Ergodic currents dual to a real tree

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ERGODIC CURRENTS DUAL TO A REAL TREE

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Abstract. Let T be an $\mathbb{R}$-tree with dense orbits in the boundary of Outer space. When the free group $\mathbb{F}_N$ acts freely on T, we prove that the number of projective classes of ergodic currents dual to T is bounded above by $3N-5$.

We combine Rips induction and splitting induction to define unfolding induction for such an $\mathbb{R}$-tree T. Given a current μ dual to T, the unfolding induction produces a sequence of approximations converging towards μ.

We also give a unique ergodicity criterion.

1. Introduction

1.1. **Main results.** Let $\mathbb{F}_N$ be the free group with N generators. M. Culler and K. Vogtmann [CV86] introduced **Outer space**: the space CV_N of projective classes of free minimal actions of $\mathbb{F}_N$ by isometries on simplicial metric trees. We denote by cv_N the space of free minimal actions of $\mathbb{F}_N$ by isometries on simplicial metric trees: cv_N is the unprojectivised Outer space. The space CV_N is a contractible simplicial complex, with missing faces. The maximal dimension of a simplex in CV_N is $3N-4$. Outer space admits a Thurston boundary, denoted by $\partial\mathrm{CV}_N$, which gives rise to a compactification $\overline{\mathrm{CV}}_N = \mathrm{CV}_N \uplus \partial\mathrm{CV}_N$ of CV_N. This compactification consists of projective classes of $\mathbb{R}$-trees with a minimal, very small action of $\mathbb{F}_N$ by isometries [CL95, BF95]. Again, we denote by $\partial\mathrm{cv}_N$ and $\overline{\mathrm{cv}}_N$ the corresponding unprojectivized spaces.

The group $\mathrm{Out}(\mathbb{F}_N)$ of outer automorphisms of the free group $\mathbb{F}_N$ acts on CV_N. One basically considers that the Outer space plays the same role for $\mathrm{Out}(\mathbb{F}_N)$ as the Teichmüller space of a surface S for the mapping class group $\mathrm{MCG}(S)$ of S – see for instance [BV06] where the analogy is carried on.

Another space on which $\mathrm{Out}(\mathbb{F}_N)$ naturally acts, and which appears to capture slightly different information from $\mathrm{Out}(\mathbb{F}_N)$, is the space $\mathbb{P}\mathrm{Curr}(\mathbb{F}_N)$ of projective classes of currents [Kap06, Kap05]. Let us recall what a current is.

The space $\partial\mathbb{F}_N$ of ends of the free group $\mathbb{F}_N$ is a Cantor set, equipped with an action by homeomorphisms of $\mathbb{F}_N$. The space $\partial\mathbb{F}_N$ is also the Gromov boundary of $\mathbb{F}_N$. The action of $\mathbb{F}_N$ on itself by left multiplication extends continuously to $\partial\mathbb{F}_N$. We denote by $\partial^2\mathbb{F}_N = (\partial\mathbb{F}_N)^2 \smallsetminus \Delta$ the double boundary of $\mathbb{F}_N$, where Δ stands for the diagonal. The double boundary inherits a product topology from $\partial\mathbb{F}_N$, and the action of $\mathbb{F}_N$ on $\partial\mathbb{F}_N$ gives rise to a diagonal action on $\partial^2\mathbb{F}_N$. The involution $(X,Y) \mapsto (Y,X)$ of $\partial^2\mathbb{F}_N$ is called the flip map. A **current** μ is a $\mathbb{F}_N$-invariant, flip-invariant, Radon measure (that is to say a Borel measure which is finite on compact sets) on $\partial^2\mathbb{F}_N$. We notice that a linear combination, with non-negative coefficients, of currents is still a current.

The space $\mathrm{Curr}(\mathbb{F}_N)$ of currents of $\mathbb{F}_N$ is equipped with the weak-$*$-topology: it is a locally compact space. The group $\mathrm{Out}(\mathbb{F}_N)$ naturally acts on $\mathrm{Curr}(\mathbb{F}_N)$ – see for instance [Kap06] for a detailed description of this action. The space $\mathbb{P}\mathrm{Curr}(\mathbb{F}_N)$ of projective classes of (non-zero) currents equipped with the quotient topology is a compact set, and the action of $\mathrm{Out}(\mathbb{F}_N)$ on $\mathrm{Curr}(\mathbb{F}_N)$ induces an action on $\mathbb{P}\mathrm{Curr}(\mathbb{F}_N)$.

The spaces $\mathrm{Curr}(\mathbb{F}_N)$ and cv_N can be viewed, to some extent, as dual spaces. Indeed, M. Lustig and I. Kapovich [KL09] defined a kind of duality bracket:

$$\begin{array}{rccc} \langle\cdot,\cdot\rangle : & \overline{\mathrm{cv}}_N \times \mathrm{Curr}(\mathbb{F}_N) & \to & \mathbb{R}_{\geq 0} \\ & (T,\mu) & \mapsto & \langle T,\mu\rangle. \end{array}$$

This bracket is characterized as being the unique continuous $\mathrm{Out}(\mathbb{F}_N)$-equivariant map $\overline{\mathrm{cv}}_N \times \mathrm{Curr}(\mathbb{F}_N) \to \mathbb{R}_{\geq 0}$ which is $\mathbb{R}_{\geq 0}$-homogeneous in the first entry and $\mathbb{R}_{\geq 0}$-linear in the second one, and which satisfies that

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for all $T \in \overline{\mathrm{cv}}_N$ and for all rational current μ_w induced by a conjugacy class of a primitive element w of $\mathbb{F}_N$, $\langle T, \mu_w \rangle$ is the translation length of the conjugacy class of w in T – see [KL09]. The number $\langle T, \mu \rangle$ is the **intersection number** of T and μ.

A tree $T \in \overline{\mathrm{cv}}_N$ and a current $\mu \in \mathrm{Curr}(\mathbb{F}_N)$ are **dual** when $\langle T, \mu \rangle = 0$. We notice that the nullity of the intersection number $\langle T, \mu \rangle = 0$ only depends on the projective classes of T and μ. The main goal of this paper is to explain how to build all the currents dual to a given tree. We remark that the set of projective classes of currents dual to a given tree T is convex: the extremal points of this set are **ergodic** currents.

Theorem 1.1. *Let T be an $\mathbb{R}$-tree with a free, minimal action of $\mathbb{F}_N$ by isometries with dense orbits. There are at most $3N-5$ projective classes of ergodic currents dual to T.*

The duality between trees and currents can also be understood by considering laminations. An $\mathbb{R}$-tree T with an action of $\mathbb{F}_N$ by isometries with dense orbits has a dual lamination $L(T)$ [CHL08b]. I. Kapovich and M. Lustig [KL10] proved that a current is dual to T if and only if it is carried by the dual lamination. Thus we can rephrase our main Theorem to:

Theorem 1.2. *Let T be an $\mathbb{R}$-tree with a free, minimal action of $\mathbb{F}_N$ by isometries with dense orbits. The dual lamination $L(T)$ carries at most $3N-5$ projective classes of ergodic currents.*

1.2. **The surface case.** Generally speaking, the present work is inspired by the situation of hyperbolic surfaces which we recall here.

Let $S = S_g$ be an oriented surface of genus g with negative Euler characteristic $\chi(S) = 2-2g < 0$. The mapping class group of S acts on Teichmüller space $\mathrm{Teich}(S)$ which is homeomorphic to a ball of dimension $6g-6$. Teichmüller space can be compactified by adding its Thurston boundary $\partial\mathrm{Teich}(S)$ which is homeomorphic to a sphere of dimension $6g-7$. There are several useful models to describe $\partial\mathrm{Teich}(S)$, see for instance [Pau88]. In particular, points in $\partial\mathrm{Teich}(S)$ can be seen alternatively as projective classes of:

- $\mathbb{R}$-trees with a minimal small action by isometries of the fundamental group of the surface,
- measured geodesic laminations on S,
- measured singular foliations on S (up to Whitehead equivalence).
- currents (see [Bon88])

A geodesic lamination may carry more than one transverse measure (even if the lamination is minimal). The simplex of measures carried by a given geodesic lamination embeds in the boundary of Teichmüller space and thus has dimension less than $6g-5$. In fact, G. Levitt [Lev83] proved that an orientable foliation carries at most $3g-3$ distinct projective ergodic measures, see also [Pap86].

It is not easy to exhibit a non-uniquely ergodic minimal foliation. Examples come from interval exchange transformations (IET). Indeed, the mapping torus of an IET gives rise to a foliated surface (through zippering process). H. Keynes and D. Newton [KN76] and M. Keane [Kea77] gave examples of non-uniquely ergodic minimal IET (and thus of non-uniquely ergodic minimal foliations on a surface). E. Sataev [Sat75] constructed minimal IET with exactly k ergodic measures for $1 \leq k \leq 2g$. The main tool to investigate ergodic properties of the foliation of an IET is to use a suitable induction such as Rauzy-Veech induction.

In the case of Outer space, there are fewer models available to describe points in the boundary $\partial\mathrm{CV}_N$. In particular, I. Kapovich and M. Lustig [KL07] proved that there is no $\mathrm{Out}(\mathbb{F}_N)$-equivariant continuous embedding of $\partial\mathrm{CV}_N$ in the space of currents. Although D. Gaboriau and G. Levitt [GL95] proved that the dimension of $\partial\mathrm{CV}_N$ is $3N-5$ (see also V. Guirardel [Gui00] where this result is reinterpreted in terms of length measures), we cannot deduce from this fact a bound on the number of ergodic currents dual to an $\mathbb{R}$-tree. This leads us to follow the alternative strategy of analyzing the ergodic properties via an induction.

1.3. **Outline of the proof.** The general strategy to prove Theorem 1.1 consists in mimicking what has been done for interval exchange transformations. However, a tree in $\partial\mathrm{cv}_N$ is not necessarily transverse to the foliation of an interval exchange transformation. Thus we cannot use the Rauzy-Veech induction. Instead, for an $\mathbb{R}$-tree T in $\partial\mathrm{CV}_N$ with dense orbits and a basis A of $\mathbb{F}_N$, we defined, together with M. Lustig [CHL09], a kind of a band complex $S_A = (K_A, A)$: the mapping torus of the system of isometries on the compact heart given by A. For such a band complex we can use the unfolding induction which consists in either the Rips induction [CH14] or the splitting induction [CHR11].

First, let us recall that a current can be seen as a Kolmogorov function. Given a graph Γ and a marking isomorphism $\mathbb{F}_N \xrightarrow{\sim} \pi_1(\Gamma)$, the universal cover $\widetilde{\Gamma}$ is a (simplicial) tree with an action of $\mathbb{F}_N$. Points in $\partial\mathbb{F}_N$

correspond to points in the boundary of $\partial\widetilde{\Gamma}$ and elements $(X,Y) \in \partial^2\mathbb{F}_N$ are represented by bi-infinite lines in $\widetilde{\Gamma}$. The topology on $\partial^2\mathbb{F}_N$ is given by the cylinders: sets of lines that share a common subpath. Currents are completely described by the measures of the cylinders: For a current μ and a finite path γ in Γ (or rather its lift in $\widetilde{\Gamma}$), the Kolmogorov function associated to μ, assigns to γ the measure $\mu(\gamma)$ of the cylinder defined by γ. (Note that the fuzziness about the absence of base-point in the marking isomorphism or the choice of a lift of γ in $\widetilde{\Gamma}$ is justified by the $\mathbb{F}_N$-invariance of currents.)

For a marked graph Γ and a current μ we consider the non-negative vector $\mu_\Gamma = (\mu(e))_{e\in E(\Gamma)}$ of the μ-measure of the cylinders defined by the edges of Γ. The unfolding induction, starting from an indecomposable $\mathbb{R}$-tree T in $\partial\mathrm{CV}_N$, produces a sequence (Γ_n) of marked graphs and a sequence of non-negative integer matrices (M_n) such that for any current μ dual to T:

(1) the sequence of vectors (μ_{Γ_n}) completely determines μ,
(2) $\mu_{\Gamma_n} = M_n\mu_{\Gamma_{n+1}}$.

Using properties (1) and (2), we derive that the bound on the the dimension of the simplex of currents dual to T is given by the Euler characteristic of the graphs Γ_n which remains constant through splitting induction.

In fact, the sequence of marked graphs (Γ_n) given by the unfolding induction limits to the support of the currents dual to T. It also gives a decomposition of the cylinders of the dual lamination $L(T)$ defined by the edges of Γ_n similar to the Kakutani-Rokhlin towers approximations of a dynamical system – see for instance [Dur10, Definition 6.4.1] .

In Section 6, we address the question of unique ergodicity: an $\mathbb{R}$-tree in $\partial\mathrm{CV}_N$ is uniquely ergodic if it is dual to a unique projective current. In the context of IET, there is a famous sufficient condition known as Masur criterion [Mas82]. This criterion can be understood using the sequence of matrices of the Rauzy-Veech induction, see for instance [Yoc05]. We derive such a criterion for $\mathbb{R}$-trees in $\partial\mathrm{CV}_N$ in terms of the sequence of matrices of the splitting induction.

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2. Preliminaries

2.1. **Trees and dual laminations.** An $\mathbb{R}$-tree T is a 0-hyperbolic metric space. It has a Gromov boundary ∂T. We denote by $\widehat{T} = \overline{T} \cup \partial T$ the union of the metric completion of T and its Gromov boundary. This is a topological space which is not compact in general. A **direction** d at a point P in $\widehat{T}$ is a connected component of $\widehat{T} \smallsetminus \{P\}$. We weaken the topology on $\widehat{T}$ by considering the set of directions as a sub-basis of open sets, we denote by $\widehat{T}^{\mathrm{obs}}$ the resulting topological space which is Hausdorff, compact and has exactly the same connected subsets as $\widehat{T}$ [CHL07]. Indeed $\widehat{T}^{\mathrm{obs}}$ is a dendrite in the terminology of B. Bowditch [Bow99].

We denote by $\partial\mathbb{F}_N$ the Gromov-boundary of $\mathbb{F}_N$: $\partial\mathbb{F}_N$ is a Cantor set. Fixing a basis A of $\mathbb{F}_N$, elements of $\mathbb{F}_N$ are finite reduced words in $A^{\pm 1}$ and elements of $\partial\mathbb{F}_N$ are infinite reduced words in $A^{\pm 1}$. Let T be an $\mathbb{R}$-tree in $\partial\mathrm{cv}_N$ with dense orbits. For a point P in T the orbit map $\mathbb{F}_N \to T$, $u \mapsto uP$, has a unique continuous extension to a map $\mathcal{Q} : \partial\mathbb{F}_N \to \widehat{T}^{\mathrm{obs}}$. This map $\mathcal{Q}$ does not depend on the choice of P [LL03, LL08, CHL07].

We denote by $\partial^2\mathbb{F}_N = (\partial\mathbb{F}_N)^2 \smallsetminus \Delta$ the double boundary of $\mathbb{F}_N$, where Δ stands for the diagonal. The **dual lamination** [CHL08a, CHL08b] of the tree T is

$$L(T) = \{(X,Y) \in \partial^2\mathbb{F}_N \mid \mathcal{Q}(X) = \mathcal{Q}(Y)\}.$$

This is a closed, $\mathbb{F}_N$-invariant, flip-invariant, subset of $\partial^2\mathbb{F}_N$.

The map $\mathcal{Q}$ induces a continuous [CHL09] map $\mathcal{Q}^2 : L(T) \to \overline{T}$ (here the topology on $\overline{T}$ is the metric topology). The **limit set** $\Omega = \mathcal{Q}^2(L(T))$ of T is the image of the lamination by $\mathcal{Q}$. The tree T is of **Levitt type** if the limit set is totally disconnected.

2.2. **Compact heart.** Let A be a basis of $\mathbb{F}_N$. Elements of $\mathbb{F}_N$ are vertices of the Cayley graph of $\mathbb{F}_N$ with respect to the basis A. Elements of $\partial\mathbb{F}_N$ are infinite reduced paths starting at 1. Elements of $\partial^2\mathbb{F}_N$ are identified with bi-infinite reduced paths in the Cayley graph indexed by $\mathbb{Z}$. The **unit cylinder** of $\partial^2\mathbb{F}_N$ is the set of bi-infinite reduced paths going through 1 at index 0:

$$C_A(1) = \{(X,Y) \in \partial^2\mathbb{F}_N \mid X_0 \neq Y_0\}$$

where X_0 is the first letter of the infinite word X. The unit cylinder $C_A(1)$ is a compact set. We denote by $L_A(T) = L(T) \cap C_A(1)$ the symbolic lamination relative to A. It is a compact set. Its image by $\mathcal{Q}^2$ is the **compact limit set** relative to A:

$$\Omega_A = \mathcal{Q}^2(L_A(T)) = \mathcal{Q}^2(L(T) \cap C_A(1)) \subseteq \overline{T}.$$

The convex hull of Ω_A is the **compact heart** $K_A \subseteq \overline{T}$ of the tree T relative to A.

For each element $a \in A$ of the basis we consider its restriction to K_A as a partial isometry. By allowing inverses and composition we get a pseudo-action of $\mathbb{F}_N$ on K_A. We denote by $S_A = (K_A, A)$ this system of isometries [CHL09].

An element u of $\mathbb{F}_N$ is **admissible** if it is non-empty as a partial isometry. An infinite reduced word $X \in \partial\mathbb{F}_N$ is **admissible** if all its prefixes are admissible. In this case the domains of the prefixes form a nested sequence of non-empty compact subtrees of the compact heart K_A and the **domain** of X is their intersection, it is exactly the image of X by the map $\mathcal{Q}$:

$$\{\mathcal{Q}(X)\} = \mathrm{dom}X = \bigcap_{n\in\mathbb{N}} \mathrm{dom}(X_n)$$

where X_n is the prefix of length n. A bi-infinite reduced word Z has two halves and we write $Z = (Z_-, Z_+) \in C_A(1)$. It is **admissible** if all its finite factors are admissible, or equivalently, if its two halves are admissible with the same domain. In this case, the **domain** of Z is the common domain of its two halves:

$$\mathrm{dom}(Z) = \{\mathcal{Q}(Z)\} = \{\mathcal{Q}(Z_+)\} = \{\mathcal{Q}(Z_-)\}.$$

The set of admissible bi-infinite words of the system of isometries $S_A = (K_A, A)$ is the **admissible lamination** $L(S_A)$. This is a shift-invariant symmetric closed subset of the full-shift of bi-infinite reduced words in $A^{\pm 1}$. We remark that a bi-infinite reduced word Z corresponds to a pair (Z_-, Z_+) in $\partial^2\mathbb{F}_N$, and for a pair $(X, Y) \in \partial^2\mathbb{F}_N$, we can define the bi-infinite reduced word $X^{-1}Y$. This correspondence is rather between a shift-orbit of bi-infinite reduced words in $A^{\pm 1}$ and an $\mathbb{F}_N$-orbit in $\partial^2\mathbb{F}_N$. By abuse of notations, using the above correspondence, we have:

Proposition 2.1 ([CHL09])**.** *Let T be an $\mathbb{R}$-tree with a minimal, very small action of $\mathbb{F}_N$ by isometries with dense orbits. Let A be a basis of $\mathbb{F}_N$ and let $S_A = (K_A, A)$ be the associated system of isometries.*

Then, the admissible lamination of S_A is equal to the dual lamination of T: $L(S_A) = L(T)$.

For a word $w \in \mathbb{F}_N$, the **cylinder** $C_A(w) \subseteq L(S_A)$ is the set of bi-infinite admissible words Z in S_A which reads w at index 0: w is a prefix of the positive half Z_+. The shifts of the cylinders form a sub-basis of open sets of the admissible lamination.

2.3. **Systems of isometries.** More generally a **system of isometries** $S = (F, A)$ is a compact forest F together with a finite set A of partial isometries of F. Here a **compact forest** is a disjoint union of finitely many compact $\mathbb{R}$-trees. A partial isometry is a non-empty isometry between two compact subtrees. The graph $\Gamma = \Gamma(S)$ of such a system of isometries has the connected components of F as vertices and the partial isometries in A as edges. The edge $a \in A$ goes from the connected component that contains its domain to the connected component of that contains its image. We always assume that Γ is connected.

A finite reduced path γ in Γ is **admissible** if it is non-empty as a partial isometry. Its domain $\mathrm{dom}(\gamma)$ is a non-empty compact subtree of F. The infinite reduced path X in Γ is **admissible** if all its subpaths are admissible. Its domain $\mathrm{dom}(X)$ is the intersection of the nested domains of the initial subpaths of X. A bi-infinite reduced path $Z = \cdots Z_{-1}Z_0Z_1Z_2\cdots$ has to halves $Z_- = Z_0^{-1}Z_{-1}^{-1}\cdots$ and $Z_+ = Z_1Z_2\cdots$. It is **admissible** if all its subpaths are admissible or equivalently if its two halves are admissible and its **domain** $\mathrm{dom}(Z) = \mathrm{dom}(Z_-) \cap \mathrm{dom}(Z_+)$ is a non-empty compact subtree of F.

The **admissible lamination** $L(S)$ of the system of isometries $S = (F, A)$ is the set of bi-infinite admissible paths in Γ. For a finite reduced path γ in Γ, the **cylinder** $C(\Gamma, \gamma)$ is the set of bi-infinite admissible paths Z that goes through γ at index 0: γ is a prefix of the positive half Z_+.

Following D. Gaboriau [Gab97], a system of isometries has **independent generators** if the domain $\mathrm{dom}(X)$ of any infinite admissible path X in Γ contains a single point which we denote by $\mathcal{Q}(X)$.

2.4. **Currents.** A **current** for the free group $\mathbb{F}_N$ is an $\mathbb{F}_N$-invariant, flip-invariant, Radon measure (that is to say a Borel measure which is finite on compact sets) on $\partial^2\mathbb{F}_N$. The support of a current is a lamination.

Let T be an $\mathbb{R}$-tree with a minimal very small action of $\mathbb{F}_N$ by isometries with dense orbits. We denote by $\mathbb{P}\mathrm{Curr}(T)$ the simplex of currents carried on the dual lamination $L(T)$. Let A be a basis of $\mathbb{F}_N$ and $S_A = (K_A, A)$ be the system of isometries on the compact heart of T. Recall that we identify the dual lamination $L(T)$ with the admissible lamination $L(S_A)$ which is the shift of bi-infinite admissible words in $A^{\pm 1}$. In this setting a current if a shift-invariant, flip-invariant finite Borel measure on $L(S_A)$ [Kap06, CHL08c].

For a current $\mu \in \mathbb{P}\mathrm{Curr}(T)$ and a word $w \in \mathbb{F}_N$, we denote by $\mu(w) = \mu(C_A(w))$ the finite measure of the cylinder of w. As the cylinders generate the topology, the current μ is completely determined by the values $\mu(w)$, for $w \in \mathbb{F}_N$.

Proposition 2.2. *Let T be an $\mathbb{R}$-tree with a free minimal action of $\mathbb{F}_N$ by isometries with dense orbits. Let $\mu \in \mathbb{P}Curr(T)$ be a current carried by the dual lamination of T. Then μ has no atoms.*

Proof. Let A be a basis of $\mathbb{F}_N$ and let $S_A = (K_A, A)$ be the associated system of isometries. Let Z be a bi-infinite admissible word such that $\mu(Z) > 0$. Then, as μ is shift-invariant, denoting by σ the shift-map

$$\mu(\{\sigma^n Z \mid n \in \mathbb{Z}\}) = \mu(Z) \cdot \left|\{\sigma^n Z \mid n \in \mathbb{Z}\}\right| \leq \mu(L(S_A)) < \infty.$$

Thus the shift-orbit of Z is finite and there exists $n > 0$ such that $\sigma^n Z = Z$. Let u be the prefix of the positive half Z_+ of length n. We have $u^{-1}Z_+ = Z_+$ and using the equivariant map $\mathcal{Q}$

$$\mathcal{Q}(Z_+) = u^{-1}\mathcal{Q}(Z_+).$$

But the action of $\mathbb{F}_N$ on T is free, a contradiction. □

2.5. **Non-negative matrices.** We recall here basic facts that are folklore in ergodic theory. The statements here are suitable in our context. In the analog context of interval exchange transformations we refer to the course of J.-C. Yoccoz [Yoc05] (in particular Corollary III.5 and the proof of Proposition IV.10).

For integers $d \geq 1$, $1 \leq i \leq d$ and $0 \leq j \leq d$ we consider the matrices $A_d^i \in M_{d\times(d+1)}(\mathbb{Z}_{\geq 0})$ and $B_d^{i,j} \in M_{(d+1)\times d}(\mathbb{Z}_{\geq 0})$:

$$A_d^i = \left(\begin{array}{c|c} I_d & C_d^i \end{array}\right) \text{ et } B_d^{i,j} = \left(\begin{array}{c|c} I_j & 0 \\ \hline L_d^i & \\ \hline 0 & I_{d-j} \end{array}\right)$$

where C_d^i is the column vector of height d with 0 coefficients except the i-th coefficient which is 1 and $L_d^i = {}^t C_d^i$ is the line vector of length d with 0 coefficients except the i-th coefficient which is 1.

Let (d_n) be a sequence of positive integers and (M_n) be a sequence of $d_n \times d_{n+1}$ integer non-negative matrices of the form $A_{d_n}^{i_n}$ or $B_{d_n}^{i_n,j_n}$. For such a sequence we denote by

$$D = \liminf d_n.$$

The **positive cone** of the sequence $(M_n)_{n\in\mathbb{N}}$ is the set

$$C = \{(v_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N},\ v_n \in \mathbb{R}_{\geq 0}^{d_n},\ \text{and } v_n = M_n v_{n+1}\}.$$

Lemma 2.3. *The positive cone C has projective dimension at most $D - 1 = \liminf d_n - 1$.*

Proof. By linearity, the dimension of C is bounded above by $\liminf d_n$ and thus the projective dimension is bounded above by $\liminf d_n - 1$. □

We now state a Lemma that will be used to get a unique ergodicity criterion in Section 6.

Lemma 2.4. *Assume that there exists $L \geq 1$ and infinitely many n such that*

1. *$d_n = d_{n+L} = D$,*
2. *the square matrix $M_{[n,n+L-1]} = M_n \cdots M_{n+L-1}$ has strictly positive entries.*

Then, the projective positive cone defined by the sequence (M_n) contains exactly one point.

Proof. The matrices $M_{[n,n+L-1]}$ are non-negative integer matrices. The entries of $M_{[n,n+L-1]}$ are bounded by some constant depending only on L. The matrices $M_{[n,n+L-1]}$ uniformly contract the Hilbert distance in the positive cone. Thus C has zero diameter. □

3. Levitt case

3.1. **Rips induction.** Let $S = (F, A)$ be a system of isometries with graph Γ. Let F' be the set of elements of F which belongs to the domains of at least two partial isometries in $A^{\pm 1}$. The set F' is the union of the intersections of the domains of all possible pairs of distinct elements in $A^{\pm 1}$:

$$F' = \{P \mid \exists a \neq b \in A^{\pm 1},\ P \in \mathrm{dom}(a) \cap \mathrm{dom}(b)\} = \cup_{a \neq b \in A^{\pm 1}} \mathrm{dom}(a) \cap \mathrm{dom}(b).$$

Thus F' is also a compact forest. Let A' be the set of all possible non-empty restrictions of elements of A to pairs of connected components of F'. The system of isometries $S' = (F', A')$ is obtained from S by **Rips induction**.

Let Γ' be the graph of S' and let $\tau : \Gamma' \to \Gamma$ map a vertex K' of Γ' (which is a connected component of F') to the connected component $\tau(K') = K$ of F that contains K'. Similarly, τ maps the edge $a' \in A'^{\pm 1}$ to the edge $a \in A^{\pm 1}$ of which a' is a restriction.

The system of isometries $S = (F, A)$ is **reduced** [CH14] if

(1) the graph Γ is connected,
(2) it has independent generators,
(3) for each $P \in F$ there exists at least one infinite reduced admissible path X such that $\mathcal{Q}(X) = P$ and,
(4) for each partial isometry $a \in A^{\pm 1}$, and each extremal point P in $\mathrm{dom}(a)$, P is in F'.

We remark that if the system of isometries S is reduced then the graph Γ does not have vertices of valence 1.

We summarize our previous work in the following Proposition:

Proposition 3.1 ([CH14, Propostions 3.12, 3.13 and 5.6])**.** *Let S be a reduced system of isometries and S' be the system of isometries obtained by Rips induction. Then S' is reduced and the map $\tau : \Gamma' \to \Gamma$ is a homotopy equivalence. Moreover, the map τ induces a one-to-one correspondence between bi-infinite admissible paths in S and S': For any bi-infinite admissible path $Z' \in L(S')$, the bi-infinite path $\tau(Z')$ is reduced and admissible and, for any bi-infinite admissible path Z in Γ there exists a unique bi-infinite admissible path $Z' \in L(S')$ such that $\tau(Z') = Z$. By abuse of notations we write*

$$L(S) = L(S').$$

□

We now proceed to analyze cylinders of the lamination.

Proposition 3.2. *Let $S = (F, A)$ be a reduced system of isometries with graph Γ. Let S' be obtained by Rips induction from S. Let Γ' be the graph of S' and $\tau : \Gamma' \to \Gamma$ be the graph map.*

Then, for each edge e of Γ

$$C(\Gamma, e) = \biguplus_{e'} C(\Gamma', e').$$

where the disjoint union is taken over all edges e' of Γ' such that $\tau(e') = e$. More generally, for any finite admissible path w in Γ

$$C(\Gamma, w) = \biguplus_{w'} C(\Gamma', w').$$

where the disjoint union is taken over all finite reduced paths w' of Γ' such that $\tau(w') = w$.

Proof. First recall that the equalities in the Proposition are understood through the identification of $L(S)$ and $L(S')$ via the map τ.

From Proposition 3.1, for each bi-infinite admissible path $Z \in C(\Gamma, w)$ there exists a unique bi-infinite admissible path $Z' \in L(S')$ such that $\tau(Z') = Z$. Let w' be the prefix of Z' of length $|w|$, then $\tau(w') = w$. This proves that Z' is in $C(\Gamma', w')$ and that w' is unique. Conversely for any path w' of Γ' such that $\tau(w') = w$ and any bi-infinite reduced path $Z' \in C(\Gamma', w')$, $\tau(Z')$ is a bi-infinite reduced admissible path in $C(\Gamma, w)$. □

3.2. **Analysis of the lamination.** Let T be an $\mathbb{R}$-tree with a free, minimal, action of the free group $\mathbb{F}_N$ with dense orbits. Let A be a basis for $\mathbb{F}_N$, and let $S_0 = S_A = (K_A, A)$ be the corresponding system of isometries. Recall that S_0 is reduced [CH14, Proposition 5.6]. We perform inductively Rips induction to get a sequence $S_n = (F_n, \Gamma_n)$ of reduced systems of isometries, together with maps $\tau_n : \Gamma_{n+1} \to \Gamma_n$. By Propositions 2.1 and 3.1, for each n we have that $L(T) = L(S_0) = L(S_n)$.

For a vertex v_n in Γ_n we denote by $C(\Gamma_n, v_n)$ the set of bi-infinite admissible paths that goes through v_n at index 0. From Proposition 3.1 we get that $\tau_{n-1}(C(\Gamma_n, v_n)) \subseteq C(\Gamma_{n-1}, \tau_{n-1}(v_n))$. Again by abuse of notations we simply write $C(\Gamma_n, v_n) \subseteq C(\Gamma_{n-1}, \tau_{n-1}(v_n))$. Recall also that v_n is a connected component of the forest F_n and that for each $Z \in C(\Gamma_n, v_n)$, $\mathcal{Q}(Z)$ is a point in v_n.

Let now $(v_n)_{n\in\mathbb{N}}$ be a sequence such that for each $n \in \mathbb{N}$, v_n is a vertex in Γ_n and $\tau_n(v_{n+1}) = v_n$. The connected components v_n of F_n are nested and we denote by $v_\infty \subset K_A$ their intersection. We proved [CH14] that the tree T is of Levitt type if and only if for each such sequence v_∞ consists of a single point. The cylinders $C(\Gamma_n, v_n)$ are also nested and their intersection is the set of bi-infinite admissible paths Z in Γ_0 such that $\mathcal{Q}(Z) \in v_\infty$. As the action on T is free, the map $\mathcal{Q}$ is finite-to-one [CH14, Corollary 5.4] and thus we proved

Proposition 3.3. *Let T be an $\mathbb{R}$-tree in ∂cv_N with dense orbits and of Levitt type. Let A be a basis of $\mathbb{F}_N$ and (S_n) be the sequence of systems of isometries obtained from $S_0 = S_A = (K_A, A)$ by Rips induction. Let (v_n) be sequence of vertices of the graphs Γ_n such that $\tau_n(v_{n+1}) = v_n$. Then, the nested intersection of the compact subtrees v_n of K_A is a singleton and the nested intersection of the cylinders $C(\Gamma_n, v_n)$ is finite.* □

3.3. **Rips induction and currents.** Let T be an $\mathbb{R}$-tree with a free action of $\mathbb{F}_N$ by isometries with dense orbits. Let A be a basis of $\mathbb{F}_N$ and $S_0 = S_A = (K_A, A)$ be the associated system of isometries. Let $(S_n)_{n\in\mathbb{N}}$ be the sequence of systems of isometries obtained from S_0 by Rips induction. By Proposition 3.1 for each $n \in \mathbb{N}$, we have $L(S_n) = L(S)$. Let Γ_n be the graph of S_n and for each edge e recall that $C(\Gamma_n, e)$ is the set of bi-infinite admissible paths in Γ_n that goes through e at index 0.

For a current μ supported on the dual lamination $L(T) = L(S_0) = L(S_n)$, for each $n \in \mathbb{N}$, for each edge e of Γ_n we consider the finite number $\mu(e) = \mu(C(\Gamma_n, e))$. If e and e' are two consecutive edges of Γ_n separated by a valence 2 vertex, any path through e goes through e', and thus $\mu(e) = \mu(e')$. A **generalized edge** $\widehat{e}$ of Γ_n is a maximal reduced path such that all inner vertices have valence exactly 2. Recall that the maps τ_n are homotopy equivalences, that Γ_0 is the rose with N petals and that the graphs Γ_n are connected and do not have vertices of valence 1. The graph Γ_n has at most $3N - 3$ generalized edges. We denote by $\mathrm{GE}(\Gamma_n)$ the set of generalized edges of Γ_n. We pick-up arbitrarily an edge e in each generalized edge $\widehat{e}$ in $\mathrm{GE}(\Gamma_n)$. As the system of isometries S_n is reduced, there is a bi-infinite admissible path Z going through each edge e of Γ_n and thus through each generalized edge $\widehat{e}$. By Proposition 3.1, $\tau_0 \circ \cdots \circ \tau_{n-1}(Z)$ is a bi-infinite reduced path in Γ_0 and in particular its finite subpath $\tau_0 \circ \cdots \circ \tau_{n-1}(\widehat{e})$ is reduced.

The **incidence matrix** M_n of τ_n is the non-negative matrix such that for each pair of generalized edges $\widehat{e} \in \mathrm{GE}(\Gamma_n)$ and $\widehat{e}' \in \mathrm{GE}(\Gamma_{n+1})$, the coefficient $M_n(\widehat{e}, \widehat{e}')$ is the number of occurrences of e in the reduced finite path $\tau_n(\widehat{e}')$. We remark that this matrix depends on the choice of the edge e in the generalized edge $\widehat{e}$.

To the current μ carried by $L(S)$, for each $n \in \mathbb{N}$, we associate the non-negative vector $\mu_n = (\mu(\widehat{e}))_{\widehat{e}\in\mathrm{GE}(\Gamma_n)}$.

Proposition 3.4. *Let T be an $\mathbb{R}$-tree with a free action of $\mathbb{F}_N$ by isometries with dense orbits. Let A be a basis of $\mathbb{F}_N$ and $S_0 = S_A = (K_A, A)$ be the associated system of isometries. Let $(S_n)_{n\in\mathbb{N}}$ be the sequence of systems of isometries obtained from S_0 by Rips induction. Let Γ_n be the graph of S_n and $GE(\Gamma_n)$ its set of generalized edge and, let M_n be the incidence matrix of $\tau_n : \Gamma_{n+1} \to \Gamma_n$.*

Then, for each current μ carried by $L(T)$

$$\mu_n = M_n \mu_{n+1}$$

where $\mu_n = (\mu(C(\Gamma_n, \widehat{e})))_{\widehat{e}\in GE(\Gamma_n)}$ is the non-negative vector associated to μ at step n.

Proof. From Proposition 3.2, we have

$$\mu(\widehat{e}) = \mu(e) = \sum_{e'} \mu(e') = \sum_{e'} \mu(\widehat{e}'),$$

where the sum is taken over all edges e' such that $\tau(e') = e$. Grouping together the generalized edges of Γ', proves the Proposition. □

We now use the fact that the tree is of Levitt type:

Proposition 3.5. *With the above notations, let $(v_n)_{n\in\mathbb{N}}$ be a sequence such that for each n, v_n is a vertex of Γ_n and $\tau_n(v_{n+1}) = v_n$. Then the sequence $(\mu(C(\Gamma_n, v_n)))_{n\in\mathbb{N}}$ is non-increasing and converges to 0.*

Proof. Recall from Section 3.2 that the cylinder $C(\Gamma_n, v_n)$ is the set of bi-infinite reduced admissible paths that goes through v_n at index 0. The cylinders $C(\Gamma_n, v_n)$ are nested and from Proposition 3.3 their intersection $C(v_\infty)$ is finite. By Proposition 2.2 the current μ has no atoms and $\mu(C(v_\infty))$ is null, which proves the Proposition. □

3.4. **The simplex of currents.** We are now ready to prove Theorem 1.1 in the case where the Rips induction completely decomposes the tree.

Theorem 3.6. *Let T be an $\mathbb{R}$-tree with a minimal free action of $\mathbb{F}_N$ by isometries with dense orbits. Assume that T is a tree of Levitt type.*

Then the simplex $\mathbb{P}Curr(T)$ of currents carried by the dual lamination $L(T)$ has dimension at most $3N-4$ (and projective dimension at most $3N-5$).

Proof. Let A be a basis of $\mathbb{F}_N$ and $S_0 = S_A = (K_A, A)$ be the associated system of isometries. We perform the Rips induction to get a sequence of systems of isometries S_n together with maps $\tau_n : \Gamma_{n+1} \to \Gamma_n$.

For each $\mu \in \mathbb{P}\mathrm{Curr}(T)$, we consider the sequence of vectors $(\mu_n)_{n\in\mathbb{N}}$, where $\mu_n = (\mu_n(\widehat{e}))_{\widehat{e}\in \mathrm{GE}(\Gamma_n)}$. We denote by $C = \{\ (\mu_n)_{n\in\mathbb{N}} \mid \mu \in \mathbb{P}\mathrm{Curr}(T)\ \}$ the positive cone spanned by these vectors. For each n, let M_n be the incidence matrix of the map τ_n. From Proposition 3.4, C is the cone associated to the sequence of matrices (M_n) as in Section 2.5.

The incidence matrices M_n are products of matrices of the form A_d^i and $B_d^{i,j}$ (see Section 2.5). Indeed, the matrix A_d^i is that of splitting the i-th edge of Γ_n into two edges and the matrix $B_d^{i,j}$ is that of replacing the reduced path made of the the i-th and j-th edges of Γ_{n+1} when they form a generalized edge. Edges splitted in the Rips induction are incident to branch vertices of Γ_n and the number of resulting splitted edges is at most the valence of that branch vertex. Moreover, these splittings creates as many vertices as the number of new edges. Thus, both the numbers of matrices A_d^i and B_d^j in each M_n is bounded by $2N-2$. Because T is of Levitt type, the Rips machine goes for ever and the number of edges of Γ_n goes to infinity.

From Lemma 2.3, the projective dimension of C is at most $D-2$ where D is the inferior limit of the number of generalized edges in Γ_n. As Γ_n is homotopic to the rose with N-petals, the number of generalized edges is bounded above by $3N-3$. The Rips induction applied to a graph Γ_n with $3N-3$ generalized edges eventually produces a graph with strictly less than $3N-3$ generalized edges. Thus we get that $D < 3N-3$ and the projective dimension of C is at most $3N-5$.

Recall from Section 2.4 that for a finite reduced word w in $A^{\pm 1}$ the cylinder $C_A(w) \subseteq \partial^2\mathbb{F}_N$ is the clopen set of bi-infinite reduced words in $A^{\pm 1}$ that reads w at index 0. The set of translates of all such cylinders is a sub-basis of open sets of the shift of bi-infinite reduced words in $A^{\pm 1}$. Thus a current is completely determined by the measures of these cylinders.

Recall that a a generalized edge $\widehat{e}$ in Γ_n is mapped by $\tau_0 \circ \cdots \tau_{n-1}$ to a finite reduced admissible path in Γ_0 and that Γ_0 is the rose with N petals labeled by A.

We denote by $\langle \widehat{e} | w \rangle$ the number of occurrences of the word w in the reduced word $\tau_0 \circ \cdots \tau_{n-1}(\widehat{e})$. We claim that for a current $\mu \in \mathbb{P}\mathrm{Curr}(T)$,

$$\mu(w) = \lim_{n\to+\infty} \sum_{\widehat{e}\in \mathrm{GE}(\Gamma_n)} \langle \widehat{e} | w \rangle \mu(\widehat{e}).$$

This formula proves that μ is completely determined by the image of μ in C and thus proves the Proposition. We now prove the above formula.

Using Proposition 3.1, we get

$$C(\Gamma_0, w) = \biguplus_{w'} C(\Gamma_n, w')$$

where the disjoint union is taken over all reduced paths w' in Γ_n with label w. Passing to the current we get

$$\mu(w) = \sum_{w'} \mu(C(\Gamma_n, w')).$$

If w' is a subpath of a generalized edge $\widehat{e}$ of Γ_n, then all bi-infinite reduced paths through w' goes through $\widehat{e}$ and thus $\mu(C(\Gamma_n, w')) = \mu(\widehat{e})$. Taking into account only these occurrences of the label w inside generalized edges, we get the inequality

$$\mu(w) \geq \sum_{\widehat{e} \in \mathrm{GE}(\Gamma_n)} \langle \widehat{e}|w\rangle \mu(\widehat{e}).$$

The difference between the two terms above is exactly the measure of the cylinders of the occurrences of the label w that are not inside a generalized edge. Those occurrences cross a point of valence at least 3. For such a vertex v in Γ_n of valence at least 3, we denote by $\langle v|w\rangle$ the number of reduced paths in Γ_n with label w that pass through v. We remark that $\langle v|w\rangle$ is bounded above by $(|w|-1)(2N-1)^{|w|}$, where $|w|-1$ is the number of vertices inside a path of length $|w|$ and $2N$ is the maximal valence of a vertex in Γ_n. (Note that this bound is very loose but sufficient for our proof). We also use the notation $\mu(v) = \mu(C(\Gamma_n, v))$. The difference of the two terms in the above inequality is bounded by

$$0 \leq \mu(w) - \sum_{\widehat{e} \in \mathrm{GE}(\Gamma_n)} \langle \widehat{e}|w\rangle \mu(e) \leq \sum_v \langle v, w\rangle \mu(v) \leq (|w|-1)(2N-1)^{|w|} \sum_v \mu(v)$$

where v ranges over all branch-points of Γ_n. Now Γ_n is homotopy equivalent to Γ_0 and does not have vertices of valence 1, thus the number of branch-points in Γ_n is bounded above by $2N-2$. We get

$$0 \leq \mu(w) - \sum_{\widehat{e} \in \mathrm{GE}(\Gamma_n)} \langle \widehat{e}|w\rangle \mu(e) \leq (|w|-1)(2N-1)^{|w|}(2N-2) \max_v \mu(v)$$

By Proposition 3.5, the last factor goes to 0 when n goes to infinity which concludes the proof. □

4. Splitting induction

There are trees in the boundary of Outer space where the Rips induction is useless: the trees of surface type [CH14]. For these trees (and in general) we define another kind of induction, which we call splitting induction. It is a little more involved to use it for analyzing the dual lamination as we are going to miss finitely many leaves. But this gives us the same results for currents when they have no atoms.

4.1. **Splitting.** This Section recalls the definitions and results of our previous work with P. Reynolds [CHR11, Sections 4.2 and 4.3].

Let $S = (F, A)$ be a reduced system of isometries. A point $P \in F$ is in the **interior** of a tree $K \subseteq F$ if P is in the interior of a segment contained in K. A point P is **extremal** in a tree K if it is not in the interior.

Let x be an interior point of a connected component K_x of F. Let $\pi_0(K_x \smallsetminus \{x\}) = L \biguplus R$ be a partition of the set of directions at x in two non-empty subsets. Then (x, L, R) is a **splitting partition** of the system of isometries $S = (F, A)$ if

(1) there exists exactly one partial isometry $a_0 \in A^{\pm 1}$ defined at x whose domain $\mathrm{dom}(a_0)$ meets both sets of directions L and R and,
(2) for each direction $d \in L \cup R$ at x, there is at least another partial isometry $a \in A^{\pm 1} \smallsetminus \{a_0\}$ defined at d: $x \in \mathrm{dom}(a)$ and $\mathrm{dom}(a) \cap d \neq \emptyset$.

For a splitting partition (x, L, R), let F' be the forest obtained by splitting K_x into two disjoint compact trees $\overline{L}$ and $\overline{R}$:

$$F' = (F \smallsetminus K_x) \biguplus (L \cup \{x\}) \biguplus (R \cup \{x\}).$$

We insist that there are two distinct copies of x in F', which we denote by x_L and x_R.

We also split the partial isometries in A. The partial isometry a_0 is replaced by its two restrictions: $a_0^L = {}_{L \cup \{x_L\}]}a_0$ and $a_0^R = {}_{R \cup \{x_R\}]}a_0$. Let $a \in A^{\pm 1}$ be a partial isometry distinct from a_0 and a_0^{-1}. If $\mathrm{dom}(a)$ is not reduced to a point, we let a' be the closure of the restriction (at the source and the target) of a to $F \smallsetminus \{x\}$. For instance, if x is in the domain of a then by definition of a splitting point, x is extremal in $\mathrm{dom}(a)$ and $\mathrm{dom}(a)$ meets either L or R, say L. In this case $\mathrm{dom}(a') = \{x_L\} \cup (\mathrm{dom}(a) \cap L)$. Finally, if

$\mathrm{dom}(a)$ is a singleton $\{y\}$ and $y.a = y'$ we define a' arbitrarily by letting

$$\begin{cases} y.a' = y' \text{ if } x \notin \{y, y'\} \\ x_L.a' = x_L \text{ if } y = y' = x \\ x_L.a' = y' \text{ if } y = x \text{ and } y' \neq x \\ y.a' = x_L \text{ if } y \neq x \text{ and } y' = x. \end{cases}$$

Considering the graphs Γ and Γ' of the systems of isometries S and S', as above there is a natural map $\tau : \Gamma' \to \Gamma$ which is one-to-one, except that it maps both vertices $\overline{L}$ and $\overline{R}$ of Γ' to the vertex K_x of Γ and, except that it maps both edges a_0^L and a_0^R of Γ' to the edge a_0 of Γ. Indeed the map τ is a folding.

The splitting induction affects the admissible paths that contain the subpaths $a_0^{R\,-1}a_0^L$ or $a_0^{L\,-1}a_0^R$. We thus restrict to a sublamination. A finite admissible path u in Γ is **regular** if $\mathrm{dom}(u)$ contains strictly more than one point. The **regular lamination** $L'(S)$ of a system of isometries S is the set of bi-infinite admissible paths Z in Γ such that all finite subpaths u of Z are regular. This regular lamination is also the derived set $L'(S)$ of non-isolated leaves in $L(S)$:

Proposition 4.1 ([CHR11])**.** *Let T be an $\mathbb{R}$-tree with a free minimal action of $\mathbb{F}_N$ by isometries with dense orbits. Let A be a basis of $\mathbb{F}_N$ and $S_A = (K_A, A)$ be the associated system of isometries. Recall that the dual lamination $L(T)$ of T is equal to the admissible lamination of S_A.*

Then, the regular lamination $L'(S_A)$ is the derived space of $L(S_A)$. □

We summarize our previous work with P. Reynolds in the following Proposition:

Proposition 4.2 ([CHR11, Lemma 4.9 and 4.10])**.** *Let S be a reduced system of isometries and let S' be a system of isometries obtained by splitting. Then S' is reduced and the map $\tau : \Gamma' \to \Gamma$ is a homotopy equivalence.*

For any bi-infinite reduced regular path $Z' \in L(S')$ in Γ', the bi-infinite path $\tau(Z')$ is reduced, admissible and regular. For any bi-infinite reduced regular path $Z \in L(S)$ in Γ, there exists a unique bi-infinite reduced regular path $Z' \in L(S')$ such that $\tau(Z') = Z$.

Through the map τ, we identify the regular laminations:

$$L'(S) = L'(S').$$

We now write the effect of splitting on the cylinders of the lamination of S. As in Section 3, for a finite reduced path u of Γ (u can be a vertex or an edge), the cylinder $C(\Gamma, u)$ is the set of bi-infinite admissible paths that goes through u at index 0. But here we rather consider the regular cylinder $C'(\Gamma, u)$: the set of bi-infinite reduced admissible regular paths that goes through u at index 0. Exactly as in Proposition 3.2 we have:

Proposition 4.3. *Let $S = (F, A)$ be a reduced system of isometries with graph Γ. Let S' be obtained by splitting induction from S. Let Γ' be the graph of S' and $\tau : \Gamma' \to \Gamma$ be the graph map. For each edge e of Γ:*

$$C'(\Gamma, e) = \biguplus_{e'} C'(\Gamma', e')$$

where the disjoint union is taken over all edges e' of Γ' such that $\tau(e') = e$. More generally, for every finite regular path u in Γ

$$C'(\Gamma, u) = \biguplus_{u'} C'(\Gamma', u')$$

where the disjoint union is taken over all finite reduced paths u' of Γ' such that $\tau(u') = u$. □

4.2. **Existence of splittings.** To use the splitting induction to analyze laminations and currents we first need to prove that we can perform it: there exists splitting partitions. From our previous work with P. Reynolds [CHR11] we know that this is the case, at least when we cannot perform Rips induction. A system of isometries $S = (F, A)$ is of **surface type** if any point x of F belongs to the domains of at least two partial isometries $a \neq b \in A^{\pm 1}$ (equivalently Rips induction does nothing to S).

Proposition 4.4 ([CHR11, Proposition 4.7])**.** *Let $S = (F, A)$ be a reduced system of isometries of surface type, then there exists a splitting partition for S.* □

5. Unfolding

5.1. **Unfolding induction and currents.** Let T be an $\mathbb{R}$-tree with a free minimal action of $\mathbb{F}_N$ by isometries with dense orbits. Let $L(T)$ be the dual lamination of T. By Proposition 2.2, no leaf of $L(T)$ is periodic and, any current $\mu \in \mathbb{P}\mathrm{Curr}(T)$ carried by $L(T)$ has no atoms. By definition of the derived space we get:

Proposition 5.1. *Let T be an $\mathbb{R}$-tree with a free minimal action of $\mathbb{F}_N$ by isometries with dense orbits. Then every current $\mu \in \mathbb{P}Curr(T)$ carried by the dual lamination $L(T)$ is carried by the regular lamination $L'(T)$.* □

Let A be a basis of $\mathbb{F}_N$ and $S_A = (K_A, A)$ the associated system of isometries. Let $S_n = (F_n, A_n)$ be a sequence of systems of isometries obtained from $S_0 = S_A$ by **unfolding induction**: at each step n either Rips or splitting induction is performed. Let $\tau_n : \Gamma_{n+1} \to \Gamma_n$ be the map between the graphs. As S_0 is reduced, τ_n is a homotopy equivalence, Γ_n is connected and does not have vertices of valence 1. The graph Γ_n has at most $3N-3$ generalized edges in $\mathrm{GE}(\Gamma_n)$.

For each generalized edge $\widehat{e}' \in \mathrm{GE}(\Gamma_{n+1})$ and each edge e in Γ_n we consider $\langle \widehat{e}', e\rangle$ the number of occurrences of e (without taking into account orientation) in the finite path $\tau(\widehat{e})$. The **incidence matrix** M_n of the map τ_n has entry $\langle \widehat{e}', e\rangle$ for each pair $(\widehat{e}, \widehat{e}') \in \mathrm{GE}(\Gamma_n) \times \mathrm{GE}(\Gamma_{n+1})$. Again, we remark that the incidence matrix depends on the choice of the edge e in the generalized edge $\widehat{e}$.

For a generalized edge $\widehat{e}$ of Γ_n we consider the non-negative number $\mu(C'(\Gamma_n, \widehat{e}))$, where $C'(\Gamma_n, \widehat{e})$ is the cylinder of regular bi-infinite admissible paths in Γ_n that passes through $\widehat{e}$ at index 0. We consider the non-negative vector $\mu_n = (\mu(C'(\Gamma_n, \widehat{e})))_{\widehat{e} \in \mathrm{GE}(\Gamma_n)}$.

Proposition 5.2. *Let T be an $\mathbb{R}$-tree with a free action of $\mathbb{F}_N$ by isometries with dense orbits. Let A be a basis of $\mathbb{F}_N$ and $S_0 = S_A = (K_A, A)$ be the associated system of isometries. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of systems of isometries obtained from S_0 by unfolding induction. Let Γ_n be the graph of S_n and $GE(\Gamma_n)$ its set of generalized edge and, let M_n be the incidence matrix of $\tau_n : \Gamma_{n+1} \to \Gamma_n$.*

Then, for each current μ carried by $L(T)$

$$\mu_n = M_n \mu_{n+1}$$

where $\mu_n = (\mu(C'(\Gamma_n, \widehat{e})))_{\widehat{e} \in GE(\Gamma_n)}$ is the non-negative vector associated to μ at step n.

Proof. The proof is the same as the proof of Proposition 3.4, using Proposition 4.3 if we use splitting induction at step n. □

We can now prove Theorem 1.1 when the induction completely analyzes the lamination.

Theorem 5.3. *Let T be an $\mathbb{R}$-tree with a minimal, free, action of $\mathbb{F}_N$ by isometries with dense orbits. Let A be a basis of $\mathbb{F}_N$ and let $S_A = (K_A, A)$ be the associated system of isometries. Let $S_n = (F_n, A_n)$ be a sequence of systems of isometries obtained from $S_0 = S_A$ by unfolding induction.*

Assume that each nested intersection of connected components v_n of F_n is a singleton.

Then, the simplex of currents $\mathbb{P}Curr(T)$ carried by the dual lamination $L(T)$ has dimension at most $3N-4$ (and projective dimension at most $3N-5$).

Proof. Again the proof is the same as the proof of Theorem 3.6, using the regular lamination instead of the lamination and Proposition 5.2. □

5.2. **Decomposable case.** The hypothesis of Theorem 5.3 are not always satisfied: there exist $\mathbb{R}$-trees with no induction sequence that completely analyzes the lamination. Those trees are decomposable in the sense of V. Guirardel [Gui08]. In this case however, the induction procedure ends up with a subtree with an action of a subgroup of $\mathbb{F}_N$ with rank strictly smaller than N. And this allows us to conclude the proof of Theorem 1.1 by induction on N.

Let T be an $\mathbb{R}$-tree with a minimal free action of $\mathbb{F}_N$ with dense orbits. Let A be a basis of $\mathbb{F}_N$ and let $S_A = (K_A, A)$ be the associated system of isometries. Let $S_n = (F_n, A_n)$ be an infinite sequence of systems of isometries obtained from $S_0 = S_A$ by unfolding induction. By Proposition 4.4, this is always possible. Let $\tau_n : \Gamma_{n+1} \to \Gamma_n$ be the map between the graphs. As S_n is reduced, Γ_n is connected, does not have vertices of valence 1 and τ_n is a homotopy equivalence. The number of edges of Γ_n is strictly increasing with n. The inverse limit of (Γ_n, τ_n) is an infinite graph $\widehat{\Gamma}$. Vertices of $\widehat{\Gamma}$ are sequences $(v_n)_{n\in\mathbb{N}}$ such that v_n is a vertex

of Γ_n and $\tau_n(v_{n+1}) = v_n$, thus $(v_n)_{n\in\mathbb{N}}$ is a sequence of nested compact subtrees of K_A. Edges of $\widehat{\Gamma}$ are sequences $(a_n)_{n\in\mathbb{N}}$ such that a_n is an edge of Γ_n and $\tau_n(a_{n+1}) = a_n$, thus a_{n+1} is a restriction of the partial isometry a_n.

By our previous work [CH14], the index of $\widehat{\Gamma}$ is finite and thus it contains a finite **core graph** Γ_∞: the union of all reduced loops in $\widehat{\Gamma}$. We remark that Γ_∞ can be empty and that it can fail to be connected.

First case: Γ_∞ is empty. Equivalently, each connected component of $\widehat{\Gamma}$ is a tree. Moreover, as the index of $\widehat{\Gamma}$ is finite, each connected component of $\widehat{\Gamma}$ has a finite number of ends. Let (v_n) be a vertex of $\widehat{\Gamma}$, this is a sequence of vertices of Γ_n which are nested connected components of the forest F_n. As before, we denote by v_∞ the nested intersection of (v_n). There are finitely many infinite reduced paths in $\widehat{\Gamma}$ starting from v_∞. For each $x \in v_\infty$ there exists an infinite path in $\widehat{\Gamma}$ starting at (v_n) which reads an admissible reduced regular word X such that $\mathcal{Q}(X) = x$. We get that v_∞ is finite and thus a singleton. Theorem 5.3 applies in this case.

Second case: Γ_∞ is non-empty. As Γ_∞ is finite, there exists $s \in \mathbb{N}$ such that for all $n \geq s$, Γ_∞ is a subgraph of Γ_n and the map τ_n restricts to the identity on Γ_∞. We denote by $\Gamma^1_\infty, \dots, \Gamma^r_\infty$ the connected components of Γ_∞ in this case $r \geq 1$. For each $i = 1, \dots, r$, let F^i_∞ be the compact forest whose connected components are the nested intersections of the vertices of Γ^i_∞, and let A^i_∞ be the partial isometries of F^i_∞ which are the nested intersections of edges of Γ^i_∞.

Lemma 5.4. *The system of isometries $S^i_\infty = (F^i_\infty, A^i_\infty)$ has graph Γ^i_∞ and, is reduced.*

Let N_i be the rank of the free group $\pi_1(\Gamma^i_\infty)$. Let T_i be the $\mathbb{R}$-tree associated to S^i_∞. The action of F_{N_i} on T_i is free, minimal by isometries and with dense orbits. □

For any finite admissible non-singular subpath u in Γ^i_∞, we consider the regular cylinder $C'(\Gamma^i_\infty, u)$ of bi-infinite admissible regular paths that passes through u at index 0. The translates of such cylinders form a basis of open sets of the regular dual lamination $L'(T_i)$. We insist that we consider $L'(T_i)$ as a lamination with respect to $F_{N_i} = \pi_1(\Gamma^i_\infty)$ (and not with respect to the original $\mathbb{F}_N$). Any current $\mu \in \mathbb{P}\mathrm{Curr}(T)$ carried by the dual lamination $L(T)$ induces a current $\mu^i \in \mathbb{P}\mathrm{Curr}(T_i)$ by letting

$$\mu^i(C(\Gamma^i_\infty, u)) = \mu(C(\Gamma^i_\infty, u)).$$

For each $n > s$, as the homotopy equivalence τ_n fixes the subgraph Γ_∞ the incidence matrix M_n is reducible and we denote by M^{GE}_n its submatrix corresponding to generalized edges in $\mathrm{GE}(\Gamma_n \smallsetminus \Gamma_\infty)$. Let C be the positive cone of non-negative vectors $((\mu_n(\widehat{e}))_{\widehat{e}\in \mathrm{GE}(\Gamma_n\smallsetminus\Gamma_\infty)})_{n\geq s}$.

Lemma 5.5. *The map*

$$\begin{array}{rcl} \mathbb{P}Curr(T) & \to & \mathbb{P}Curr(T_1) \times \cdots \times \mathbb{P}Curr(T_r) \times C \\ \mu & \mapsto & (\mu^1, \dots, \mu^r, ((\mu_n(\widehat{e}))_{\widehat{e}\in GE(\Gamma_n\smallsetminus\Gamma_\infty)})_{n\geq s}) \end{array}$$

is injective.

Proof. Indeed for any finite reduced word w in $A^{\pm 1}$, we claim that

$$\mu(w) = \mu(C(\Gamma_0, w)) = \sum_{i=1}^{i=r} \sum_{w' \text{ occurence of } w \text{ in } \Gamma^i_\infty} \mu^i(w') + \lim_{n\to\infty} \sum_{\widehat{e}\in \mathrm{GE}(\Gamma_n\smallsetminus\Gamma_\infty)} \langle w, \widehat{e}\rangle \mu(\widehat{e}).$$

As in the proof of Theorem 3.6, Propositions 3.1 and 4.2 prove that for each n:

$$\mu(w) \geq \sum_{i=1}^{i=r} \sum_{w' \text{ occurence of } w \text{ in } \Gamma^i_\infty} \mu^i(w') + \sum_{\widehat{e}\in \mathrm{GE}(\Gamma_n\smallsetminus\Gamma_\infty)} \langle w, \widehat{e}\rangle \mu(\widehat{e}).$$

and that the difference is given by

$$\Delta_n = \sum_{w'} \mu(C'(\Gamma_n, w'))$$

where the sum is taken over all occurrences of w as a label of a path w' in Γ_n that contains both vertices of Γ_∞ and edges of $\Gamma_n \smallsetminus \Gamma_\infty$. By definition of Γ_∞, all vertices of Γ_∞ have valence at least 2 and thus those occurrences of w passes over one of the branch points of Γ_n. Therefore the number of such occurrences is uniformly bounded (by a constant which depends only on N). Hence, there are finitely many occurrences of

w in the inverse limit $\widehat{\Gamma}$ which contains both vertices of Γ_∞ and edges out of Γ_∞. The vertices out of the core graph of $\widehat{\Gamma}$ corresponds to singletons in K_A and as the action of $\mathbb{F}_N$ on T is free there are only finitely many bi-infinite admissible regular paths Z in $\widehat{\Gamma}$ that passes through these occurrences of w at index 0.

We thus get that $\lim_{n\to\infty} \Delta_n = 0$ □

By Proposition 5.2 and by definition of the cone C, for all $(v_n)_{n\geq s} \in C$ and for $n \geq s$, we have $v_n = M_n^{\mathrm{GE}} v_{n+1}$. Thus, the cone C has finite dimension at most the number of generalized edges in $\Gamma_n \smallsetminus \Gamma_\infty$ minus one. By induction on the rank of the free group, $\mathbb{P}\mathrm{Curr}(T_i)$ has dimension at most $3N_i - 5$. Let Γ_n^{GE} be the graph obtained from Γ_n by replacing generalized edges by edges (so that Γ_n^{GE} has no vertex of valence 2). Contracting the image of each component Γ_∞^i in Γ_n^{GE} to a vertex, we get a connected graph G_n. By construction G_n has no vertex of valence 1, at most r vertices of valence 2 and $|\mathrm{GE}(\Gamma_n \smallsetminus \Gamma_\infty)|$ edges. Moreover G_n has Euler characteristic $1 - N + N_1 + \cdots + N_r$. Using that the number of edges of a connected graph without vertex of valence 1 is bounded above by three times the Euler characteristic plus the number of vertices of valence 2, we get

$$|\mathrm{GE}(\Gamma_n \smallsetminus \Gamma_\infty)| \leq 3(N - N_1 - \cdots - N_r) - 3 + r$$

and thus

$$(3N_1 - 4) + \cdots + (3N_r - 4) + |\mathrm{GE}(\Gamma_n \smallsetminus \Gamma_\infty)| - 1 \leq 3N - 4 - 3r$$

which concludes the proof of Theorem 1.1 since $r \geq 1$.

6. Unique ergodicity

Recall that an $\mathbb{R}$-tree in $\partial \mathrm{CV}_N$ is uniquely ergodic if it is dual to a unique projective current. There are known examples of uniquely ergodic trees: As the transition matrix of a train-track of an iwip outer automorphisms is a primitive integer matrix (up to passing to a power), Perron-Frobenius Theorem implies that the attracting tree in $\partial \mathrm{CV}_N$ of a non-geometric iwip outer automorphisms is uniquely ergodic – see for instance [CHL08c].

In this Section we give a more general concrete criterion of unique ergodicity.

Let T be an $\mathbb{R}$-tree in $\partial \mathrm{CV}_N$ with dense orbits. Let A be a basis of $\mathbb{F}_N$, and (S_n) a sequence of systems of isometries obtained from $S_0 = S_A = (K_A, A)$ by unfolding induction. As in Section 3.2 and 5.1, let (Γ_n) be the sequence of associated graphs and $\tau_n : \Gamma_{n+1} \to \Gamma_n$ the homotopy equivalences. We denote by D the inferior limit of the number d_n of generalized edges of Γ_n:

$$D = \liminf d_n.$$

We say that **all edges are fully splitted** between steps n and $n + L$ if the image of each generalized edge $\widehat{e}$ of Γ_{n+L} in Γ_n is an edge-path that covers Γ_n. This is equivalent to saying that the incidence matrix (for generalized edges) of $\tau_n \circ \cdots \circ \tau_{n+L-1} : \Gamma_{n+L} \to \Gamma_n$ has strictly positive entries.

Theorem 6.1. *Assume that there exists L and infinitely many n such that the number of generalized edges of Γ_n is D and between steps n and $n + L$ all edges are fully splitted.*

Then, T is uniquely ergodic.

Proof. Indeed with the hypothesis above the matrices $M_n \cdots M_{n+L-1}$ are square matrices with strictly positive and bounded coefficients. Thus we can use Lemma 2.4. □

7. Approximation of the dual lamination

7.1. **Indecomposable trees.** We saw in the proof of Theorem 1.1 that our main tool is to approximate the dual lamination by the sequence of unfoldings $(\Gamma_n, \tau_n)_{n\in\mathbb{N}}$ obtained by the unfolding induction. In this section, we focus on this approximation of $L(T)$ by $(\Gamma_n, \tau_n)_{n\in\mathbb{N}}$.

Let T be an indecomposable tree. In our previous work [CH14] we proved that there are two cases: Levitt type or surface type. According to this dichotomy we describe the unfolding induction used in the present paper.

- If T is of Levitt type the unfolding induction consists only in Rips induction as defined in [CH14].
- If T is of surface type the unfolding induction consists only in splitting induction introduced in our paper with P. Reynolds [CHR11].

In both cases the induction produces a projective system of graphs $(\Gamma_n)_{n\in\mathbb{N}}$. If the action on the tree T is free, the inverse limit is a lamination. If T is of Levitt type then the inverse limit is precisely the dual lamination of T. If T is of surface type then it misses some diagonal leaves: the inverse limit is a sublamination of the dual lamination of T and it contains the regular sublamination of T.

Proposition 7.1 ([CH14, CHR11]). *Let $T \in \partial cv_N$ be a free indecomposable tree. Let $(\Gamma_n)_{n\in\mathbb{N}}$ be the sequence of graphs obtained by unfolding induction. Then,*

- *either T is of Levitt type and $\varprojlim \Gamma_n = L(T)$,*
- *or T is of surface type and $L'(T) \subseteq \varprojlim \Gamma_n \subseteq L(T)$.*

We describe now an elementary way to understand this inverse limit of graphs. As explained in [CHL08a], a lamination $L \in \partial^2\mathbb{F}_N$ can be alternatively thought as a symbolic lamination, i.e. a subshift of the shift of bi-infinite reduced words in $A^{\pm 1}$, where A denotes a basis of $\mathbb{F}_N$.

Let A be the basis of $\mathbb{F}_N$ that labels the edges of the rose Γ_0. A bi-infinite line in the graph Γ_n reads off a (not necessarily reduced) bi-infinite word in $A^{\pm 1}$. The set of reduced bi-infinite words obtained in that way is the **admissible lamination** L_n of Γ_n. The laminations L_n are nested and their intersection is the inverse limit of $(\Gamma_n)_{n\in\mathbb{N}}$:

$$\varprojlim \Gamma_n = \bigcap_{n\in\mathbb{N}} L_n.$$

Proposition 7.1 only deals with free indecomposable trees. Another class of interesting trees are the arational trees. They are used by M. Bestvina and P. Reynolds [BR12] to describe the boundary of the complex of free factors. As proved by P. Reynolds [Rey12] a tree is arational if and only if it is indecomposable and either free or transverse to a minimal filling measured geodesic lamination on a once-punctured surface.

We already addressed the case of indecomposable free trees. Let T be an arational tree which is transverse to a minimal filling measured geodesic lamination on a once-punctured surface. Then the action of $\mathbb{F}_N$ on T is not free: the simple closed curved γ around the puncture fixes a point in T. Running the unfolding induction on T (in this case a splitting induction), the loop γ will not be splitted open and will remain in the inverse limit of the graphs $(\Gamma_n)_{n\in\mathbb{N}}$. However the nested intersection of the admissible laminations L_n is again in between the regular lamination and the dual lamination of T.

Proposition 7.2. *Let T be an arational tree. Let $(\Gamma_n)_{n\in\mathbb{N}}$ be the sequence of graphs given by unfolding induction. Let L_n be the admissible lamination of Γ_n. Then*

$$L'(T) \subseteq \bigcap_{n\in\mathbb{N}} L_n \subseteq L(T).$$

7.2. **An example of a decomposable tree.** In Section 5.2 we warn that, for a decomposable tree, the unfolding induction does not always converge to the dual lamination. Here is a concrete example.

We consider an $\mathbb{R}$-tree T with a free action of F_5. This tree is given by its compact heart described in Figure 1: T is transverse to (the lift to the universal cover of) the foliation of the suspension of the system of isometries.

This system of isometries was designed so that:

- the first step of the unfolding induction has to be Rips induction (on a subsegment of EJ)
- the lengths and the point G are chosen such that inductively each step of the unfolding induction has to be Rips induction (on a subsegment of AB, CH, ID or EF)
- at the end of this infinite Rips induction process, we are left with
  - Cantor sets lying in the segments AB, EF, CH and DI
  - a surface system of isometries on the segment GJ (which is an interval exchange transformation of two intervals)
- the unfolding induction then proceed with splitting inductions.

The key point of our example is that the point G is a regular point of the limit set. We checked that this is the case when for instance α is any irrational number in $(0;1)$, $\beta = 0.311\ldots$ is the positive root of $\beta^3 - \beta^2 - 3\beta + 1 = 0$, and $\gamma = \frac{1}{2} + \beta + \frac{1}{2}\beta^2 = 0.859\ldots$. The system of isometries made of a, b and c is inspired by the interval translation mapping of M. Boshernitzand and I. Kornfeld [BK95].

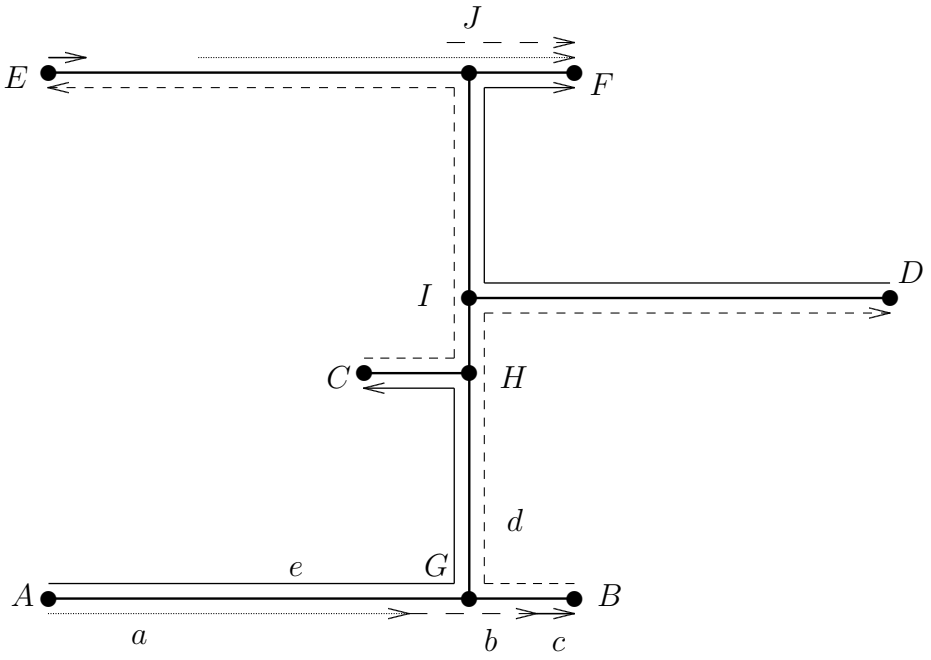

FIGURE 1. The compact heart of the tree T with the partial isometries. In bold the compact subtree has six extremal points $A, \dots, F$ and four points of valence 3: G, H, I, J. The five partial isometries $a, b, \dots, e$ are defined on intervals that are represented by arrows (orientation indicates the isometries between domains and ranges, the label of each partial isometry is located next to its domain). Length are given by

$$AB = EF = GJ = 1, \quad AG = DI = EJ = \gamma, \quad BG = CH = FJ = 1 - \gamma,$$
$$GI = HJ = \alpha, \quad GH = IJ = 1 - \alpha$$

The widths of the partial isometries are

$$\text{width}(a) = 1 - \beta, \quad \text{width}(b) = \beta - \beta^2, \quad \text{width}(c) = \beta^2$$

Note that in the same spirit, we could have built an example of a decomposable tree for which the unfolding induction consists only of splitting inductions but the process is twice infinite, see Figure 2. Such a tree is surface in our terminology [CH14].

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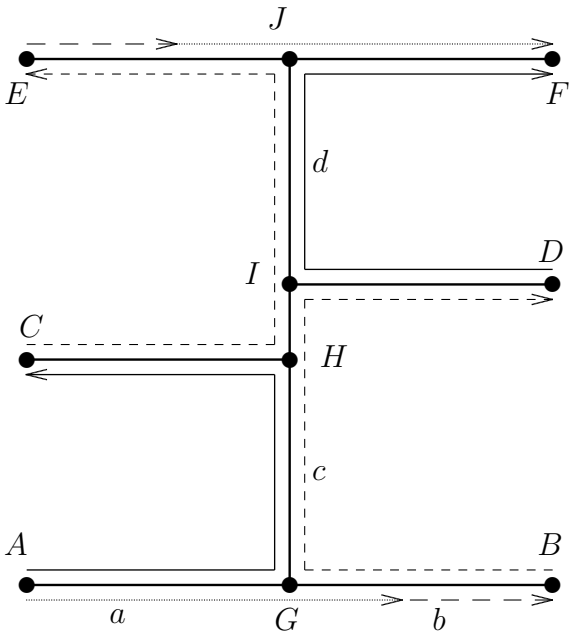

FIGURE 2. A compact heart of a tree dual to a foliation on a surface, for which the splitting induction does not exhaust the foliation.

Length are given by

$$AB = GJ = EF = 1, \quad AG = BG = CH = DI = EJ = FJ = \tfrac{1}{2}$$
$$GI = \alpha, \quad IJ = 1 - \alpha$$

The widths β of the partial isometry a and the width α of c are any irrational numbers in $(0; 1)$.

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