Weighted Sobolev Inequalities in CD(0,N) spaces

David Tewodrose

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Abstract

In this note, we prove global weighted Sobolev inequalities on non-compact $\mathrm{CD}(0,N)$ spaces satisfying a suitable growth condition, extending to possibly non-smooth structures a previous result of Minerbe, stated on Riemannian manifolds with non-negative Ricci curvature and an adequate reverse doubling condition. We use then this result in the context of Ahlfors regular $\mathrm{RCD}(0,N)$ spaces to get a uniform bound of the corresponding weighted heat kernel via a weighted Nash inequality.

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Contents

1 Introduction

Riemannian manifolds with non-negative Ricci curvature enjoy strong analytic properties, like local Sobolev inequalities and parabolic Harnack inequalities (see [SC02] for a nice account on this topic) or estimates for heat kernels and Green functions [LY86]. It has been well-known for quite a long time that to establish this properties, the non-negativity of the Ricci curvature could be replaced by two of its consequences, namely the doubling (2.2) and Poincaré (2.3) properties. This observation allows to forget about the smooth structure of the space under consideration. Following this path, Sturm provided Gaussian estimates for the fundamental solution of parabolic operators [St95], and parabolic Harnack inequalities [St96], in the setting of PI doubling spaces endowed with a local, regular and strongly regular Dirichlet form. Here and in the whole article by PI doubling space we mean a metric measure space with doubling and Poincaré properties. Afterwards, general doubling spaces with Poincaré type inequalities were studied at length by Hajlasz and Koskela [HK00]: in this context, they proved local Sobolev type inequalities, a Trudinger inequality, a Rellich-Kondrachov theorem, and discussed many related results.

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Contrary to Riemannian manifolds with non-negative Ricci curvature, PI doubling spaces are stable with respect to measured Gromov-Hausdorff convergence. But the doubling and Poincaré properties do not retain enough curvature features to be regarded as an appropriate extension of non-negativity of the Ricci curvature to non-smooth spaces. For instance, the Heisenberg group $\mathbb{H}^n$ is a PI doubling space which can be approximated with Riemannian manifolds, however it follows from [Ju09] that any such approximating sequence cannot satisfy a uniform bound of the Ricci curvature.

Approximately a decade ago, Sturm [St06] and Lott and Villani [LV09] independently proposed the curvature-dimension condition $\mathrm{CD}(0, N)$, for $N \in [1, +\infty)$, as an extension of non-negativity of the Ricci curvature and bound above by N of the dimension for possibly non-smooth metric measure spaces. Together with the infinitesimal Hilbertiannity property introduced later on by Ambrosio, Gigli and Savaré [AGS14b], the $\mathrm{CD}(0, N)$ condition becomes $\mathrm{RCD}(0, N)$ condition. The class of $\mathrm{RCD}(0, N)$ spaces has been extensively studied over the past years, and it is by now well-known that it contains the measured Gromov-Hausdorff closure of Riemannian manifolds with non-negative Ricci curvature and dimension lower than N, as well as $\mathrm{CAT}(0)$ spaces (also called Alexandrov spaces) with n-dimensional Hausdorff measure, n being lower that N. Moreover, $\mathrm{RCD}(0, N)$ spaces are PI doubling spaces endowed with a suitable Dirichlet form: the works of Sturm [St95, St96] imply the parabolic Harnack inequality and Gaussian estimates for the heat kernel. In the broader context of $\mathrm{RCD}(K, N)$ spaces, $K \in \mathbb{R}$ standing for a bound by below of the Ricci curvature in the sense of Lott-Sturm-Villani, the Li-Yau estimates with sharp constants have been proved recently by Jiang, Li and Zhang [JLZ16], building on the Harnack inequalities previously established by Garofalo and Mondino [GM14] (see also [Ji15]). Note that most previous works on RCD spaces made use of the $\mathrm{RCD}^*(K, N)$ condition, which has been a posteriori proved equivalent to the $\mathrm{RCD}(K, N)$ condition.

The aim of this note is to provide another analytic result, namely a global weighted Sobolev inequality, for a particular class of non-compact $\mathrm{CD}(0, N)$ spaces.

In [LV07], global Sobolev type inequalities were obtained for $\mathrm{CD}(K, N)$ spaces with $K > 0$. Nonetheless, a generalized Bonnet-Myers theorem holds in the CD context [St06, Cor. 2.6], implying that the support of the measure of a $\mathrm{CD}(K, N)$ space with $K > 0$ is compact. A L^1-Sobolev inequality is also known for essentially non-branching $\mathrm{CD}(K, N)$ spaces with $K < 0$ [Vi09, Th. 30.23]. Here we replace this essentially non-branching assumption by a growth condition, and get a Sobolev inequality of different nature, by very different means. The sharp global Sobolev inequality has been established by Profeta [P15] on $\mathrm{RCD}(K, N)$ spaces with $K > 0$ and $N > 2$. Finally, Cavaletti and Mondino proved in their critical work [CM17, Th. 1.11] a global Sobolev inequality with sharp constant for bounded essentially non-branching $\mathrm{CD}^*(K, N)$ spaces, taking into account a bound on the diameter.

We recall the definition of upper gradient in Section 2. Here is our main result.

Theorem 1 (Weighted Sobolev inequalities)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathrm{CD}(0, N)$ space with $N > 2$. Assume that there exists $1 < \eta < N$ such that*

$$0 < \Theta_{inf} := \liminf_{r \to +\infty} \frac{V(o, r)}{r^\eta} \le \Theta_{sup} := \limsup_{r \to +\infty} \frac{V(o, r)}{r^\eta} < +\infty \tag{1.1}$$

for some $o \in X$. Then for any $1 \le p < \eta$, there exists a constant $C = C(N, \eta, \Theta_{inf}, \Theta_{sup}, p) > 0$, depending only on N, η, Θ_{inf}, Θ_{sup} and p, such that for any function $u : X \to \mathbb{R}$ admitting an upper gradient $g \in L^p(X, \mathfrak{m})$,

$$\left(\int_X |u|^{p^*} d\mu \right)^{\frac{1}{p^*}} \le C \left(\int_X g^p d\mathfrak{m} \right)^{\frac{1}{p}}$$

where $p^* = Np/(N-p)$ and μ is the measure absolutely continuous with respect to $\mathfrak{m}$ with density $w_o = V(o, \mathsf{d}(o,\cdot))^{p/(N-p)}\mathsf{d}(o,\cdot)^{-Np/(N-p)}$.

Note that the growth condition (1.1) impose a dimensional restriction at infinity. For instance if $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathrm{RCD}(0, N)$ space, the Mondino-Naber decomposition [MN14] forces η to be an integer. This dimensional issue will be discussed in a larger context in [T].

Our proof relies on an abstract procedure (Theorem 2) which permits to patch local inequalities into a global one by means of a discrete Poincaré inequality. This technique, based upon ideas of Grigor'yan and Saloff-Coste [GS05], was successfully applied by Minerbe [Min09] to get a weighted L^2-Sobolev inequality on smooth Riemannian manifolds with non-negative Ricci curvature satisfying a suitable reverse doubling condition. Hein subsequently extended Minerbe's result [He11] to smooth Riemannian manifolds with an appropriate polynomial growth condition and whose Ricci curvature satisfies a lower bound $\mathrm{Ric} \geq -Cr^{-2}$, where r is the distance function to a reference point of the manifold.

We exploit a remark from Minerbe, which says that on Riemannian manifolds, besides the assumed reverse doubling condition, the weighted L^2-Sobolev inequality he established follows only from the doubling and Poincaré properties. We simply show that this goes also for non-smooth structures, and that the proof works for any exponent $p \in (1, N]$.

The paper is organized as follows. In Section 2, we introduce the tools of non-smooth analysis that we shall use throughout the article. We also define the $\mathrm{CD}(0, N)$ and $\mathrm{RCD}(0, N)$ conditions, and present the aforementioned patching procedure. In Section 3 we explain how to prove Theorem 1. We conclude with an application in Section 4, namely a control of the weighted heat kernel (Theorem 5) via a weighted Nash inequality (Theorem 4), in the context of Ahlfors regular $\mathrm{RCD}(0, N)$ spaces. Note that some weighted Nash inequalities were also considered in [BBGL12], but to the best understanding of the author, they are fundamentally different from ours.

Several constants will appear in this work. For better readability, if a constant C depends only on parameters $a_1, a_2, \cdots$ we will always write $C = C(a_1, a_2, \cdots)$ for its first occurrence, and then write more simply C if there is no ambiguity.

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2 Preliminaries

Unless otherwise mentioned, in the whole article $(X, \mathsf{d}, \mathfrak{m})$ will denote a triple such that (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a Borel measure, positive and finite on bounded sets, and without any loss of generality, we assume $\mathrm{supp}(\mathfrak{m}) = X$. We will use standard notations for functional spaces: $\mathrm{Lip}(X, \mathsf{d})$ for the space of d-Lipschitz functions and $L^p(X, \mathfrak{m})$ for the space of p-integrable functions, for any $1 \leq p \leq +\infty$. We will write $B_r(x)$ for the ball centered at $x \in X$ with radius $r > 0$, and $V(x, r)$ for the quantity $\mathfrak{m}(B_r(x))$. For any $\lambda > 0$, if B denotes a ball of radius $r > 0$, we write λB for the ball with same center than B and radius λr. For any Borel set $A \subset X$ and any function $u : X \to \mathbb{R}$, we denote by u_A or $\fint_A u \, \mathrm{d}\mathfrak{m}$ the mean value $\frac{1}{\mathfrak{m}(A)} \int_A u \, \mathrm{d}\mathfrak{m}$, and by $\langle u \rangle_A$ the mean value $\frac{1}{\mu(A)}\int_A u\,\mathrm{d}\mu$, where μ is as in Theorem 1.

Non-smooth analysis.
Let us recall that a continuous function $\gamma : [0, L] \to X$ is called *rectifiable curve* if its length $L(\gamma)$, defined as

$$L(\gamma) := \left\{ \sum_{i=1}^{n} \mathsf{d}(\gamma(x_i), \gamma(x_{i-1})) \;:\; n \in \mathbb{N}\backslash\{0\},\; 0 = x_0 < \cdots < x_n = 1 \right\},$$

is finite. For any rectifiable curve $\gamma : [0, L] \to X$, there exists a continuous function $\bar{\gamma} : [0, L(\gamma)] \to X$, called arc-length parametrization of γ, such that $\mathsf{d}(\bar{\gamma}(s), \bar{\gamma}(t)) = |t - s|$ for all $0 \le t \le s \le L(\gamma)$, and a non-decreasing continuous map $\varphi : [0, L] \to [0, L(\gamma)]$, such that $\gamma = \bar{\gamma} \circ \varphi$ (see e.g. [BBI01, Prop. 2.5.9]). When $L = L(\gamma)$ and $\varphi \equiv \mathrm{Id}$, we say that γ is parametrized by arc-length. This allows to introduce upper gradients, which can be regarded as extensions of the norm of the gradient for measurable functions defined on X.

Definition 1. *Let $u : X \to [-\infty, +\infty]$ be an extended real-valued function. A measurable function $g : X \to [0, +\infty]$ is called upper gradient of u if for any rectifiable curve $\gamma : [0, L] \to X$ parametrized by arc-length,*

$$|u(\gamma(L)) - u(\gamma(0))| \le \int_0^L g(\gamma(s))\,\mathrm{d}s.$$

The classical gradient of a Lipschitz function vanishes on the sets on which the function is constant. The following truncation property is an extension of this fact.

Definition 2. *Let $u : X \to [-\infty, +\infty]$ and $g : X \to [0, +\infty]$ be two measurable functions. For any $0 < t_1 < t_2$ and any function $v : X \to \mathbb{R}$, we denote by $v_{t_1}^{t_2}$ the truncated function $\min(\max(0, u - t_1), t_2 - t_1)$. We say that (u, g) satisfies the truncation property if for any $0 < t_1 < t_2$, any $b \in \mathbb{R}$ and any $\varepsilon \in \{-1, 1\}$, $g\chi_{t_1 < u < t_2}$ is an upper gradient of $(\varepsilon(u - b))_{t_1}^{t_2}$.*

It can be easily checked that the couple (u, g) made of a measurable function u and any of its upper gradients g satisfy the truncation property.

Let us recall now the notion of Sobolev spaces for general metric measure spaces introduced by Cheeger [Ch99].

Definition 3. *Let $1 \le p < +\infty$. The Sobolev $(1, p)$ norm of a function $u \in L^p(X, \mathfrak{m})$ is by definition*

$$\|u\|_{W^{1,p}} := \left(\|u\|_{L^p}^p + \inf \liminf_{i \to \infty} \|g_i\|_{L^p}^p \right)^{1/p}$$

where the infimum is taken over all the sequences of functions $(u_i)_{i\in\mathbb{N}}$, $(g_i)_{i\in\mathbb{N}}$ such that g_i is an upper gradient of u_i for every $i \in \mathbb{N}$ and $\|u_i - u\|_{L^p} \to 0$. The Sobolev space $W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ is then defined as the closure of $\mathrm{Lip}(X, \mathsf{d}) \cap L^p(X, \mathfrak{m})$ with respect to $\|\cdot\|_{W^{1,p}}$.

The above relaxation process can be achieved by using slopes of bounded Lipschitz functions instead of upper gradients of L^p functions (Lemma 4). Recall that the slope of a Lipschitz function f is defined by:

$$|Df|(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}, \quad \forall x \in X.$$

We will use the next notion to turn weak inequalities into strong inequalities.

Definition 4. *Let Ω be a bounded open set of X. Ω is called a John domain if there exists $x_0 \in \Omega$ and $C > 0$ such that for every $x \in \Omega$, there exists a Lipschitz curve $\gamma : [0, L] \to \Omega$ parametrized by arc-length such that $\gamma(0) = x$, $\gamma(L) = x_0$ and for any $t \in [0, L]$,*

$$C \leq \frac{d(\gamma(t), X \backslash \Omega)}{t}. \tag{2.1}$$

Let us point out that condition (2.1) prevents John domains to have cusps on their boundary, as one can easily understand from a simple example. Take $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi/2, |y| < e^{-\tan x}\}$. Then (2.1) fails at the cuspidal point $(1, 0)$: define $x_\varepsilon = (\pi/2 - \varepsilon, 0)$ for any $0 < \varepsilon < \pi/4$, then for any Lipschitz curve γ starting from x_ε parametrized by arc-length and with length larger than ε,

$$\frac{d(\gamma(\varepsilon), \mathbb{R}^2 \backslash \Omega)}{\varepsilon} \leq \frac{d(x_{2\varepsilon}, \mathbb{R}^2 \backslash \Omega)}{\varepsilon} = \frac{e^{-\tan(\pi/2 - 2\varepsilon)}}{\varepsilon} \xrightarrow{\varepsilon \to 0} 0.$$

Finally, let us recall that the space $(X, \mathsf{d}, \mathfrak{m})$ is called doubling if the following condition holds:

$$\exists C_D \geq 1 \ : \ V(x, 2r) \leq C_D V(x, r), \quad \forall x \in X, \forall r > 0, \tag{2.2}$$

and we say that it satisfies a weak local (p, p) Poincaré inequality if there exists $\lambda > 1$ such that:

$$\exists C_P > 0 \ : \ \int_B |u - u_B|^p \, \mathrm{d}\mathfrak{m} \leq C_P r^p \int_{\lambda B} g^p \, \mathrm{d}\mathfrak{m}, \quad \forall B = B(x, r), \tag{2.3}$$

for any locally $\mathfrak{m}$-integrable function u and any of its upper gradients $g \in L^p(X, \mathfrak{m})$. If the same inequality holds with $\lambda = 1$, we say that a *strong* (p, p) Poincaré inequality holds.

The $\mathrm{CD}(0, N)$ and $\mathrm{RCD}(0, N)$ conditions.
Let us give the definition of the curvature-dimension conditions $\mathrm{CD}(0, N)$ and $\mathrm{RCD}(0, N)$. For the more general condition $\mathrm{CD}(K, N)$ with $K \in \mathbb{R}$, we refer to [Vi09, Chap. 29, 30].

Recall that a curve $\gamma : [0, 1] \to X$ is called a geodesic if $\mathsf{d}(\gamma(s), \gamma(t)) = |t - s| \mathsf{d}(\gamma(0), \gamma(1))$ for any $0 \leq s \leq t \leq 1$. The space (X, d) is called geodesic if for any couple of points $x_0, x_1 \in X$ there exists a geodesic γ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We denote by $\mathcal{P}(X)$ the set of probability measures on X and by $\mathcal{P}_2(X)$ the set of probability measures μ on X with finite second moment, i.e. such that there exists $x_o \in X$ for which $\int_X \mathsf{d}^2(x_o, x) \, \mathrm{d}\mu(x) < +\infty$. The Wasserstein distance between two measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ is by definition

$$W_2(\mu_0, \mu_1) = \inf \left(\int_{X \times X} \mathsf{d}(x_0, x_1)^2 \, \mathrm{d}\pi(x_0, x_1) \right)^{1/2}$$

where the infimum is taken over the set of probability measures π on $X \times X$ with first marginal equal to μ_0 and second marginal equal to μ_1. A standard result of optimal transport theory states that if the space (X, d) is geodesic, then the metric space $(\mathcal{P}_2, W_2)$ is geodesic too. Let us introduce the Rényi entropy.

Definition 5. *Given $N \in (1, +\infty)$, the N-Rényi entropy $S_N(\cdot|\mathfrak{m})$ relative to $\mathfrak{m}$ is defined as follows:*

$$S_N(\mu|\mathfrak{m}) = -\int_X \rho^{1 - \frac{1}{N}} \, \mathrm{d}\mathfrak{m} \qquad \forall \mu \in \mathcal{P}(X),$$

where $\mu = \rho \mathfrak{m} + \mu^{sing}$ is the Lebesgue decomposition of μ with respect to $\mathfrak{m}$.

We are now in a position to introduce the $\mathrm{CD}(0,N)$ condition, which could be summarized as weak geodesical convexity of N'-Rényi entropies for any $N' \geq N$.

Definition 6. *Given $N \in (1,+\infty)$, $(X,\mathsf{d},\mathfrak{m})$ satisfies the $CD(0,N)$ condition if for any $N' \geq N$, the N'-Rényi entropy is weakly geodesically convex, meaning that for every couple of measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ between μ_0 and μ_1 such that for any $t \in [0,1]$,*

$$S_N(\mu_t|\mathfrak{m}) \leq (1-t)S_N(\mu_0|\mathfrak{m}) + tS_N(\mu_1|\mathfrak{m}).$$

Any space satisfying the $\mathrm{CD}(0,N)$ *condition is called a* $\mathrm{CD}(0,N)$ *space.*

The Bishop-Gromov theorem holds on $\mathrm{CD}(0,N)$ spaces [Vi09, Th. 30.11], and as a direct consequence, the doubling condition holds too, with optimal doubling constant $C_D = 2^N$. Moreover, Rajala proved the following uniform weak local $(1,1)$ Poincaré inequality [Raj12, Th. 1.1] .

Proposition 1. *Assume that $(X,\mathsf{d},\mathfrak{m})$ is a $\mathrm{CD}(0,N)$ space. Then for any measurable function $u : X \to [-\infty,+\infty]$ admitting an upper gradient $g \in L^1(X,\mathfrak{m})$, for any ball $B \subset X$ with radius r such that $2B \subset X$,*

$$\int_B |u - u_B| \,\mathrm{d}\mathfrak{m} \leq 4r \int_{2B} g \,\mathrm{d}\mathfrak{m}.$$

Unfortunately, the $\mathrm{CD}(0,N)$ condition does not distinguish Riemannian and non-Riemannian structures. Indeed, $\mathbb{R}^n$ equipped with the distance induced by the infinite norm and the Lebesgue measure satisfies the $\mathrm{CD}(0,N)$ condition [Vi09], however it is not a Riemannian structure, as the infinite norm is not induced by any scalar product. To focus on more Riemannian-like structures, Ambrosio, Gigli and Savaré added to the theory the notion of infinitesimal Hilbertianity, leading to the so-called RCD condition, R standing for *Riemannian* [AGS14b].

Definition 7. *Define the Cheeger energy of a function $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ as*

$$\mathsf{Ch}(f) := \frac{1}{2}(\|f\|^2_{W^{1,2}} - \|f\|^2_{L^2}).$$

$(X,\mathsf{d},\mathfrak{m})$ is called infinitesimally Hilbertian if Ch is a quadratic form. If in addition $(X,\mathsf{d},\mathfrak{m})$ is a $\mathrm{CD}(0,N)$ space, it is said to satisfy the $\mathrm{RCD}(0,N)$ condition, or more simply it is called a $\mathrm{RCD}(0,N)$ space.

Note that $(X,\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian if and only if $W^{1,2}(X,\mathsf{d},\mathfrak{m})$ is a Hilbert space, whence the terminology. This condition allows to apply the general theory of gradient flows on Hilbert spaces and to canonically associate to Ch its $L^2(X,\mathfrak{m})$ gradient flow, denoted by $(h_t)_{t\geq 0}$ and called *heat flow* of $(X,\mathsf{d},\mathfrak{m})$. The infinitesimal Hilbertianity implies that this heat flow is a linear, continuous, self-adjoint and Markovian contraction semigroup in $L^2(X,\mathfrak{m})$. The terminology 'heat flow' comes from the characterization of $(h_t)_{t\geq 0}$ as the only semi-group of operators such that $t \mapsto h_t f$ is locally absolutely continuous in $(0,+\infty)$ with values in $L^2(X,\mathfrak{m})$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t f = \Delta h_t f \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,+\infty),$$

for every $f \in L^2(X,\mathfrak{m})$, the Laplace operator Δ being defined in this context by:

$$f \in D(\Delta) \iff \exists h := \Delta f \in L^2(X,\mathfrak{m}) \text{ s.t. } \mathsf{Ch}(f,g) = -\int_X hg \,\mathrm{d}\mathfrak{m} \;\; \forall g \in W^{1,2}(X,\mathsf{d},\mathfrak{m}).$$

Let us close this quick overview by saying that on infinitesimally Hilbertian spaces, Ch admits an integral representation

$$2\mathsf{Ch}(f) = \int_X \Gamma(f,f)\,\mathrm{d}\mathfrak{m} \quad \forall f \in W^{1,2}(X,\mathsf{d},\mathfrak{m}),$$

where $\Gamma : W^{1,2}(X,\mathsf{d},\mathfrak{m}) \times W^{1,2}(X,\mathsf{d},\mathfrak{m}) \to L^1(X,\mathfrak{m})$ is the *Carré du champ* operator defined by a classical polarization procedure.

Patching procedure

Let us present now the patching procedure [GS05, Min09] that we shall apply to get Theorem 1. Recall that μ is the Borel measure absolutely continuous with respect to $\mathfrak{m}$ with density $w_o = V(o,\mathsf{d}(o,\cdot))^{p/(N-p)}\mathsf{d}(o,\cdot)^{-Np/(N-p)}$. For a given set $\{\cdot\}$, we denote by $Card\{\cdot\}$ its cardinality.

Definition 8. *A countable family $(U_i, U_i^*, U_i^{\#})_{i\in I}$ of Borel subsets of X with finite $\mathfrak{m}$-measure is called a good covering of (X,d) with respect to $(\mu,\mathfrak{m})$ if:*

1. *for every $i \in I$, $U_i \subset U_i^* \subset U_i^{\#}$;*
2. *there exists a $\mathfrak{m}$-negligible Borel set $E \subset X$ such that $X \backslash E \subset \bigcup_i U_i$;*
3. ***(overlapping condition at level 3)***
   there exists $Q_1 > 0$ such that for every $i_0 \in I$, $Card(\{i \in I : U_{i_0}^{\#} \cap U_i^{\#} \neq \emptyset\}) \leq Q_1$;
4. ***(embracing condition between level 1 and 2)***
   for every $(i,j) \in I^2$ such that $\overline{U_i} \cap \overline{U_j} \neq \emptyset$, there exists $k(i,j) \in I$ such that $U_i \cup U_j \subset U^*_{k(i,j)}$;
5. ***(measure control of the embracing condition)***
   there exists $Q_2 > 0$ such that for every $i,j \in I$, if $\overline{U_i} \cap \overline{U_j} \neq \emptyset$,

$$\begin{array}{ll} (i) & \mu(U^*_{k(i,j)}) \leq Q_2 \min(\mu(U_i),\mu(U_j)); \\ (ii) & \mathfrak{m}(U^*_{k(i,j)}) \leq Q_2 \min(\mathfrak{m}(U_i),\mathfrak{m}(U_j)). \end{array}$$

Assume that $(U_i, U_i^*, U_i^{\#})_{i\in I}$ is a good covering of (X,d) with respect to $(\mu,\mathfrak{m})$. Let us explain how to define out of $(U_i, U_i^*, U_i^{\#})_{i\in I}$ a canonical weighted graph $(\mathcal{V},\mathcal{E},\mu)$, where $\mathcal{V}$ is the set of vertices of the graph, $\mathcal{E}$ is the set of edges, and μ is a weight on the graph (i.e. a function $\mu : \mathcal{V} \sqcup \mathcal{E} \to \mathbb{R}$). We define $\mathcal{V}$ by associating to each U_i a vertex i (informally, we put a point i on each U_i). Then we define $\mathcal{E}$ as

$$\mathcal{E} := \{(i,j) \in \mathcal{V} \times \mathcal{V} : i \neq j \text{ and } \overline{U_i} \cap \overline{U_j} \neq \emptyset\}.$$

We will write $i \sim j$ whenever $(i,j) \in \mathcal{E}$. Note that two vertices are linked if the associated pieces of the covering intersect. But in practical, we will always consider good coverings such that $\mathring{U}_i \cap \mathring{U}_j = \emptyset$ for every $i \neq j$, so roughly speaking, we are just linking two vertices i and j if they correspond to adjacent pieces U_i and U_j. Afterwards we weight the vertices of the graph setting $\mu(i) := \mu(U_i)$ for every $i \in \mathcal{V}$ (the repeated use of the letter "μ" won't cause any trouble), and the edges setting $\mu(i,j) := \max(\mu(i),\mu(j))$ for every $(i,j) \in \mathcal{E}$.

The patching theorem (Theorem 2) states that if some local inequalities are true on the pieces of the good covering, and if a discrete inequality holds on the associated canonical weighted graph, then the local inequalities can be patched into a global one. Let us give the precise definitions.

Definition 9. *We say that the good covering* $(U_i, U_i^*, U_i^\#)_{i\in I}$ *satisfies local continuous* L^p *Sobolev-Neumann inequalities if there exists a constant* $S_c > 0$ *such that for every* $i \in I$*,*

1. ***(levels 1-2)*** *for any measurable function* $u : U_i^* \to \mathbb{R}$ *and any upper gradient* $g \in L^p(U_i^*, \mathfrak{m})$ *of* u*,*

$$\left(\int_{U_i} |u - \langle u\rangle_{U_i}|^{p^*} \,\mathrm{d}\mu\right)^{\frac{1}{p^*}} \le S_c \left(\int_{U_i^*} g^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}};$$

2. ***(levels 2-3)*** *for any measurable function* $u : U_i^\# \to \mathbb{R}$ *and any upper gradient* $g \in L^p(U_i^\#, \mathfrak{m})$ *of* u*,*

$$\left(\int_{U_i^*} |u - \langle u\rangle_{U_i^*}|^{p^*} \,\mathrm{d}\mu\right)^{\frac{1}{p^*}} \le S_c \left(\int_{U_i^\#} g^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}}.$$

Definition 10. *We say that the weighted graph* $(\mathcal{V}, \mathcal{E}, \mu)$ *satisfies a discrete* (p, p) *Poincaré inequality if there exists a constant* $S_d > 0$ *such that for every* $f \in L^p(\mathcal{V}, \mu)$*,*

$$\left(\sum_{i\in\mathcal{V}} |f(i)|^p \mu(i)\right)^{\frac{1}{p}} \le S_d \left(\sum_{\{i,j\}\in\mathcal{E}} |f(i) - f(j)|^p \mu(i,j)\right)^{\frac{1}{p}}.$$

Remark 1. *Note that here we differ from Minerbe's terminology, which call the above inequality a discrete* L^p *Sobolev-Dirichlet inequality of order* ∞*. More generally, we say that a discrete* L^p *Sobolev-Dirichlet inequality of order* k *holds if there exists a constant* S_d *such that for every* $f \in L^p(\mathcal{V}, \mu)$*,*

$$\left(\sum_{i\in\mathcal{V}} |f(i)|^{\frac{pk}{k-p}} \mu(i)\right)^{\frac{k-p}{k}} \le S_d \sum_{\{i,j\}\in\mathcal{E}} |f(i) - f(j)|^p \mu(i,j).$$

As we don't need this general definition, we choose the appellation "Poincaré" which seems more natural.

In the following statements, we consider $1 \le q < +\infty$.

Definition 11. *A good covering* $(U_i, U_i^*, U_i^\#)_{i\in I}$ *is called a* (p, q) *patchwork if it satisfies the local continuous* L^p *Sobolev-Neumann inequalities and if the associated weighted graph* $(\mathcal{V}, \mathcal{E}, m)$ *satisfies the discrete* (q, q) *Poincaré inequality.*

We are now in a position to state the patching theorem.

Theorem 2. *Assume that* X *admits a* (p, q) *patchwork. Then there exists a constant* $C = C(p, q, Q_1, Q_2, S_c, S_d) > 0$ *such that for any measurable function* $u : X \to \mathbb{R}$ *admitting an upper gradient* $g \in L^p(X, \mathfrak{m})$*,*

$$\left(\int_X |u|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} \le C \left(\int_X g^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}}.$$

The proof of Theorem 2 can be copied *verbatim* from [Min09, Th. 1.8], replacing norm of gradients by upper gradients, thus we omit it. Nonetheless, let us stress that this proof does not require any extra assumption on $(X, \mathsf{d}, \mathfrak{m})$.

Note that a similar statement holds if we replace X by a subset A with finite measure and the discrete (q,q) Poincaré inequality by a discrete (q,q) Poincaré-Neumann inequality: there exists a constant $S_d > 0$ such that for every $f : \mathcal{V} \to \mathbb{R}$ with finite support,

$$\left(\sum_{i\in\mathcal{V}} |f(i) - \mathfrak{m}(f)|^p \mu(i)\right)^{\frac{1}{p}} \le S_d \left(\sum_{\{i,j\}\in\mathcal{E}} |f(i) - f(j)|^p \mu(i,j)\right)^{\frac{1}{p}},$$

where $\mathfrak{m}(f) = \left(\sum_{i\,:\,f(i)\neq 0} \mathfrak{m}(i)\right)^{-1} \sum_i f(i)\mathfrak{m}(i)$. See [Min09, Th. 1.10] for the proof.

Theorem 3. *Let A be a subset of X with $\mathfrak{m}(A) < +\infty$. Assume that A admits a finite good covering $(U_i, U_i^*, U_i^\#)_{i\in I}$ satisfying the local continuous L^p Sobolev-Neumann inequalities and whose associated weighted graph $(\mathcal{V}, \mathcal{E}, m)$ satisfies the above discrete (q,q) Poincaré-Neumann inequality. Then there exists a constant $C = C(p, q, Q_1, Q_2, S_c, S_d) > 0$ such that for any $u \in L^1(A, \mu)$ admitting an upper gradient $g \in L^p(A, \mathfrak{m})$,*

$$\left(\int_A |u - \langle u\rangle_A|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} \le C \left(\int_A g^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}}.$$

3 Proof of the main result

In this section, we prove Theorem 1. Let $(X, \mathsf{d}, \mathfrak{m})$ be a non-compact $\mathrm{CD}(0, N)$ space with $N \ge 3$. Take $1 \le p < N$ and $p < \eta < N$, and assume that the growth condition (1.1) holds. Let us recall that $p^* = pN/(N-p)$ and that μ is the measure absolutely continuous with respect to $\mathfrak{m}$ with density $w_o = V(o, \mathsf{d}(o,\cdot))^{p/(N-p)} \mathsf{d}(o,\cdot)^{-Np/(N-p)}$.

As pointed out by Minerbe [Min09], on Riemannian manifolds, the local continuous L^2 Sobolev-Neumann inequalities can be derived from the doubling condition and a uniform local $(2,2)$ strong Poincaré inequality, both implied by the non-negativity of the Ricci curvature. However, the discrete $(2^*, 2^*)$ Poincaré inequality requires the addition of a reverse doubling condition (3.1), which is an immediate consequence of the growth condition (1.1).

Lemma 1. *There exists $A > 0$ and $C_{RD} > 0$ such that*

$$\frac{V(o,R)}{V(o,r)} \ge C_{RD} \left(\frac{R}{r}\right)^\eta \qquad \forall\, A < r \le R. \tag{3.1}$$

Proof. The growth condition (1.1) implies the existence of $A > 0$ such that for any $r \ge A$, $\Theta_{inf}/2 \le r^{-\eta} V(o,r) \le 2\Theta_{sup}$. Take $R \ge r$. Then $R^{-\eta} V(o,R) \ge \Theta_{inf}/2$, whence the result with $C_{RD} = \Theta_{inf}/(4\Theta_{sup})$. □

Remark 2. *With no loss of generality, we can (and will) assume that $A = 1$.*

We shall need the following result, namely a local L^p-Sobolev inequality, which is a well-known consequence of the doubling and Poincaré properties of $(X, \mathsf{d}, \mathfrak{m})$.

Proposition 2. *There exists a constant $C = C(N,p) > 0$ such that for any function $u \in L^1_{loc}(X, \mathfrak{m})$, any upper gradient g of u, and any ball $B = B_R(x) \subset X$,*

$$\left(\fint_B |u - u_B|^{p^*} \,\mathrm{d}\mathfrak{m}\right)^{1/p^*} \le CR \left(\fint_B g^p \,\mathrm{d}\mathfrak{m}\right)^{1/p} \tag{3.2}$$

or, alternatively,

$$\left(\int_B |u - u_B|^{p^*} \,\mathrm{d}\mathfrak{m}\right)^{1/p^*} \le C \frac{R}{V(x,R)^{1/N}} \left(\int_B g^p \,\mathrm{d}\mathfrak{m}\right)^{1/p}. \tag{3.3}$$

Proof. Assume that $u : X \to \mathbb{R}$ is a locally $\mathfrak{m}$-measurable function admitting an upper gradient $g \in L^p(X, \mathfrak{m})$. Assume that B is a ball of X with radius R. Using Hölder's inequality, Proposition 1 gives

$$\fint_B |u - u_B| \, \mathrm{d}\mathfrak{m} \leq 2^{N+2} r \left(\fint_{2B} g^p \, \mathrm{d}\mu \right)^{1/p}.$$

As $\eta > p$ we are in a position to apply 1. of [HK00, Th. 5.1], which implies

$$\left(\fint_B |u - u_B|^{p^*} \, \mathrm{d}\mathfrak{m} \right)^{1/p^*} \leq Cr \left(\fint_{10B} g^p \, \mathrm{d}\mathfrak{m} \right)^{1/p}.$$

To turn this weak inequality into a strong one, let us apply [HK00, Th. 9.7] to the ball B. As $(X, \mathsf{d}, \mathfrak{m})$ is a $CD(0, N)$ space, the metric structure (X, d) is proper and geodesic, then all the balls of X are John domain with a common constant [HK00, Cor. 9.5]. The fact that there exists a constant $C > 0$ such that for every ball $B(x, \rho) \subset B$ with $\rho < 2r$,

$$\mathfrak{m}(B(x, r)) \geq C \left(\frac{\rho}{2r} \right)^\eta \mathfrak{m}(B),$$

is easily verified using the doubling condition. Then [HK00, Th. 9.7] applies and gives the result. □

Remark 3. *Theorem 9.7 of [HK00] is stated for weak John domains, a generalization of the notion of John domain to structures without enough rectifiable curves (especially fractals, see [HK00, p.39] for details). However being a John domain implies being a weak John domain, and $CD(0, N)$ spaces are geodesics and therefore they contain enough rectifiable curves.*

Finally, let us state a result whose proof can be taken from [Min09, Prop. 2.8], replacing smooth functions by measurable ones, norm of gradients by upper gradients, and the strong local $(2, 2)$ Poincaré inequality used there by Proposition 1. Notice that even if Proposition 1 provides only a weak inequality, one can harmlessly substitute it to the strong one used in the smooth case, because it is applied to a function f which is Lipschitz on a ball B and extended by 0 outside of B.

Proposition 3. *There exists $\kappa_0 = \kappa_0(N, \eta, p) > 1$ such that for every $R > 0$, for any couple of points x, y in the geodesic sphere $S(o, R)$, there exists a rectifiable curve from x to y that remains inside $B(o, R) \backslash B(o, \kappa_0^{-1} R)$.*

Remark 4. *It is worth pointing out that the conclusion of Proposition 3 can be understood as a connectedness property, as it implies that any annulus $B(o, \kappa_o^{i+2}) \backslash B(o, \kappa_o^{i-1})$ must be connected. Moreover, the proof can be carried out with only the doubling and Poincaré properties, thus the conclusion holds for any PI doubling space.*

Let us prove now Theorem 1.

STEP 1: The good covering.

Let us give explain in a few words on how to construct a good covering on $(X, \mathsf{d}, \mathfrak{m})$. We refer to [Min09, Section 2.3.1] for the details. Define κ as the square-root of the constant κ_0 given by Proposition 3. Then for any $R > 0$, two connected components of $B(o, \kappa R) \backslash B(o, R)$ are always contained in one component of $B(o, \kappa R) \backslash B(o, \kappa^{-1} R)$. Let us write $A_i = B(o, \kappa^i) \backslash B(o, \kappa^{i-1})$ for any $i \in \mathbb{N}$.

Let γ be a line starting at o, i.e. a continuous function $\gamma : [0, +\infty) \to X$ such that $\gamma(o) = 0$ and $\mathsf{d}(\gamma(t), \gamma(s)) = |t - s|$ for any $s, t \geq 0$. Such a line can be obtained as follows. For $x_1 \in S(o, 1)$, let $\gamma_1 : [0, 1] \to X$ be a geodesic between o and x_1. Define then recursively $x_n := \text{argmin}\{\mathsf{d}(x_{n-1}, x) : x \in S(o, n)\}$ and γ_n geodesic between x_{n-1} and x_n for any $n \geq 1$. The concatenation of all the γ_n provides the desired γ.

Then for any integer i, denote by $(U'_{i,a})_{0 \leq a \leq h'_i}$ the connected components of A_i, $U'_{i,0}$ being the one which intersects γ. Let us prove that the number h'_i is uniformly bounded. This was stated without proof in [Min09].

Lemma 2. *There exists a constant $h = h(N, \kappa) < \infty$ such that $\sup_i h'_i \leq h$.*

Proof. Take $i \in \mathbb{N}$. For every $0 \leq a \leq h'_i$, pick x_a in $U_{i,a} \cap S(o, (\kappa^i + \kappa^{i-1})/2)$. As the balls $V(x_a, (\kappa^i - \kappa^{i-1})/4)$, $0 \leq a \leq h'_i$, are disjoints and all included in $V(o, \kappa_i)$,

$$h'_i \min_{0 \leq a \leq h'_i} V(x_a, (\kappa^i - \kappa^{i-1})/4) \leq \sum_{0 \leq a \leq h'_i} V(x_a, (\kappa^i - \kappa^{i-1})/4) \leq V(o, \kappa^i).$$

Assume for simplicity that $\min\limits_{0 \leq a \leq h'_i} V(x_a, (\kappa^i - \kappa^{i-1})/4) = V(x_0, (\kappa^i - \kappa^{i-1})/4)$. Notice that $d(o, x_0) \leq \kappa_i$. Then

$$h'_i \leq \frac{V(o, \kappa^i)}{V(x_0, (\kappa^i - \kappa^{i-1})/4)} \leq \frac{V(x_0, \kappa^i + \mathsf{d}(o, x_0))}{V(x_0, (\kappa^i - \kappa^{i-1})/4)} \leq \left(\frac{8\kappa^i}{\kappa^i - \kappa^{i-1}} \right)^N$$

by the doubling condition. Whence the result with $h = \left(\frac{8\kappa}{\kappa - 1}\right)^N$. □

Define then the covering $(U'_{i,a}, U'^*_{i,a}, U'^\#_{i,a})_{i \in \mathbb{N}, 0 \leq a \leq h'_i}$ where $U'^*_{i,a}$ is by definition the union of the sets $U'_{j,b}$ such that $\overline{U'_{j,b}} \cap \overline{U'_{i,a}} \neq \emptyset$, and $U'^\#_{i,a}$ is by definition the union of the sets $U'^*_{j,b}$ such that $\overline{U'^*_{j,b}} \cap \overline{U'^*_{i,a}} \neq \emptyset$. Note that $(U'_{i,a}, U'^*_{i,a}, U'^\#_{i,a})_{i \in \mathbb{N}, 0 \leq a \leq h'_i}$ is not necessarily a good covering, as there is no reason a priori that it satisfies the measure control of the overlapping condition: pieces $U'_{i,a}$ may be arbitrary small compared to their neighbors. Thus whenever $\overline{U'_{i,a}} \int S(o, \kappa^i) = \emptyset$, we define $U_{i-1,a} = U'_{i,a} \cup U'_{i-1,a'}$ where a' is such that $\overline{U'_{i,a}} \cap \overline{U_{i-1,a'}} \neq \emptyset$; otherwise we define $U_{i,a} = U'_{i,a}$. In other words, we incorporate small pieces $U'_{i,a}$ into the adjacent piece $U'_{i-1,a'}$. Then we define $U^*_{i,a}$ and $U^\#_{i,a}$ in a similar way than $U'^*_{i,a}$ and $U'^\#_{i,a}$. Using the doubling condition, one can show that $(U_{i,a}, U^*_{i,a}, U^\#_{i,a})_{i \in \mathbb{N}, 0 \leq a \leq h_i}$ is a good covering on (X, d) with respect to $(\mu, \mathfrak{m})$, with constants Q_1 and Q_2 depending only on N.

<u>STEP 2:</u> The discrete (p^*, p^*) Poincaré inequality.

Let us denote by $(\mathcal{V}, \mathcal{E}, \mu)$ the weighted graph obtained from the good covering $(U_{i,a}, U^*_{i,a}, U^\#_{i,a})_{i \in \mathbb{N}, 0 \leq a \leq h_i}$. Define the degree $\deg(i)$ of a vertex i as the number of vertices j such that $i \sim j$. As a consequence of Lemma 2, $\sup \deg(i) : i \in \mathcal{V} \leq 2h$. Moreover, the doubling condition implies easily the existence of a number $C \geq 1$ such that for every $i, j \in E$, $C^{-1}\mathfrak{m}(i) \leq \mathfrak{m}(j) \leq C\mathfrak{m}(j)$. Thus by [Min09, Prop. 1.12], the discrete $(1, 1)$ Poincaré inequality implies the (q, q) one for every $q \geq 1$. But the discrete $(1, 1)$ Poincaré inequality is equivalent to the isoperimetric inequality ([Min09, Prop. 1.14]): there exists a constant $\mathcal{I} > 0$ such that for any $\Omega \subset \mathcal{V}$ with finite measure,

$$\frac{\mu(\Omega)}{\mu(\partial\Omega)} \leq \mathcal{I}$$

where $\partial\Omega := \{(i,j) \in \mathcal{E} : i \in \Omega, j \notin \Omega\}$. The only ingredients to prove this isoperimetric inequality are the doubling and reverse doubling conditions, see Section 2.3.3 in [Min09]. Then the discrete (q,q) Poincaré inequality holds for any $q \geq 1$, with a constant S_d depending only on q, η, Θ_{inf}, Θ_{sup} and on the doubling and Poincaré constants of $(X, \mathsf{d}, \mathfrak{m})$, i.e. on N.

STEP 3: The local continuous L^p Sobolev-Neumann inequalities.

Let us explain how to get the local continuous L^p Sobolev-Neumann inequalities. We start by deriving from the local L^p-Sobolev inequality (3.2) a crucial technical result, namely a L^p-Sobolev-type inequality on connected Borel subsets of annuli.

Lemma 3. *Let $R > 0$ and $\alpha > 1$. Let A be a connected Borel subset of $B(o, \alpha R) \backslash B(o, R)$. For $0 < \delta < 1$, denote by $(A)_\delta$ the δ-neighborhood of A, i.e. $(A)_\delta = \bigcup_{x \in A} B_\delta(x)$. Then there exists a constant $C = C(N, \delta, \alpha, p) > 0$ such that for any measurable function $u : (A)_\delta \to [-\infty, +\infty]$ and any upper gradient $g \in L^p((A)_\delta, \mathfrak{m})$,*

$$\left(\int_A |u - u_A|^{p^*} \, \mathrm{d}\mathfrak{m} \right)^{1/p^*} \leq C \frac{R^p}{V(o,R)^{p/N}} \left(\int_{(A)_\delta} g^p \, \mathrm{d}\mathfrak{m} \right)^{1/p}.$$

Proof. Define $s = \delta R$ and choose $(x_j)_{j \in J}$ an s-lattice of A (a maximal set of points whose distance between two of them is at least s). Set $V_i = B(x_i, s)$ and $V_i^* = V_i^\# = B(x_i, 3s)$. Using the doubling condition, there is no difficulty in proving that $(V_i, V_i^*, V_i^\#)$ is a good covering of (X, d) with respect to $(\mathfrak{m}, \mathfrak{m})$. A discrete (p^*, p^*) Poincaré-Neumann inequality holds on the associated weighted graph, as one can easily checked following the lines of [Min09, Lem. 2.10]. The local continuous L^p Sobolev-Neumann inequalities stem from the proof of [Min09, Lem. 2.11], where we replace (14) there by Proposition 2. Then Theorem 3 gives the result. □

Let us prove that Lemma 3 implies the local continuous L^p Sobolev-Neumann inequalities with a constant S_c depending only on N, η and p. Take a piece of the good covering $U_{i,a}$. Choose $\delta = (1 - \kappa^{-1})/2$ so that $(U_{i,a})_\delta \subset U_{i,a}^*$. Take a measurable function $u : U_{i,a}^* \to [-\infty, +\infty]$ and an upper gradient g of u. By the triangle inequality and the elementary fact $|x + y|^{p^*} \leq 2^{p^*-1}(|x| + |y|)$ holding for any $x, y \in \mathbb{R}$,

$$\int_{U_{i,a}} |u - \langle u \rangle_{U_{i,a}}|^{p^*} \, \mathrm{d}\mu \leq 2^{p^*} \inf_{c \in \mathbb{R}} \int_{U_{i,a}} |u - c|^{p^*} \, \mathrm{d}\mu \leq 2^{p^*} \int_{U_{i,a}} |u - u_{U_{i,a}}|^{p^*} w_o \, \mathrm{d}\mathfrak{m}.$$

As w_o is a radial function, let us define $\bar{w}_o(r) = w_o(x)$ for $r = \mathsf{d}(o,x)$. Note that by Bishop-Gromov theorem, $\bar{w}_o$ is a decreasing function, so

$$\int_{U_{i,a}} |u - \langle u \rangle_{U_{i,a}}|^{p^*} \, \mathrm{d}\mu \leq 2^{p^*} \bar{w}_o(\kappa^{i-1}) \int_{U_{i,a}} |u - u_{U_{i,a}}|^{p^*} \, \mathrm{d}\mathfrak{m}.$$

Applying Lemma 3 with $A = U_{i,a}$, $R = \kappa^{i-1}$ and $\alpha = \kappa^2$, we get

$$\begin{aligned} \int_{U_{i,a}} |u - \langle u \rangle_{U_{i,a}}|^{p^*} \, \mathrm{d}\mu &\leq C^{p^*} 2^{p^*} \frac{\kappa^{p^*(i-1)}}{V(o, \kappa^{i-1})^{p^*/N}} \bar{w}_o(\kappa^{i-1}) \left(\int_{U_{i,a}^*} g^p \, \mathrm{d}\mathfrak{m} \right)^{p^*/p} \\ &\leq C \left(\int_{U_{i,a}^*} g^p \, \mathrm{d}\mathfrak{m} \right)^{p^*/p} \end{aligned}$$

where we used the same letter C to denote different constants depending only on N, p, and κ. As κ depends only on N, η and p, we get the result.

An analogous argument implies the inequalities between levels 2 and 3.

STEP 4: Conclusion.

Apply Theorem 2 to get the result.

4 Weighted Nash inequality and bound of the corresponding heat kernel

In this section, we apply Theorem 1 in the setting of $\mathrm{RCD}(0,N)$ spaces to derive a corresponding weighted Nash inequality, from which we deduce a uniform bound on a weighted heat kernel when the spaces are k-Ahlfors regular for some integer k between 1 and N. Recall that $(X,\mathsf{d},\mathfrak{m})$ is called k-Ahlfors regular if there exists a constant $C>0$ such that

$$C^{-1} \leq \frac{V(x,r)}{r^k} \leq C, \qquad \forall x \in X, \forall r > 0.$$

Note that if $(X,\mathsf{d},\mathfrak{m})$ is k-Ahlfors regular and satisfies the growth condition (1.1) for some η, then $\eta = k$.

Theorem 4 (Weighted Nash inequality)**.** *Assume that $(X,\mathsf{d},\mathfrak{m})$ is a $\mathrm{RCD}(0,N)$ space, with $N>2$, satisfying (1.1) with $\eta > 2$. Then there exists a constant $C = C(N,\Theta_{inf},\Theta_{sup}) > 0$ such that for any $u \in L^1(X,\mu) \cap W^{1,2}(X,\mathsf{d},\mathfrak{m})$,*

$$\|u\|_{L^2(X,\mu)}^{2+\frac{4}{N}} \leq C \|u\|_{L^1(X,\mu)}^{\frac{4}{N}} \mathsf{Ch}(u).$$

To prove this theorem, we need a standard lemma, consequence of [ACDM15, Section 3], which states that the relaxation procedure defining Ch can be achieved with slopes of bounded Lipschitz functions instead of upper gradients. We give a proof for the reader's convenience.

Lemma 4. *Let $u \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$. Then*

$$\mathsf{Ch}(u) = \inf \left\{ \liminf_{n\to\infty} \int_X |Du_n|^2 \,\mathrm{d}\mathfrak{m} : (u_n)_n \subset \mathrm{Lip_b}(X) \cap L^2(X,\mathfrak{m}), \|u_n - u\|_{L^2(X,\mu)} \to 0 \right\}.$$

Proof. Choose a point $o \in X$ and for every $n \in \mathbb{N}^*$, let χ_n be a Lipschitz function constant equal to 1 on $B(o,n)$, to 0 on $X \backslash B(o,n+1)$ and such that $|D\chi_n| \leq 2$. Take $f \in \mathrm{Lip}(X) \cap L^2(X,\mathfrak{m})$ and define, for every $n \in \mathbb{N}$, $f_n = f\chi_n$. Using the chain rule and Young's inequality for some $\varepsilon > 0$, denoting by $\mathrm{Lip}(\chi_n)(\leq 2)$ the Lipschitz constant of χ_n, we get

$$\begin{aligned} |Df_n|^2 &\leq \left(\chi_n |Df| + f \,\mathrm{Lip}(\chi_n) 1_{B(0,n+1)\backslash B(0,n)}\right)^2 \\ &\leq (1+\varepsilon)|Df|^2 + 4(1+1/\varepsilon) f^2 1_{B(0,n+1)\backslash B(0,n)}. \end{aligned}$$

Integrating over X and taking the limit superior, it implies

$$\limsup_{n\to\infty} \int_X |Df_n|^2 \,\mathrm{d}\mathfrak{m} \leq (1+\varepsilon) \int_X |Df|^2 \,\mathrm{d}\mathfrak{m},$$

and letting ε go to 0 leads to

$$\limsup_{n\to\infty}\int_X |Df_n|^2\,\mathrm{d}\mathfrak{m} \le \int_X |Df|^2\,\mathrm{d}\mathfrak{m}.$$

Then for $u \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$, for any sequence $(u_k)_k \subset \mathrm{Lip}(X) \cap L^2(X,\mathfrak{m})$ $L^2(X,\mathfrak{m})$-converging to u, considering for any $k \in \mathbb{N}$ a sequence $(v_{k,n})_n \subset \mathrm{Lip_b}(X)$ built as above, a diagonal argument provides a sequence $(v_{k,n(k)})_k$ such that

$$\liminf_{k\to\infty}\int_X |Dv_{k,n(k)}|^2\,\mathrm{d}\mathfrak{m} \le \liminf_{k\to\infty}\int_X |\nabla u_k|^2\,\mathrm{d}\mathfrak{m}.$$

Taking the infimum among all sequences $(u_k)_k$ L^2-converging to u leads to the result. □

We can now prove Theorem 4. The proof presented here is the standard way to deduce a Nash inequality from a Sobolev inequality, see for instance [BBGL12].

Proof. By the previous lemma it is sufficient to prove the result for $u \in \mathrm{Lip_b}(X)$. By Hölder's inequality,

$$\|u\|_{L^2(X,\mu)} \le \|u\|^{\theta}_{L^1(X,\mu)}\|u\|^{1-\theta}_{L^{2^*}(X,\mu)}$$

where $\frac{1}{2} = \frac{\theta}{1} + \frac{1-\theta}{2^*}$ i.e. $\theta = \frac{2}{N+2}$. Then by Theorem 1 applied for $p = 2 < \eta$,

$$\|u\|_{L^2(X,\mu)} \le C\|u\|^{\frac{2}{N+2}}_{L^1(X,\mu)}\||Df|\|^{\frac{N}{N+2}}_{L^2(X,\mathfrak{m})}.$$

As $u \in \mathrm{Lip_b}(X)$, $\mathsf{Ch}(f) = \||Df|\|^2_{L^2(X,m)}$. The result follows from the previous inequality raised to the power $2(N+2)/N$. □

We deduce now from Theorem 4 a bound on the weighted heat kernel of k-Ahlfors regular $\mathrm{RCD}(0,N)$ spaces. Let us explain which weighted heat kernel we are dealing with. We consider $w_o = V(o,\mathsf{d}(o,\cdot))^{2/(N-2)}\mathsf{d}(o,\cdot)^{-2N/(N-2)}$, i.e. the case $p = 2$. Define the Dirichlet form Q on $L^2(X,\mu)$ as the restriction of Ch to $\mathcal{D}(Q) := W^{1,2}(X,\mathsf{d},\mathfrak{m}) \cap L^2(X,\mu)$. Denote by $(h^\mu_t)_{t>0}$ the semi-group associated to Q. This semi-group admits a self-adjoint generator $-A$ defined on a set $\mathcal{D}(A)$ dense in $\mathcal{D}(Q)$ and characterized by:

$$Q(f,g) = \int_X (Af)g\,\mathrm{d}\mu \qquad \forall f \in \mathcal{D}(A),\, \forall g \in \mathcal{D}(Q).$$

As for any $f \in \mathcal{D}(\Delta)$ and any $g \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$,

$$\mathsf{Ch}(f,g) = \int_X (-\Delta f)g\,\mathrm{d}\mathfrak{m} = \int_X (-w_o^{-1}\Delta f)g\,\mathrm{d}\mu,$$

$\mathcal{D}(A) = \mathcal{D}(\Delta) \cap L^2(X,\mu)$ and $A = w_o^{-1}\Delta$. The semi-group $(h^\mu_t)_{t>0}$ is then characterized by the fact that for any $f \in L^2(X,\mu)$, $t \to h^\mu_t f$ is locally absolutely continuous on $(0,+\infty)$ with values in $L^2(X,\mu)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}h^\mu_t f = -Ah^\mu_t f \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,+\infty).$$

By the Markov property, each h^μ_t can be uniquely extended from $L^2(X,\mu) \cap L^1(X,\mu)$ to a contraction from $L^1(X,\mu)$ to itself. Note that if $1 \le p,q \le +\infty$ and L is a bounded operator from $L^p(X,\mu)$ to $L^q(X,\mu)$, we denote by $\|L\|_{L^p(X,\mu)\to L^q(X,\mu)}$ its norm.

Theorem 5 (Bound of the weighted heat kernel)**.** *Assume that $N \geq 3$, and let $k \in (2, N] \cap \mathbb{N}$. Assume that $(X, \mathsf{d}, \mathfrak{m})$ is a k-Ahlfors regular* $\mathrm{RCD}(0, N)$ *space satisfying the growth condition (1.1). Then there exists $C = C(N, \Theta_{inf}, \Theta_{sup}) > 0$ such that*

$$\|h_t^\mu\|_{L^1(X,\mu)\to L^\infty(X,\mu)} \leq \frac{C}{t^{N/2}}, \qquad \forall t > 0,$$

or equivalently, for any $t > 0$, h_t^μ admits a kernel p_t^μ with respect to μ such that for every $x, y \in X$,

$$p_t^\mu(x, y) \leq \frac{C}{t^{N/2}}.$$

To prove this theorem we follow closely the lines of [SC02, Th. 4.1.1.]. The constant C may differ from line to line, note however that it will always depend only on N, Θ_{inf} and Θ_{sup}. For better readability, we will write $L^p(\mu)$ instead of $L^p(X, \mu)$.

Proof. Take $f \in L^1(\mu) \cap L^2(\mu) \cap \mathcal{D}(A)$ with $\|f\|_{L^1(\mu)} = 1$. Then by contraction $\|h_t^\mu f\|_{L^1(\mu)} \leq 1$ for any $t > 0$, so by Theorem 4,

$$\|h_t^\mu f\|_{L^2(\mu)}^{2+4/N} \leq C\mathsf{Ch}(h_t^\mu f). \tag{4.1}$$

As $h_t^\mu f \in \mathcal{D}(A)$ for any $t > 0$,

$$\mathsf{Ch}(h_t^\mu f) = Q(h_t^\mu f, h_t^\mu f) = \int_X (Ah_t^\mu f) h_t^\mu f \,\mathrm{d}\mathfrak{m} = -\int_X \left(\frac{\mathrm{d}}{\mathrm{d}t} h_t^\mu f\right) h_t^\mu f \,\mathrm{d}\mathfrak{m} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h_t^\mu f\|_{L^2(\mu)}^2.$$

Therefore, (4.1) becomes $u(t)^{1+2/N} \leq -\frac{C}{2}u'(t)$, where $u(t) = \|h_t^\mu f\|_{L^2(\mu)}^2$. Write $v(t) = \frac{N}{2}u(t)^{-2/N}$ to get $\frac{2}{C} \leq -v'(t)$ and thus $\frac{2}{C}t \leq v(t) - v(0)$. As $v(0) = \frac{N}{2}\|h_t^\mu f\|_{L^2(\mu)}^{-4/N} \geq 0$, one gets $\frac{2}{C}t \leq v(t)$, leading to

$$\|h_t^\mu f\|_{L^2(\mu)} \leq \frac{C}{t^{N/4}}.$$

Therefore $\|h_t^\mu\|_{L^1(\mu)\to L^2(\mu)} \leq \frac{C}{t^{N/4}}$. Using the self-adjointness of $h_t f$, one deduces by duality $\|h_t^\mu\|_{L^2(\mu)\to L^\infty(\mu)} \leq \frac{C}{t^{N/4}}$. Finally the semi-group property

$$\|h_t^\mu\|_{L^1(\mu)\to L^\infty(\mu)} \leq \|h_{t/2}^\mu\|_{L^1(\mu)\to L^2(\mu)}\|h_{t/2}^\mu\|_{L^2(\mu)\to L^\infty(\mu)}$$

implies the result. □

Remark 5. *The author spent quite a long time in trying to extend [Min09, Th. 0.5] to the* $\mathrm{RCD}(0, N)$ *context, in which the L^2-cohomology of a space makes sense [G15b, Section 3.5.2]. However, giving a meaning in a non-smooth context to the critical integrability assumption*

$$\int_M |\mathrm{Rm}|^{n/2}\,\mathrm{d}V_g < +\infty$$

where Rm *denotes the curvature tensor of a Riemannian manifold (M, g), seems presently hardly feasible.*

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