

Controllability and optimal control of the transport equation with a localized vector field*

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Abstract—We study controllability of a Partial Differential Equation of transport type, that arises in crowd models. We are interested in controlling such system with a control being a Lipschitz vector field on a fixed control set ω .

We prove that, for each initial and final configuration, one can steer one to another with such class of controls only if the uncontrolled dynamics allows to cross the control set ω .

We also prove a minimal time result for such systems. We show that the minimal time to steer one initial configuration to another is related to the condition of having enough mass in ω to feed the desired final configuration.

I. INTRODUCTION

In recent years, the study of systems describing a crowd of interacting autonomous agents has draw a great interest from the control community (see e.g. the Cucker-Smale model [4]). A better understanding of such interaction phenomena can have a strong impact in several key applications, such as road traffic and egress problems for pedestrians. Beside the description of interaction, it is now relevant to study problems of **control of crowds**, i.e. of controlling such systems by acting on few agents, or on the crowd localized in a small subset of the configuration space.

Two main classes are widely used to model crowds of interacting agents. In **microscopic models**, the position of each agent is clearly identified; the crowd dynamics is described by a large dimensional ordinary differential equation, in which couplings of terms represent interactions. In **macroscopic models**, instead, the idea is to represent the crowd by the spatial density of agents; in this setting, the evolution of the density solves a partial differential equation of transport type. This is an example of a **distributed parameter system**. Some nonlocal terms can model the interactions between the agents. In this article, we focus on this second approach.

To our knowledge, there exist few studies of control of this kind of equations. In [7], the authors provide approximate alignment of a crowd described by the Cucker-Smale model [4]. The control is the acceleration, and it is localized in a control region ω which moves in time. In a similar

situation, a stabilization strategy has been established in [2], by generalizing the Jurdjevic-Quinn method to distributed parameter systems.

In this article, we study a partial differential equation of transport type, that is widely used for modeling of crowds. Let ω be a nonempty open connected subset of \mathbb{R}^d ($d \geq 1$), being the portion of the space on which the control is allowed to act. Let $v : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a vector field assumed Lipschitz and uniformly bounded. Consider the following linear transport equation

$$\begin{cases} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u)\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \quad (1)$$

where $\mu(t)$ is the time-evolving measure representing the crowd density and μ^0 is the initial data. The control is the function $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$. The function $v + \mathbb{1}_\omega u$ represents the velocity field acting on μ . System (1) is a first approximation for crowd modeling, since the uncontrolled vector field v is given, and it does not describe interactions between agents. Nevertheless, it is necessary to understand controllability properties for such simple equation. Indeed, the results contained in this article will be instrumental to a forthcoming paper, where we will study more complex crowd models, with a non-local term $v[\mu]$.

We now recall the precise notion of approximate controllability for System (1). We say that System (1) is *approximately controllable* from μ^0 to μ^1 on the time interval $(0, T)$ if for each $\varepsilon > 0$ there exists $\mathbb{1}_\omega u$ such that the corresponding solutions to System (1) satisfies $W_p(\mu(T), \mu^1) \leq \varepsilon$. The definition of the Wasserstein distance W_p is recalled in Section II.

To control System (1), from a geometrical point of view, the uncontrolled vector field v needs to send the support of μ^0 to ω forward in time and the support of μ^1 to ω backward in time. This idea is formulated in the following Condition:

Condition 1 (Geometrical condition) Let μ^0, μ^1 be two probability measures on \mathbb{R}^d satisfying:

- (i) For all $x^0 \in \text{supp}(\mu^0)$, there exists $t^0 > 0$ such that $\Phi_{t^0}^v(x^0) \in \omega$, where Φ_t^v is the *flow* associated to v , i.e. the solution to the Cauchy problem

$$\begin{cases} \dot{x}(t) = v(x(t)) \text{ for a.e. } t > 0, \\ x(0) = x^0. \end{cases}$$

- (ii) For all $x^1 \in \text{supp}(\mu^1)$, there exists $t^1 > 0$ such that $\Phi_{-t^1}^v(x^1) \in \omega$.

Remark 1 Condition 1 is the minimal one that we can expect to steer any initial condition to any targets. Indeed, if the first

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item of Condition 1 is not satisfied, there exists a whole sub-population of the measure μ_0 that never intersects the control region, thus, we cannot act on it.

We denote by \mathcal{U} the set of admissible controls, that are functions $\mathbb{1}_{\omega}u : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ Lipschitz in space, measurable in time and uniformly bounded. If we impose the classical Carathéodory condition of $\mathbb{1}_{\omega}u$ being in \mathcal{U} , then the flow $\Phi_t^{v+\mathbb{1}_{\omega}u}$ is an homeomorphism (see [1, Th. 2.1.1]). As a result, one cannot expect exact controllability, since for general measures there exists no homeomorphism sending one to another. We then have the following result of approximate controllability.

Theorem 1 *Let μ^0, μ^1 be two probability measures on \mathbb{R}^d compactly supported absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1. Then there exists $T > 0$ such that System (1) is approximately controllable at time T from μ^0 to μ^1 with a control $\mathbb{1}_{\omega}u$ in \mathcal{U} .*

The proof of this result will be given in Section III. After having proven approximate controllability for System (1), we aim to study the minimal time problem, i.e. the minimal time to send μ_0 to μ_1 . We have the following result.

Theorem 2 *Let μ^0, μ^1 be two probability measures, with compact support, absolutely continuous with respect to the Lebesgue measure and satisfying Condition 1.*

We say that T^ is an admissible time if it satisfies*

(a) *For each $x^0 \in \text{supp}(\mu^0)$*

$$T^* \geq \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \omega\}.$$

(b) *For each $x^1 \in \text{supp}(\mu^1)$*

$$T^* \geq \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x^1) \in \omega\}.$$

(c) *There exists a sequence $(u_k)_k$ of C^∞ -functions equal to 0 in ω^c such that*

$$\lim_{k \rightarrow \infty} [\Phi_t^{v+u_k} \# \mu^0](\omega) \geq 1 - \lim_{k \rightarrow \infty} [\Phi_{t-T^*}^{v+u_k} \# \mu^1](\omega). \quad (2)$$

Let T_0 be the infimum of such T^ . Then, for all $T > T_0$, System (1) is approximately controllable from μ^0 to μ^1 at time T .*

The proof of this Theorem is given in Section IV.

Remark 2 The meaning of condition (2) is the following: functions u_k are used to store the mass in ω . Thus, condition (2) means that at each time t there is more mass that has entered ω than mass that has exited. This is the minimal condition that we can expect in this setting, since control can only move masses, without creating them.

This paper is organized as follows. In Section II, we recall some properties of the continuity equation and the Wasserstein distance. Sections III and IV are devoted to prove Theorems 1 and 2, respectively. We conclude with some numerical examples in Section V.

II. THE CONTINUITY EQUATION AND THE WASSERSTEIN DISTANCE

In this section, we recall some properties of the continuity equation (1) and of the Wasserstein distance, which will be used all along this paper.

We denote by $\mathcal{P}_c(\mathbb{R}^d)$ the space of probability measures in \mathbb{R}^d with compact support, and by $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ the subset of $\mathcal{P}_c(\mathbb{R}^d)$ of measures which are absolutely continuous with respect to the Lebesgue measure. First of all, we give the definition of the push-forward of a measure and of the Wasserstein distance.

Definition 1 Denote by Γ the set of the Borel maps $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$. For a $\gamma \in \Gamma$, we define the push-forward $\gamma \# \mu$ of a measure μ of \mathbb{R}^d as follows:

$$(\gamma \# \mu)(E) := \mu(\gamma^{-1}(E)),$$

for every subset E such that $\gamma^{-1}(E)$ is μ -measurable.

Definition 2 Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. Define

$$W_p(\mu, \nu) = \inf_{\gamma \in \Gamma} \left\{ \left(\int_{\mathbb{R}^d} |\gamma(x) - x|^p d\mu \right)^{1/p} : \gamma \# \mu = \nu \right\}. \quad (3)$$

Proposition 1 W_p is a distance on $\mathcal{P}_c^{ac}(\mathbb{R}^d)$, called the Wasserstein distance.

The Wasserstein distance can be extended to all pairs of measures μ, ν compactly supported with the same mass $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d) \neq 0$, by the formula

$$W_p(\mu, \nu) = |\mu|^{1/p} W_p \left(\frac{\mu}{|\mu|}, \frac{\nu}{|\nu|} \right).$$

For more details about the Wasserstein distance, in particular for its definition on the whole space of measures $\mathcal{P}_c(\mathbb{R}^d)$, we refer to [8, Chap. 7].

We now recall a standard result for the continuity equation:

Theorem 3 (see [8]) *Let $T \in \mathbb{R}$, $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ and w be a vector field uniformly bounded, Lipschitz in space and measurable in time. Then the system*

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d \end{cases} \quad (4)$$

admits a unique solution¹ μ in $C^0([0, T]; \mathcal{P}_c^{ac}(\mathbb{R}^d))$. Moreover, it holds $\mu(\cdot, t) = \Phi_t^w \# \mu^0$ for all $t \in \mathbb{R}$, where the flow $\Phi_t^w(x^0)$ is the unique solution at time t to

$$\begin{cases} \dot{x}(t) = w(x(t), t) \text{ for a.e. } t \geq 0, \\ x(0) = x^0. \end{cases} \quad (5)$$

In the rest of the paper, the following properties of the Wasserstein distance will be helpful.

Property 1 (see [6]) *Let $\mu, \nu \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$. Let $w : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ be a vector field uniformly bounded, Lipschitz in space and measurable in time. For each $t \in \mathbb{R}$, it holds*

$$W_p^p(\Phi_t^w \# \mu, \Phi_t^w \# \nu) \leq e^{(p+1)L|t|} W_p^p(\mu, \nu), \quad (6)$$

where L is the Lipschitz constant of w .

¹Here, $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ is equipped with the weak topology, that coincides with the topology induced by the Wasserstein distance W_p , see [8, Thm 7.12].

Property 2 Let μ, ν, ρ, η some positive measures satisfying $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ and $\rho(\mathbb{R}^d) = \eta(\mathbb{R}^d)$. It then holds

$$W_p^p(\mu + \rho, \nu + \eta) \leq W_p^p(\mu, \nu) + W_p^p(\rho, \eta). \quad (7)$$

Using the properties of Wasserstein distance given in Section 1 of [6], we can replace W_p by W_1 in the definition of the approximate controllability.

III. PROOF OF THEOREM 1

In this section, we prove approximate controllability of System (1). The proof is based on three approximation steps, corresponding to Proposition 2, 3, and 4. The proof is then given at the end of the section.

In a first step, we suppose that the open connected control subset ω contains the support of both μ^0, μ^1 .

Proposition 2 Let $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ be such that $\text{supp}(\mu^0) \subset \omega$ and $\text{supp}(\mu^1) \subset \omega$. Then, for all $T > 0$, System (1) is approx. contr. at time T with $\mathbb{1}_{\omega} u$ in U .

Proof: We assume that $d := 2, T := 1$ and $\omega := (0, 1)^2$, but the reader will see that the proof can be clearly adapted to any space dimension. Fix $n \in \mathbb{N}^*$. Define $a_0 := 0, b_0 := 0$ and the points a_i, b_i for all $i \in \{1, \dots, n\}$ by induction as follows: suppose that for $i \in \{0, \dots, n-1\}$ the points a_i and b_i are given, then a_{i+1} and b_{i+1} are the smallest values satisfying

$$\int_{(a_i, a_{i+1}) \times \mathbb{R}} d\mu^0 = \frac{1}{n} \quad \text{and} \quad \int_{(b_i, b_{i+1}) \times \mathbb{R}} d\mu^1 = \frac{1}{n}.$$

Again, for all $i \in \{0, \dots, n-1\}$, we define $a_{i,0} := 0, b_{i,0} := 0$ and supposing that for a $j \in \{0, \dots, n-1\}$ the points $a_{i,j}$ and $b_{i,j}$ are already defined, $a_{i,j+1}$ and $b_{i,j+1}$ are the smallest values such that

$$\int_{A_{ij}} d\mu^0 = \frac{1}{n^2} \quad \text{and} \quad \int_{B_{ij}} d\mu^1 = \frac{1}{n^2},$$

where $A_{ij} := (a_i, a_{i+1}) \times (a_{ij}, a_{i(j+1)})$ and $B_{ij} := (b_i, b_{i+1}) \times (b_{ij}, b_{i(j+1)})$. Since μ^0 and μ^1 have a mass equal to 1 and are supported in $(0, 1)^2$, then $a_n, b_n \leq 1$ and $a_{i,n}, b_{i,n} \leq 1$ for all $i \in \{0, \dots, n-1\}$. We give in Figure 1 an example of such decomposition.

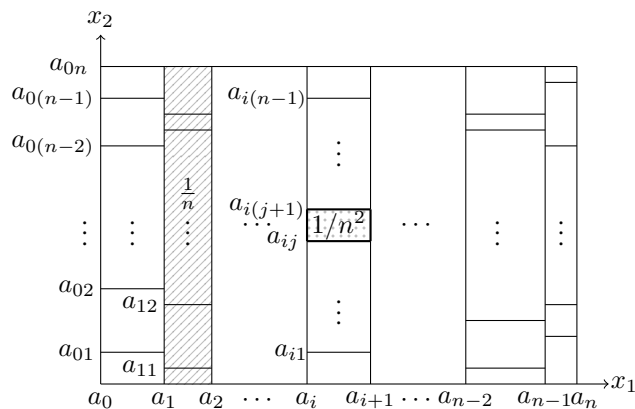


Fig. 1. Example of a decomposition of μ^0 .

If one aims to define a vector field sending each A_{ij} to B_{ij} , then some shear stress is naturally introduced to the

interfaces of the cells. To overcome this problem, we first define sets $\tilde{A}_{ij} \subset\subset A_{ij}$ and $\tilde{B}_{ij} \subset\subset B_{ij}$ for all $i, j \in \{0, \dots, n-1\}$. We then send the mass of μ^0 from each \tilde{A}_{ij} to each \tilde{B}_{ij} , while we do not control the mass contained in $A_{ij} \setminus \tilde{A}_{ij}$. More precisely, for all $i, j \in \{0, \dots, n-1\}$, we define, $a_i^-, a_i^+, a_{ij}^-, a_{ij}^+$ the smallest values such that

$$\int_{(a_i, a_i^-) \times (a_{ij}, a_{i(j+1)})} d\mu^0 = \int_{(a_i^+, a_{i+1}) \times (a_{ij}, a_{i(j+1)})} d\mu^0 = \frac{1}{n^3}$$

$$\begin{aligned} \int_{(a_i^-, a_i^+) \times (a_{ij}, a_{ij}^-)} d\mu^0 &= \int_{(a_i^-, a_i^+) \times (a_{ij}^+, a_{i(j+1)})} d\mu^0 \\ &= \frac{1}{n} \times \left(\frac{1}{n^2} - \frac{2}{n^3} \right). \end{aligned}$$

We similarly define $b_i^+, b_i^-, b_{ij}^+, b_{ij}^-$. We finally define

$$\tilde{A}_{ij} := [a_i^-, a_i^+] \times [a_{ij}^-, a_{ij}^+] \quad \text{and} \quad \tilde{B}_{ij} := [b_i^-, b_i^+] \times [b_{ij}^-, b_{ij}^+].$$

The goal is to build a solution to System (1) such that the corresponding flow Φ_t^u satisfies

$$\Phi_T^u(\tilde{A}_{ij}) = \tilde{B}_{ij}, \quad (8)$$

for all $i, j \in \{0, \dots, n-1\}$. We observe that we do not take into account the displacement of the mass contained in $A_{ij} \setminus \tilde{A}_{ij}$. We will show that the corresponding term $W_1(\sum_{ij} \Phi_T^{v+u} \# \mu^0|_{A_{ij} \setminus \tilde{A}_{ij}}, \sum_{ij} \mu^1|_{B_{ij} \setminus \tilde{B}_{ij}})$ tends to zero when n goes to the infinity. The rest of the proof is divided into two steps. In a first step, we build a flow and a velocity field such that its flow satisfies (8). In a second step, we compute the Wasserstein distance between μ^1 and $\mu(T)$ showing that it converges to zero when n goes to infinity.

Step 1: We first build a flow satisfying (8). For all $i \in \{0, \dots, n-1\}$, we denote by c_i^- and c_i^+ the linear functions equal to a_i^- and a_i^+ at time $t = 0$ and equal to b_i^- and b_i^+ at time $t = T = 1$, respectively i.e.

$$c_i^-(t) = (b_i^- - a_i^-)t + a_i^- \quad \text{and} \quad c_i^+(t) = (b_i^+ - a_i^+)t + a_i^+.$$

Similarly, for all $i, j \in \{0, \dots, n-1\}$, we denote by c_{ij}^- and c_{ij}^+ the linear functions equal to a_{ij}^- and a_{ij}^+ at time $t = 0$ and equal to b_{ij}^- and b_{ij}^+ at time $t = T = 1$, respectively, i.e.

$$c_{ij}^-(t) = (b_{ij}^- - a_{ij}^-)t + a_{ij}^- \quad \text{and} \quad c_{ij}^+(t) = (b_{ij}^+ - a_{ij}^+)t + a_{ij}^+.$$

Consider the application being the following linear combination of c_i^-, c_i^+ and c_{ij}^-, c_{ij}^+ in \tilde{A}_{ij} , i.e.

$$x(x^0, t) := \begin{pmatrix} \frac{a_i^+ - x_1^0}{a_i^+ - a_i^-} c_i^-(t) + \frac{x_1^0 - a_i^-}{a_i^+ - a_i^-} c_i^+(t) \\ \frac{a_{ij}^+ - x_2^0}{a_{ij}^+ - a_{ij}^-} c_{ij}^-(t) + \frac{x_2^0 - a_{ij}^-}{a_{ij}^+ - a_{ij}^-} c_{ij}^+(t) \end{pmatrix}, \quad (9)$$

when $x^0 \in \tilde{A}_{ij}$. Let us prove that an extension of the application $(x^0, t) \mapsto \Phi_t^u(x^0) := x(x^0, t)$ is a flow associated to a velocity field u . We remark that $t \mapsto x(x^0, t)$ is C^1 and is solution to

$$\begin{cases} \frac{dx_1(x^0, t)}{dt} = \alpha_i(t)x_1(x^0, t) + \beta_i(t) & \forall t \in [0, T], \\ \frac{dx_2(x^0, t)}{dt} = \alpha_{ij}(t)x_2(x^0, t) + \beta_{ij}(t) & \forall t \in [0, T], \end{cases}$$

where for all $t \in [0, 1]$

$$\begin{cases} \alpha_i(t) = \frac{b_i^+ - b_i^- + a_i^- - a_i^+}{c_i^+(t) - c_i^-(t)}, & \beta_i(t) = \frac{a_i^+ b_i^- - a_i^- b_i^+}{c_i^+(t) - c_i^-(t)}, \\ \alpha_{ij}(t) = \frac{b_{ij}^+ - b_{ij}^- + a_{ij}^- - a_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}, & \beta_{ij}(t) = \frac{a_{ij}^+ b_{ij}^- - a_{ij}^- b_{ij}^+}{c_{ij}^+(t) - c_{ij}^-(t)}. \end{cases}$$

For all $t \in [0, 1]$, consider the set $C_{ij}(t) := [c_i^-(t), c_i^+(t)] \times [c_{ij}^-(t), c_{ij}^+(t)]$. We remark that $C_{ij}(0) = \tilde{A}_{ij}$ and $C_{ij}(T) = \tilde{B}_{ij}$. On $C_{ij} := \{(x, t) : t \in [0, T], x \in C_{ij}(t)\}$, we then define the velocity field u by

$u_1(x, t) = \alpha_i(t)x_1 + \beta_i(t)$ and $u_2(x, t) = \alpha_{ij}(t)x_2 + \beta_{ij}(t)$, for all $(x, t) \in C_{ij}$ ($x = (x_1, x_2)$). We extend u by a \mathcal{C}^∞ and uniformly bounded function outside $\cup_{ij} C_{ij}$, then having $u \in \mathcal{U}$. Then, System (1) admits a unique solution and the flow on C_{ij} is given by the expression (9).

Step 2: We now prove that the refinement of the grid provides convergence to the target μ_1 , *i.e.*

$$W_1(\mu^1, \mu(T)) \xrightarrow{n \rightarrow \infty} 0. \quad (10)$$

We remark that

$$\int_{\tilde{B}_{ij}} d\mu(T) = \int_{\tilde{B}_{ij}} d\mu^1 = \frac{(n-2)^2}{n^4}.$$

Hence, by defining $R := (0, 1)^2 \setminus \bigcup_{ij} \tilde{B}_{ij}$, we also have

$$\int_R d\mu(T) = \int_R d\mu^1 = 1 - \frac{(n-2)^2}{n^2}.$$

It comes that

$$\begin{aligned} W_1(\mu^1, \mu(T)) &\leq \sum_{i,j=1}^n W_1(\mu^1 \times \mathbb{1}_{\tilde{B}_{ij}}, \mu(T) \times \mathbb{1}_{\tilde{B}_{ij}}) \\ &\quad + W_1(\mu^1 \times \mathbb{1}_R, \mu(T) \times \mathbb{1}_R). \end{aligned} \quad (11)$$

We estimate each term in the right-hand side. Since we deal with absolutely continuous measures, using Proposition 2, there exist measurable maps $\gamma_{ij} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, for all $i, j \in \{0, \dots, n-1\}$, and $\bar{\gamma} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\gamma_{ij} \# (\mu^1 \times \mathbb{1}_{\tilde{B}_{ij}}) = \mu(T) \times \mathbb{1}_{\tilde{B}_{ij}}$$

and

$$\bar{\gamma} \# (\mu^1 \times \mathbb{1}_R) = \mu(T) \times \mathbb{1}_R.$$

In the first term, for each $i, j \in \{0, \dots, n-1\}$, observe that γ_{ij} moves masses inside B_{ij} only. Thus

$$\begin{aligned} W_1(\mu^1 \times \mathbb{1}_{\tilde{B}_{ij}}, \mu(T) \times \mathbb{1}_{\tilde{B}_{ij}}) &= \int_{\tilde{B}_{ij}} |x - \gamma_{ij}(x)| d\mu^1(x) \\ &\leq (b_i^+ - b_i^- + b_{ij}^+ - b_{ij}^-) \frac{(n-2)^2}{n^4}. \end{aligned} \quad (12)$$

Concerning the second term in (11), observe that $\bar{\gamma}$ moves a small mass in the bounded set ω . Thus it holds

$$\begin{aligned} W_1(\mu^1 \times \mathbb{1}_R, \mu(T) \times \mathbb{1}_R) &\leq \int_R |x - \bar{\gamma}(x)| d\mu^1(x) \\ &\leq \sqrt{2} \left(1 - \frac{(n-2)^2}{n^2}\right) = 4\sqrt{2} \frac{n-1}{n^2}. \end{aligned} \quad (13)$$

We thus have (10) by combining (11), (12) and (13). \blacksquare

In the rest of the section, we remove the constraints $\text{supp}(\mu^0) \subset \omega$ and $\text{supp}(\mu^1) \subset \omega$, now imposing Condition 1. First of all, we give a consequence of Condition 1.

Lemma 1 *If Condition 1 is satisfied for μ^0 , $\mu^1 \in \mathcal{P}_c(\mathbb{R}^d)$, then the following Condition 2 is satisfied too:*

Condition 2 There exist two real numbers $T_0^*, T_1^* > 0$ and a non-empty open set $\omega_0 \subset \subset \omega$ such that

- (i) For all $x^0 \in \text{supp}(\mu^0)$, there exists $t^0 \in [0, T_0^*]$ such that $\Phi_{t^0}^v(x^0) \in \omega_0$.
- (ii) For all $x^1 \in \text{supp}(\mu^1)$, there exists $t^1 \in [0, T_1^*]$ such that $\Phi_{-t^1}^v(x^1) \in \omega_0$.

Proof: We use a compactness argument. Let $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ and assume that Condition 1 holds. Let $x^0 \in \text{supp}(\mu^0)$. Using Condition 1, there exists $t^0(x^0) > 0$ such that $\Phi_{t^0(x^0)}^v(x^0) \in \omega$. Choose $r(x^0) > 0$ such that $B_{r(x^0)}(\Phi_{t^0(x^0)}^v(x^0)) \subset \subset \omega$, that exists since ω is open. By continuity of the application $x^1 \mapsto \Phi_{t^0(x^0)}^v(x^1)$ (see [1, Th. 2.1.1]), there exists $\hat{r}(x^0)$ such that

$$x^1 \in B_{\hat{r}(x^0)}(x^0) \Rightarrow \Phi_{t^0(x^0)}^v(x^1) \in B_{r(x^0)}(\Phi_{t^0(x^0)}^v(x^0)).$$

Since μ^0 is compactly supported, we can find a set $\{x_1^0, \dots, x_N^0\} \subset \text{supp}(\mu^0)$ such that

$$\text{supp}(\mu^0) \subset \bigcup_{i=1}^N B_{\hat{r}(x_i^0)}(x_i^0).$$

Thus the first item of Lemma 1 is satisfied for

$$T_0^* := \max\{t^0(x_i^0)\} \quad \text{and} \quad \omega_0 := \bigcup_{i=1}^N B_{r(x_i^0)}(\Phi_{t^0(x_i^0)}^v(x_i^0)).$$

The proof of the existence of T_1^* is similar. \blacksquare

We now prove that we can store nearly the whole mass of μ_0 in ω , under Condition 2.

Proposition 3 *Let $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ satisfying the first item of Condition 2. Then there exists $\mathbb{1}_\omega u \in \mathcal{U}$ such that*

$$\text{supp}(\mu(T_0^*)) \subset \omega. \quad (14)$$

Proof: Let $k \in \mathbb{N}^*$. We denote by $\alpha := d(\omega, \omega_0)$, $\omega_1 := \{x^0 \in \mathbb{R}^d : d(x^0, \omega_0) < \alpha/2\}$ and $S_k := \{x^0 \in \mathbb{R}^d : d(x^0, \omega_0) < \alpha/2k\}$. We define θ_k a cutoff function on ω of class \mathcal{C}^∞ satisfying

$$\begin{cases} 0 \leq \theta_k \leq 1, \\ \theta_k = 1 \text{ in } S_k^c, \\ \theta_k = 0 \text{ in } \omega_0. \end{cases} \quad (15)$$

Define

$$u_k := (\theta_k - 1)v. \quad (16)$$

We remark that the support of u_k is included in ω . Let $x^0 \in \text{supp}(\mu^0)$. Define

$$t^*(x^0) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \bar{\omega}_0\} \leq T_0^*.$$

Consider the flow $y := \Phi_t^v(x^0)$ associated to x^0 without control, *i.e.* the solution to

$$\begin{cases} \dot{y}(t) = v(y(t)), \\ y(0) = x^0 \end{cases}$$

and the flow $z_k := \Phi_t^{u_k+v}(x^0)$ associated to x^0 with the control u_k given in (16), *i.e.* the solution to

$$\begin{cases} \dot{z}_k(t) = (v + u_k)(z_k(t)) = \theta_k(z_k(t)) \times v(z_k(t)), \\ z_k(0) = x^0. \end{cases} \quad (17)$$

We now prove that the range of z_k for $t \geq 0$ is included in the range of y for $t \geq 0$. Consider the solution γ_k to the following system

$$\begin{cases} \dot{\gamma}_k(t) = \theta_k(y(\gamma_k(t))), \quad t \geq 0, \\ \gamma_k(0) = 0. \end{cases} \quad (18)$$

Since θ_k and y are Lipschitz, then System (18) admits a solution defined for all times. We remark that $\xi_k := y \circ \gamma_k$ is solution to System (17). Indeed for all $t \geq 0$

$$\begin{cases} \dot{\xi}_k(t) = \dot{\gamma}_k(t) \times y(\gamma_k(t)) = \theta_k(\xi_k(t)) \times v(\xi_k(t)), \\ \xi_k(0) = y(\gamma_k(0)) = y(0). \end{cases}$$

By uniqueness of the solution to System (17), we obtain

$$y(\gamma_k(t)) = z_k(t) \text{ for all } t \geq 0.$$

Using the fact that $0 \leq \theta \leq 1$ and the definition of γ_k , we have

$$\begin{cases} \gamma_k \text{ increasing,} \\ \gamma_k(t) \leq t & \forall t \in [0, t^*(x^0)], \\ \gamma_k(t) \leq t^*(x^0) & \forall t \geq t^*(x^0). \end{cases}$$

We deduce that, for all $x^0 \in \text{supp}(\mu^0)$,

$$\{z_k(t) : t \geq 0\} \subset \{y(s) : s \in [0, t^*(x^0)]\}.$$

We now prove that for all k large enough, there exists $t \in (0, t^*(x^0))$ such that for all $s > t$, then $\Phi_s^{u_k+v}(x^0) \in \omega_1$. Consider $B := B_{\alpha/2}(\Phi_{t^*}^v(x^0)) \subset \omega_1$. By continuity, there exists $\beta > 0$ such that $\Phi_t^v(x^0) \in B$ for all $t \in (t^* - \beta, t^*)$. For all $s \in [0, t^* - \beta]$, we can find $r(s) > 0$ such that $B_{r(s)}(\phi_s^v(x^0)) \subset \omega_0^c$. By compactness of $\{\phi_s^v(x^0) : s \in [0, t^* - \beta]\}$, there exists a finite subcover $\{B_{r(s_i)}(\phi_{s_i}^v(x^0))\}_{1 \leq i \leq n}$ of $\{\phi_s^v(x^0) : s \in [0, t^* - \beta]\}$. We denote by $R := \frac{1}{2} \min\{r(s_i)\}$. Let k be such that $\alpha/2k < R$. Thus

$$\begin{cases} \Phi_s^{v+u_k}(x^0) = \Phi_s^v(x^0), & \text{for all } s \leq t^* - \beta, \\ \Phi_s^{v+u_k}(x^0) \in B \subset \omega_1, & \text{for all } s > t^* - \beta. \end{cases}$$

There exists a ball $B_r(x^0)$, such that $\Phi_s^{v+u_k}(x^1) \in \omega_1$ for all $x^1 \in B_r(x^0)$ and $s > t^* - \beta$. Thus, by compactness of $\text{supp}(\mu^0)$, for k large enough, $\Phi_{T_0^*}^{v+u_k}(x^0) \in \omega_1$ for all $x^0 \in \text{supp}(\mu^0)$. ■

The third step of the proof is to restrict a measure contained in ω to a measure contained in a hypercube $S \subset \omega$.

Proposition 4 *Let $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ satisfying $\text{supp}(\mu^0) \subset \omega$. Define S an open hypercube strictly included in ω and choose $\delta > 0$. Then there exists $\mathbb{1}_\omega u \in \mathcal{U}$ such that the corresponding solution to System (1) satisfies*

$$\text{supp}(\mu(\delta)) \subset S.$$

Proof: From [5, Lemma 1.1, Chap. 1] and [3, Lemma 2.68, Chap. 2], there exists a function $\eta \in \mathcal{C}^2(\bar{\omega})$ satisfying $\kappa_0 \leq |\nabla \eta| \leq \kappa_1$ in $\omega \setminus S$, $\eta > 0$ in ω and $\eta = 0$ on $\partial\omega$, with $\kappa_0, \kappa_1 > 0$. We extend η by zero outside of ω . $S \subset \subset \omega_k$. We denote by

$$u_k := k \nabla \eta.$$

Let $x^0 \in \text{supp}(\mu^0)$. Consider the flow $z_k(t) = \Phi_t^{v+u_k}(x^0)$ associated to x^0 , i.e. the solution to system

$$\begin{cases} \dot{z}_k(t) = v(z(t)) + u_k(z_k(t)), & t \geq 0, \\ z_k(0) = x^0. \end{cases}$$

The properties of η imply that $n \cdot \nabla \eta < C < 0$ on $\partial\omega$, where n represents that exterior normal vector to $\partial\omega$. We deduce that, for k large enough, $n \cdot (v + k \nabla \eta) < 0$ on $\partial\omega$. Thus $z_k(t) \in \omega$ for all $t \geq 0$.

We now prove that there exists $K \in \mathbb{N}^*$ and $T \in (0, \delta)$ such that for all $k > K$ and $t \in [T, \delta]$, $z_k(t) \in S$ for all $x^0 \in \text{supp}(\mu^0)$. By contradiction, assume that there exists three sequences $\{k_n\}_{n \in \mathbb{N}^*} \subset \mathbb{N}^*$, $\{t_n\}_{n \in \mathbb{N}^*} \subset (0, \delta)$ and $\{x_n^0\}_{n \in \mathbb{N}^*} \in \text{supp}(\mu^0)$ satisfying $k_n \rightarrow \infty$, $t_n \rightarrow \delta$ and

$$z_{k_n}(x_n^0, t_n) \in S^c. \quad (19)$$

Consider the function f_n defined for all $t \in [0, \delta]$ by

$$f_n(t) := \eta(z_{k_n}(t)).$$

Its time derivative is given by

$$\dot{f}_n(t) = k_n |\nabla \eta(z_{k_n}(t))|^2 + v(z_{k_n}(t)) \cdot \nabla \eta(z_{k_n}(t)).$$

Then, using (19) and the properties of η , it holds

$$f_n(t_n) \geq (k_n \kappa_0^2 - \|v\|_{L^\infty(\omega)} \kappa_1) t_n,$$

which is in contradiction for n large enough with

$$f_{k_n}(t_n) \leq \|\eta\|_\infty.$$

Thus we deduce that, for a $K \in \mathbb{N}^*$ and a $T \in [0, \delta]$, $\Phi_t^{v+u_k}(x^0) \in S$ for all $x^0 \in \text{supp}(\mu^0)$, $t \in (T, \delta)$ and $k > K$. ■

We now have all the tools to prove Theorem 1. The idea is the following: we first send μ_0 inside ω with a control u_1 , then from ω to an hypercube S with a control u_2 . On the other side, we send μ_1 inside ω backward in time with a control u_5 , then from ω to S with a control u_4 . When both the source and the target are in S , we send one to the other with a control u_3 .

Proof of Theorem 1: Consider μ^0, μ^1 satisfying Condition 1. Then, by Lemma 1, there exist T_0^*, T_1^* for which μ^0, μ^1 satisfy Condition 2. Define $T := T_0^* + T_1^* + \delta$ with $\delta > 0$.

Choose $\varepsilon > 0$ and denote by $T_1 := T_0^*, T_2 := T_0^* + \delta/3, T_5 := T_1^*$ and $T_4 := T_1^* + \delta/3$. Using Propositions 3 and 4, there exists some controls $u^1, u^2, u^4, u^5 \in \mathcal{U}$ and a square $S \subset \omega$ such that the solutions to

$$\begin{cases} \partial_t \rho_0 + \nabla \cdot ((v + \mathbb{1}_\omega u^1) \rho_0) = 0 & \text{in } \mathbb{R}^d \times [0, T_1], \\ \partial_t \rho_0 + \nabla \cdot ((v + \mathbb{1}_\omega u^2) \rho_0) = 0 & \text{in } \mathbb{R}^d \times [T_1, T_2], \\ \rho_0(0) = \mu^0 & \text{in } \mathbb{R}^d \end{cases} \quad (20)$$

and

$$\begin{cases} \partial_t \rho_1 + \nabla \cdot ((v + \mathbb{1}_\omega u^5) \rho_1) = 0 & \text{in } \mathbb{R}^d \times [-T_5, 0], \\ \partial_t \rho_1 + \nabla \cdot ((v + \mathbb{1}_\omega u^4) \rho_1) = 0 & \text{in } \mathbb{R}^d \times [-T_4, -T_5], \\ \rho_1(0) = \mu^1 & \text{in } \mathbb{R}^d, \end{cases} \quad (21)$$

satisfy $\text{supp}(u_i) \subset \omega$, $\rho_0(T_2)(S) > 1 - \varepsilon$ and $\rho_1(-T_4)(S) > 1 - \varepsilon$.

We now apply Proposition 2 to approximately steer $\rho_0(T_2)$ to $\rho_1(-T_4)$ inside S : this gives a control u_3 on the time interval $[0, \frac{\delta}{3}]$. Thus, concatenating u_1, u_2, u_3, u_4, u_5 on the time interval $[0, T]$, we approximately steer μ_0 to μ_1 . ■

IV. PROOF OF THEOREM 2

In this section, we prove Theorem 2 about minimal time. To achieve controllability in this setting, one needs to store the mass coming from μ^0 in ω and to send it out with a rate adapted to approximate μ^1 .

Let T_0 be the infimum satisfying Condition (2), and fix $s > 0$. We now prove that System (1) is approximately controllable at time $T := T_0 + s$. Consider $N \in \mathbb{N}^*$, $\tau := T_0/N$, $\delta < \tau$, $\xi := \tau - \delta$ and $\tau_i := i \times \tau$. Define

$$\begin{cases} A_i := \{x^0 \in \text{supp}(\mu^0) : t^0(x^0) \in [0, \tau_i]\}, \\ B_i := \{x^1 \in \text{supp}(\mu^1) : T_0 - t^1(x^1) \in [\tau_i, \tau_{i+1}]\}, \end{cases}$$

where $\begin{cases} t^0(x^0) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x^0) \in \omega\}, \\ t^1(x^1) := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x^1) \in \omega\}. \end{cases}$

We remark that $\mu^0 \times \mathbb{1}_{A_i}$ represents the mass of μ^0 which has entered ω at time τ_i and $\mu^1 \times \mathbb{1}_{B_i}$ the mass of μ^1 which need

to exit ω in the time interval (τ_i, τ_{i+1}) . Then, by hypothesis of the Theorem, there exists K such that

$$(\Phi_{\tau_i}^{v+u_K} \#(\mu^0 \times \mathbb{1}_{A_i^0}))(\omega) \geq 1 - (\Phi_{\tau_i-T}^{v+u_K} \#(\mu^1 \times \mathbb{1}_{A_i^1}))(\omega) - \varepsilon.$$

The function u_K can be then used to store the mass of μ^0 in ω . The meaning of the previous equation is that the stored mass is sufficient to fill the required mass for μ^1 .

We now define the control achieving approximate controllability at time T as follows: First of all, using the same strategy as in the Proof of Theorem 1, we can send a part of $\phi_{s-\xi}^{v+u_K} \#(\mu^0 \times \mathbb{1}_{A_0})$ approximately to $\phi_{-T^*}^{v+u_K} \#(\mu^1 \times \mathbb{1}_{B_0})$ during the time interval $(s-\xi, s)$. More precisely, we replace T_0^* and T_1^* by $s-\xi$ and ξ in the proof of Theorem 1. Thus, we send the mass of μ^0 contained in A_0 near to the mass of μ^1 contained in B_0 . We repeat this process on each time interval (τ_i, τ_{i+1}) for A_i to B_i . Thus, the mass of μ^0 is globally sent close to the mass of μ^1 in time T .

V. EXAMPLE OF MINIMAL TIME PROBLEM

In this section, we give explicit controls realizing the approximate minimal time in one simple example. The interest of such example is to show that the minimal time can be realized by non-Lipschitz controls, that are unfeasible.

We study an example on the real line. We consider a constant initial data $\mu_0 = \mathbb{1}_{[0,1]}$ and a constant uncontrolled vector field $v = 1$. The control set is $\omega = [2, 3]$. Our first target is the measure $\mu_1 = \frac{1}{2} \mathbb{1}_{[4,6]}$. We now consider the following control strategy:

$$u(t, x) = \begin{cases} 0 & t \in [0, \frac{4}{3}] \cap [\frac{7}{3}, \frac{13}{3}], \\ \psi(2 + (t - \frac{4}{3}), \frac{7}{3} + 2(t - \frac{4}{3})) & t \in [\frac{4}{3}, \frac{5}{3}), \\ \psi(2 + (t - \frac{5}{3}), \frac{7}{3} + 2(t - \frac{5}{3})) & t \in [\frac{5}{3}, 2), \\ \psi(2 + (t - 2), \frac{7}{3} + 2(t - 2)) & t \in [2, \frac{7}{3}], \end{cases} \quad (22)$$

where $\psi(a, b)$ is defined as follows:

$$\psi(a, b)(x) = \begin{cases} \frac{x-a}{b-a} & x \in [a, b], \\ 0 & x \notin [a, b]. \end{cases} \quad (23)$$

The choice of $\psi(a, b)$ given above has the following meaning: the vector field $\psi(a, b)$ is linearly increasing on the interval, thus an initial measure with constant density $k \mathbb{1}_{[\alpha_0, \beta_0]}$ with $a \leq \alpha_0 \leq \beta_0 \leq b$ will be transformed to a measure with constant density, supported in $[\alpha(t), \beta(t)]$, where $\alpha(t)$ is the unique solution of the ODE

$$\begin{cases} \dot{x} = v + u, \\ x(0) = \alpha_0, \end{cases}$$

and similarly for $\beta(t)$. As a consequence, we can easily describe the solution $\mu(t)$ of (1) with control (22) and initial data μ_0 . For simplicity, we only describe the measure evolution and the vector field on the time interval $[1, \frac{4}{3}]$ in Figure 2. One can observe that the linearly increasing time-varying control allows to rarefy the mass.

Two remarks are crucial:

- The vector field $v + \psi(a, b)$ is not Lipschitz, since it is discontinuous. Thus, one needs to regularize such vector field with a Lipschitz mollificator. As a consequence, the final state does not coincide with μ_1 , but it can be chosen arbitrarily close to it;

- The strategy presented here cuts the measure in three slices of mass $\frac{1}{3}$, and rarefying each of them separately. Its total time is $4 + \frac{1}{3}$. One can apply the same strategy with a larger number n of slices, and rarefying the mass in $[2, 2 + \frac{1}{n}]$ by choosing the control $\psi(2 + t, 2 + \frac{1}{n} + 2t)$. With this method, one can reduce the total time to $4 + \frac{1}{n}$, then being approximately close to the minimal time $T_0 = 4$ given by Theorem 2.

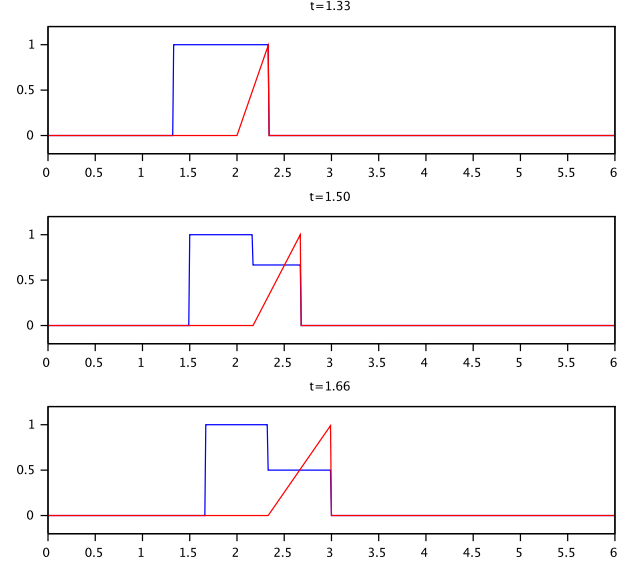


Fig. 2. Blue: density of the measure. Red: control vector field.

VI. CONCLUSION

In this article, we studied the control of a transport equation, where the control is a Lipschitz vector field in a fixed set ω . We proved that approximate controllability can be achieved under reasonable geometric conditions for the uncontrolled systems. We also proved a result of minimal time control from one configuration to another. Future research directions include the study of more general transport equations, namely when the uncontrolled dynamics presents interaction terms, such as in models for crowds and opinion dynamics.

REFERENCES

- [1] A. Bressan and B. Piccoli. *Introduction to the mathematical theory of control*, volume 2 of *AIMS Series on Applied Mathematics*. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2007.
- [2] M. Caponigro, B. Piccoli, F. Rossi, and E. Trélat. Mean-field sparse jurdjevic-quinn control. *Submitted*, 2017.
- [3] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.
- [4] F. Cucker and S. Smale. Emergent behavior in flocks. *IEEE Trans. Automat. Control*, 52(5):852–862, 2007.
- [5] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
- [6] B. Piccoli and F. Rossi. Transport equation with nonlocal velocity in Wasserstein spaces: convergence of numerical schemes. *Acta Appl. Math.*, 124:73–105, 2013.
- [7] B. Piccoli, F. Rossi, and E. Trélat. Control to flocking of the kinetic Cucker-Smale model. *SIAM J. Math. Anal.*, 47(6):4685–4719, 2015.

- [8] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.