideal curve defined by the differential equation which they will eventually study more abstractly.

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