Simultaneous global exact controllability in projection of infinite 1D bilinear Schrödinger equations

Alessandro Duca

Simultaneous global exact controllability in projection of infinite 1D bilinear Schrödinger equations

Alessandro Duca

Institut Fourier, Université Grenoble Alpes
100 Rue des Mathématiques, 38610 Gières, France
alessandro.duca@univ-grenoble-alpes.fr

Abstract

The aim of this work is to study the controllability of infinite bilinear Schrödinger equations on a segment. In particular, we consider the equations (BSE) $i\partial_t\psi^j = -\Delta\psi^j + u(t)B\psi^j$ in the Hilbert space $L^2((0,1),\mathbb{C})$ for every $j \in \mathbb{N}^*$. The Laplacian $-\Delta$ is equipped with Dirichlet homogeneous boundary conditions, B is a bounded symmetric operator and $u \in L^2((0,T),\mathbb{R})$ with $T > 0$. First, we show that simultaneously controlling infinite (BSE) by projecting onto suitable N dimensional spaces is equivalent to the simultaneous controllability of N equations (without projecting). Second, we prove the simultaneous local and global exact controllability of infinite bilinear Schrödinger equations in projection. The local controllability is guaranteed for any positive time and both the outcomes can be ensured for explicit B. In conclusion, we rephrase the results in terms of density matrices.

AMS subject classifications: 35Q41, 93C20, 93B05, 81Q15.

Keywords: Schrödinger equation, simultaneous control, global exact controllability, moment problem, perturbation theory, density matrices.

1 Introduction

1.1 The problem

In this work, we consider infinite particles constrained in a one-dimensional bounded region and subjected to an external control field. A suitable choice for such setting is to model the dynamics of these particles by infinitely many bilinear Schrödinger equations in the Hilbert space $\mathscr{H} = L^2((0,1),\mathbb{C})$

$$\text{(BSE)} \qquad \begin{cases} i\partial_t\psi_j(t) = A\psi_j(t) + u(t)B\psi_j(t), & t \in (0,T),\ T > 0, \\ \psi_j(0) = \psi_j^0 \in L^2((0,1),\mathbb{C}), & j \in \mathbb{N}^*. \end{cases}$$

The Laplacian $A = -\Delta$ is equipped with homogeneous Dirichlet boundary conditions such that

$$D(A) = H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C}).$$

The bounded symmetric operator B models the action of the external field, while the control function $u \in L^2((0,T),\mathbb{R})$ represents its intensity.

We study the controllability of the infinite bilinear Schrödinger equations (BSE) at the same time T with one unique control u by projecting onto suitable finite dimensional subspaces of $\mathscr{H}$.

In order to detail the purpose of the work, we need to introduce the following notations. First, we denote by Γ_t^u the unitary propagator in $\mathscr{H}$ generated by the dynamics of the (BSE) in a given time interval $[0,t]$ (when it is defined). Second, we consider an orthonormal system $\Psi := \{\psi_j\}_{j\in\mathbb{N}^*}$ and we call $\pi_N(\Psi)$ with $N \in \mathbb{N}^*$ the orthogonal projector such that

$$\pi_N(\Psi) : \mathscr{H} \longrightarrow span\overline{\{\psi_j \ : \ j \leq N\}}^{\,L^2}. \tag{1}$$

Third, we said that two sequences of functions $\{\psi_j^1\}_{j\in\mathbb{N}^*}, \{\psi_j^2\}_{j\in\mathbb{N}^*} \subseteq \mathscr{H}$ are unitarily equivalent when there exists $\Gamma \in U(\mathscr{H})$ (the space of the unitary operators in $\mathscr{H}$) such that

$$\psi_j^1 = \Gamma\psi_j^2, \qquad \forall j \in \mathbb{N}^*.$$

We investigate the existence of orthonormal systems Ψ so that, for any suitable $\{\psi_j^1\}_{j\in\mathbb{N}^*}$ and $\{\psi_j^2\}_{j\in\mathbb{N}^*}$ unitarily equivalent, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_n(\Psi)\,\Gamma_T^u\psi_j^1 = \pi_n(\Psi)\,\psi_j^2, \qquad \forall j \in \mathbb{N}^*. \tag{2}$$

If we denote by $\langle\cdot,\cdot\rangle_{L^2}$ the usual L^2-scalar product, then the identities (2) become

$$\langle\psi_k, \Gamma_T^u\psi_j^1\rangle_{L^2} = \langle\psi_k, \psi_j^2\rangle_{L^2}, \qquad \forall j,k \in \mathbb{N}^*,\ k \le N.$$

In order to achieve the result, we show that the simultaneous global exact controllability in projection onto a suitable N dimensional space is equivalent to the controllability of N problems (without projecting).

1.2 Main results

Let $\|\cdot\|_{L^2}$ be the norm of the Hilbert space $\mathscr{H} = L^2((0,1),\mathbb{C})$ such that $\|\cdot\|_{L^2} = \sqrt{\langle\cdot,\cdot\rangle_{L^2}}$. Let

$$\{\phi_j\}_{j\in\mathbb{N}^*}, \qquad \{\lambda_j\}_{j\in\mathbb{N}^*}$$

respectively be the eigenfunctions and the eigenvalues of A such that

$$\phi_j(x) = \sqrt{2}\sin(j\pi x), \qquad \lambda_j = \pi^2 j^2, \qquad \forall j \in \mathbb{N}^*. \tag{3}$$

We notice that $\{\phi_j\}_{j\in\mathbb{N}^*}$ forms an Hilbert basis of $\mathscr{H}$ and we consider the spaces

$$H^3_{(0)} = D(|A|^{\frac{3}{2}}), \qquad \|\cdot\|_{(3)} = \|\cdot\|_{H^3_{(0)}} = \Big(\sum_{k=1}^{\infty} |k^3\langle\cdot,\phi_k\rangle_{L^2}|^2\Big)^{\frac{1}{2}},$$

$$\ell^\infty(H^3_{(0)}) = \big\{\{\psi_j\}_{j\in\mathbb{N}^*} \subset H^3_{(0)} \big|\ \sup_{j\in\mathbb{N}^*}\|\psi_j\|_{(3)} < \infty\big\}.$$

Theorem 1.1. *Let Γ_t^u be the unitary propagator in $\mathscr{H}$ generated by the dynamics of the* (BSE) *in the time interval $[0,t]$. The two following results are equivalent with $N \in \mathbb{N}^*$.*

(1) *Let $\{\psi_j^1\}_{j\in\mathbb{N}^*}$ and $\{\psi_j^2\}_{j\in\mathbb{N}^*} \subset H^3_{(0)}$ be a couple of complete orthonormal systems of $\mathscr{H}$. Let $\widehat{\Gamma}$ be the unitary operator such that $\{\widehat{\Gamma}\psi_j^2\}_{j\in\mathbb{N}^*} = \{\psi_j^1\}_{j\in\mathbb{N}^*}$. For any $\Psi := \{\psi_j\}_{j\le N} \subset H^3_{(0)}$ orthonormal system of $\mathscr{H}$ such that $\{\widehat{\Gamma}\psi_j\}_{j\le N} \subset H^3_{(0)}$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that*

$$\langle\psi_k, \Gamma_T^u\psi_j^1\rangle_{L^2} = \langle\psi_k, \psi_j^2\rangle_{L^2}, \qquad \forall j,k \in \mathbb{N}^*,\ k \le N.$$

In other words, the following identities are satisfied (with $\pi_N(\Psi)$ defined in (1)*):*

$$\pi_N(\Psi)\Gamma_T^u\psi_j^1 = \pi_N(\Psi)\psi_j^2, \qquad \forall j \in \mathbb{N}^*.$$

(2) *Let $\{\psi_j^1\}_{j\le N}$ and $\{\psi_j^2\}_{j\le N} \subset H^3_{(0)}$ be a couple of orthonormal systems in $\mathscr{H}$. There exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that*

$$\Gamma_T^u\psi_j^1 = \psi_j^2, \qquad \forall j \le N.$$

Theorem 1.1 ensures the equivalence between the controllability in projection of infinite bilinear Schrödinger equations and the controllability of finitely many (BSE) without projecting.

Before providing the other main results of the work, we need to introduce further notations. For $s > 0$, we call $H^s := H^s((0,1),\mathbb{C})$, $H_0^s := H_0^s((0,1),\mathbb{C})$ and, for $N \in \mathbb{N}^*$, we define

$$I^N := \{(j,k) \in \mathbb{N}^* \times \{1,...,N\} : j \ne k\}. \tag{4}$$

Assumptions I. The bounded symmetric operator B satisfies the following conditions.

1. For any $N \in \mathbb{N}^*$, there exists $C_N > 0$ so that $|\langle \phi_k, B\phi_j \rangle_{L^2}| \geq \frac{C_N}{k^3}$ for every $j \leq N$ and $k \in \mathbb{N}^*$.
2. $Ran(B|_{H^2_{(0)}}) \subseteq H^2_{(0)}$ and $Ran(B|_{H^3_{(0)}}) \subseteq H^3 \cap H^1_0$.
3. For every $N \in \mathbb{N}^*$ and $(j,k),(l,m) \in I^N$ such that $(j,k) \neq (l,m)$ and $j^2 - k^2 - l^2 + m^2 = 0$,
$$\langle \phi_j, B\phi_j \rangle_{L^2} - \langle \phi_k, B\phi_k \rangle_{L^2} - \langle \phi_l, B\phi_l \rangle_{L^2} + \langle \phi_m, B\phi_m \rangle_{L^2} \neq 0.$$

The first condition in Assumptions I quantifies how much B "mixes" eigenstates, while the second fixes its regularity. The third condition instead is required in order to decouple, through perturbation theory techniques, the eigenvalues resonances appearing in the proof of the following result.

The next theorem states the second main outcome of the work which ensures the simultaneous local exact controllability in projection for any positive time.

Theorem 1.2. *Let Γ^u_t be the unitary propagator in $\mathscr{H}$ generated by the dynamics of the* (BSE) *in the time interval $[0,t]$ with B satisfying Assumptions I. Let $N \in \mathbb{N}^*$. For every $T > 0$, there exist a neighborhood O in $\ell^\infty(H^3_{(0)})$ and $\Psi := \{\psi_j\}_{j\in\mathbb{N}^*} \in O$ such that the following result is verified. For every $\{\psi^1_j\}_{j\in\mathbb{N}^*} \in O$, there exist $\{\theta_j\}_{j\leq N} \subset \mathbb{R}$ and $u \in L^2((0,T),\mathbb{R})$ such that*

$$\begin{cases} \langle \psi_k, \Gamma^u_T \psi_j \rangle_{L^2} = e^{i\theta_j} \langle \psi_k, \psi^1_j \rangle_{L^2}, & \forall j,k \in \mathbb{N}^*,\ j \leq N,\ k \leq N, \\ \langle \psi_k, \Gamma^u_T \psi_j \rangle_{L^2} = \langle \psi_k, \psi^1_j \rangle_{L^2}, & \forall j,k \in \mathbb{N}^*,\ j > N,\ k \leq N. \end{cases}$$

In other words, the following identities are satisfied (with $\pi_N(\Psi)$ defined in (1)*):*

$$\begin{cases} \pi_N(\Psi)\Gamma^u_T \psi_j = e^{i\theta_j} \pi_N(\Psi)\psi^1_j, & \forall j \in \mathbb{N}^*,\ j \leq N, \\ \pi_N(\Psi)\Gamma^u_T \psi_j = \pi_N(\Psi)\psi^1_j, & \forall j \in \mathbb{N}^*,\ j > N. \end{cases}$$

Theorem 1.2 allows to locally control infinite bilinear Schrodinger equations in any positive $T > 0$ with one single u by projecting onto suitable a finite dimensional subspace of $\mathscr{H}$. The statement is a simplification of Proposition 4.2 where the family $\{\psi_j\}_{j\in\mathbb{N}^*}$ and the neighborhood O are specified.

The next theorem states the third main result of the work that is the simultaneous global exact controllability in projection.

Theorem 1.3. *Let Γ^u_t be the unitary propagator in $\mathscr{H}$ generated by the dynamics of the* (BSE) *in the time interval $[0,t]$ with B satisfying Assumptions I. Assume that $\Psi := \{\psi_j\}_{j\in\mathbb{N}^*} \subset H^3_{(0)}$ is an orthonormal system of $\mathscr{H}$. Let $\{\psi^1_j\}_{j\in\mathbb{N}^*}$ and $\{\psi^2_j\}_{j\in\mathbb{N}^*} \subset H^3_{(0)}$ be complete orthonormal systems of $\mathscr{H}$ and $\widehat{\Gamma} \in U(\mathscr{H})$ be such that $\{\widehat{\Gamma}\psi^2_j\}_{j\in\mathbb{N}^*} = \{\psi^1_j\}_{j\in\mathbb{N}^*}$. If the following condition is satisfied*

$$\{\widehat{\Gamma}\psi_j\}_{j\leq N} \subset H^3_{(0)} \tag{5}$$

with $N \in \mathbb{N}^*$, then there exist $T > 0$, $u \in L^2((0,T),\mathbb{R})$ and $\{\theta_k\}_{k\leq N} \subset \mathbb{R}$ such that

$$\langle \psi_k, \Gamma^u_T \psi^1_j \rangle_{L^2} = e^{i\theta_k} \langle \psi_k, \psi^2_j \rangle_{L^2}, \qquad \forall j,k \in \mathbb{N}^*,\ k \leq N. \tag{6}$$

Theorem 1.3 allows to control with a single u and at the same time T any finite number of components of infinitely many solutions of the problems (BSE). We notice that the outcome is guaranteed when the orthonormal system $\{\psi_j\}_{j\in\mathbb{N}^*}$ verifies a "$H^3_{(0)}-$compatibility condition" exposed in (5) and is ensured up to phases in the components which prevents to formulate the result in terms of projectors. Although, if $\Psi^3 = \Psi^2$, then $\{\widehat{\Gamma}\psi^3_j\}_{j\leq N} = \{\psi^1_j\}_{j\leq N} \subset H^3_{(0)}$ and, since $e^{i\theta_k}\langle \psi^2_k, \psi^2_j \rangle_{L^2} = e^{i\theta_k}\delta_{k,j} = e^{i\theta_j}\langle \psi^2_k, \psi^2_j \rangle_{L^2}$ for every $j,k \in \mathbb{N}^*$, the relations (6) become

$$\begin{aligned} \pi_N(\Psi^2)\Gamma^u_T \psi^1_j &= \pi_N(\Psi^2)\ e^{i\theta_j}\psi^2_j, & \forall j \leq N, \\ \pi_N(\Psi^2)\Gamma^u_T \psi^1_j &= \pi_N(\Psi^2)\ \psi^2_j, & \forall j > N. \end{aligned}$$

As Ψ^2 is composed by orthogonal elements, the next corollary follows.

Corollary 1.4. *Let Γ_t^u be the unitary propagator in $\mathscr{H}$ generated by the dynamics of the* (BSE) *in the time interval $[0,t]$ with B satisfying Assumptions I. Let $\Psi^1 := \{\psi_j^1\}_{j\in\mathbb{N}^*}$, $\Psi^2 := \{\psi_j^2\}_{j\in\mathbb{N}^*} \subset H^3_{(0)}$ be complete orthonormal systems of $\mathscr{H}$. For every $N \in \mathbb{N}^*$, there exist $T > 0$, $u \in L^2((0,T),\mathbb{R})$ and $\{\theta_j\}_{j\leq N} \subset \mathbb{R}$ such that*

$$\begin{cases} \Gamma_T^u \psi_j^1 = e^{i\theta_j}\psi_j^2, & \forall j \leq N, \\ \pi_N(\Psi^2)\ \Gamma_T^u \psi_j^1 = 0, & \forall j > N. \end{cases}$$

Here, one can notice the parallelism between our results with the ones provided in [MN15] by Morancey and Nersesyan. On the one hand, Corollary 1.4 implies the controllability of any finite number of bilinear Schrödinger equations (such as [MN15, Main Theorem]). On the other hand, similar statements to Theorem 1.2 and Theorem 1.3 can be ensured by using Theorem 1.1 with the outcomes from [MN15]. Nevertheless, in this case the hypotheses on the operator B would not be easy to confirm. More precisely, the mentioned work studies the simultaneous global exact controllability of any finite number of bilinear Schrödinger equations defined by the time-dependent Hamiltonian

$$H(t) = -\Delta + V + u(t)M_\mu, \qquad u \in L^2((0,T),\mathbb{R}), \qquad D(-\Delta) = H^2 \cap H_0^1,$$

with M_μ a multiplication operator for a function μ and V a suitable potential. One of the technical conditions imposed on V assures that the eigenvalues of $-\Delta + V$ are rationally independent (condition (C_7) from [MN15, p. 20]). The adopted hypotheses are not usually easy to validate and, for this reason, they prove the existence a residual set $\mathcal{Q}$ in H^4 such that the controllability holds for every $\mu \in \mathcal{Q}$ (where a residual set is an intersection of countably many sets with dense interiors).

From this perspective, our purpose is different. We aim to provide explicit conditions on the problem, such as Assumptions I, ensuring the controllabilty. To this end, we develop a new set of techniques from the ones adopted in the mentioned work. For instance, we provide an alternative strategy to the "Coron's return method" (used in [MN15]) to prove the local controllability ensured by Theorem 1.2. For further details on our approach, we refer to Section 4.1. In Example 2.2, we present specific control fields B satisfying Assumptions I, *i.e.* $B : \psi \in \mathscr{H} \mapsto x^2\psi$.

In Section 6, we use the controllability result provided by Corollary 1.4 in order to ensure the global exact controllability for suitable density matrices by projecting onto suitable finite dimensional spaces.

1.3 A brief bibliography

Global approximate controllability results for the bilinear Schrödinger equation are provided with different techniques in literature. For instance, adiabatic arguments are considered by Boscain, Chittaro, Gauthier, Mason, Rossi and Sigalotti in [BCMS12] and [BGRS15]. The result is achieved with Lyapunov techniques by Mirrahimi in [Mir09] and by Nersesyan in [Ner10]. Lie-Galerking arguments are used by Boscain, Boussaïd, Caponigro, Chambrion, Mason and Sigalotti in [CMSB09], [BCCS12] and [BdCC13].

The exact controllability of the bilinear Schrödinger equation (BSE) is in general a more delicate matter as a consequence of the results provided by Ball, Mardsen and Slemrod in [BMS82]. In particular, they ensure that the equation is not exactly controllable in $\mathscr{H}$ and $D(A)$ with controls in $L^2_{loc}(\mathbb{R}^+,\mathbb{R})$ when $B : D(A) \to D(A)$, even though the (BSE) is well-posed in such spaces.

Despite this non-controllability result, many authors have addressed the problem for weaker notions of controllability by considering suitable subspaces of $D(A)$. This idea was preliminarly introduced by Beauchard in [Bea05] and popularized by the work in [BL10]. Let M_μ be the multiplication operator for a function $\mu \in \mathscr{H}$. In [BL10], Beauchard and Laurent prove the local exact controllability of (BSE) in a neighborhood of the first eigenfunction of A in $S \cap H^3_{(0)}$ when $B = M_\mu$ for a suitable $\mu \in H^3$.

Another important work on the subject is [Mor14] where Morancey ensures the simultaneous local exact controllability in $S \cap H^3_{(0)}$ for at most three problems (BSE) and up to phases, when $B = M_\mu$ for suitable $\mu \in H^3$. In [MN15] (mentioned before), Morancey and Nersesyan extend the previous result. They prove the existence of a residual set of functions $\mathcal{Q}$ in H^4 so that, for $B = M_\mu$ and $\mu \in \mathcal{Q}$, the simultaneous global exact controllability is verified for any finite number of (BSE) in $H^4_{(V)} := D(|A+V|^2)$ for any $V \in H^4$.

1.4 Scheme of the work

In Section 2, we fix the notations considered in the work and we present some preliminary features of the problem such as the well-posedness of the (BSE) in the space $H^3_{(0)}$ ensured by [BL10].
In Section 3, we prove Theorem 1.1 by showing that the simultaneous global exact controllability in projection onto a suitable N dimensional space is equivalent to the controllability of N problems .
In Section 4, we provide Proposition 4.2 and its proof. The proposition extends Theorem 1.2 and it states the simultaneous local exact controllability in projection for any positive time up to phases. In order to motivate the modification of the problem, we emphasize the obstructions to overcome.
In Section 5, we ensure the simultaneous global exact controllability of N (BSE) by gathering a global approximate controllability and a local exact controllability. The outcome leads to Theorem 1.3.
In Section 6, we rephrase our results in terms of density matrices, while in Section 7, we provide some conclusive comments on the results of our work.
In Appendix A, we briefly discuss the solvability of the so-called "moment problems", while in Appendix B, we develop the perturbation theory techniques adopted in the work.

2 Auxiliary results

2.1 Notations and preliminaries

We denote $\mathscr{H} = L^2((0,1),\mathbb{C})$, its norm $\|\cdot\|_{L^2}$ and its scalar product $\langle\cdot,\cdot\rangle_{L^2}$ such that

$$\langle f, g\rangle_{L^2} = \int_0^1 \overline{f}(x)g(x)dx, \qquad \forall f, g \in \mathscr{H}.$$

Let $\mathscr{B}$ a Banach space. We introduce for $s > 0$,

$$\begin{aligned} H^s_{(0)} =& D(|A|^{\frac{s}{2}}), \qquad \|\cdot\|_{(s)} = \|\cdot\|_{H^s_{(0)}} = \Big(\sum_{k=1}^{\infty} |k^s\langle\cdot,\phi_k\rangle_{L^2}|^2\Big)^{\frac{1}{2}}, \\ & h^s(\mathscr{B}) = \Big\{\{\psi_j\}_{j\in\mathbb{N}^*} \subset \mathscr{B} \Big| \sum_{j=1}^{\infty}(j^s\|\psi_j\|_{\mathscr{B}})^2 < \infty\Big\}. \end{aligned} \tag{7}$$

We recall that $\{\phi_j\}_{j\in\mathbb{N}^*}$ is the Hilbert basis composed by eigenfunctions of A defined in (3) and related to the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}^*}$. Let

$$\Psi := \{\psi_j\}_{j\in\mathbb{N}^*} \subset \mathscr{H}, \qquad \mathscr{H}_N(\Psi) := \text{span}\{\psi_j : j \leq N\}.$$

We define $\pi_N(\Psi)$ the orthogonal projector such that

$$\pi_n(\Psi) : \mathscr{H} \longrightarrow \mathscr{H}_N(\Psi). \tag{8}$$

Remark 2.1. *If a bounded operator B satisfies Assumptions I, then $B \in L(H^2_{(0)}, H^2_{(0)})$. Indeed, B is closed in $\mathscr{H}$, so for every $\{u_n\}_{n\in\mathbb{N}^*} \subset \mathscr{H}$ such that $u_n \xrightarrow{\mathscr{H}} u$ and $Bu_n \xrightarrow{\mathscr{H}} v$, we have $Bu = v$. Now, for every $\{u_n\}_{n\in\mathbb{N}^*} \subset H^2_{(0)}$ such that $u_n \xrightarrow{H^2_{(0)}} u$ and $Bu_n \xrightarrow{H^2_{(0)}} v$, the convergences with respect to the $\mathscr{H}$-norm are implied and $Bu = v$. Hence, the operator B is closed in $H^2_{(0)}$ and $B \in L(H^2_{(0)}, H^2_{(0)})$. The same argument leads to $B \in L(H^3_{(0)}, H^3 \cap H^1_0)$ since $Ran(B|_{H^3_{(0)}}) \subseteq H^3 \cap H^1_0$.*

Example 2.2. *Assumptions I are satisfied for $B : \psi \mapsto x^2\psi$. Indeed, the condition 1) is guaranteed as*

$$\begin{cases} |\langle\phi_j, x^2\phi_k\rangle_{L^2}| = \left|\frac{(-1)^{j-k}}{(j-k)^2\pi^2} - \frac{(-1)^{j+k}}{(j+k)^2\pi^2}\right|, & \forall j,k \in \mathbb{N},\ j \neq k, \\ |\langle\phi_k, x^2\phi_k\rangle_{L^2}| = \left|\frac{1}{3} - \frac{1}{2k^2\pi^2}\right|, & \forall k \in \mathbb{N}^*. \end{cases}$$

The point 2) is trivially true, while the 3) holds since for $(j,k),(l,m) \in I^N$ so that $(j,k) \neq (l,m)$

$$j^2 - k^2 - l^2 + m^2 = 0 \quad \Longrightarrow \quad j^{-2} - k^{-2} - l^{-2} + m^{-2} \neq 0.$$

We notice that the same properties are valid for other control operators such as

$$B : \psi \in \mathscr{H} \longmapsto \sin\Big(\frac{\pi}{2}x\Big)\psi \qquad or \qquad B : \psi \in \mathscr{H} \longmapsto x^3\psi.$$

2.2 Well-posedness

In the current subsection, we cite an important result of well-posedness for the following problem in $\mathscr{H}$

$$\begin{cases} i\partial_t\psi(t) = A\psi(t) + u(t)\mu\psi(t), & t \in (0,T), \\ \psi(0) = \psi^0 \in L^2((0,1),\mathbb{C}). \end{cases} \tag{9}$$

Proposition 2.3. [*BL10, Proposition* 2] *Let $\mu \in H^3$, $T > 0$, $\psi^0 \in H^3_{(0)}$ and $u \in L^2((0,T),\mathbb{R})$. There exists a unique mild solution of (9) in $H^3_{(0)}$, i.e. $\psi \in C^0([0,T],H^3_{(0)})$ so that*

$$\psi(t) = e^{-iAt}\psi^0 - i\int_0^t e^{-iA(t-s)}u(s)\mu\psi(s)ds, \qquad \forall t \in [0,T]. \tag{10}$$

Moreover, for every $R > 0$, there exists $C = C(T,\mu,R) > 0$ such that, if $\|u\|_{L^2((0,T),\mathbb{R})} < R$, then, for every $\psi^0 \in H^3_{(0)}$, the solution satisfies

$$\|\psi\|_{C^0([0,T],H^3_{(0)})} \leq C\|\psi^0\|_{(3)}, \qquad \|\psi(t)\|_{L^2} = \|\psi^0\|_{L^2}, \qquad \forall t \in [0,T].$$

The outcome of Proposition 2.3 is not only valid for multiplication operators, but also for other suitable operators B. Indeed, the same proofs of [BL10, *Lemma* 1] and [BL10, *Proposition* 2] lead to the well-posedness of the (BSE) when B is a bounded symmetric operator such that

$$B \in L(H^3_{(0)}, H^3 \cap H^1_0), \qquad B \in L(H^2_{(0)}),$$

which are verified if B satisfies Assumptions I, thanks to Remark 2.1. As a consequence, for every $\{\psi_j\}_{j\in\mathbb{N}^*} \in \ell^\infty(H^3_{(0)})$, it follows $\{\Gamma^u_T\psi_j\}_{j\in\mathbb{N}^*} \in \ell^\infty(H^3_{(0)})$. We refer to (7) for the definition of $\ell^\infty(H^3_{(0)})$.

We denote Γ^u_t the unitary propagator in $\mathscr{H}$ generated by the (BSE) in the time interval $[0,t]$ and, for any mild solution ψ_j in $L^2((0,1),\mathbb{C})$ of the j-th problem (BSE) with $j \in \mathbb{N}^*$, we have

$$\Gamma^u_t\psi_j(0) = \psi_j(t)\ .$$

2.3 Time reversibility

An important feature of the bilinear Schrödinger equation is the time reversibility. If we consider $\psi(t) = \Gamma^u_t\psi^0$ and we substitute t with $T-t$ for $T > 0$ in a bilinear Schrödinger equation, then we have

$$\begin{cases} i\partial_t\Gamma^u_{T-t}\psi^0 = -A\Gamma^u_{T-t}\psi^0 - u(T-t)B\Gamma^u_{T-t}\psi^0, & t \in (0,T), \\ \Gamma^u_{T-0}\psi^0 = \Gamma^u_T\psi^0 = \psi^1. \end{cases}$$

We define the operator $\widetilde{\Gamma}^{\widetilde{u}}_t$ such that $\Gamma^u_{T-t}\psi^0 = \widetilde{\Gamma}^{\widetilde{u}}_t\psi^1$ for $\widetilde{u}(t) := u(T-t)$ and

$$\begin{cases} i\partial_t\widetilde{\Gamma}^{\widetilde{u}}_t\psi^1 = (-A - \widetilde{u}(t)B)\widetilde{\Gamma}^{\widetilde{u}}_t\psi^1, & t \in (0,T), \\ \widetilde{\Gamma}^{\widetilde{u}}_0\psi^1 = \psi^1 \in L^2((0,1),\mathbb{C}). \end{cases} \tag{11}$$

As $\psi^0 = \widetilde{\Gamma}^{\widetilde{u}}_T\Gamma^u_T\psi^0$ and $\psi^1 = \Gamma^u_T\widetilde{\Gamma}^{\widetilde{u}}_T\psi^1$, it follows $\widetilde{\Gamma}^{\widetilde{u}}_T = (\Gamma^u_T)^{-1} = (\Gamma^u_T)^*$. The operator $\widetilde{\Gamma}^{\widetilde{u}}_t$ describes the reversed dynamics of Γ^u_t induced by the system (11) and generated by the Hamiltonian $(-A - \widetilde{u}(t)B)$.

3 Proof of Theorem 1.1

(2) $\Longrightarrow$ (1) Let $\Psi^3 := \{\psi^3_j\}_{j\in\mathbb{N}^*} \in H^3_{(0)}$ be an orthonormal system. We consider $\{\psi^1_j\}_{j\in\mathbb{N}^*}$, $\{\psi^2_j\}_{j\in\mathbb{N}^*} \subset H^3_{(0)}$ complete orthonormal systems. Let $\widehat{\Gamma} \in U(\mathscr{H})$ be such that $\widehat{\Gamma}\psi^2_j = \psi^1_j$ and $\widehat{\Gamma}\psi^3_k \in H^3_{(0)}$ for every $k \leq N$. We notice that the controllability stated in the point **(2)** of Theorem 1.1 is also valid for the reversed dynamics discussed in Section 2.3. Hence, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\widetilde{\Gamma}^u_T\psi^3_k = \widehat{\Gamma}\psi^3_k, \quad \forall k \leq N, \qquad \Longrightarrow \qquad \langle\widetilde{\Gamma}^{\widetilde{u}}_T\psi_k, \psi^1_j\rangle_{L^2} = \langle\widehat{\Gamma}\psi^3_k, \psi^1_j\rangle_{L^2}, \qquad \forall j,k \in \mathbb{N}^*,\ k \leq N.$$

Let $\widetilde{u}$ be introduced in Section 2.3. The claim is proved since

$$\langle \Gamma_T^{\widetilde{u}}\psi_j^1, \psi_k^3\rangle_{L^2} = \langle \psi_j^1, \widetilde{\Gamma}_T^{\widetilde{u}}\psi_k^3\rangle_{L^2} = \langle \psi_j^1, \widehat{\Gamma}\psi_k^3\rangle_{L^2} = \langle \psi_j^2, \psi_k^3\rangle_{L^2}, \qquad \forall j,k\in\mathbb{N}^*,\ k\leq N.$$

(1) $\Longrightarrow$ (2) Let $\{\psi_j^1\}_{j\leq N}$, $\{\psi_j^2\}_{j\leq N}\subset H^3_{(0)}$ be two orthonormal systems of $\mathscr{H}$. We complete them by defining two complete orthonormal systems $\{\psi_j^1\}_{j\in\mathbb{N}^*}$, $\{\psi_j^2\}_{j\in\mathbb{N}^*}\subset H^3_{(0)}$. Now, thanks to the point **(1)**, there exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Psi^2)\Gamma_T^u\psi_j^1 = \pi_N(\Psi^2)\ \psi_j^2, \qquad \forall j\in\mathbb{N}^*.$$

As Ψ^2 is composed by orthogonal elements and Γ_T^u is unitary, the claim is proved since

$$\begin{cases}\Gamma_T^u\psi_j^1 = \psi_j^2, & \forall j\leq N,\\ \pi_N(\Psi^2)\ \Gamma_T^u\psi_j^1 = 0, & \forall j>N.\end{cases}$$

4 Simultaneous local exact controllability in projection

4.1 Introductive discussion

In this section, we examine the simultaneous local exact controllability in projection and we start by explaining why we need to modify the problem in order to achieve the result. Let $\Phi=\{\phi_j\}_{j\in\mathbb{N}^*}$ be an Hilbert basis composed by eigenfunctions of A. For every $j\in\mathbb{N}^*$, we denote $\phi_j(t,x)=e^{-i\lambda_j t}\phi_j(x)$ with $t>0$. From now on, we adopt the notation $\phi_j(t)=\phi_j(t,\cdot)$. Let $\epsilon>0$ and $T>0$. We consider the space

(12)
$$O_{\epsilon,T} := \Big\{\{\psi_j\}_{j\in\mathbb{N}^*}\in\ell^\infty(H^3_{(0)})\Big|\ \langle\psi_j,\psi_k\rangle_{L^2}=\delta_{j,k};\ \sup_{k\leq N}\sum_{j\in\mathbb{N}^*}k^6|\langle\psi_j,\phi_k(T)\rangle_{L^2}-\langle\phi_j(T),\phi_k(T)\rangle_{L^2}|^2<\epsilon\Big\}.$$

We would like to prove to validity of Theorem (1.2) in the neighborhood $O_{\epsilon,T}$ with respect to the projector $\pi_N(\Phi)$ (defined in (8)) and with $T>0$ large enough. Let

$$\Gamma_t^u\phi_j = \sum_{k=1}^{\infty}\phi_k(T)\langle\phi_k(T),\Gamma_t^u\phi_j\rangle_{L^2} \qquad \text{with} \quad t\in[0,T],\ \ \phi_j(T)=e^{-i\lambda_j T}\phi_j, \qquad j\in\mathbb{N}^*$$

be the solution of the j-th (BSE) with initial data ϕ_j. We consider the infinite matrix $\alpha(u)$ such that

$$\alpha_{k,j}(u)=\langle\phi_k(T),\Gamma_T^u\phi_j\rangle_{L^2},, \qquad \forall k,j\in\mathbb{N}^*\ k\leq N.$$

We would like to ensure the existence of $\epsilon>0$ such that for any $\{\psi_j\}_{j\in\mathbb{N}^*}\in O_{\epsilon,T}$,

$$\exists u\in L^2((0,T),\mathbb{R}) \quad : \quad \pi_N(\Phi)\Gamma_T^u\phi_j=\pi_N(\Phi)\psi_j, \qquad \forall j\in\mathbb{N}^*.$$

This outcome can be proved by studying the local surjectivity of α for $T>0$. To this purpose, we want to use the Generalized Inverse Function Theorem ([Lue69, Theorem 1; p. 240]) and study the surjectivity of the Fréchet derivative of α the infinite matri x$\gamma(v):=(d_u\alpha(0))\cdot\ v$ such that, for $j,k\in\mathbb{N}^*$ and $k\leq N$,

$$\gamma_{k,j}(v):=\left\langle\phi_k(T),-i\int_0^T e^{-iA(T-s)}v(s)Be^{-iAs}\phi_j ds\right\rangle_{L^2}=-i\int_0^T v(s)e^{-i(\lambda_j-\lambda_k)s}ds B_{k,j},$$

for $B_{k,j}=\langle\phi_k,B\phi_j\rangle_{L^2}=\langle B\phi_k,\phi_j\rangle_{L^2}=\overline{B_{j,k}}$. The surjectivity of γ consists in proving the solvability of the moment problem

$$\frac{x_{k,j}}{B_{k,j}}=-i\int_0^T u(s)e^{-i(\lambda_j-\lambda_k)s}ds, \qquad \forall j,k\in\mathbb{N}^*,\ k\leq N, \tag{13}$$

for each infinite matrix $\mathbf{x}:=\{x_{k,j}\}_{\substack{j,k\in\mathbb{N}^*\\ k\leq N}}$ belonging to a suitable space. To this end, one would use Haraux Theorem as explained in Remark A.1 but an obstruction appears. The terms $\{\lambda_j-\lambda_k\}_{\substack{j,k\in\mathbb{N}^*\\ k\leq N}}$ in the

moment problem (13) present the so-called "eigenvalues resonances". Formally, for some $j,k,n,m \in \mathbb{N}^*$ such that $j \neq k$, $n \neq m$, $(j,k) \neq (n,m)$ and $k,m \leq N$, there holds $\lambda_j - \lambda_k = \lambda_n - \lambda_m$, which implies

$$\frac{x_{k,j}}{B_{k,j}} = -i\int_0^T u(s)e^{-i(\lambda_j-\lambda_k)s}ds = -i\int_0^T u(s)e^{-i(\lambda_n-\lambda_m)s}ds = \frac{x_{n,m}}{B_{n,m}}. \tag{14}$$

An example of eigenvalues resonance is $\lambda_7 - \lambda_1 = \lambda_8 - \lambda_4$, but many others can be listed. For instance, all the diagonal terms of γ since $\lambda_j - \lambda_k = 0$ for $j = k$. The relation (14) represents a constraint on the considered matrices $\mathbf{x}$ which is not naturally satisfied in our framework.

In order to avoid this phenomenon, we adopt the following strategy. First, we consider the Hamiltonian characterizing the bilinear Schrödinger equations (BSE) and we act the following decomposition

$$A + u(t)B = (A + u_0B) + u_1(t)B, \qquad u_0 \in \mathbb{R},\ u_1 \in L^2((0,T),\mathbb{R}).$$

Second, we consider $A + u_0B$ instead of A. We repeat the previous steps by considering $\{\phi_j^{u_0}\}_{j\in\mathbb{N}^*}$ an Hilbert basis of $\mathscr{H}$ composed by eigenfunctions of $A+u_0B$ and $\{\lambda_j^{u_0}\}_{j\in\mathbb{N}^*}$ the corresponding eigenvalues. By using u_0B as a perturbating term in $A + u_0B$, we modify the eigenvalues gaps

$$\lambda_j^{u_0} - \lambda_k^{u_0}, \qquad \forall j,k \in \mathbb{N}^*,\ k \leq N$$

in order to remove all the non-diagonal resonances. Afterwards, we consider $\widehat{\alpha}$ depending on the parameter u_0 (instead of α) such that it is defined by the elements $\widehat{\alpha}_{k,j}(u) = \langle e^{-i\lambda_k^{u_0}T}\phi_k^{u_0}, \Gamma_T^u\phi_j^{u_0}\rangle_{L^2}$ with $k,j \in \mathbb{N}^*$ and $k \leq N$. Now, we rotate the terms of $\widehat{\alpha}$ in order to remove the resonances on the diagonal terms. We denote by α^{u_0} the obtained map, which is defined by the elements

$$\alpha_{k,j}^{u_0}(u) = \frac{\overline{\widehat{\alpha}_{j,j}(u)}}{|\widehat{\alpha}_{j,j}(u)|}\widehat{\alpha}_{k,j}(u), \qquad \forall j,k \in \mathbb{N}^*,\ k \leq N.$$

In conclusion, we use the Generalized Inverse Function Theorem with respect to the map α^{u_0}.

4.2 The modified problem

Let $N \in \mathbb{N}^*$ and $u(t) = u_0 + u_1(t)$, for u_0 and $u_1(t)$ real. We introduce the following Cauchy problems

$$\begin{cases} i\partial_t\psi_j(t) = (A + u_0B)\psi_j(t) + u_1(t)B\psi_j(t), & t \in (0,T), \\ \psi_j^0 = \psi_j(0), & j \in \mathbb{N}^*. \end{cases} \tag{15}$$

As B is bounded, $A + u_0B$ has pure discrete spectrum. We recall that $\{\lambda_j^{u_0}\}_{j\in\mathbb{N}^*}$ are the eigenvalues of $A + u_0B$ that correspond to an Hilbert basis of $\mathscr{H}$ made by eigenfunctions $\Phi^{u_0} := \{\phi_j^{u_0}\}_{j\in\mathbb{N}^*}$. We set

$$O_{\epsilon_0,T}^{u_0} := \Big\{\{\psi_j\}_{j\in\mathbb{N}^*} \in \ell^\infty(H^3_{(0)})\big|\ \langle\psi_j,\psi_k\rangle_{L^2} = \delta_{j,k};\ \sup_{k\leq N}\sum_{j\in\mathbb{N}^*} k^6|\langle\psi_j,\phi_k^{u_0}(T)\rangle_{L^2} - \langle\phi_j^{u_0}(T),\phi_k^{u_0}(T)\rangle_{L^2}|^2 < \epsilon_0\Big\}, \tag{16}$$

with $\phi_j^{u_0}(T) := e^{-i\lambda_j^{u_0}T}\phi_j^{u_0}$. We choose $|u_0|$ small so that $\lambda_k^{u_0} \neq 0$ for every $k \in \mathbb{N}^*$ (Lemma $B.4$, Appendix B). The introduction of the new Hilbert basis imposes to define the space

$$\widetilde{H}^3_{(0)} := D(|A + u_0B|^{\frac{3}{2}}), \qquad \|\cdot\|_{\widetilde{H}^3_{(0)}} = \Big(\sum_{k=1}^{\infty}\big||\lambda_k^{u_0}|^{\frac{3}{2}}\langle\cdot,\phi_k\rangle_{L^2}\big|^2\Big)^{\frac{1}{2}}.$$

Although, from now on, we consider u_0 in the neighborhood provided Lemma $B.6$ and $\widetilde{H}^3_{(0)} \equiv H^3_{(0)}$. As introduced in the previous subsection, we define the infinite matrix $\widehat{\alpha}$ with elements $\widehat{\alpha}_{k,j}(u_1) = \langle\phi_k^{u_0}(T), \Gamma_T^{u_0+u_1}\phi_j^{u_0}\rangle_{L^2}$ for $k \leq N$ and $j \in \mathbb{N}^*$. Now, the map α^{u_0} is the infinite matrix with elements

$$\begin{cases} \alpha_{k,j}^{u_0}(u_1) = \frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|}\widehat{\alpha}_{k,j}(u_1), & \forall j,k \in \mathbb{N}^*,\ j,k \leq N, \\ \alpha_{k,j}^{u_0}(u_1) = \widehat{\alpha}_{k,j}(u_1), & \forall j,k \in \mathbb{N}^*,\ j > N,\ k \leq N. \end{cases} \tag{17}$$

Let $\gamma^{u_0}(v) = ((d_{u_1}\alpha^{u_0})(0)) \cdot v$ be the Fréchet derivative of α^{u_0} and $B^{u_0}_{k,j} = \langle \phi^{u_0}_k, B\phi^{u_0}_j\rangle_{L^2}$ for $k \leq N$ and $j \in \mathbb{N}^*$. Defined $\widehat{\gamma}_{k,j}(v) = ((d_{u_1}\widehat{\alpha})(0)) \cdot v$, we compute the elements of $\gamma^{u_0}(v)$ such that

$$\begin{cases} \gamma^{u_0}_{k,j} = \big(\overline{\widehat{\gamma}_{j,j}}\delta_{k,j} + \widehat{\gamma}_{k,j} - \delta_{k,j}\Re(\widehat{\gamma}_{j,j})\big), & \forall j,k \in \mathbb{N}^*,\ j,k \leq N, \\ \gamma^{u_0}_{k,j} = \widehat{\gamma}_{k,j}, & \forall j,k \in \mathbb{N}^*,\ k \leq N,\ j > N, \end{cases}$$

$$(18) \qquad \Longrightarrow \qquad \begin{cases} \gamma^{u_0}_{k,j} = \widehat{\gamma}_{k,j} = -i\int_0^T u_1(s) e^{-i(\lambda^{u_0}_j - \lambda^{u_0}_k)s} ds B^{u_0}_{k,j}, & \forall j,k \in \mathbb{N}^*,\ k \neq j, \\ \gamma^{u_0}_{k,k} = \Re(\widehat{\gamma}_{k,k}) = 0, & \forall k \in \mathbb{N}^*. \end{cases}$$

The relation $\gamma^{u_0}_{k,k} = 0$ is due to the fact that $(i\widehat{\gamma}_{k,k}) \in \mathbb{R}$ since $\widehat{\gamma}_{k,j} = -\overline{\widehat{\gamma}_{j,k}}$ for $j,k \leq N$. Hence, the diagonal elements of γ^{u_0} are all equal to 0 due to the rotations adopted in the definition α^{u_0}.

Remark 4.1. *As $O^{u_0}_{\epsilon_0,T}$ is composed by orthonormal elements, we have*

$$T_{\Phi^{u_0}} O^{u_0}_{\epsilon_0,T} = \big\{ \{\psi_j\}_{j\in\mathbb{N}^*} \subset \ell^\infty(H^3_{(0)}) \big|\ \langle \phi^{u_0}_k, \psi_j\rangle_{L^2} = -\overline{\langle \phi^{u_0}_j, \psi_k\rangle_{L^2}} \big\}.$$

We refer to (7) for the definition of the space $\ell^\infty(H^3_{(0)})$. Let $\widetilde{\Gamma}^{\widetilde{u}}_t$ be the propagator defined in Section 2.3 for $t \in [0,T]$ and $u \in L^2((0,T),\mathbb{R})$ with $T > 0$. For every $k \in \mathbb{N}^*$, $u_1 \in \mathbb{R}$ and $L^2((0,T),\mathbb{R})$, from Proposition 2.3 and Lemma B.6, there exists $C > 0$ so that

$$\sum_{j=1}^{+\infty} j^6 |\alpha^{u_0}_{k,j}|^2 = \sum_{j=1}^{+\infty} j^6 |\langle \widetilde{\Gamma}^{u_0+\widetilde{u}_1}_T \phi^{u_0}_k, \phi^{u_0}_j\rangle_{L^2}|^2 = \|\widetilde{\Gamma}^{u_0+\widetilde{u}_1}_T \phi^{u_0}_k\|^2_{\widetilde{H}^3_{(0)}} \leq C \|\widetilde{\Gamma}^{u_0+\widetilde{u}_1}_T \phi^{u_0}_k\|^2_{(3)} < \infty.$$

Now, each $\{\alpha^{u_0}_{k,j}\}_{j\in\mathbb{N}^*} \in h^3(\mathbb{C})$ (defined in (7)). The maps α^{u_0} and γ^{u_0} take respectively values in

$$Q^N := \big\{ \{x_{k,j}\}_{\substack{k,j\in\mathbb{N}^* \\ k\leq N}} \in (h^3(\mathbb{C}))^N \big|\ x_{k,k} \in \mathbb{R},\ \ k \leq N \big\},$$
$$G^N := \big\{ \{x_{k,j}\}_{\substack{k,j\in\mathbb{N}^* \\ k\leq N}} \in (h^3(\mathbb{C}))^N \big|\ \ x_{k,j} = -\overline{x_{j,k}},\ x_{k,k} = 0\ j,k \leq N \big\}.$$

The third point of Remark B.9 implies that the controllability in $O^{u_0}_{\epsilon_0,T}$ (defined in (16)) with $\epsilon_0 > 0$ ensures the controllability in $O_{\epsilon,T}$ (defined in (12)) for suitable $\epsilon > 0$. Let $\{\psi_j\}_{j\in\mathbb{N}^*} \in O_{\epsilon,T}$ and $\widehat{\Gamma} \in U(\mathscr{H})$ be so that $\{\widehat{\Gamma}\psi_j\}_{j\in\mathbb{N}^*} = \{\phi^{u_0}_j\}_{j\in\mathbb{N}^*}$ and satisfying $\{\widehat{\Gamma}\phi^{u_0}_j\}_{j\leq N} \subset H^3_{(0)}$. There exists $C > 0$ so that

$$\sum_{j\in\mathbb{N}^*} |j^3 \langle \phi^{u_0}_k, \psi_j\rangle_{L^2}|^2 = \sum_{j\in\mathbb{N}^*} |j^3 \langle \widehat{\Gamma}\phi^{u_0}_k, \phi^{u_0}_j\rangle_{L^2}|^2 \leq C \|\widehat{\Gamma}\phi^{u_0}_k\|_{(3)} < \infty, \qquad \forall k \leq N,$$

thanks to Lemma B.4 and Lemma B.6. Fixed $e^{i\theta^1_j} := \frac{\overline{\langle \phi^{u_0}_j(T), \psi_j\rangle_{L^2}}}{|\langle \phi^{u_0}_j(T), \psi_j\rangle_{L^2}|}$ for $j \leq N$ and $e^{i\theta^1_j} := 1$ for $j > N$,

$$\{e^{i\theta^1_j}\langle \phi^{u_0}_k(T), \psi_j\rangle_{L^2}\}_{\substack{k\leq N \\ j,k\in\mathbb{N}^*}} \in Q^N_\epsilon := \{\{x_{k,j}\}_{\substack{k,j\in\mathbb{N}^* \\ k\leq N}} \in Q^N \mid \sup_{k\leq N} \sum_{j\in\mathbb{N}^*} k^6 |x_{k,j} - \delta_{k,j}|^2 \leq \epsilon\}.$$

When α^{u_0} is surjective in Q^N_ϵ, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$(19) \quad \{e^{i\theta^1_j}\langle \phi^{u_0}_k, \psi_j\rangle_{L^2}\}_{\substack{k\leq N \\ j,k\in\mathbb{N}^*}} = \{e^{i\theta^2_j}\langle \phi^{u_0}_k, \Gamma^u_T \phi^{u_0}_j\rangle_{L^2}\}_{\substack{k\leq N \\ j,k\in\mathbb{N}^*}} \quad \textit{with} \quad \begin{cases} e^{i\theta^2_j} := \frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|}, & \forall j \leq N, \\ e^{i\theta^2_j} := 1, & \forall j > N. \end{cases}$$

Thus, the surjectivity of the map α^{u_0} in Q^N_ϵ ensures the validity of Theorem 1.2 with respect the projector $\pi_N(\Phi^{u_0})$ in $O_{\epsilon,T}$ (also in $O^{u_0}_{\epsilon_0,T}$ for a suitable $\epsilon_0 > 0$).

4.3 Proof of Theorem 1.2

In the next proposition, we state the simultaneous local exact controllability in projection for any $T > 0$ up to phases. The result implies Theorem 1.2.

Proposition 4.2. *Let $N \in \mathbb{N}^*$ and B satisfy Assumptions I. For every $T > 0$, there exist $\epsilon > 0$ and $u_0 \in \mathbb{R}$ such that the following result is verified. Let $\{\psi_j^1\}_{j\in\mathbb{N}^*} \in O_{\epsilon,T}$ (defined in (12)) and $\widehat{\Gamma} \in U(\mathscr{H})$ be such that $\{\widehat{\Gamma}\psi_j^1\}_{j\in\mathbb{N}^*} = \{\phi_j^{u_0}\}_{j\in\mathbb{N}^*}$. If $\{\widehat{\Gamma}\phi_j^{u_0}\}_{j\leq N} \subset H^3_{(0)}$, then there exist a sequence of real numbers $\{\theta_j\}_{j\in\mathbb{N}^*} = \big\{\big\{\widehat{\theta}_j\big\}_{j\leq N}, 0, ...\big\}$ and $u \in L^2((0,T),\mathbb{R})$ such that*

$$\pi_N(\Phi^{u_0})\psi_j = \pi_N(\Phi^{u_0})e^{i\theta_j}\Gamma_T^u\phi_j^{u_0}, \qquad \forall j \in \mathbb{N}^*.$$

Proof. **1)** Let u_0 belong to the neighborhoods defined in Appendix B by Lemma $B.4$, Lemma $B.5$, Lemma $B.6$ and Remark $B.9$. As discussed in Remark 4.1, the surjectivity in Q_ϵ^N of the map α^{u_0} guarantees the simultaneous local exact controllability in projection up to phases in $O_{\epsilon,T}$. We consider Generalized Inverse function Theorem ([Lue69, Theorem 1; p. 240]) since Q^N and G^N are real Banach spaces. If γ^{u_0} is surjective in G^N, then the surjectivity of α^{u_0} in Q_ϵ^N is ensured for ϵ small enough. Now, the map γ^{u_0} is surjective when the following moment problem is solvable

$$x_{k,j}^{u_0}/B_{k,j}^{u_0} = -i\int_0^T u(s)e^{-i(\lambda_j^{u_0}-\lambda_k^{u_0})s}ds, \qquad \forall j,k \in \mathbb{N}^*,\ k \leq N. \tag{20}$$

for every $\big\{x_{k,j}^{u_0}\big\}_{\substack{j,k\in\mathbb{N}^*\\ k\leq N}} \in G^N$. We notice that the equations (20) for $k = j$ are redundant as $\gamma_{k,k}^{u_0} = 0$ and $x_{k,k}^{u_0} = 0$ for every $k \leq N$ and $\{x_{k,j}^{u_0}\}_{\substack{k,j\in\mathbb{N}^*\\ k\leq N}} \in G^N$. The same is true for $j,k \leq N$ such that $j < k$ since

$$\{x_{j,k}\}_{j,k\leq N}, \qquad \{\gamma_{j,k}(u)\}_{j,k\leq N} \qquad \text{with} \qquad u \in L^2((0,T),\mathbb{R}),$$

are skew-hermitian matrices. Thus, we can prove the solvability of (20) for $k < j$ and $j = k = 1$. Now, we have $\big\{x_{k,j}^{u_0}\big\}_{\substack{j,k\in\mathbb{N}^*\\ k\leq N}} \in (h^3)^N$ and $\big\{\gamma_{k,j}^{u_0}\big\}_{\substack{j,k\in\mathbb{N}^*\\ k\leq N}} \in (h^3)^N$. Lemma $B.5$ (Appendix B) yields that

$$\big\{x_{k,j}^{u_0}/B_{k,j}^{u_0}\big\}_{\substack{j,k\in\mathbb{N}^*\\ k\leq N}} \in (\ell^2(\mathbb{C}))^N, \qquad \big\{\gamma_{k,j}^{u_0}/B_{k,j}^{u_0}\big\}_{\substack{j,k\in\mathbb{N}^*\\ k\leq N}} \in (\ell^2(\mathbb{C}))^N.$$

Thanks to Lemma $B.8$ (Appendix B), for I^N defined in (4), there exists

$$\mathscr{G} := \sup_{A\subset I^N}\Big(\inf_{\substack{(j,k),(n,m)\in I^N\setminus A\\ (j,k)\neq(n,m)}}|\lambda_j^{u_0}-\lambda_k^{u_0}-\lambda_n^{u_0}+\lambda_m^{u_0}|\Big) \geq \mathscr{G}' := \inf_{\substack{(j,k),(n,m)\in I^N\\ (j,k)\neq(n,m)}}|\lambda_j^{u_0}-\lambda_k^{u_0}-\lambda_n^{u_0}+\lambda_m^{u_0}| > 0$$

where A runs over the finite subsets of I^N (see the next point for further details on $\mathscr{G}$). The solvability of the moment problem (20) is guaranteed from Remark $A.1$ by considering the sequence of numbers

$$\big\{\lambda_j^{u_0}-\lambda_k^{u_0}\big\}_{\substack{j,k\in\mathbb{N}^*,\ k\leq N\\ k<j \text{ or } j=k=1}}.$$

Indeed, $x_{1,1}^{u_0} = 0$ and Remark $B.9$ ensures that $\lambda_j^{u_0}-\lambda_k^{u_0} \neq \lambda_l^{u_0}-\lambda_m^{u_0}$ for every $(j,k),(l,m) \in I^N$ (see (4)) such that $(j,k) \neq (n,m)$. In conclusion, the solvability of the moment problem implies the surjectivity of γ^{u_0} and the Generalized Inverse function Theorem ([Lue69, Theorem 1; p. 240]) ensures the surjectivity of α^{u_0} in Q_ϵ^N for $T > 0$ large and suitable ϵ. The proof is achieved as discussed in Remark 4.1.

2) We show that the controllability ensured in **1)** is valid for every time $T > 0$ since $\mathscr{G} = +\infty$. Let

$$A^M := \{(j,n) \in (\mathbb{N}^*)^2|\ j,n \geq M;\ j \neq n\}, \qquad M \in \mathbb{N}^*.$$

Thanks to the relation (38) in the proof of Lemma $B.4$ (Appendix B), for $|u_0|$ small enough, we have

$$|\lambda_j^{u_0}-\lambda_n^{u_0}| \geq |\lambda_j-\lambda_n| - O(|u_0|) \geq 2\pi^2\min\{\lambda_{j+1}-\lambda_j, \lambda_{n+1}-\lambda_n\} - O(|u_0|).$$

Hence, for every $K \in \mathbb{R}$, there exists $M_K > 0$ large enough such that

$$\inf_{(j,n)\in A^{M_K}}|\lambda_j^{u_0}-\lambda_n^{u_0}| > K \qquad \Longrightarrow \qquad \mathscr{G} \geq \sup_{M\in\mathbb{N}^*}\big(\inf_{(j,n)\in A^M}|\lambda_j^{u_0}-\lambda_n^{u_0}| - 2\lambda_N^{u_0}\big) > 0.$$

In conclusion, for $|u_0|$ small enough, Lemma $B.4$ implies the existence of $C > 0$ such that

$$\mathscr{G} \geq C\big(\lim_{M\to\infty}\inf_{(j,n)\in A^M}|\lambda_j-\lambda_n| - 2\lambda_N\big) \geq C\lim_{M\to\infty}(\lambda_{M+2}-\lambda_{M+1}-2N^2\pi^2) = +\infty. \qquad \square$$

5 Simultaneous global exact controllability in projection

5.1 Preliminaries

The common approach adopted in order to prove global exact controllability results consists in gathering the global approximate controllability and the local exact controllability. Nevertheless, this strategy can not be used to prove the controllability in projection as the propagator Γ_T^u does not preserve the space $\pi_N(\Psi)H^3_{(0)}$ for any $\Psi := \{\psi_j\}_{j\in\mathbb{N}^*} \subset H^3_{(0)}$. For instance, let $\Psi = \{\psi_j\}_{j\in\mathbb{N}^*}$ be an orthonormal system and $\psi^1, \psi^2 \in H^3_{(0)}$ be unitarily equivalent. When we have

$$\pi(\Psi)\Gamma^{u_1}_{T_1}\psi_1 = \pi(\Psi)\Gamma^{u_2}_{T_2}\psi_2, \qquad T_1, T_2 > 0,\ u_1 \in L^2((0,T_1),\mathbb{R}),\ u_2 \in L^2((0,T_1),\mathbb{R}),$$

it is not guaranteed that there exists $T > 0$ and a control $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi(\Psi)\Gamma^u_T\psi_1 = \pi(\Psi)\psi_2.$$

To this purpose, we adopt an alternative strategy based on the "transposition argument" introduced by Theorem 1.1. We prove the simultaneous global exact controllability for N problems (BSE) (without projection) in $(H^3_{(0)})^N$. In such space, we can concatenate and reverse dynamics since it is preserved by the dynamics. Second, we ensure Theorem 1.3 thanks to Theorem 1.1.

5.2 Simultaneous approximate controllability

In this section, we prove the simultaneous global approximate controllability for finite number of (BSE).

Definition 5.1. The problems (BSE) are said to be simultaneously globally approximately controllable in $H^3_{(0)}$ when, for every $N \in \mathbb{N}^*$, $\psi_1, ..., \psi_N \in H^3_{(0)}$, $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi_1, ..., \widehat{\Gamma}\psi_N \in H^3_{(0)}$ and $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\|\Gamma^u_T\psi_k - \widehat{\Gamma}\psi_k\|_{(3)} < \epsilon$ for every $1 \le k \le N$.

Theorem 5.2. *Let B satisfy Assumptions I. The problems* (BSE) *are simultaneously globally approximately controllable in $H^3_{(0)}$.*

Proof. In the point **1)** of the proof, we suppose that (A, B) admits a non-degenerate chain of connectedness (see [BdCC13, Definition 3]). We treat the general case in the point **2)** of the proof.

1) Preliminaries. Let π_m be the orthogonal projector $\pi_m : \mathscr{H} \to \mathscr{H}_m := span\overline{\{\phi_j\ :\ j \le m\}}^{\,L^2}$ for every $m \in \mathbb{N}^*$. Up to reordering of $\{\phi_k\}_{k\in\mathbb{N}^*}$, the couples $(\pi_m A\pi_m, \pi_m B\pi_m)$ for $m \in \mathbb{N}^*$ admit non-degenerate chains of connectedness in $\mathscr{H}_m$. Let $\|\cdot\|_{BV(T)} = \|\cdot\|_{BV((0,T),\mathbb{R})}$ and $|||\cdot|||_{(s)} := |||\cdot|||_{L(H^s_{(0)}, H^s_{(0)})}$ for $s > 0$. Thanks to the validity of Assumptions I, we have $B : H^2_{(0)} \to H^2_{(0)}$.

> **Claim.** For every $\epsilon > 0$, there exist $N_1 \in \mathbb{N}^*$ and $\widetilde{\Gamma}_{N_1} \in U(\mathscr{H})$ such that $\pi_{N_1}(\Phi)\widetilde{\Gamma}_{N_1}\pi_{N_1}(\Phi) \in SU(\mathscr{H}_{N_1})$ and
>
> $$\|\widetilde{\Gamma}_{N_1}\phi_j - \widehat{\Gamma}\phi_j\|_{L^2} < \epsilon, \qquad \forall j \le N. \tag{21}$$

Let $N_1 \in \mathbb{N}^*$ be such that $N_1 \ge N$. We apply the orthonormalizing Gram-Schmidt process to $\{\pi_{N_1}(\Phi)\widehat{\Gamma}\phi_j\}_{j\le N}$ and we define the sequence $\{\widetilde{\phi}_j\}_{j\le N}$ that we complete in $\{\widetilde{\phi}_j\}_{j\le N_1}$, an orthonormal basis of $\mathscr{H}_{N_1}$. The operator $\widetilde{\Gamma}_{N_1}$ is the unitary map such that $\widetilde{\Gamma}_{N_1}\phi_j = \widetilde{\phi}_j$, for every $j \le N_1$. In conclusion, we consider N_1 large enough so that the statement is verified.

Finite dimensional controllability. Let T_{ad} be the set of $(j,k) \in \{1, ..., N_1\}^2$ such that $B_{j,k} := \langle\phi_j, B\phi_k\rangle_{L^2} \neq 0$ and $|\lambda_j - \lambda_k| = |\lambda_m - \lambda_l|$ with $m, l \in \mathbb{N}^*$ implies $\{j,k\} = \{m,l\}$ for $B_{m,l} = 0$. For every $(j,k) \in \{1, ..., N_1\}^2$ and $\theta \in [0, 2\pi)$, we define $E^\theta_{j,k}$ the $N_1 \times N_1$ matrix with elements $(E^\theta_{j,k})_{l,m} = 0$, $(E^\theta_{j,k})_{j,k} = e^{i\theta}$ and $(E^\theta_{j,k})_{k,j} = -e^{-i\theta}$ for $(l,m) \in \{1, ..., N_1\}^2 \setminus \{(j,k),(k,j)\}$. Let $E_{ad} = \left\{E^\theta_{j,k}\ :\ (j,k) \in T_{ad},\ \theta \in [0, 2\pi)\right\}$ and $Lie(E_{ad})$. Fixed v a piecewise constant control taking value in E_{ad} and $\tau > 0$, we introduce the control system on $SU(\mathscr{H}_{N_1})$

$$\begin{cases} \dot{x}(t) = x(t)v(t), & t \in (0,\tau), \\ x(0) = Id_{SU(\mathscr{H}_{N_1})}. \end{cases} \tag{22}$$

Claim. (22) is controllable, *i.e.* for $R \in SU(\mathscr{H}_{N_1})$, there exist $p \in \mathbb{N}^*$, $M_1, ..., M_p \in E_{ad}$, $\alpha_1, ..., \alpha_p \in \mathbb{R}^+$ such that $R = e^{\alpha_1 M_1} \circ ... \circ e^{\alpha_p M_p}$.

For every $(j,k) \in \{1, ..., N_1\}^2$, we define the $N_1 \times N_1$ matrices $R_{j,k}$, $C_{j,k}$ and D_j as follow. For $(l,m) \in \{1, ..., N_1\}^2 \setminus \{(j,k),(k,j)\}$,we have $(R_{j,k})_{l,m} = 0$ and $(R_{j,k})_{j,k} = -(R_{j,k})_{k,j} = 1$, while $(C_{j,k})_{l,m} = 0$ and $(C_{j,k})_{j,k} = (C_{j,k})_{k,j} = i$. Moreover, for $(l,m) \in \{1, ..., N_1\}^2 \setminus \{(1,1),(j,j)\}$, $(D_j)_{l,m} = 0$ and $(D_j)_{1,1} = -(D_j)_{j,j} = i$. We consider the basis of $su(\mathscr{H}_{N_1})$

$$\mathbf{e} := \{R_{j,k}\}_{j,k \leq N_1} \cup \{C_{j,k}\}_{j,k \leq N_1} \cup \{D_j\}_{j \leq N_1}.$$

Thanks to [Sac00, Theorem 6.1], the controllability of (22) is equivalent to prove that $Lie(E_{ad}) \supseteq su(\mathscr{H}_{N_1})$ for $su(\mathscr{H}_{N_1})$ the Lie algebra of $SU(\mathscr{H}_{N_1})$. The claim si valid as it is possible to obtain the matrices $R_{j,k}$, $C_{j,k}$ and D_j for every $j,k \leq N_1$ by iterated Lie brackets of elements in E_{ad}.

Finite dimensional estimates. Thanks to the previous claim and to the fact that $\pi_{N_1}(\Phi)\widetilde{\Gamma}_{N_1}\pi_{N_1}(\Phi) \in SU(\mathscr{H}_{N_1})$, there exist $p \in \mathbb{N}^*$, $M_1, ..., M_p \in E_{ad}$ and $\alpha_1, ..., \alpha_p \in \mathbb{R}^+$ such that

$$\pi_{N_1}(\Phi)\widetilde{\Gamma}_{N_1}\pi_{N_1}(\Phi) = e^{\alpha_1 M_1} \circ ... \circ e^{\alpha_p M_p}. \tag{23}$$

Claim. For every $l \leq p$ and $e^{\alpha_l M_l}$ from (23), there exist $\{T_n^l\}_{l \in \mathbb{N}^*} \subset \mathbb{R}^+$ and $\{u_n^l\}_{n \in \mathbb{N}^*}$ such that $u_n^l : (0, T_n^l) \to \mathbb{R}$ for every $n \in \mathbb{N}^*$ and

$$\lim_{n \to \infty} \|\Gamma_{T_n^l}^{u_n^l}\phi_k - e^{\alpha_l M_l}\phi_k\|_{L^2} = 0, \qquad \forall k \leq N_1, \tag{24}$$

$$\begin{gathered} \sup_{n \in \mathbb{N}^*} \|u_n^l\|_{BV(T_n)} < \infty, \qquad \sup_{n \in \mathbb{N}^*} \|u_n^l\|_{L^\infty((0,T_n),\mathbb{R})} < \infty, \\ \sup_{n \in \mathbb{N}^*} T_n \|u_n^l\|_{L^\infty((0,T_n),\mathbb{R})} < \infty. \end{gathered} \tag{25}$$

We consider the results developed in [Cha12, *Section* 3.1 & *Section* 3.2] by Chambrion and leading to [Cha12, *Proposition* 6] since (A, B) admits a non-degenerate chain of connectedness ([BdCC13, Definition 3]). Each $e^{\alpha_l M_l}$ is a rotation in a two dimensional space for every $l \in \{1, ..., p\}$ and this work allows to explicit $\{T_n^l\}_{l \in \mathbb{N}^*} \subset \mathbb{R}^+$ and $\{u_n^l\}_{n \in \mathbb{N}^*}$ satisfying (25) such that $u_n^l : (0, T_n^l) \to \mathbb{R}$ for every $n \in \mathbb{N}^*$ and

$$\lim_{n \to \infty} \|\pi_{N_1}(\Phi)\Gamma_{T_n^l}^{u_n^l}\phi_k - e^{\alpha_l M_l}\phi_k\|_{L^2} = 0, \qquad \forall k \leq N_1. \tag{26}$$

As $e^{\alpha_l M_l} \in SU(\mathscr{H}_{N_1})$, we have $\lim_{n \to \infty} \|\Gamma_{T_n^l}^{u_n^l}\phi_k - e^{\alpha_l M_l}\phi_k\|_{L^2} = 0$ for $k \leq N_1$.

Infinite dimensional estimates.

Claim. There exist $K_1, K_2, K_3 > 0$ such that for every $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\|\Gamma_T^u\phi_k - \widehat{\Gamma}\phi_k\|_{L^2} \leq \epsilon$ for every $k \leq N_1$ and

$$\|u\|_{BV(T)} \leq K_1, \qquad \|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_2, \qquad T\|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_3. \tag{27}$$

We assume $p = 2$, but the following result is valid for any $p \in \mathbb{N}^*$. Thanks to (24), there exists $n \in \mathbb{N}^*$ large enough such that, for every $k \leq N_1$,

$$\begin{aligned} \|\Gamma_{T_n^2}^{u_n^2}\Gamma_{T_n^1}^{u_n^1}\phi_k - e^{\alpha_2 M_2}e^{\alpha_1 M_1}\phi_k\|_{L^2} &\leq \|\Gamma_{T_n^1}^{u_n^1}\phi_k - e^{\alpha_1 M_1}\phi_k\|_{L^2} + \sum_{l=1}^{N_1} \|\big(\Gamma_{T_n^2}^{u_n^2}\phi_l - e^{\alpha_2 M_2}\phi_l\big)\langle\phi_l, e^{\alpha_1 M_1}\phi_k\rangle_{L^2}\|_{L^2} \\ &\leq \|\Gamma_{T_n^1}^{u_n^1}\phi_k - e^{\alpha_1 M_1}\phi_k\|_{L^2} + \Big(\sum_{l=1}^{N_1} \|\Gamma_{T_n^2}^{u_n^2}\phi_l - e^{\alpha_2 M_2}\phi_l\|_{L^2}^2\Big)^{\frac{1}{2}} \leq \epsilon. \end{aligned}$$

In the previous inequality, we considered that $e^{\alpha_1 M_1}\phi_k \in \mathscr{H}_{N_1}$. The identity (23) leads to the existence of $K_1, K_2, K_3 > 0$ such that for every $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\|\Gamma_T^u\phi_k - \widetilde{\Gamma}_{N_1}\phi_k\|_{L^2} < \epsilon$ for every $k \leq N_1$ and

$$\|u\|_{BV(T)} \leq K_1, \qquad \|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_2, \qquad T\|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_3. \tag{28}$$

The relation (21) and the triangular inequality achieve the claim.

Global approximate controllability with respect to the L^2-norm. Let us recall that $\{\psi_j\}_{j\leq N}\subset H^3_{(0)}$ and $\widehat{\Gamma}\in U(\mathscr{H})$ satisfies $\{\widehat{\Gamma}\psi_j\}_{j\leq N}\subset H^3_{(0)}$.

Claim. There exist $K_1,K_2,K_3>0$ such that for every $\epsilon>0$, there exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that $\|\Gamma^u_T\psi_k-\widehat{\Gamma}\psi_k\|_{L^2}\leq\epsilon$ for every $k\leq N$ and

$$\|u\|_{BV(T)}\leq K_1,\qquad \|u\|_{L^\infty((0,T),\mathbb{R})}\leq K_2,\qquad T\|u\|_{L^\infty((0,T),\mathbb{R})}\leq K_3. \tag{29}$$

We assume that $\|\psi_j\|_{L^2}=1$ for every $j\in\mathbb{N}^*$, but the same proof is also valid for the generic case. From the previous claim, there exist two controls respectively steering $\{\phi_j\}_{j\leq N}$ close to $\{\psi_j\}_{j\leq N}$ and $\{\phi_j\}_{j\leq N}$ close to $\{\widehat{\Gamma}\psi_j\}_{j\leq N}$ thanks to the fact that $N_1\geq N$. Vice versa, thanks to the time reversibility (see Section 2.3), there exists a control steering $\{\psi_j\}_{j\leq N}$ close to $\{\phi_j\}_{j\leq N}$. In other words, there exist $T_1,T_2>0$, $u_1\in L^2((0,T_1),\mathbb{R})$ and $u_2\in L^2((0,T_2),\mathbb{R})$ such that

$$\|\Gamma^{u_1}_{T_1}\psi_j-\phi_j\|_{L^2}\leq\frac{\epsilon}{2},\qquad \|\Gamma^{u_2}_{T_2}\phi_j-\widehat{\Gamma}\psi_j\|_{L^2}\leq\frac{\epsilon}{2},\qquad \forall j\leq N.$$

The chosen controls u_1 and u_2 satisfy (29). The claim is proven as

$$\|\Gamma^{u_2}_{T_2}\Gamma^{u_1}_{T_1}\psi_j-\widehat{\Gamma}\psi_j\|_{L^2}\leq\|\Gamma^{u_2}_{T_2}\Gamma^{u_1}_{T_1}\psi_j-\Gamma^{u_2}_{T_2}\phi_j\|_{L^2}+\|\Gamma^{u_2}_{T_2}\phi_j-\widehat{\Gamma}\psi_j\|_{L^2}\leq\epsilon,\qquad\forall j\leq N.$$

Global approximate controllability with respect to the $H^3_{(0)}$-norm.

Claim. There exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that $\|\Gamma^u_T\psi_k-\widehat{\Gamma}\psi_k\|_{(3)}\leq\epsilon$ for every $k\leq N$.

We consider the propagation of regularity developed by Kato in [Kat53]. We notice that $i(A+u(t)B-ic)$ is maximal dissipative in $H^2_{(0)}$ for suitable $c:=\|u\|_{L^\infty((0,T),\mathbb{R})}$. Let $\lambda>c$ and $\widehat{H}^4_{(0)}:=D(A(i\lambda-A))\equiv H^4_{(0)}$. We know that $B:\widehat{H}^4_{(0)}\subset H^2_{(0)}\to H^2_{(0)}$ and the arguments of Remark 2.1 imply that $B\in L(\widehat{H}^4_{(0)},H^2_{(0)})$. For $T>0$ and $u\in BV((0,T),\mathbb{R})$, we have $|||\,u(t)B(i\lambda-A)^{-1}\,|||_{(2)}<1$ and

$$M:=\sup_{t\in[0,T]}|||\,(i\lambda-A-u(t)B)^{-1}\,|||_{L(H^2_{(0)},\widehat{H}^4_{(0)})}\leq\sup_{t\in[0,T]}\sum_{l=1}^{+\infty}|||\,(u(t)B(i\lambda-A)^{-1})^l\,|||_{(2)}<+\infty.$$

We know $\|k+f(\cdot)\|_{BV((0,T),\mathbb{R})}=\|f\|_{BV((0,T),\mathbb{R})}$ for $f\in BV((0,T),\mathbb{R})$ and $k\in\mathbb{R}$. Equivalently,

$$N:=|||\,i\lambda-A-u(\cdot)B\,|||_{BV\left([0,T],L(\widehat{H}^4_{(0)},H^2_{(0)})\right)}=\|u\|_{BV(T)}\,|||\,B\,|||_{L(\widehat{H}^4_{(0)},H^2_{(0)})}<+\infty.$$

We call $C_1:=|||\,A(A+u(T)B-i\lambda)^{-1}\,|||_{(2)}<\infty$ and U^u_t the propagator generated by $A+uB-ic$ such that $U^u_t\psi=e^{-ct}\Gamma^u_t\psi$. Thanks to [Kat53, *Section* 3.10], for every $\psi\in H^4_{(0)}$, it follows

$$\|(A+u(T)B-i\lambda)U^u_t\psi\|_{(2)}\leq Me^{MN}\|(A-i\lambda)\psi\|_{(2)}\quad\Longrightarrow\quad\|\Gamma^u_T\psi\|_{(4)}\leq C_1Me^{MN+cT}\|\psi\|_{(4)}.$$

For every $T>0$, $u\in BV((0,T),\mathbb{R})$ and $\psi\in H^4_{(0)}$, there exists $C(K)>0$ depending on $K=\left(\|u\|_{BV(T)},\|u\|_{L^\infty((0,T),\mathbb{R})},T\|u\|_{L^\infty((0,T),\mathbb{R})}\right)$ so that $\|\Gamma^u_T\psi\|_{(4)}\leq C(K)\|\psi\|_{(4)}$. From (25), there exists $C>0$ such that

$$|||\,\Gamma^{u_n^l}_{T_n^l}\,|||_{(4)}\leq C. \tag{30}$$

For every $\psi\in H^4_{(0)}$, from the Cauchy-Schwarz inequality, we have $\|A\psi\|^2_{L^2}\leq\|A^2\psi\|_{L^2}\|\psi\|_{L^2}$ and $\|A^{\frac{3}{2}}\psi\|^4_{L^2}\leq\left(\langle A^2\psi,A\psi\rangle_{L^2}\right)^2\leq\|A^2\psi\|^2_{L^2}\|A\psi\|^2_{L^2}$, which imply

$$\|\psi\|^8_{(3)}\leq\|\psi\|^2_{L^2}\|\psi\|^6_{(4)}. \tag{31}$$

In conclusion, the previous claim, the relation (30) and the relation (31) ensure the claim.

2) Conclusion. Let (A,B) do not admit a non-degenerate chain of connectedness. We decompose

$$A+u(\cdot)B=(A+u_0B)+u_1(\cdot)B, \qquad u_0\in\mathbb{R}, \quad u_1\in L^2((0,T),\mathbb{R}).$$

We notice that, if (A,B) satisfies Assumptions I, then Remark $B.7$ and Remark $B.9$ (Appendix B) are valid. We consider u_0 belonging to the neighborhoods provided by such remarks and we denote $\{\phi_k^{u_0}\}_{k\in\mathbb{N}^*}$ a Hilbert basis of $\mathscr{H}$ made by eigenfunctions of $A+u_0B$. The step **1)** of the proof can be repeated by considering the sequence $\{\phi_k^{u_0}\}_{k\in\mathbb{N}^*}$ instead of $\{\phi_k\}_{k\in\mathbb{N}^*}$ and the spaces $D(|A+u_0B|^{\frac{3}{2}})$ in substitution of $H^3_{(0)}$. The claim is equivalently proved since, thanks to Remark $B.7$, there exist $C_1,C_2>0$ such that

$$C_1\big\||A+u_0B|^{\frac{3}{2}}\psi\big\|\leq\|\psi\|_{(3)}\leq C_2\big\||A+u_0B|^{\frac{3}{2}}\psi\big\|, \qquad \forall\psi\in H^3_{(0)}\equiv D(|A+u_0B|^{\frac{3}{2}}). \quad \square$$

5.3 Proofs of Theorem 1.3

In the current subsection, we provide the proof of Theorem 1.3 which requires the following proposition.

Proposition 5.3. *Let $N\in\mathbb{N}^*$ and B satisfy Assumptions I. For any $\{\psi_k^1\}_{k\leq N}$, $\{\psi_k^2\}_{k\leq N}\subset H^3_{(0)}$ orthonormal systems, there exist $T>0$, $u\in L^2((0,T),\mathbb{R})$ and $\{\theta_k\}_{k\leq N}\subset\mathbb{R}$ such that*

$$\Gamma_T^u\psi_k^1=e^{i\theta_k}\psi_k^2, \qquad \forall k\leq N.$$

Proof. Let $N\in\mathbb{N}^*$ and let $u_0\in\mathbb{R}$ belong to the neighborhoods provided by Lemma $B.5$, Lemma $B.6$ and Remark $B.9$ (Appendix B). Let $\widetilde{\alpha}^{u_0}$ be the map with elements

$$\begin{cases}\frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|}\widehat{\alpha}_{k,j}(u_1), & \forall j,k\in\mathbb{N}^*,\ j,k\leq N,\\ \widehat{\alpha}_{k,j}(u_1), & \forall j,k\in\mathbb{N}^*,\ k>N,\ j\leq N.\end{cases}$$

The proof of Proposition 4.2 can be repeated in order to prove the local surjectivity of $\widetilde{\alpha}^{u_0}$ for every $T>0$, instead of α^{u_0} introduced in (17). The discussion from Remark 4.1 implies that this result corresponds to the simultaneous local exact controllability up to phases of N problems (BSE) in the neighborhood

$$O^N_{\epsilon,T}:=\Big\{\{\psi_j\}_{j\leq N}\subset H^3_{(0)}\big|\ \langle\psi_j,\psi_k\rangle_{L^2}=\delta_{j,k};\ \ \sup_{j\leq N}\|\psi_j-\phi_j^{u_0}\|^2_{(3)}<\epsilon\Big\}$$

with $\epsilon>0$. Hence, for any $\{\psi_k\}_{k\leq N}\in O^N_{\epsilon,T}$, there exist $u\in L^2((0,T),\mathbb{R})$ and $\{\theta_j\}_{j\leq N}\subset\mathbb{R}$ so that

$$\Gamma_T^u\phi_j^{u_0}=e^{i\theta_j}\psi_j, \qquad \forall j\leq N.$$

Thanks to Theorem 5.2, we have the following result. For any $\{\psi_j^1\}_{j\leq N}\subset H^3_{(0)}$ composed by orthonormal elements, there exist $T_1>0$ and $u_1\in L^2((0,T_1),\mathbb{R})$ such that, for all $j\leq N$,

$$\|\Gamma_{T_1}^{u_1}\psi_j^1-\phi_j^{u_0}\|_{(3)}<\frac{\epsilon}{N} \qquad\Longrightarrow\qquad \{\Gamma_{T_1}^{u_1}\psi_j^1\}_{j\leq N}\in O^N_{\epsilon,T}.$$

The local controllability is also valid for the reversed dynamics (see Section 2.3) and for every $T>0$, there exist $u\in L^2((0,T),\mathbb{R})$ and $\{\theta_j\}_{j\leq N}\subset\mathbb{R}$ so that

$$\{\Gamma_{T_1}^{u_1}\psi_j^1\}_{j\leq N}=\{e^{i\theta_j}\widetilde{\Gamma}_T^u\phi_j^{u_0}\}_{j\leq N} \quad\Longrightarrow\quad \{e^{-i\theta_j}\Gamma_T^{\widetilde{u}}\Gamma_{T_1}^{u_1}\psi_j^1\}_{j\leq N}=\{\phi_j^{u_0}\}_{j\leq N}.$$

Then, there exist $T_2>0$ and $u_2\in L^2((0,T_2),\mathbb{R})$ such that $\{e^{-i\theta_j}\Gamma_{T_2}^{u_2}\psi_j^1\}_{j\leq N}=\{\phi_j^{u_0}\}_{j\leq N}$. Now, the same property is valid for the reversed dynamics of (11) and, for every $\{\psi_j^2\}_{j\leq N}\subset H^3_{(0)}$ composed by orthonormal elements, there exist $T_3>0$, $u_3\in L^2((0,T_3),\mathbb{R})$ and $\{\theta_j'\}_{j\leq N}\subset\mathbb{R}$ such that $\{e^{-i\theta_j'}\widetilde{\Gamma}_{T_3}^{u_3}\psi_j^2\}_{j\leq N}=\{\phi_j^{u_0}\}_{j\leq N}$. In conclusion, for $\widetilde{u}_3(\cdot)=u_3(T_3-\cdot)$, the proof is achieved as

$$\{e^{-i(\theta_j-\theta_j')}\Gamma_{T_3}^{\widetilde{u}_3}\Gamma_{T_2}^{u_2}\psi_j^1\}_{j\leq N}=\{\psi_j^2\}_{j\leq N}. \quad \square$$

Proof of Theorem 1.3. Theorem 1.3 is proved by repeating the proof of Theorem 1.1. In particular, it follows from the proof of **(2)** $\Longrightarrow$ **(1)** by keeping in mind the validity of Proposition 5.3. $\square$

6 Global exact controllability in projection for density matrices

Let ψ^1, $\psi^2 \in \mathscr{H}$. We define the rank one operator $|\psi^1\rangle\langle\psi^2|$ such that $|\psi^1\rangle\langle\psi^2|\psi = \psi^1\langle\psi^2,\psi\rangle_{L^2}$ for every $\psi \in \mathscr{H}$. For any $\widehat{\Gamma} \in U(\mathscr{H})$, we have

$$\widehat{\Gamma}|\psi^1\rangle\langle\psi^2| = |\widehat{\Gamma}\psi^1\rangle\langle\psi^2|, \qquad |\psi^1\rangle\langle\psi^2|\widehat{\Gamma}^* = |\psi^1\rangle\langle\widehat{\Gamma}\psi^2|.$$

Let $\mathscr{H}$ be an infinite dimensional Hilbert space. In quantum mechanics, any statistical ensemble can be described by a wave function (pure state) or by a density matrix (mixed state) which is a positive operator of trace 1. For any density matrix ρ, there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}^*} \subset \mathscr{H}$ such that

$$\rho = \sum_{j\in\mathbb{N}^*} l_j|\psi_j\rangle\langle\psi_j|, \qquad \sum_{j\in\mathbb{N}^*} l_j = 1, \qquad l_j \geq 0, \quad \forall j \in \mathbb{N}^*. \tag{32}$$

The sequence $\{\psi_j\}_{j\in\mathbb{N}^*}$ is a set of eigenvectors of ρ and $\{l_j\}_{j\in\mathbb{N}^*}$ are the corresponding eigenvalues. If there exists $j_0 \in \mathbb{N}^*$ such that $l_{j_0} = 1$ and $l_j = 0$ for each $j \neq j_0$, then the corresponding density matrix represents a pure state up to a phase. For this reason, the density matrices formalism is said to be an extension of the common formulation of the quantum mechanics in terms of wave function. We also notice that for any density matrix ρ and a complete orthonormal system $\{\psi_j\}_{j\in\mathbb{N}^*}$ in $\mathscr{H}$, there exists a positive hermitian matrix $\{\rho_{j,k}\}_{j,k\in\mathbb{N}^*}$ such that

$$\rho = \sum_{j,k\in\mathbb{N}^*} \rho_{j,k}|\psi_j\rangle\langle\psi_k|. \tag{33}$$

Now, for any other density matrix $\widetilde{\rho}$, there exists an orthonormal system $\{\widetilde{\psi}_j\}_{j\in\mathbb{N}^*}$, such that

$$\widetilde{\rho} = \sum_{j,k\in\mathbb{N}^*} \rho_{j,k}|\widetilde{\psi}_j\rangle\langle\widetilde{\psi}_k|. \tag{34}$$

Let us consider $T > 0$ and a time dependent self-adjoint operator $H(t)$ (called Hamiltonian) for $t \in (0,T)$. The dynamics of a general density matrix ρ is described by the Von Neumann equation

$$\begin{cases} i\frac{d\rho}{dt}(t) = [H(t),\rho(t)], & t \in (0,T), \\ \rho(0) = \rho^0, & ([H,\rho] = H\rho - \rho H), \end{cases} \tag{35}$$

for ρ^0 the initial solution of the problem. The solution is $\rho(t) = U_t\rho(0)U_t^*$, where U_t is the unitary propagator generated by $H(t)$. In the present work, we have $\mathscr{H} = L^2((0,1),\mathbb{C})$, $H(t) = A + u(t)B$ and U_t corresponds to Γ_t^u. In this framework, the problem (35) is said to be globally exactly controllable if, for any couple of density matrices ρ^1 and ρ^2, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\rho^2 = \Gamma_T^u\rho^1(\Gamma_T^u)^*.$$

Thanks to the decomposition (32), the controllability of (35) is equivalent (up to phases) to the simultaneous controllability of the infinite bilinear Schrödinger equations (BSE). This idea is behind the following theorem which follows from Corollary 1.4.

Theorem 6.1. *Let B satisfy Assumptions I. Let ρ^1 and ρ^2 be two density matrices with eigenfunctions in $H^3_{(0)}$ and $\widehat{\Gamma} \in U(\mathscr{H})$ be the unitary opeator so that*

$$\rho^1 = \widehat{\Gamma}\rho^2\widehat{\Gamma}^*.$$

1) *Let $\Psi := \{\psi_j\}_{j\in\mathbb{N}^*}$ be an orthonormal system composed by the eigenfunctions of ρ^2. For any $N \in \mathbb{N}^*$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that*

$$\pi_N(\Psi)\ \Gamma_T^u\rho^1(\Gamma_T^u)^*\ \pi_N(\Psi) = \pi_N(\Psi)\ \rho^2\ \pi_N(\Psi).$$

2) *Let $\Psi := \{\psi_j\}_{j\leq N} \subset H^3_{(0)}$ be an orthonormal system such that $\{\widehat{\Gamma}\psi_j\}_{j\leq N} \subset H^3_{(0)}$ with $N \in \mathbb{N}^*$. Let $\{\rho_{j,k}\}_{j,k\leq N}$ be the positive hermitian matrix such that*

$$\pi_N(\Psi)\rho^2\pi_N(\Psi) = \sum_{j,k\leq N} \rho_{j,k}|\psi_j\rangle\langle\psi_k|.$$

There exist $T > 0$, $u \in L^2((0,T),\mathbb{R})$ *and* $\{\theta_{j,k}\}_{j,k\leq N}$ *such that*

$$\pi_N(\Psi)\ \Gamma^u_T\rho^1(\Gamma^u_T)^*\ \pi_N(\Psi) = \sum_{j,k\leq N} e^{i\theta_{j,k}}\rho_{j,k}|\psi_j\rangle\langle\psi_k|.$$

Proof. **1)** Let $\{\psi^1_j\}_{j\in\mathbb{N}^*} \subset H^3_{(0)}$ be an orthonormal system made by eigenfunctions of ρ^1. We have

$$\rho^1 = \sum_{j=1}^{\infty} l_j|\psi^1_j\rangle\langle\psi^1_j|, \qquad \rho^2 = \sum_{j=1}^{\infty} l_j|\psi_j\rangle\langle\psi_j|.$$

The sequence $\{l_j\}_{j\in\mathbb{N}^*} \subset \mathbb{R}^+$ corresponds to the spectrum of ρ^1 and ρ^2. Now, thanks to Corollary 1.4, there exist $T > 0$, $u \in L^2((0,T),\mathbb{R})$ and $\{\theta_j\}_{j\leq N}$ such that $\pi_N(\Psi)\ \Gamma^u_T\psi^1_j = e^{i\theta_j}\pi_N(\Psi)\ \psi_j$ for every $j \leq N$, while $\pi_N(\Psi)\ \Gamma^u_T\psi^1_j = \pi_N(\Psi)\ \psi_j$ for every $j > N$. Thus,

$$\pi_N(\Psi)\ \Gamma^u_T\rho^1(\Gamma^u_T)^*\pi_N(\Psi) = \sum_{j=1}^{N} l_j|e^{i\theta_j}\pi_N(\Psi)\ \Gamma^u_T\psi^1_j\rangle\langle\psi^1_j\Gamma^u_T\pi_N(\Psi)\ e^{i\theta_j}|$$
$$+ \sum_{j=N+1}^{\infty} l_j|\pi_N(\Psi)\ \Gamma^u_T\psi^1_j\rangle\langle\psi^1_j\Gamma^u_T\pi_N(\Psi)\ | = \sum_{j=1}^{\infty} l_j\pi_N(\Psi)\ |\psi_j\rangle\langle\psi_j|\pi_N(\Psi) = \pi_N(\Psi)\ \rho^2\pi_N(\Psi).$$

2) A similar approach can be used in order to prove the second point of the theorem. In particular, the statement follows by decomposing ρ^2 with respect to $\{\psi_j\}_{j\in\mathbb{N}^*}$ as done in (33). Such step provides a positive hermitian matrix $\{\rho_{j,k}\}_{j,k\in\mathbb{N}^*}$. Now, we define $\{\psi^1_j\}_{j\in\mathbb{N}^*}$ as the orthonormal system such that (34) is valid for the density matrix ρ^1. The claim is proved by simultaneously steering $\{\psi^1_j\}_{j\in\mathbb{N}^*}$ in $\{\psi_j\}_{j\in\mathbb{N}^*}$ with respect to the projector $\pi(\Psi)$ by using Corollary 1.4. □

7 Conclusion

In this manuscript, we study the controllability of the infinite bilinear Schrödinger equations (BSE) at the same time T with one unique control u by projecting onto suitable finite dimensional subspaces of $\mathscr{H}$. The first result of the work is Theorem 1.1 which shows that the simultaneous global exact controllability of the (BSE) in projection onto a suitable N dimensional space is equivalent to the controllability of N problems (BSE) (without projecting). Our second outcome is the simultaneous local exact controllability of infinite bilinear Schrodinger equations in projection and in any positive $T > 0$. The property is stated by Theorem 1.2 and Proposition 4.2. Afterwards, we prove Theorem 1.3 that allows to control with a single u and at the same time T any finite number of components of infinitely many solutions of the problems (BSE). The outcome is guaranteed when the orthogonal projector is defined by an orthonormal systems verifying a "$H^3_{(0)}-$compatibility condition" exposed in (5). In conclusion, we rephrase the main results in terms of density matrices.

As explained in Section 1.2, one can notice the parallelism between our results with the ones provided in [MN15] by Morancey and Nersesyan. From this point of view, our purpose is to add a contribution to the theory presented in the mentioned paper. More precisely, even though Theorem 1.1 and the theory from [MN15] allow to prove similar results to the ones of our work, such outcomes would be guaranteed under the validity of hypotheses on the operator B which are not easy to confirm. From this perspective, we aim to provide explicit conditions on the problem, such as Assumptions I, ensuring the controllabilty and to this end, we develop a new set of techniques from the ones adopted in [MN15]. For instance, we present an alternative strategy to the "Coron's return method" (used in [MN15]) to prove the local controllability ensured by Theorem 1.2 (see Section 4.1). Having explicit conditions on the control field allows to consider specific B satisfying Assumptions I (see Example 2.2) and for which the controllability is guaranteed.

Here, one could wonder if the techniques developed in this manuscript can be applied to study the controllability of infinite (BSE) (without projecting). Although, a direct application is not possible. Indeed, one of the crucial points of our strategy is the possibility of decoupling with a uniform gap the eigenvalues resonances (Lemma B.8) appearing in the proof of Theorem 1.2 (see Section 4.1 for

further details). Such property is provided via perturbation theory techniques when the third condition of Assumptions I is verified and a uniform gap is guaranteed when the eigenvalues resonances are finite. Thus, projecting the dynamics onto finite dimensional spaces allows to have only finitely many resonances which is crucial for our strategy.

In any case, a possible approach that might lead to the controllability of infinite (BSE) is the following. As already done in our work, one could act a perturbation in order to decouple the eigenvalues resonances appearing in the proof of the simultaneous local exact controllability. In such framework, we do not expect to have a uniform spectral gap and then the Haraux's Thoerem [KL05, *Theorem* 4.6] can not be applied. As a consequence, the solvability of the moment problem (such as (20)) appearing in this proof can not be achieved in ℓ^2. Nevertheless, we do not exclude the possibility of proving its solvability in some spaces h^s with $s \in [0,1)$ (defined in 7) by using more raffinate techniques as the Beurling's Theorem [KL05, *Theorem* 9.2] (see also [AI95, Chapter I.2]). If such result would be valid, then the well-posedness of the (BSE) can be provided in $H^{3+s}_{(0)}$ by imposing slightly more regularity on the operator B and we might conclude the proof as done in the current work.

Acknowledgments. The author thanks Thomas Chambrion for suggesting him the problem and Nabile Boussaïd for the periodic discussions. He is also grateful to Morgan Morancey for the explanation about the works [Mor14] and [MN15].

A Moment problem

In this appendix, we briefly adapt some results concerning the solvability of the moment problems (as (13) and (20)). Let [BL10, *Proposition* 19; 2)] be satisfied and $\{f_k\}_{k\in\mathbb{Z}}$ be a Riesz basis (see [BL10, *Definition* 2]) in $X = \overline{span\{f_k : k\in\mathbb{Z}\}}^{\mathscr{H}} \subseteq \mathscr{H}$, with $\mathscr{H}$ and Hilbert space. For $\{v_k\}_{k\in\mathbb{Z}}$ the unique biorthogonal family to $\{f_k\}_{k\in\mathbb{Z}}$ ([BL10, *Remark* 7]), $\{v_k\}_{k\in\mathbb{Z}}$ is also a Riesz basis of X ([BL10, *Remark* 9]). Thanks to [BL10, *Proposition* 19; 2)], there exist $C_1, C_2 > 0$ such that

$$C_1\sum_{k\in\mathbb{Z}}|x_k|^2 \leq \|u\|^2_{\mathscr{H}} \leq C_2\sum_{k\in\mathbb{Z}}|x_k|^2, \qquad u(t)=\sum_{k\in\mathbb{Z}}x_k v_k(t),\ \{x_k\}_{k\in\mathbb{N}^*}\in\ell^2(\mathbb{C}).$$

Moreover, for every $u\in X$, we know that $u=\sum_{k\in\mathbb{Z}}v_k\langle f_k,u\rangle_{\mathscr{H}}$ since $\{f_k\}_{k\in\mathbb{Z}}$ and $\{v_k\}_{k\in\mathbb{Z}}$ are reciprocally biorthonoromal (see [BL10, *Remark* 9]) and

$$C_1\sum_{k\in\mathbb{Z}}|\langle f_k,u\rangle_{\mathscr{H}}|^2 \leq \|u\|^2_{\mathscr{H}} \leq C_2\sum_{k\in\mathbb{Z}}|\langle f_k,u\rangle_{\mathscr{H}}|^2. \tag{36}$$

When Haraux's Thoerem [KL05, *Theorem* 4.6] is verified, for $T>0$ large enough, $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a Riesz basis in $X=\overline{span\{e^{i\lambda_k(\cdot)} : k\in\mathbb{Z}\}}^{L^2} \subseteq L^2((0,T),\mathbb{C})$. The relation (36) is satisfied and $F: u\in X \longmapsto \left\{\langle e^{i\lambda_k(\cdot)},u\rangle_{\mathscr{H}}\right\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C})$ is invertible. For every $\{x_k\}_{k\in\mathbb{Z}}\in\ell^2(\mathbb{C})$, there exists $u\in X$ such that

$$x_k = \int_0^T u(s)e^{-i\lambda_k s}ds, \qquad k\in\mathbb{Z}.$$

Remark A.1. *Let $\{\lambda_k\}_{k\in\mathbb{N}^*}$ be an ordered sequence of pairwise distinct real numbers such that $\lambda_1=0$. Let $G:=\inf_{k\neq j}|\lambda_k-\lambda_j|>0$ and $G':=\sup_{K\subset\mathbb{N}^*}\inf_{\substack{k\neq j\\ k,j\in\mathbb{N}^*\setminus K}}|\lambda_k-\lambda_j|$, where K runs over the finite subsets of $\mathbb{Z}$. For $k\in\mathbb{N}^*$, we call $\omega_k=-\lambda_k$, while we impose $\omega_k=\lambda_{-k}$ for $-k\in\mathbb{N}^*\setminus\{1\}$. We call $\mathbb{Z}^*=\mathbb{Z}\setminus\{0\}$. The sequence $\{\omega_k\}_{k\in\mathbb{Z}^*\setminus\{-1\}}$ satisfies the hypotheses of [KL05, Theorem 4.6] for*

$$\sup_{K\subset\mathbb{Z}^*\setminus\{-1\}}\ \inf_{\substack{k\neq j\\ k,j\in\mathbb{N}^*\setminus K}}|\omega_k-\omega_j| = G'.$$

Let $\{x_k\}_{k\in\mathbb{N}^*}\in\ell^2(\mathbb{C})$. We call $\{\widetilde{x}_k\}_{k\in\mathbb{Z}^*\setminus\{-1\}}\in\ell^2(\mathbb{C})$ so that $\widetilde{x}_k=x_k$ for $k\in\mathbb{N}^*$, while $\widetilde{x}_k=\overline{x}_{-k}$ for $-k\in\mathbb{N}^*\setminus\{1\}$. For $T>2\pi/G'$, there exists $u\in L^2((0,T),\mathbb{C})$ such that, for every $k\in\mathbb{Z}^*\setminus\{-1\}$,

$$\widetilde{x}_k=\int_0^T u(s)e^{-i\omega_k s}ds, \quad\Longrightarrow\quad \begin{cases} x_k=\int_0^T u(s)e^{i\lambda_k s}ds=\int_0^T\overline{u}(s)e^{i\lambda_k s}ds, & \forall k\in\mathbb{N}^*\setminus\{1\},\\ x_1=\int_0^T u(s)ds. \end{cases}$$

In conclusion, if $x_1\in\mathbb{R}$, then solvability of the moment problem is provided with u real.

B Analytic Perturbation

Let us consider the problem (15) and the eigenvalues $\{\lambda_j^{u_0}\}_{j\in\mathbb{N}^*}$ of the operator $A+u_0B$. When B is a bounded symmetric operator satisfying Assumptions I and $A=-\Delta$ is the Laplacian with Dirichlet type boundary conditions $D(A)=H^2((0,1),\mathbb{C})\cap H_0^1((0,1),\mathbb{C})$, thanks to [Kat95, *Theorem VII*.2.6] and [Kat95, *Theorem VII*.3.9], the following proposition follows.

Proposition B.1. *Let B be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood $D(0)$ of $u=0$ in $\mathbb{R}$ small enough where the maps $u\mapsto\lambda_j^u$ are analytic for every $j\in\mathbb{N}^*$.*

The next lemma proves the existence of perturbations, which do not shrink the eigenvalues gaps.

Lemma B.2. *Let B be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood $D(0)$ in $\mathbb{R}$ of $u=0$ such that, for each $u_0\in D(0)$, there exists $r>0$ such that, for every $j\in\mathbb{N}^*$,*

$$\mu_j:=\frac{\lambda_j+\lambda_{j+1}}{2}\in\rho(A+u_0B),\qquad |\!|\!|\,(A+u_0B-\mu_j)^{-1}\,|\!|\!|\le r.$$

Proof. Let $D(0)$ be the neighborhood provided by Proposition B.1. We know $(A-\mu_j)$ is invertible in a bounded operator and $\mu_j\in\rho(A)$ (resolvent set of A). Let $\delta:=\min_{j\in\mathbb{N}^*}\{|\lambda_{j+1}-\lambda_j|\}$. We know that $|\!|\!|\,(A-\mu_j)^{-1}\,|\!|\!|\le\sup_{k\in\mathbb{N}^*}\frac{1}{|\mu_j-\lambda_k|}=\frac{2}{|\lambda_{j+1}-\lambda_j|}\le\frac{2}{\delta}$. Thus, for $u_0\in D(0)$,

$$|\!|\!|\,(A-\mu_j)^{-1}u_0B\,|\!|\!|\le|u_0|\,|\!|\!|\,(A-\mu_j)^{-1}\,|\!|\!|\,|\!|\!|\,B\,|\!|\!|\le\frac{2}{\delta}|u_0|\,|\!|\!|\,B\,|\!|\!|$$

and if $|u_0|\le\frac{\delta(1-\epsilon)}{2\,|\!|\!|\,B\,|\!|\!|}$ for $\epsilon\in(0,1)$, then $|\!|\!|\,(A-\mu_j)^{-1}u_0B\,|\!|\!|\le1-\epsilon$. The operator $(A+u_0B-\mu_j)$ is invertible and $|\!|\!|\,(A+u_0B-\mu_j)^{-1}\,|\!|\!|\le\frac{2}{\delta\epsilon}$ as $\|(A+u_0B-\mu_j)\psi\|_{L^2}\ge\|(A-\mu_j)\psi\|_{L^2}-\|u_0B\psi\|_{L^2}\ge\frac{\delta}{2}\|\psi\|_{L^2}-\frac{\delta(1-\epsilon)}{2}\|\psi\|_{L^2}$ for every $\psi\in D(A)$. The parameter r stated in the lemma corresponds to $2/(\delta\epsilon)$, while the neighborhood is $\{u_0\in D(0)\ :\ |u_0|\le\delta(1-\epsilon)/(2\,|\!|\!|\,B\,|\!|\!|\,)\}$. □

Lemma B.3. *Let B be a bounded symmetric operator satisfying Assumptions I and $P_{\phi_k}^\perp$ be the projector onto the orthogonal space of ϕ_k. There exists a neighborhood $D(0)$ of 0 in $\mathbb{R}$ such that*

$$(A+u_0P_{\phi_k}^\perp B-\lambda_k^{u_0})$$

is invertible with bounded inverse from $D(A)\cap\phi_k^\perp$ to $\phi_k^\perp$ for every $u_0\in D(0)$ and $k\in\mathbb{N}^*$.

Proof. Let $D(0)$ be the neighborhood provided by Lemma B.2. For any $u_0\in D(0)$, one can consider the decomposition $(A+u_0P_{\phi_k}^\perp B-\lambda_k^{u_0})=(A-\lambda_k^{u_0})+u_0P_{\phi_k}^\perp B$. The operator $A-\lambda_k^{u_0}$ is invertible with bounded inverse when it acts on the orthogonal space of ϕ_k and we estimate $|\!|\!|\,((A-\lambda_k^{u_0})\big|_{\phi_k^\perp})^{-1}u_0P_{\phi_k}^\perp B\,|\!|\!|$. However, for every $\psi\in D(A)\cap Ran(P_{\phi_k}^\perp)$ such that $\|\psi\|_{L^2}=1$, we have

$$\|(A-\lambda_k^{u_0})\psi\|_{L^2}\ge\min\{|\lambda_{k+1}-\lambda_k^{u_0}|,|\lambda_k^{u_0}-\lambda_{k-1}|\}\|\psi\|_{L^2}.$$

Let $\delta_k:=\min\left\{|\lambda_{k+1}-\lambda_k^{u_0}|,|\lambda_k^{u_0}-\lambda_{k-1}|\right\}$. Thanks to Lemma B.2, for $|u_0|$ small enough, $\lambda_k^{u_0}\in\left(\frac{\lambda_{k-1}+\lambda_k}{2},\frac{\lambda_k+\lambda_{k+1}}{2}\right)$ and then

$$\delta_k\ge\min\left\{\left|\lambda_{k+1}-\frac{\lambda_k+\lambda_{k+1}}{2}\right|,\left|\frac{\lambda_{k-1}+\lambda_k}{2}-\lambda_{k-1}\right|\right\}\ge\frac{(2k-1)\pi^2}{2}>k.$$

Afterwards, $|\!|\!|\,((A-\lambda_k^{u_0})\big|_{\phi_k^\perp})^{-1}u_0P_{\phi_k}^\perp B\,|\!|\!|\le\frac{1}{\delta_k}|u_0|\,|\!|\!|\,B\,|\!|\!|$ and, if $|u_0|\le(1-r)\frac{\delta_k}{|\!|\!|\,B\,|\!|\!|}\le\frac{(1-r)}{|\!|\!|\,B\,|\!|\!|}$ for $r\in(0,1)$, then it follows $|\!|\!|\,((A-\lambda_k^{u_0})\big|_{\phi_k^\perp})^{-1}u_0P_{\phi_k}^\perp B\,|\!|\!|\le(1-r)<1$. The operator $A_k:=(A-\lambda_k^{u_0}+u_0P_{\phi_k}^\perp B)$ is invertible when it acts on the orthogonal space of ϕ_k and, for every $\psi\in D(A)$ and $r=\frac{1}{2}$,

$$\|A_k\psi\|_{L^2}\ge\|(A-\lambda_k^{u_0})\psi\|_{L^2}-\|u_0P_{\phi_k}^\perp B\psi\|_{L^2}\ge\delta_k\|\psi\|_{L^2}-|\!|\!|\,u_0P_{\phi_k}^\perp B\,|\!|\!|\,\|\psi\|_{L^2}=\frac{1}{2}\|\psi\|_{L^2}.$$

In conclusion, $|\!|\!|\,((A-\lambda_k^{u_0}+u_0P_{\phi_k}^\perp B)\big|_{\phi_k^\perp})^{-1}\,|\!|\!|\le2$ for every $k\in\mathbb{N}^*$. □

Lemma B.4. *Let B be satisfy Assumptions I. There exists a neighborhood $D(0)$ of 0 in $\mathbb{R}$ such that, for any $u_0 \in D(0)$, we have $\lambda_j^{u_0} \neq 0$ and there exist two constants $C_1, C_2 > 0$ such that*

$$C_1\lambda_j \leq \lambda_j^{u_0} \leq C_2\lambda_j, \qquad \forall j \in \mathbb{N}^*.$$

Proof. Let $u_0 \in D(0)$ for $D(0)$ the neighborhood provided by Lemma $B.3$. We decompose the eigenfunction $\phi_j^{u_0} = a_j\phi_j + \eta_j$, where a_j is an orthonormalizing constant and η_j is orthogonal to ϕ_j. Hence $\lambda_k^{u_0}\phi_k^{u_0} = (A+u_0B)(a_k\phi_k+\eta_k)$ and $\lambda_k^{u_0}a_k\phi_k+\lambda_k^{u_0}\eta_k = Aa_k\phi_k+A\eta_k+u_0Ba_k\phi_k+u_0B\eta_k$. By projecting onto the orthogonal space of ϕ_k,

$$\lambda_k^{u_0}\eta_k = A\eta_k + u_0P_{\phi_k}^{\perp}Ba_k\phi_k + u_0P_{\phi_k}^{\perp}B\eta_k.$$

However, Lemma $B.3$ ensures that $A + u_0P_{\phi_k}^{\perp}B - \lambda_k^{u_0}$ is invertible with bounded inverse when it acts on the orthogonal space of ϕ_k and then

$$\eta_k = -a_k((A + u_0P_{\phi_k}^{\perp}B - \lambda_k^{u_0})\big|_{\phi_k^{\perp}})^{-1}u_0P_{\phi_k}^{\perp}B\phi_k, \tag{37}$$

$$\begin{aligned}\implies \quad \lambda_j^{u_0} &= \langle a_j\phi_j+\eta_j, (A+u_0B)(a_j\phi_j+\eta_j)\rangle_{L^2} = |a_j|^2\lambda_j + u_0\langle a_j\phi_j, Ba_j\phi_j\rangle_{L^2}\\ &+ \langle a_j\phi_j, (A+u_0B)\eta_j\rangle_{L^2} + \langle \eta_j, (A+u_0B)a_j\phi_j\rangle_{L^2} + \langle \eta_j, (A+u_0B)\eta_j\rangle_{L^2}.\end{aligned}$$

By using the relation (37),

$$\begin{aligned}\langle \eta_j, (A+u_0B)\eta_j\rangle_{L^2} &= \langle \eta_j, (A+u_0P_{\phi_k}^{\perp}B - \lambda_j^{u_0})\eta_j\rangle_{L^2} + \lambda_j^{u_0}\|\eta_j\|_{L^2}^2 = \lambda_j^{u_0}\|\eta_j\|_{L^2}^2\\ &+ \Big\langle \eta_j, -a_j(A+u_0P_{\phi_j}^{\perp}B - \lambda_j^{u_0})((A+u_0P_{\phi_j}^{\perp}B - \lambda_j^{u_0})\big|_{\phi_j^{\perp}})^{-1}u_0P_{\phi_j}^{\perp}B\phi_j\Big\rangle_{L^2}.\end{aligned}$$

However, $(A+u_0P_{\phi_j}^{\perp}B - \lambda_j^{u_0})((A+u_0P_{\phi_j}^{\perp})B - \lambda_j^{u_0})\big|_{\phi_j^{\perp}})^{-1} = Id$ and $\langle \eta_j, (A+u_0B)\eta_j\rangle_{L^2} = \lambda_j^{u_0}\|\eta_j\|_{L^2}^2 - u_0a_j\langle \eta_j, P_{\phi_j}^{\perp}B\phi_j\rangle_{L^2}$. Moreover, we have $\langle \phi_j, (A+u_0B)\eta_j\rangle_{L^2} = u_0\langle \phi_j, B\eta_j\rangle_{L^2} = u_0\langle P_{\phi_j}^{\perp}B\phi_j, \eta_j\rangle_{L^2}$ and $\langle \eta_j, (A+u_0B)\phi_j\rangle_{L^2} = u_0\langle \eta_j, P_{\phi_j}^{\perp}B\phi_j\rangle_{L^2}$. Thus

$$\lambda_j^{u_0} = |a_j|^2\lambda_j + u_0|a_j|^2B_{j,j} + \lambda_j^{u_0}\|\eta_j\|_{L^2}^2 + u_0\overline{a_j}\langle P_{\phi_j}^{\perp}B\phi_j, \eta_j\rangle_{L^2}. \tag{38}$$

One can notice that $|a_j| \in [0,1]$ and $\|\eta_j\|_{L^2}$ are uniformly bounded in j. We show that the first accumulates at 1 and the second at 0. Indeed, from the proof of Lemma $(B.3)$ and the relation (37), there exists $C_1 > 0$ such that

$$\|\eta_j\|_{L^2}^2 \leq |u_0|^2 \,|||\, ((A + u_0P_{\phi_j}^{\perp}B - \lambda_j^{u_0})\big|_{\phi_j^{\perp}})^{-1} \,|||\,^2|a_j|^2\|B\phi_j\|_{L^2}^2 \leq \frac{C_1}{j^2} \tag{39}$$

for $r \in (0,1)$, which implies that $\lim_{j\to\infty}\|\eta_j\|_{L^2} = 0$. Afterwards, by contradiction, if $|a_j|$ does not converge to 1, then there exists $\{a_{j_k}\}_{k\in\mathbb{N}^*}$ a subsequence of $\{a_j\}_{j\in\mathbb{N}^*}$ such that $|a_{j_\infty}| := \lim_{k\to\infty}|a_{j_k}| \in [0,1)$. Now, we have

$$1 = \lim_{k\to\infty}\|\phi_{j_k}^{u_0}\|_{L^2} \leq \lim_{k\to\infty}|a_{j_k}|\|\phi_{j_k}\|_{L^2} + \|\eta_{j_k}\|_{L^2} = \lim_{k\to\infty}|a_{j_k}| + \|\eta_{j_k}\|_{L^2} = |a_{j_\infty}| < 1$$

that is absurd. Then, $\lim_{j\to\infty}|a_j| = 1$. From (38), it follows that there exist two constants $C_1, C_2 > 0$ such that, for each $j \in \mathbb{N}^*$, $C_1\lambda_j \leq \lambda_j^{u_0} \leq C_2\lambda_j$ for $|u_0|$ small enough. The relation also implies that $\lambda_j^{u_0} \neq 0$ for every $j \in \mathbb{N}^*$ and $|u_0|$ small enough. □

Lemma B.5. *Let B be a bounded symmetric operator satisfying Assumptions I. For every $N \in \mathbb{N}^*$, there exist a neighborhood $D(0)$ of 0 in $\mathbb{R}$ and $\widetilde{C}_N > 0$ such that, for any $u_0 \in D(0)$, we have*

$$|\langle \phi_k^{u_0}, B\phi_j^{u_0}\rangle_{L^2}| \geq \frac{\widetilde{C}_N}{k^3}, \qquad \forall k, j \in \mathbb{N}^*,\ j \leq N.$$

Proof. We start by choosing $k \in \mathbb{N}^*$ such that $k \neq j$ and $u_0 \in D(0)$ for $D(0)$ the neighborhood provided by Lemma $B.4$. Thanks to Assumptions II, we have

$$\begin{aligned}&|\langle \phi_k^{u_0}, B\phi_j^{u_0}\rangle_{L^2}| = |\langle a_k\phi_k+\eta_k, B(a_j\phi_j+\eta_j)\rangle_{L^2}|\\ &\geq C_N\frac{\overline{a_k}a_j}{k^3} - \big|\overline{a_k}\langle \phi_k, B\eta_j\rangle_{L^2} + a_j\langle \eta_k, B\phi_j\rangle_{L^2} + \langle \eta_k, B\eta_j\rangle_{L^2}\big|.\end{aligned} \tag{40}$$

1) Expansion of $\langle \eta_k, B\phi_j\rangle_{L^2}$, $\langle \phi_k, B\eta_j\rangle_{L^2}$ and $\langle \eta_k, B\eta_j\rangle_{L^2}$: Thanks to (37), we have $\langle \eta_k, B\phi_j\rangle_{L^2} = \langle -a_k((A+u_0P^\perp_{\phi_k}B-\lambda_k^{u_0})\big|_{\phi_k^\perp})^{-1}u_0P^\perp_{\phi_k}B\phi_k, P^\perp_{\phi_k}B\phi_j\rangle_{L^2}$ for every $k\in\mathbb{N}^*$ and $j\le N$, while the operator $\big((A+u_0P^\perp_{\phi_k}B-\lambda_k^{u_0})\big|_{\phi_k^\perp}\big)^{-1}$ corresponds to $((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\sum_{n=0}^\infty\big(u_0((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}BP^\perp_{\phi_k}\big)^n$ for $|u_0|$ small enough. For $M_k:=\sum_{n=0}^\infty\big(u_0((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B\big)^nP^\perp_{\phi_k}$, we have

$$\langle \eta_k, B\phi_j\rangle_{L^2} = -u_0\langle a_kM_kB\phi_k, ((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B\phi_j\rangle_{L^2}. \tag{41}$$

Thanks to $B:D(A)\to D(A)$, for every $k\in\mathbb{N}^*$ and $j\le N$,

$$\begin{aligned}((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B\phi_j &= P^\perp_{\phi_k}B((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\phi_j - \big[P^\perp_{\phi_k}B, ((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}\big]\phi_j\\ &= P^\perp_{\phi_k}B((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\phi_j - ((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}[B,A]((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\phi_j.\end{aligned}$$

For $\widetilde{B}_k := ((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}[B,A]$, we have $((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B\phi_j = P^\perp_{\phi_k}(B+\widetilde{B}_k)(\lambda_j-\lambda_k^{u_0})^{-1}\phi_j$. and, for every $k\in\mathbb{N}^*$ and $j\le N$,

$$\langle \eta_k, B\phi_j\rangle_{L^2} = -\frac{u_0}{\lambda_j-\lambda_k^{u_0}}\langle a_kM_kB\phi_k, (B+\widetilde{B}_k)\phi_j\rangle_{L^2}. \tag{42}$$

For every $k\in\mathbb{N}^*$ and $j\le N$, we obtain

$$\begin{aligned}&|\langle \eta_k, B\eta_j\rangle_{L^2}| = |\langle B\eta_k, \eta_j\rangle_{L^2}| = |\langle u_0a_kB((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}M_kB\phi_k,\\ &u_0a_j((A-\lambda_j^{u_0})P^\perp_{\phi_j})^{-1}M_jB\phi_j\rangle_{L^2}\Big| = \Big|\frac{a_j\overline{a_k}u_0^2}{\lambda_k-\lambda_j^{u_0}}\big\langle\phi_k, L_{k,j}\phi_j\big\rangle_{L^2}\Big|\end{aligned} \tag{43}$$

with $L_{k,j} := (A-\lambda_j^{u_0})BM_k((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B((A-\lambda_j^{u_0})P^\perp_{\phi_j})^{-1}M_jB$. Now, there exists $\epsilon>0$ such that $|a_l|\in(\epsilon,1)$ for every $l\in\mathbb{N}^*$. Thanks to (42), (43) and (40), there exists $\widehat{C}_N$ such that

$$\begin{aligned}|\langle\phi_k^{u_0}, B\phi_j^{u_0}\rangle_{L^2}| &\ge \widehat{C}_N\frac{1}{k^3} - \Big|\frac{u_0}{\lambda_j-\lambda_k^{u_0}}\langle M_kB\phi_k, (B+\widetilde{B}_k)\phi_j\rangle_{L^2}\Big|\\ &- \Big|\frac{u_0}{\lambda_k-\lambda_j^{u_0}}\langle(B+\widetilde{B}_j)\phi_k, M_jB\phi_j\rangle_{L^2}\Big| - \Big|\frac{u_0^2}{\lambda_k-\lambda_j^{u_0}}\big\langle\phi_k, L_{k,j}\phi_j\big\rangle_{L^2}\Big|.\end{aligned} \tag{44}$$

2) Features of the operators M_k, $\widetilde{B}_k$ and $L_{k,j}$. Each M_k for $k\in\mathbb{N}^*$ is uniformly bounded in $L(H^2_{(0)},H^2_{(0)})$ when $|u_0|$ is small enough so that $|||\,u_0((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}BP^\perp_{\phi_k}\,|||_{L(H^2_{(0)})}<1$. The definition of $\widetilde{B}_k$ implies that $\widetilde{B}_kP^\perp_{\phi_k} = ((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B(A-\lambda_k^{u_0})P^\perp_{\phi_k} - P^\perp_{\phi_k}BP^\perp_{\phi_k}$. Hence, the operators $\widetilde{B}_k$ are uniformly bounded in k in $L\big(H^2_{(0)}\cap Ran(P^\perp_{\phi_k}), H^2_{(0)}\cap Ran(P^\perp_{\phi_k})\big)$. Third, one can notice that $B((A-\lambda_j^{u_0})P^\perp_{\phi_j})^{-1}M_jB\in L(H^2_{(0)},H^2_{(0)})$ for every $j\in\mathbb{N}^*$. Then, for every $k\in\mathbb{N}^*$ and $j\le N$,

$$\begin{aligned}&(A-\lambda_j^{u_0})BM_k((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k} = (A-\lambda_j^{u_0})B((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\\ &\sum_{n=0}^\infty\big(u_0P^\perp_{\phi_k}B((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\big)^nP^\perp_{\phi_k} = (A-\lambda_j^{u_0})((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}(\widetilde{B}_k+B)\widetilde{M}_k\end{aligned}$$

with $\widetilde{M}_k := \sum_{n=0}^\infty\big(u_0P^\perp_{\phi_k}B((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\big)^nP^\perp_{\phi_k}$. Now, the operators $\widetilde{M}_k$ are uniformly bounded in $L(H^2_{(0)},H^2_{(0)})$ as M_k. Hence $L_{k,j}$ are uniformly bounded in $L(H^2_{(0)},H^2_{(0)})$.

Let $\{F_l\}_{l\in\mathbb{N}^*}$ be an infinite uniformly bounded family of operators in $L(H^2_{(0)},H^2_{(0)})$. For every $l,j\in\mathbb{N}^*$, there exists $c_{l,j}>0$ such that $\sum_{k=1}^\infty|k^2\langle\phi_k,F_l\phi_j\rangle_{L^2}|^2<\infty$, which implies $|\langle\phi_k,F_l\phi_j\rangle_{L^2}|\le\frac{c_{l,j}}{k^2}$ for every $k\in\mathbb{N}^*$. Now, the constant $c_{l,j}$ can be assumed uniformly bounded in l since, for every $k,j\in\mathbb{N}^*$,

$$\sup_{l\in\mathbb{N}^*}|k^2\langle\phi_k,F_l\phi_j\rangle_{L^2}|^2 \le \sup_{l\in\mathbb{N}^*}\sum_{m\in\mathbb{N}^*}|m^2\langle\phi_m,F_l\phi_j\rangle_{L^2}|^2 \le \sup_{l\in\mathbb{N}^*}\|F_l\phi_j\|^2_{(2)}<\infty.$$

Thus, for every infinite uniformly bounded family of operators $\{F_l\}_{l\in\mathbb{N}^*}$ in $L(H^2_{(0)},H^2_{(0)})$ and for every $j\in\mathbb{N}^*$, there exists a constant c_j such that

$$|\langle\phi_k,F_l\phi_j\rangle_{L^2}|\le\frac{c_j}{k^2},\qquad\forall k,l\in\mathbb{N}^*. \tag{45}$$

3) Conclusion. We know that $|\lambda_j - \lambda_k^{u_0}|^{-1}$ and $|\lambda_k - \lambda_j^{u_0}|^{-1}$ asymptotically behave as k^{-2} thanks to Lemma $B.4$. From the previous point, the families of operators $\{BM_k(B+\widetilde{B}_k)\}_{k\in\mathbb{N}^*}$, $\{L_{k,j}\}_{k\in\mathbb{N}^*}$ are uniformly bounded in $L(H^2_{(0)}, H^2_{(0)})$ and $BM_j(B+\widetilde{B}_j) \in L(H^2_{(0)}, H^2_{(0)})$ for every $1 \le j \le N$. Hence, we use the relation (45) in (44) and there exist $C_1, C_2, C_3, C_4 > 0$ depending on $j \in \mathbb{N}^*$ such that, for $|u_0|$ small enough and $k \in \mathbb{N}^*$ large enough,

$$|\langle \phi_k^{u_0}, B\phi_j^{u_0}\rangle_{L^2}| \ge \widehat{C}_N \frac{1}{k^3} - \frac{C_1|u_0|}{|\lambda_j - \lambda_k^{u_0}|k^2} - \frac{C_2|u_0|}{|\lambda_k - \lambda_j^{u_0}|k^2} - \frac{C_3|u_0|^2}{|\lambda_k - \lambda_j^{u_0}|k^2} \ge C_4 \frac{1}{k^3}. \tag{46}$$

Let $K \in \mathbb{N}^*$ be so that $|\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T)\rangle_{L^2}| \ge C_4\frac{1}{k^3}$ for every $k > K$. For $j \in \mathbb{N}^*$, the zeros of the analytic map $u_0 \mapsto \{|\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T)\rangle_{L^2}|\}_{k\le K} \in \mathbb{R}^K$ are discrete. Then, for $|u_0|$ small enough, $|\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T)\rangle_{L^2}| \neq 0$ for every $k \le K$. Thus, for every $j \in \mathbb{N}^*$ and $|u_0|$ small enough, there exists $C_j > 0$ such that $|\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T)\rangle_{L^2}| \ge \frac{C_j}{k^3}$ for every $k \in \mathbb{N}^*$. In conclusion, the claim is achieved for every $k \in \mathbb{N}^*$ and $j \le N$ with $\widetilde{C}_N = \min\{C_j : \ j \le N\}$. □

Lemma B.6. *Let B be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood $D(0)$ of 0 in $\mathbb{R}$ such that, for any $u_0 \in D(0)$, there exist $C_1, C_2 > 0$ such that*

$$C_1\Big(\sum_{j=1}^{\infty} \big||\lambda_j^{u_0}|^{\frac{3}{2}}\langle \phi_j^{u_0}, \cdot\rangle_{L^2}\big|^2\Big)^{\frac{1}{2}} \le \Big(\sum_{j=1}^{\infty} |j^3\langle \phi_j, \cdot\rangle_{L^2}|^2\Big)^{\frac{1}{2}} \le C_2\Big(\sum_{j=1}^{\infty} \big||\lambda_j^{u_0}|^{\frac{3}{2}}\langle \phi_j^{u_0}, \cdot\rangle_{L^2}\big|^2\Big)^{\frac{1}{2}}.$$

Proof. Let $D(0)$ be the neighborhood provided by Lemma $B.4$. For $|u_0|$ small enough, we prove that there exist $C_1 > 0$ such that $\||A+u_0B|^{\frac{s}{2}}\psi\|_{L^2} \le C_1\||A|^{\frac{s}{2}}\psi\|_{L^2}$ for $s = 3$. We start with $s = 4$ and we recall that $B \in L(H^2_{(0)})$ thanks to Remark 2.1. For any $\psi \in H^4_{(0)}$, there exists $C_2 > 0$ such that

$$\|(A+u_0B)^2\psi\|_{L^2} \le \|A^2\psi\|_{L^2} + |u_0|^2\|B^2\psi\|_{L^2} + |u_0|\|A\psi\|_{L^2}(\,|||\,B\,|||_{(2)} + \,|||\,B\,|||\,) \le C_2\||A|^2\psi\|_{L^2}.$$

Classical interpolation arguments (see for instance the proof of [BdCC13, *Lemma* 1]) imply the validity of the relation also for $s = 3$. There exists $C > 0$ such that $\|\psi\|_{\widetilde{H}^3_{(0)}} = \||A+u_0B|^{\frac{3}{2}}\psi\|_{L^2} \le C\||A|^{\frac{3}{2}}\psi\|_{L^2} = C\|\psi\|_{H^3_{(0)}}$ for every $\psi \in H^3_{(0)}$. Now, $H^2_{(0)} = D(|A|) = D(|A+u_0B|) = \widetilde{H}^2_{(0)}$ and B preserves $\widetilde{H}^2_{(0)}$ since $B : H^2_{(0)} \longrightarrow H^2_{(0)}$. The arguments of Remark 2.1 imply that $B \in L(\widetilde{H}^2_{(0)})$ and the opposite inequality follows as above thanks to the identity $A = (A+u_0B) - u_0B$. □

Remark B.7. *Let B be a bounded symmetric operator satisfying Assumptions I. The techniques of the proof of Lemma $B.6$ also allow to prove that, for $s \in (0,3)$, there exists a neighborhood $D(0)$ of 0 in $\mathbb{R}$ such that, for any $u_0 \in D(0)$, it follows $\Big(\sum_{j=1}^{\infty}\big|(\lambda_j^{u_0})^{\frac{s}{2}}\langle \phi_j^{u_0}, \cdot\rangle_{L^2}\big|^2\Big)^{\frac{1}{2}} \asymp \Big(\sum_{j=1}^{\infty}|j^s\langle \phi_j, \cdot\rangle_{L^2}|^2\Big)^{\frac{1}{2}}$.*

Lemma B.8. *Let B be a bounded symmetric operator satisfying Assumptions I and $N \in \mathbb{N}^*$. Let $\epsilon > 0$ small enough and I^N be the set defined in* (4)*. There exists a $D_\epsilon \subset \mathbb{R}\setminus\{0\}$ such that, for each $u_0 \in D_\epsilon$,*

$$\inf_{\substack{(j,k),(n,m)\in I^N \\ (j,k)\neq(n,m)}} |\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| > \epsilon.$$

Moreover, for every $\delta > 0$ small there exists $\epsilon > 0$ such that $dist(D_\epsilon, 0) < \delta$.

Proof. Let us consider the neighborhood $D(0)$ provided by Lemma $B.3$. The maps $u \mapsto \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u$ are analytic for each $j,k,n,m \in \mathbb{N}^*$ and $u \in D(0)$. The number of elements such that

$$\lambda_j - \lambda_k - \lambda_n + \lambda_m = 0, \qquad \forall j,n,k,m \in \mathbb{N}^*, \ k,m \le N \tag{47}$$

is finite. Indeed $\lambda_k = k^2\pi^2$ and (47) corresponds to $j^2 - k^2 = n^2 - m^2$. We have $|j^2 - n^2| = |k^2 - m^2| \le N^2 - 1$, which is satisfied for a finite number of elements. Thus, for I^N (defined in (4), the following set is finite

$$R := \{((j,k),(n,m)) \in (I^N)^2 \ : \ (j,k) \neq (n,m); \ \lambda_j - \lambda_k - \lambda_n + \lambda_m = 0\}.$$

1) Let $((j,k),(n,m)) \in R$, the set $V_{(j,k,n,m)} = \{u \in D \big|\ \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = 0\}$ is a discrete subset of $D(0)$ or equal to $D(0)$. Thanks to the relation (38),

$$\begin{aligned}
&\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = |a_j|^2\lambda_j + u|a_j|^2 B_{j,j} + \lambda_j^u \|\eta_j\|_{L^2}^2 + u\overline{a_j}\langle P_{\phi_j}^{\perp} B\phi_j, \eta_j\rangle_{L^2} - |a_k|^2\lambda_k \\
&- u|a_k|^2 B_{k,k} - \lambda_k^u \|\eta_k\|_{L^2}^2 - u\overline{a_k}\langle P_{\phi_k}^{\perp} B\phi_k, \eta_k\rangle_{L^2} - |a_n|^2\lambda_n - u|a_n|^2 B_{n,n} - \lambda_n^u \|\eta_n\|_{L^2}^2 \\
&- u\overline{a_n}\langle P_{\phi_n}^{\perp} B\phi_n, \eta_n\rangle_{L^2} + |a_m|^2\lambda_m + u|a_m|^2 B_{m,m} + \lambda_m^u \|\eta_m\|_{L^2}^2 + u\overline{a_m}\langle P_{\phi_m}^{\perp} B\phi_m, \eta_m\rangle_{L^2}
\end{aligned}$$

$$\begin{aligned}
\implies \quad & \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = |a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m \\
& + (|a_j|^2 B_{j,j} - |a_k|^2 B_{k,k} - |a_n|^2 B_{n,n} + |a_m|^2 B_{m,m})u + o(u).
\end{aligned} \tag{48}$$

For $|u|$ small enough, thanks to $lim_{|u|\to 0}|a_j|^2 = 1$ and to the third point of Assumptions I, each map

$$u \mapsto \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u$$

can not be constantly equal to 0. Then, $V_{(j,k,n,m)}$ is discrete and $V = \{u \in D \big|\ \exists (j,k,n,m) \in R : \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = 0\}$ is a discrete subset of $D(0)$. As R is a finite set $\widetilde{U}_\epsilon := \{u \in D : \forall (j,k,n,m) \in R \big|\ |\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \epsilon\}$ has positive measure for $\epsilon > 0$ small enough. Moreover, for any $\delta > 0$ small, there exists $\epsilon_0 > 0$ such that $dist(0, \widetilde{U}_{\epsilon_0}) < \delta$.

2) Let $((j,k),(n,m)) \in (I^N)^2 \setminus R$ be different numbers. We know that $|\lambda_j^0 - \lambda_k^0 - \lambda_n^0 + \lambda_m^0| = \pi^2|j^2 - k^2 - n^2 + m^2| > \pi^2$. First, thanks to (38), we have $\lambda_j^u \leq |a_j|^2\lambda_j + |u|C_1$ and $\lambda_j^u \geq |a_j|^2\lambda_j - |u|C_2$ for suitable constants $C_1, C_2 > 0$ non depending on the index j. Thus

$$|\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq ||a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m| - |u|(2C_1 + 2C_2).$$

Now, $\lim_{k\to\infty} |a_k|^2 = 1$. For any u in $D(0)$ and ϵ small enough, there exists $M_\epsilon \in \mathbb{N}^*$ such that $||a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m| \geq \pi^2 - \epsilon$ for every $((j,k),(n,m)) \in R^C := (I^N)^2 \setminus R$ and $j,k,n,m \geq M_\epsilon$. However $\lim_{|u|\to 0} |a_k|^2 = 1$ uniformly in k thanks to (39) and then there exists a neighborhood $W_\epsilon \subseteq D(0)$ such that, for each $u \in W_\epsilon$, it follows $||a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m| \geq \pi^2 - \epsilon$ for every $((j,k),(n,m)) \in R^C$ and $1 \leq j,k,n,m < M_\epsilon$. Thus, for each $u \in W_\epsilon$ and $((j,k),(n,m)) \in R^C$ such that $(j,k) \neq (n,m)$, we have $|\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \pi^2 - \epsilon$.

3) The proof is achieved since, for $\epsilon_1 > 0$ small enough, $\widetilde{U}_{\epsilon_1} \cap W_\epsilon$ is a non-zero measure subset of $D(0)$. For any $u \in \widetilde{U}_{\epsilon_1} \cap W_\epsilon$ and for any $((j,k),(n,m)) \in (I^N)^2$ such that $(j,k) \neq (n,m)$, we have $|\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \min\{\pi^2 - \epsilon, \epsilon_1\}$. □

Remark B.9. *Let B be a bounded symmetric operator satisfying Assumptions I. By using the techniques of the proofs of Lemma B.5 and Lemma B.8, one can ensure the existence of a neighborhood D_1 of u_0 in $\mathbb{R}$ and D_2, a countable subset of $\mathbb{R}$ such that, for any $u_0 \in D(0) := (D_1 \setminus D_2) \setminus \{0\}$, we have:*

1. *For every $N \in \mathbb{N}^*$, $(j,k),(n,m) \in I^N$ (see (4)) such that $(j,k) \neq (n,m)$, there holds $\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0} \neq 0$.*
2. *$B_{j,k}^{u_0} = \langle \phi_j^{u_0}(T), B\phi_k^{u_0}(T)\rangle_{L^2} \neq 0$ for every $j,k \in \mathbb{N}^*$.*
3. *Let $T > 0$ and $\epsilon_0 > 0$. For $|u_0|$ small enough, the neighborhood $O_{\epsilon_0,T}^{u_0}$ (defined in (16)) contains $O_{\epsilon,T}$ (defined in (12)) for $\epsilon > 0$ sufficiently small.*

References

[AI95] Sergei A. Avdonin and Sergei A. Ivanov. *Families of exponentials.* Cambridge University Press, Cambridge, 1995. The method of moments in controllability problems for distributed parameter systems, Translated from the Russian and revised by the authors.

[BCCS12] U. Boscain, M. Caponigro, T. Chambrion, and M. Sigalotti. A weak spectral condition for the controllability of the bilinear Schrödinger equation with application to the control of a rotating planar molecule. *Comm. Math. Phys.*, 311(2):423–455, 2012.

[BCMS12] U. V. Boscain, F. Chittaro, P. Mason, and M. Sigalotti. Adiabatic control of the Schrödinger equation via conical intersections of the eigenvalues. *IEEE Trans. Automat. Control*, 57(8):1970–1983, 2012.

[BdCC13] N. Boussaï d, M. Caponigro, and T. Chambrion. Weakly coupled systems in quantum control. *IEEE Trans. Automat. Control*, 58(9):2205–2216, 2013.

[Bea05] K. Beauchard. Local controllability of a 1-D Schrödinger equation. *J. Math. Pures Appl. (9)*, 84(7):851–956, 2005.

[BGRS15] U. Boscain, J. P. Gauthier, F. Rossi, and M. Sigalotti. Approximate controllability, exact controllability, and conical eigenvalue intersections for quantum mechanical systems. *Comm. Math. Phys.*, 333(3):1225–1239, 2015.

[BL10] K. Beauchard and C. Laurent. Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control. *J. Math. Pures Appl. (9)*, 94(5):520–554, 2010.

[BMS82] J. M. Ball, J. E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control Optim.*, 20(4):575–597, 1982.

[Cha12] T. Chambrion. Periodic excitations of bilinear quantum systems. *Automatica J. IFAC*, 48(9):2040–2046, 2012.

[CMSB09] T. Chambrion, P. Mason, M. Sigalotti, and U. Boscain. Controllability of the discrete-spectrum Schrödinger equation driven by an external field. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(1):329–349, 2009.

[Kat53] T. Kato. Integration of the equation of evolution in a Banach space. *J. Math. Soc. Japan*, 5:208–234, 1953.

[Kat95] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[KL05] V. Komornik and P. Loreti. *Fourier series in control theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2005.

[Lue69] D. G. Luenberger. *Optimization by vector space methods*. John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[Mir09] M. Mirrahimi. Lyapunov control of a quantum particle in a decaying potential. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1743–1765, 2009.

[MN15] M. Morancey and V. Nersesyan. Simultaneous global exact controllability of an arbitrary number of 1D bilinear Schrödinger equations. *J. Math. Pures Appl. (9)*, 103(1):228–254, 2015.

[Mor14] M. Morancey. Simultaneous local exact controllability of 1D bilinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(3):501–529, 2014.

[Ner10] V. Nersesyan. Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(3):901–915, 2010.

[Sac00] Yu. L. Sachkov. Controllability of invariant systems on Lie groups and homogeneous spaces. *J. Math. Sci. (New York)*, 100(4):2355–2427, 2000. Dynamical systems, 8.