MATHER DISCREPANCY AS AN EMBEDDING DIMENSION IN THE SPACE OF ARCS

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Abstract. Let X be a variety over a field k and let X_∞ be its space of arcs. We study the embedding dimension of the complete local ring $\widehat{A} := \widehat{\mathcal{O}_{X_\infty,P_E}}$ where P_E is the stable point defined by a divisorial valuation ν_E on X. Assuming char $k=0$, we prove that embdim $\widehat{A} = \widehat{k}_E + 1$ where $\widehat{k}_E$ is the Mather discrepancy of X with respect to ν_E. We also obtain that $\dim \widehat{A}$ has as lower bound the Mather-Jacobian log-discrepancy of X with respect to ν_E. For X normal and complete intersection, we prove as a consequence that points P_E of codimension one in X_∞ have discrepancy $k_E \leq 0$.[1]

1. Introduction

In 1968, J. Nash introduced the space of arcs X_∞ of an algebraic variety X in order to study the singularities of X. More precisely, he wanted to understand what the various *resolutions of singularities* have in common; his work being established just after the proof of resolution of singularities in characteristic zero by H. Hironaka. Nash's work was spread by H. Hironaka and later by M. Lejeune-Jalabert.

The development of *motivic integration* gave powerful tools for studying finiteness properties in the (not of finite type) k-scheme X_∞. Two main ideas in J. Denef and F. Loeser's article [DL] appear in this work: the change of variables formula in motivic integration and the stability property, which had already appeared in Kolchin's work on differential algebra. More precisely, based on this stability property, in [Re1] and [Re2] (see also [Re3]) we introduced stable points of X_∞, which are certain fat points of finite codimension in X_∞. We proved that, if P is stable then the complete local ring $\widehat{\mathcal{O}_{X_\infty,P}}$ is a Noetherian ring. From this result we proved a Curve Selection Lemma ending at stable points of X_∞. Stable points form a natural framework whenever induced morphisms $\eta_\infty : Y_\infty \to X_\infty$ are consider, where $\eta : Y \to X$ is of finite type and locally dominant ([Re2] and [Re3]).

Mori theory is also related to the study of the space of arcs. The recent work of T. de Fernex and R. Docampo [dFD] (see also [dF2]) has confirmed this relationship. In fact, a divisorial valuation $\nu = \nu_E$ on X defines a stable point P_E on X_∞ and, assuming the existence of a resolution of singularities and applying the previous Curve Selection Lemma, we can characterize $\dim \mathcal{O}_{X_\infty,P_E} = 1$ in terms of a property of lifting wedges centered at P_E ([Re3]). Then, de Fernex and Docampo's result, which gives an approach to Nash's project, can be understood as follows: assuming char $k = 0$, we have that if ν_E is a terminal valuation then $\dim \mathcal{O}_{X_\infty,P_E} = \dim \widehat{\mathcal{O}_{X_\infty,P_E}} = 1$. On the other hand, several examples of a normal hypersurface X and an essential valuation ν_E for which the property of lifting wedges centered at P_E does not hold have been studied ([IK], [dF1], [JK]). One of

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the key points in producing such examples is to require $k_E \geq 1$ where k_E is the discrepancy of X with respect to E. This suggests a connection between $\dim \mathcal{O}_{X_\infty, P_E}$, or $\dim \widehat{\mathcal{O}_{X_\infty, P_E}}$, and geometric invariants of (X, ν_E).

Understanding the algebraic properties of the rings $\widehat{\mathcal{O}_{X_\infty, P}}$ or of $\mathcal{O}_{X_\infty, P}$, P being stable, is an important problem; it leads towards the study nonconstant families of arcs in X_∞. In particular, one of our main goals is to compute $\dim \mathcal{O}_{X_\infty, P}$. In general, for any stable point P, an *upper bound* on the dimension of $\mathcal{O}_{X_\infty, P}$ follows from the stability property: Expressed in terms of cylinders, stable points are precisely the generic points of the irreducible cylinders in X_∞ and $\dim \mathcal{O}_{X_\infty, P}$ is bounded from above by the codimension as cylinder of the closure of P in X_∞ (see (4) in 2.3). If X is nonsingular at the center of P in X, then the ring $\mathcal{O}_{X_\infty, P}$ is regular and the dimension is equal to its upper bound, but in general the inequality in the bound is strict. From the change of variables formula in motivic integration it follows that the codimension as cylinder of the closure N_E of P_E is equal to $\widehat{k}_E + 1$ where $\widehat{k}_E$ is the Mather discrepancy of X with respect to E, introduced in [dFI] (see also [I]). Hence $\dim \mathcal{O}_{X_\infty, P_E} \leq \widehat{k}_E + 1$.

In this article we study the embedding dimension of $\mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_E}$. We prove that, assuming char $k = 0$, we have

$$\text{embdim } \widehat{\mathcal{O}_{X_\infty, P_E}} = \text{embdim } \mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_E} = \widehat{k}_E + 1 \tag{1}$$

that is, the embedding dimension of $\mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_E}$ is equal to the codimension as cylinder of N_E. Moreover, we describe explicitly a minimal system of coordinates of $(X_\infty)_{\mathrm{red}}$ at P_E. Applying this, we obtain the following *lower bound*:

$$\dim \widehat{\mathcal{O}_{X_\infty, P_E}} \geq \widehat{k}_E - \nu_E(Jac_X) + 1 \tag{2}$$

where Jac_X is the Jacobian ideal of X. In particular, if X is normal and complete intersection then $\dim \widehat{\mathcal{O}_{X_\infty, P_E}} \geq k_E + 1$. Hence, in this case, $\dim \mathcal{O}_{X_\infty, P_E} = 1$, or $\dim \widehat{\mathcal{O}_{X_\infty, P_E}} = 1$, implies $k_E \leq 0$.

The graded algebra associated to the divisorial valuation ν_E plays an essential role in this study. The natural coordinates of $(X_\infty)_{\mathrm{red}}$ at P_E are obtained by specialization techniques to the graded algebra of ν_E adapted from B. Teissier ([ZT], [GT], [Te]). These techniques are applied to a general projection $X \to \mathbb{A}^d$ and the induced valuation on $\mathbb{A}^d$. Such coordinates are introduced in [Re4]. In section 3 of this paper we prove that they also provide *minimal* coordinates of $(X_\infty)_{\mathrm{red}}$ at P_E and we conclude (1). The way we obtain this proof is, with the language in [Te], embedding X in a complete intersection X' which is an overweight deformation of an affine toric variety associated to the divisorial valuation ν_E. In section 4 we prove the lower bound for $\dim \widehat{\mathcal{O}_{X_\infty, P_E}}$ in (2); for this we embed X in a general complete intersection X'. The important fact used here is that X can be substituted by X' in order to compute the local rings $\widehat{\mathcal{O}_{X_\infty, P_E}}$ ([Re3], cf. 2.3 (ii) and (ix) of this paper). All these results extend to arbitrary stable points P of X_∞.

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2. Preliminaires

2.1. In this section we will set the notation and recall some properties of the space of arcs and their stable points. For more details see [DL], [EM], [IK], [Re3].

Let k be a perfect field and let X be a k-scheme. Given a field extension $k \subseteq K$, a *K-arc* on X is a k-morphism Spec $K[[t]] \to X$. The K-arcs on X are the K-rational points of a k-scheme X_∞ called the *space of arcs* of X. More precisely, $X_\infty = \lim_{\leftarrow} X_n$, where, for $n \in \mathbb{N}$, X_n is the k-scheme of n-jets whose K-rational points are the k-morphisms Spec $K[t]/(t)^{n+1} \to X$. In fact, the projective limit is a k-scheme because the natural morphisms $X_{n'} \to X_n$, for $n' \geq n$, are affine morphisms. We denote by $j_n : X_\infty \to X_n$, $n \geq 0$, the natural projections.

For every k-algebra A, we have a natural isomorphism

$$Hom_k(\text{Spec } A,\ X_\infty) \cong Hom_k(\text{Spec } A[[t]],\ X). \tag{3}$$

Given $P \in X_\infty$, with residue field $\kappa(P)$, we denote by $h_P : \text{Spec } \kappa(P)[[t]] \to X$ the $\kappa(P)$-arc on X corresponding by (3) to the $\kappa(P)$-rational point of X_∞ defined by P. The image in X of the closed point of Spec $\kappa(P)[[t]]$, or equivalently, the image P_0 of P by $j_0 : X_\infty \to X = X_0$ is called the *center* of P. Then, we denote by ν_P the order function $\text{ord}_t h_P^\sharp : \mathcal{O}_{X,P_0} \to \mathbb{N} \cup \{\infty\}$. It also follows from (3) that a K-arc on X_∞ is equivalent to a K-wedge, i.e. a k-morphism $\Phi : \text{Spec } K[[\xi, t]] \to X$.

The space of arcs of the affine space $\mathbb{A}_k^N = \text{Spec } k[x_1, \ldots, x_N]$ is $(\mathbb{A}_k^N)_\infty =$ Spec $k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots]$ where for $n \geq 0$, $\underline{X}_n = (X_{1;n}, \ldots, X_{N;n})$ is an N-uple of variables. For any $f \in k[x_1, \ldots, x_N]$, let $\sum_{n=0}^\infty F_n\, t^n$ be the Taylor expansion of $f(\sum_n \underline{X}_n\, t^n)$, hence $F_n \in k[\underline{X}_0, \ldots, \underline{X}_n]$. Equivalently, $\sum_{n=0}^\infty F_n\, t^n$ is the image of f by the morphism of k-algebras $\mathcal{O}_{\mathbb{A}_k^N} \to \mathcal{O}_{(\mathbb{A}_k^N)_\infty}[[t]]$ induced in (3) by the identity map in $(\mathbb{A}_k^N)_\infty$. If $X \subseteq \mathbb{A}_k^N$ is affine, and $I_X \subset k[x_1, \ldots, x_N]$ is the ideal defining X in $\mathbb{A}_k^N$, then we have

$$X_\infty = \text{Spec } k[\underline{X}_0, \underline{X}_1, \ldots, \underline{X}_n, \ldots] \ / \ (\{F_n\}_{n \geq 0, f \in I_X}).$$

Analogously, if $X = \text{Spec } k[[x_1, \ldots, x_N]] \ / \ I_X$ then we have

$$X_\infty = \text{Spec } k[[\underline{X}_0]][\underline{X}_1, \ldots, \underline{X}_n, \ldots] \ / \ (\{F_n\}_{n \geq 0, f \in I_X}).$$

2.2. Let X be a separated k-scheme which is locally of finite type over some Noetherian complete local ring R_0 with residue field k. Note that X may be a reduced separated k-scheme of finite type, and it may also be a k-scheme Spec $\widehat{R}$, being $\widehat{R}$ the completion of a local ring R which is a k-algebra of finite type. In [Re3] the stable points of X_∞ were defined as follows:

First, if X is affine and irreducible and P is a point of X_∞, i.e. a prime ideal of $\mathcal{O}_{X_\infty}$, then the following conditions are equivalent:

(a) There exist $n_1 \in \mathbb{N}$, and $G \in \mathcal{O}_{X_\infty} \setminus P$, $G \in \mathcal{O}_{X_{n_1}}$ such that, for $n \geq n_1$, the map $X_{n+1} \longrightarrow X_n$ induces a trivial fibration

$$\overline{j_{n+1}(Z(P))} \cap (X_{n+1})_G \longrightarrow \overline{j_n(Z(P))} \cap (X_n)_G$$

with fiber $\mathbb{A}_k^d$, where $d = \dim X$, $(X_n)_G$ is the open subset $X_n \setminus Z(G)$ of X_n and $\overline{j_n(Z(P))}$ is the closure of $j_n(Z(P))$ in X_n with the reduced structure.

(b) There exists $G \in \mathcal{O}_{X_\infty} \setminus P$ such that the ideal $P(\mathcal{O}_{X_\infty})_G$ is the radical of a finitely generated ideal of $(\mathcal{O}_{X_\infty})_G$.

We say that the point P is stable if the previous conditions hold ([Re2] and [Re3], see also J. Denef, F. Loeser [DL], lemma 4.1, and M. Lejeune-Jalabert [Le] for the stability property on the maps $j_{n+1}(X_\infty) \to j_n(X_\infty)$).

In general, i.e. for X not necessarily irreducible, the set of stable points of X_∞ is the union of the sets of stable points of the irreducible components of X. Besides this union is disjoint (see (i) in 2.3 bellow).

Recall that a subset C of X_∞ is a *cylinder* if it is of the form $C = j_n^{-1}(S)$ for some n and some constructible subset $S \subseteq X_n$ ([EM], sec. 5). Hence, from (b) above it follows that the stable points of X_∞ are precisely the generic points of the irreducible cylinders.

2.3. The next properties of stable points will be used in the next sections. The first ones, (i) to (iv), are direct consequence of the definition of stable points and of the stability property in [DL], and property (v) applies also well-known facts of the theory of valuations:

([Re3], prop. 3.7) Let P be a stable point of X_∞, then the following holds:

- **(i)** Let X_0 be an irreducible component of X such that $P \in (X_0)_\infty$. Then, the arc $h_P : \text{Spec } \kappa(P)\ [[t]] \to X_0$ defined by P is a dominant morphism.
- **(ii)** Let U be any irreducible open affine subscheme of X which contains the image of h_P, then
$$\mathcal{O}_{X_\infty,P} = \mathcal{O}_{U_\infty,P}.$$
Moreover, there exits $X' \subseteq \mathbb{A}^N_k$ a complete intersection scheme which contains U and of dimension $\dim U$ and, for any such X', we have that
$$\mathcal{O}_{(X_\infty)_{\text{red}},P} \cong \mathcal{O}_{(U_\infty)_{\text{red}},P} \cong \mathcal{O}_{(X'_\infty)_{\text{red}},P}$$
where we also denote by P the point induced by P in $(X_\infty)_{\text{red}}$ and in $(X'_\infty)_{\text{red}}$. Therefore X_∞ is irreducible at P, i.e. the nilradical of the ring $\mathcal{O}_{X_\infty,P}$ is a prime ideal.
- **(iii)** The residue field $\kappa(P)$ of P on X_∞ is a countably pure trascendental extension of a finite extension of k. This implies that $\kappa(P)$ is a separably generated field extension of k.
- **(iv)** $\dim \mathcal{O}_{\overline{j_n(X_\infty)},P_n}$ is constant for $n >> 0$, where $\overline{j_n(X_\infty)}$ is the closure of $j_n(X_\infty)$ in X_n, with the reduced structure, and P_n is the prime ideal $P \cap \mathcal{O}_{\overline{j_n(X_\infty)}}$. Since
$$\dim \mathcal{O}_{X_\infty,P} \leq \sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)},P_n} \tag{4}$$
this implies that $\dim \mathcal{O}_{X_\infty,P} < \infty$.
- **(v)** Let ν_P be the valuation on the function field $K(X_0)$ of X_0 defined by the arc h_P, X_0 being the irreducible component of X such that $P \in (X_0)_\infty$. Then, either P_0 is the generic point of X and in this case ν_P is trivial, or ν_P is a divisorial valuation.

Property (i) is equivalent to the statement in [EM] lemma 5.1 for cylinders. In property (iv), the right hand side term in (4) is the definition of the *codimension of the cylinder* $Z(P)$ (see [EM] sec. 5); but the inequality in (4) may be strict. For property (v) in the setting of cylinders, see [dFEI] and also [ELM]. The next property compares the local rings at stable points of the space of arcs of $X =$ Spec R, where R is a local ring which is a k-algebra of finite type, and of of $\widehat{X} =$ Spec $\widehat{R}$, where $\widehat{R}$ is the completion of R:

- **(vi)** Let P be a stable point of X_∞, where $X = \text{Spec } R$ as before, whose center in X is the maximal ideal of R. Then P induces a stable point in $\widehat{X}_\infty$, that we also denote by P, and we have
$$\widehat{\mathcal{O}_{X_\infty,P}} = \widehat{\mathcal{O}_{\widehat{X}_\infty,P}}\ .$$

The following finiteness property of the stable points, which is the main result in [Re2], is expressed in terms of the local ring $\mathcal{O}_{X_\infty,P}$, or more precisely, its formal completion. It implies a Curve Selection Lemma in X_∞ ending at a stable point P ([Re2], corol. 4.8). Property (viii) below helps to understand this local ring.

Finiteness property of the stable points ([Re2] th. 4.1). Let P be a stable point of X_∞, then:

- **(vii)** The formal completion $\widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}}$ of the local ring of $(X_\infty)_{\mathrm{red}}$ at a stable point P is a Noetherian ring.
- **(viii)** Moreover, if X is affine, then there exists $G \in \mathcal{O}_{X_\infty} \setminus P$ such that the ideal $P\left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_G$ is a finitely generated ideal of $\left(\mathcal{O}_{(X_\infty)_{\mathrm{red}}}\right)_G$.
- **(ix)** *([Re3] th. 3.13 if char $k = 0$)* Moreover, we have $\widehat{\mathcal{O}_{X_\infty,P}} \cong \widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P}}$.

From this it follows that, if P is a stable point of X_∞, then the maximal ideal of $\widehat{\mathcal{O}_{X_\infty,P}}$ is $P\widehat{\mathcal{O}_{X_\infty,P}}$, and even more,

$$\text{embdim}\ \widehat{\mathcal{O}_{X_\infty,P}} = \text{embdim}\ \mathcal{O}_{(X_\infty)_{\mathrm{red}},P}. \tag{5}$$

(see [Bo] cap. III, sec. 2, no. 12, corol. 2).

Stable points behave well under birational proper k-morphisms and, if we assume that char $k = 0$, then also under k-morphisms locally of finite type which are locally dominant:

- **(x)** *([Re3] prop. 4.1)* Let $\pi : Y \to X$ be a birational and proper k-morphism, then the morphism $\pi_\infty : Y_\infty \to X_\infty$ induces a one to one map between the stable points of Y_∞ and the stable points of X_∞. Besides, if Q is a stable point of Y_∞ and P its image, then the induced morphism $\widehat{\mathcal{O}_{X_\infty,P}} \to \widehat{\mathcal{O}_{Y_\infty,Q}}$ is surjective and induces an isomorphism on the residue fields $\kappa(P) \cong \kappa(Q)$.
- **(xi)** *([Re3] prop. 4.5)* Suppose that char $k = 0$. Let $\eta : Y \to X$ be a k-morphism locally dominant, then the morphism $\eta_\infty : Y_\infty \to X_\infty$ induces a map from the set of stable points of Y_∞ to the set of stable points of X_∞ Besides, if Q is a stable point of Y_∞ and P its image by the previous map, then the induced morphism $(\mathcal{O}_{X_\infty,P})_{\mathrm{red}} \to (\mathcal{O}_{Y_\infty,Q})_{\mathrm{red}}$ is an injective local morphism.

  Moreover, if η is finite and dominant, then $\widehat{\mathcal{O}_{X_\infty,P}} \to \widehat{\mathcal{O}_{Y_\infty,Q}}$ is unramified at $Q\ \widehat{\mathcal{O}_{Y_\infty,Q}}$, that is $P\ \widehat{\mathcal{O}_{Y_\infty,Q}} = Q\ \widehat{\mathcal{O}_{Y_\infty,Q}}$, and it indices a finite extension $\kappa(P) \subseteq \kappa(Q)$ on the residue fields.
- **(xii)** *([Re4] prop. 2.5)* Let $\eta : Y \to X$ be an étale k-morphism. Then Y_∞ is étale over X_∞ and, if Q is a stable point of Y_∞ and P its image, then $\widehat{\mathcal{O}_{Y_\infty,Q}} \cong \widehat{\mathcal{O}_{X_\infty,P}} \otimes_{\kappa(P)} \kappa(Q)$.

2.4. Suppose that there exists a resolution of singularities $\pi : Y \to X$ of X, i.e. a proper, birational k-morphism, with Y is smooth, such that the induced morphism $Y \setminus \pi^{-1}(Sing\ X) \to X \setminus Sing\ X$ is an isomorphism. Let E be a divisor on Y and let Y_∞^E be the inverse image of E by the natural projection $j_0^Y : Y_\infty \to Y$. Then Y_∞^E is an irreducible subset of Y_∞ whose generic point P_E^Y is a stable point of Y_∞. Besides, the image P_E^X of P_E^Y by the morphism $\pi_\infty : Y_\infty \to X_\infty$ is a stable point of X_∞ (see (x) above). We will denote $P_E = P_E^X$ if there is no possible ambiguity. Note that P_E only depends on the divisorial valuation ν_E defined by E, more precisely, if $\pi' : Y' \to X$ is another resolution of singularities such that the center E' of ν_E in Y' is a divisor, then the stable point $P_{E'}$ defined by E' coincides with P_E. Note also that the order function ν_{P_E} is equal to the restriction of the divisorial valuation ν_E to the local ring of X at the generic point of $\pi(E)$.

The set Y_∞^E is also denoted by $\mathrm{Cont}^1(E)$. More generally

$$\mathrm{Cont}^e(E) := \{Q' \in Y_\infty \ / \ \nu_{Q'}(I_E) = e\} \quad \text{for every } e \geq 1$$

where I_E is the ideal defining E in an open affine subset of Y. We also have that $\mathrm{Cont}^e(E)$ is an irreducible subset of Y_∞ whose generic point P_{eE}^Y is a stable point of Y_∞, and the image P_{eE}^X (also denoted by P_{eE}) of P_{eE}^Y by π_∞ is a stable point of X_∞.

Example 2.5. Note that there are stable points which are not of the type P_{eE} where ν_E is a divisorial valuation on X. For instance, let $X = \mathbb{A}^1$ and let P be the prime ideal (x_0, x_3) of $\mathcal{O}_{X_\infty} = k[x_0, x_1, \ldots]$. Then ν_P is the divisorial valuation ν_E defined by $\nu_E(x) = 1$, hence it is the multiplicity in $k[x]$, but $P \neq P_E$.

2.6. If $\pi : Y \to X$ is a resolution of singularities dominating the Nash blowing up of X, then the image of the canonical homomorphism $d\pi : \pi^*(\wedge^d \Omega_X) \to \wedge^d \Omega_Y$ is an invertible sheaf. That is, there exists an effective divisor $\widehat{K}_{Y/X}$ with support in the exceptional locus of π such that $d\pi(\pi^*(\wedge^d \Omega_X)) = \mathcal{O}_Y(-\widehat{K}_{Y/X}) \ \wedge^d \Omega_Y$. For any prime divisor E on Y, we define the *Mather discrepancy* to be

$$\widehat{k}_E := \mathrm{ord}_E(\widehat{K}_{Y/X}).$$

Note that $\widehat{k}_E \neq 0$ implies that E is contained in the exceptional locus of π, and that $\widehat{k}_E$ only depends on the divisorial valuation ν_E defined by E. We have $\sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)},(P_{eE})_n} = e\,(\widehat{k}_E+1)$ ([DL], lemma 3.1, [dFEI], theorem 3.9). Hence the inequality (4) states that

$$\dim \mathcal{O}_{X_\infty, P_{eE}} \leq e\ (\widehat{k}_E + 1).$$

On the other hand, if X is normal and $\mathbb{Q}$-Gorenstein (for instance X is a normal complete intersection), the *discrepancy of* X with respect to E is defined to be the coefficient of E in the divisor $K_{Y/X}$ with exceptional support which is linearly equivalent to $K_Y - \pi^*(K_X)$. If X is nonsingular then $\widehat{k}_E = k_E$ ([EM], appendix). Moreover, we have:

(xiii) *([Re3] prop. 4.2 and [Re4] corol. 2.9)* If X is nonsingular at the center P_0 of a stable point P of X_∞, then $\mathcal{O}_{X_\infty,P}$ is a regular ring of dimension $\dim \mathcal{O}_{X_\infty,P} = \sup_n \dim \mathcal{O}_{\overline{j_n(X_\infty)},P_n}$. In particular, taking $P = P_{eE}$, we have $\dim \mathcal{O}_{X_\infty,P_{eE}} = e(k_E + 1)$.

In theorem 3.8 will prove that, also in the case that X is singular at P_0, we have that $e\ (\widehat{k}_E + 1)$ is the embedding dimension of $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_{eE}}$.

Example 2.7. Let X be an irreducible formal plane curve over a field k of characteristic zero. Let us consider a (primitive) Puiseux parametrization

$$x = u^{\beta_0}$$
$$y = \sum_{\beta_0 \leq i} \lambda_i\ u^i$$

where $\lambda_i \in k$ for every $i \geq \beta_0$. Set $e_0 := \beta_0$ and,

$$\begin{aligned} \beta_{r+1} &:= min\,\{i \ / \ \lambda_i \neq 0 \text{ and } g.c.d.\{\beta_0, \ldots, \beta_r, i\} < e_r\ \}, \\ e_{r+1} &:= g.c.d.\{\beta_0, \ldots, \beta_{r+1}\} \end{aligned}$$

for $1 \leq r \leq g-1$, being g such that $e_g = 1$. Let $n_0 = 1$ and $n_r := \frac{e_{r-1}}{e_r}$ for $1 \leq r \leq g$ and let $\overline{\beta}_0 = \beta_0$ and $\overline{\beta}_r$, $1 \leq r \leq g+1$, be defined by

$$\overline{\beta}_r - n_{r-1}\overline{\beta}_{r-1} = \beta_r - \beta_{r-1}, \tag{6}$$

hence we have

$\overline{\beta}_r > n_{r-1}\,\overline{\beta}_{r-1}$ for $1 \leq r \leq g$, and $\overline{\beta}_{g+1} \geq n_g\,\overline{\beta}_g$;
$n_r\overline{\beta}_r$ belongs to the semigroup generated by $\overline{\beta}_0, \ldots, \overline{\beta}_{r-1}$, $1 \leq r \leq g+1$.

Let us consider $q_0, q_1, \ldots, q_g \in k[x,y]$ and $q_{g+1} \in k[[x,y]]$ such that q_{g+1} defines an equation of the branch, i.e. $X = \text{Spec } k[[x,y]] \;/\; (q_{g+1})$, and $q_1, \ldots, q_g$ are its approximate roots (see [ZT], appendix). More precisely, $q_0, \ldots, q_{g+1}$ can be defined as follows:

$$q_0 = x \qquad q_1 = y - \sum_{i<\overline{\beta}_1} \lambda_i \; q_0^{\frac{i}{\overline{\beta}_0}}$$

with $\text{ord}_u(q_1) = \overline{\beta}_1$ and, for $1 \leq r \leq g$,

$$(7) \qquad q_{r+1} = q_r^{n_r} - c_r\; q_0^{b_{r,0}} \ldots q_{r-1}^{b_{r,r-1}} - \sum_{\gamma=(\gamma_0,\ldots,\gamma_r)} c_\gamma\; q_0^{\gamma_0} \ldots q_r^{\gamma_r}, \quad 1 \leq r \leq g$$

with $\text{ord}_u(q_{r+1}) = \overline{\beta}_{r+1}$ (resp. ∞) for $1 \leq r < g$ (resp. $r = g$), where $\{b_{r,i}\}_{i=0}^{r-1}$ are the unique nonnegative numbers satisfying $b_{r,i} < n_i$ for $1 \leq i \leq r-1$ and $n_r\overline{\beta}_r = \sum_{0 \leq i < r} b_{r,i}\overline{\beta}_i$, for each sequence γ of nonnegative integers in the right hand side we have $n_r\overline{\beta}_r < \sum_{i=0}^{r}\gamma_i\overline{\beta}_i < \overline{\beta}_{r+1}$ (resp. $n_r\overline{\beta}_r < \sum_{i=0}^{r}\gamma_i\overline{\beta}_i$) if $1 \leq r < g$ (resp. if $r = g+1$) and $c_r, c_\gamma \in k$ and $c_r \neq 0$. For more details on approximate roots and the space of arcs of a plane branch see [Mo] and [LMR].

Let $\nu = \nu_E$ be the divisorial valuation on X given by ord_u, and let $P = P_E$ be the stable point in X_∞ defined by ν as in 2.4. Considering the projection $\eta : X \to \mathbb{A}^1_k$, $(x,y) \mapsto x$, and applying prop. 4.5 in [Re3] ((xi) in 2.3) we conclude that

$$P\widehat{\mathcal{O}_{X_\infty,P}} \;=\; (X_0, \ldots, X_{\beta_0 - 1})\;\widehat{\mathcal{O}_{X_\infty,P}}.$$

We will next describe the ring $\widehat{\mathcal{O}_{X_\infty,P}}$, and we will see that embdim $\widehat{\mathcal{O}_{X_\infty,P}} = \beta_0$, which is equal to the multiplicity of X (see [Re3], corol. 5.7).

First note that $P\mathcal{O}_{X_\infty,P}$ is generated by $\mathcal{Q} := \{Q_{r;n}\}_{0 \leq r \leq g,\; n_{r-1}\overline{\beta}_{r-1} \leq n < \overline{\beta}_r}$, even more, there exists $G \in \mathcal{O}_{X_\infty} \setminus P$ such that $P(\mathcal{O}_{X_\infty})_G = (\mathcal{Q})(\mathcal{O}_{X_\infty})_G$ (we may take $G := \prod_{0 \leq r \leq g} Q_{r;\overline{\beta}_r}$). More precisely, $(\mathcal{Q})$ defines a prime ideal in $(\mathcal{O}_{(\mathbb{A}^2)_\infty})_G$ (see [Re4], prop 4.5) whose extension to $(\mathcal{O}_{X_\infty})_G$ is $P(\mathcal{O}_{X_\infty})_G$. Note that, setting $f := q_{g+1} \in k[[x,y]]$, the following holds:

(i) $\nu(Jac(f)) = \nu(\frac{\partial f}{\partial y}) = n_g\overline{\beta}_g - \beta_g$. Set $\epsilon := n_g\overline{\beta}_g - \beta_g$,

(ii) for all $n \geq 0$, the class of $\frac{\partial F_{\epsilon+n}}{\partial Y_n}$ in $\mathcal{O}_{X_\infty,P}$ is a unit and, for $n' > n$, the class of $\frac{\partial F_{\epsilon+n}}{\partial Y_{n'}}$ in $\mathcal{O}_{X_\infty,P}$ belongs to $P\mathcal{O}_{X_\infty,P}$.

(iii) $F_0, \ldots, F_{\epsilon-1}$ belong to $(\mathcal{Q})^2\mathcal{O}_{(\mathbb{A}^2_k)_\infty}$.

From this it follows that

$$\kappa(P) \;\cong\; k(X_{\beta_0+1}, \ldots, X_n, \ldots)\,[\{W_r\}_{r=0}^g] \;\Big/\; \Big(\{W_r^{n_r} - c_r W_0^{b_{r,0}} \cdots W_{r-1}^{b_{r,r-1}}\}_{r=1}^g\Big)$$

where W_r is the class of $Q_{r;\overline{\beta}_r}$. We consider the embedding $\kappa(P) \hookrightarrow \widehat{\mathcal{O}_{X_\infty,P}}$ which sends X_n, $n \geq \beta_0$, (resp. W_0) to $X_n \in \widehat{\mathcal{O}_{X_\infty,P}}$ (resp. $X_{\beta_0} \in \widehat{\mathcal{O}_{X_\infty,P}}$) and recursively, for $1 \leq r \leq g$, sends W_r to a n_r-root of the image in $\widehat{\mathcal{O}_{X_\infty,P}}$ of $c_r W_0^{b_{r,0}} \cdots W_{r-1}^{b_{r,r-1}}$, that exists by Hensel's lemma. In particular, for each $n \geq 0$ we have defined $Y_n^{(0)} \in \kappa(P)$ such that $Y_n - Y_n^{(0)} \in (\mathcal{Q})$. Arguing recursively on $m \geq 1$ and $n \geq 0$, with the lexicographic order on (m,n), from $\{F_{\epsilon+n}\}_{n \geq 0}$, applying (ii) and Hensel's lemma, and reasoning as in corol. 5.6 in [Re3] it follows that, for $m, n \geq 0$, there exists $Y_n^{(m)} \in \kappa(P)[X_0, \ldots, X_{\beta_0-1}]$ such that,

$$F_{\epsilon+n} \equiv L_\epsilon\;(Y_n - Y_n^{(m)}) \quad \mod\; (\mathcal{Q})^m$$

in the ring $\mathcal{O}_{(\mathbb{A}^2)_\infty,(\mathcal{Q})}$, where $l := \frac{\partial f}{\partial y}$, hence L_ϵ is a unit. Therefore, the previous equalities define series $\widetilde{Y}_n \in \kappa(P)\,[[X_0,\ldots,X_{\beta_0-1}]]$, $n \geq 0$, and we conclude that

$$\widehat{\mathcal{O}_{X_\infty,P}} \;\cong\; \kappa(P)\,[[X_0,\ldots,X_{\beta_0-1}]] \;\Big/\; \left(\{\widetilde{F}_n\}_{0\leq n\leq\epsilon-1}\right)$$

where, for $0 \leq n \leq \epsilon-1$, $\widetilde{F}_n$ is obtained from F_n by substituting $Y_{n'}$ by $\widetilde{Y}_n$, $0 \leq n' \leq n$. Since, for $0 \leq r \leq g, n_{r-1}\overline{\beta}_{r-1} \leq n < \overline{\beta}_r$, the series obtained from Q_n by substituting $Y_{n'}$ by $\widetilde{Y}_n$, $0 \leq n' \leq n$, belongs to $(X_0,\ldots,X_{\beta_0-1})$, from (iii) it follows that $\widetilde{F}_n \in (X_0,\ldots,X_{\beta_0-1})^2$ for $0 \leq n \leq \epsilon-1$. Therefore embdim $\widehat{\mathcal{O}_{X_\infty,P}} = \beta_0$.

Remark 2.8. Let X be an algebraic plane curve over a field k of characteristic zero, and suppose that it is analytically irreducible. Then, there exists an étale morphism $X' \to X$ such that the curve X' has a Puiseux parametrization

$$\begin{array}{l} x' = (u')^{\beta_0} \\ y' = \sum_{\beta_0\leq i\leq m} \lambda'_i\ (u')^i \end{array} \tag{8}$$

where $\lambda'_i \in k$ for $\beta_0 \leq i \leq m$, i.e. the image of y' has a finite number of terms. Equivalently, the element q'_{g+1} obtained as in (7) from the previous parametrization, which defines an equation of the curve X', is a polynomial.

In fact, consider a Puiseux parametrization $x = u^{\beta_0}$, $y = \sum_{\beta_0\leq i} \lambda_i\ u^i$ of X and keep the notation in example 2.7. Note that the series $\sum_{\beta_0\leq i} \lambda_i\ u^i$ belongs to the hensalization $k < u >$ of $k[u]_{(u)}$ and also that the element q_{g+1} in (7) belongs to $k < x, y >$. Since X is analytically irreducible, there exists $\gamma \in k < x, y >$, γ a unit, such that $\gamma\ q_{g+1}$ is a polynomial in $k[x,y]$. Then taking $x' = (\gamma)^{\frac{1}{\beta_1}} x$, $y' = (\gamma)^{\frac{1}{\beta_0}} y$ and $u' = (\gamma)^{\frac{1}{n_1\overline{\beta}_1}} u$, we obtain (8). Recall that $n_1\overline{\beta}_1$ is the least common multiple of $\overline{\beta}_0$ and $\overline{\beta}_1$. Since char $k = 0$, adding a $n_1\overline{\beta}_1$-root of γ defines an étale morphism $X' \to X$.

Example 2.9. Let $X \subset \mathbb{A}^5_k$ be the hypersurface singularity in [IK], defined by $x_1^3+x_2^3+x_3^3+x_4^3+x_5^6 = 0$ over a field of characteristic $\neq 2,3$. The blowing up X' of X at the origin has a unique singular point, and its exceptional locus E_β is irreducible and defines an *essential* valuation ν_β (i.e. the center of ν_β on any resolution of singularities $p : \widetilde{X} \to X$ is an irreducible component of the exceptional locus of p). The blowing up Y of X' at its singular point is nonsingular, and its exceptional locus is irreducible and defines an essential valuation ν_α, $\nu_\alpha \neq \nu_\beta$. Let $\pi : Y \to X$ be the induced resolution of singularities. Let P_α, P_β be the stable points of X_∞ defined by ν_α and ν_β respectively, and set $N_\alpha := \overline{\{P_\alpha\}}, N_\beta := \overline{\{P_\beta\}}$ and X_∞^{Sing} the inverse image of Sing X by $j_0 : X_\infty \to X$. We have $N_\alpha \subset N_\beta = X_\infty^{\mathrm{Sing}}$ ([IK], theorem 4.3).

Let $\Pi : \widetilde{\mathcal{Z}} \to \mathbb{A}^5_k$ be the embedded resolution of singularities of X whose restriction to X is π. There exists a divisor $\widetilde{E}$ on $\widetilde{\mathcal{Z}}$ whose intersection with Y is E_β. Note that $b_{\widetilde{E}} := \mathrm{ord}_{\widetilde{E}} K_{\widetilde{\mathcal{Z}}/\mathbb{A}^5}$ is equal to 4 and $a_{\widetilde{E}} := \mathrm{ord}_{\widetilde{E}}\Pi^*(X)$ is equal to 3. Since, by the adjunction formula, $k_{E_\beta} = b_{\widetilde{E}} - a_{\widetilde{E}}$, we have $k_{E_\beta} = 1$. Hence, $\widehat{k}_{E_\beta} = k_{E_\beta} + \nu_\beta(Jac_X) = 1 + 2 = 3$ (see [EM], remark 9.6).

On the other hand, we have

$$P_\beta\,(\mathcal{O}_{X_\infty})_{X_{1;1}} = (X_{1;0}, X_{2;0}, X_{3;0}, X_{4;0}, X_{5;0})\ \ (\mathcal{O}_{X_\infty})_{X_{1;1}}.$$

In fact, $(X_{1;0},\ldots,X_{5;0})$ is the prime ideal in $\mathcal{O}_{(\mathbb{A}^5)_\infty}$ defined by $\nu_{\widetilde{E}}$, hence its minimal number of generators is $b_{\widetilde{E}}+1 = 5$ (see (xiii) in 2.6). Besides, the ring $\widehat{\mathcal{O}}_{X_\infty,P_\beta}$

has been described in [Re3] remark 5.16 as follows:

$$\widehat{\mathcal{O}}_{X_\infty,P_\beta} \;\cong\; \kappa(P_\beta)[[X_{1;0},X_{2;0},X_{3;0},X_{4;0},X_{5;0}]]\;/\;(\widetilde{F}_0,\widetilde{F}_1,\widetilde{F}_2)$$

where, being $f = x_1^3+x_2^3+x_3^3+x_4^3+x_5^6$ and being $\overline{F}_n$ the class of F_n modulo $(X_{1;0},\ldots,X_{5;0})$, we have that $3 = a_{\widetilde{E}}$ is the minimal n such that $\overline{F}_n \neq 0$, in fact $\overline{F}_3 = X_{1;1}^3+X_{1;2}^3+X_{1;3}^3+X_{1;4}^3$ and

$$\kappa(P_\beta) \;\cong\; k(\{X_{i;1},\ldots X_{i;n},\ldots\}_{2\leq i\leq 4})[X_{1;1}]\;/\;(\overline{F}_3).$$

Besides we have $\widetilde{F}_0,\widetilde{F}_1 \in (X_{1;0},\ldots,X_{5;0})^2$ and the initial form $\mathrm{in}(\widetilde{F}_2)$ of $\widetilde{F}_2$ in $\kappa(P_\beta)[[X_{1;0},\ldots,X_{5;0}]]$ is $3\overline{X}_{1;1}^2X_{1;0}+3X_{2;1}^2X_{2;0}+3X_{3;1}^2X_{3;0}+3X_{4;1}^2X_{4;0}$ where $\overline{X}_{1;1}$ is the class of $X_{1;1}$ in $\kappa(P_\beta)$. Note that $\nu_\beta(Jac_X)=2$, even more, for $1\leq i\leq 4$, if $f^i := \frac{\partial f}{\partial x_i}$ then $\nu_\beta(f^i)=2$, i.e. $F_0^i, F_1^i \in P_\beta$, $F_2^i \notin P_\beta$, and the coefficient in $X_{i;0}$ of $\mathrm{in}(\widetilde{F}_2)$ is the class of F_2^i in $\kappa(P_\beta)$. From this it follows that

$$\mathrm{embdim}\,\widehat{\mathcal{O}_{X_\infty,P_\beta}} = b_{\widetilde{E}}+1-(a_{\widetilde{E}}-\nu_\beta(Jac_X)) = k_{E_\beta}+1+\nu_\beta(Jac_X) = \widehat{k}_{E_\beta}+1$$

which equals 4. Moreover, in this case

$$\dim\widehat{\mathcal{O}_{X_\infty,P_\beta}} = b_{\widetilde{E}}+1-a_{\widetilde{E}} = k_{E_\beta}+1 = 2.$$

The argument to compute $\mathrm{embdim}\,\widehat{\mathcal{O}_{X_\infty,P_\beta}}$ showed in example 2.9 can be generalized to monomial valuations restricted to a normal hyperssurface over a perfect field of any characteristic. But, although, given a variety X and a divisorial valuation ν_E, there always exists a complete intersection X' containig X of the same dimension and we have $\widehat{\mathcal{O}_{X_\infty,P_E}} \cong \widehat{\mathcal{O}_{X'_\infty,P_E}}$ (see (ii) and (ix) in 2.3), X' is not normal in general. So, there is no hope to extend the result $\mathrm{embdim}\,\widehat{\mathcal{O}_{X_\infty,P_E}} = \widehat{k}_E+1$ applying this argument. For $\dim\widehat{\mathcal{O}_{X_\infty,P_E}}$, even if X is a normal hypersurface it is not true in general that $\dim\widehat{\mathcal{O}_{X_\infty,P_E}}$ equals k_E+1, but we will show that $\dim\widehat{\mathcal{O}_{X_\infty,P_E}} \geq k_E+1$.

3. Defining minimal coordinates at stable points of the space of arcs

Let X be a (singular) reduced separated scheme of finite type over a field k of characteristic zero. Let ν be a divisorial valuation on an irreducible component X_0 of X whose center lies in $Sing\ X$ and let $e\in\mathbb{N}$.

Let us consider the stable point P_{eE} of X_∞ defined by ν and e, i.e. we consider any resolution of singularities $\pi : Y\to X$ such that the center of ν on Y is a divisor E, and define $P_{eE} = P_{eE}^X$ to be the image by π_∞ of the generic point P_{eE}^Y of Y_∞^E (see 2.4). In order to study the ring $\mathcal{O}_{X_\infty,P_{eE}}$, or its completion $\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$, we may suppose that X is affine, let $X\subseteq\mathbb{A}_k^N = \mathrm{Spec}\ k[y_1,\ldots,y_N]$. We may also suppose that $\pi : Y\to X$ dominates the Nash blowing up of X and that, if x_i denotes the class of y_i in $\mathcal{O}_X$, $1\leq i\leq N$, then, after reordering the x_i's, we have

$$\mathrm{ord}_E\ \pi^*(dx_1\wedge\ldots\wedge dx_d) = \widehat{k}_E. \tag{9}$$

where $d=\dim X_0$.

Let $\rho : X\to\mathbb{A}_k^d$ be the projection on the first d coordinates, let $\eta : Y\to\mathbb{A}^d$ be the composition $\eta=\rho\circ\pi$ and let $P_{eE}^{\mathbb{A}^d}$ be the image of P_{eE}^Y by η_∞. Then the discrepancy $k_E(\mathbb{A}_k^d)$ of $\mathbb{A}_k^d$ with respect to the valuation induced by ν_E is equal to $\widehat{k}_E$ by (9). Besides, we know that the local ring $\mathcal{O}_{(\mathbb{A}^d)_\infty,P_{eE}^{\mathbb{A}^d}}$ is a regular ring of dimension $e(k_E(\mathbb{A}_k^d)+1)$ (see (xiii) in 2.6). From this, and applying [Re3], prop. 4.5

(see (xi) in 2.3) it follows that, if $\mathcal{Q}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ (hence $\sharp\mathcal{Q} = e(\widehat{k}_E + 1)$) then we have

(10) $P_{eE}\ \widehat{\mathcal{O}_{X_\infty, P_{eE}}} = (\mathcal{Q})\ \widehat{\mathcal{O}_{X_\infty, P_{eE}}}$ and $P_{eE}\ \mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_{eE}} = (\mathcal{Q})\ \mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_{eE}}$

in fact, the last assertion follows from the first one by Nakakama's lemma. Therefore embdim $\mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_{eE}}$ = embdim $\widehat{\mathcal{O}_{X_\infty, P_{eE}}} \leq e\ (\widehat{k}_E + 1)$ ([Re4], corol. 4.10).

Remark 3.1. The previous reasoning does not assure an analogous statement to (10) for $P^X_{eE}\mathcal{O}_{X_\infty, P^X_{eE}}$ since, in general the P^X_{eE}-adic topology on $\mathcal{O}_{X_\infty, P^X_{eE}}$ is not separated (see [Re3] example 3.16 and theorem 3.13).

3.2. Moreover, in [Re4] we have described a regular system of parameters $\mathcal{Q}$ of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$. We will next recall how we proceeded. First, since char $k = 0$, there exists an open subset U of Y with nonempty intersection with E, an étale morphism $\widetilde{U} \to U$ and $\{u_1, \ldots, u_d\} \subset \mathcal{O}_{\widetilde{U}}$, $\{x_1, \ldots, x_d\} \subset \mathcal{O}_V$, where V is an open subset of X, such that the following holds: for all closed points y_0 in an open subset of the strict transform $\widetilde{E}$ of E in $\widetilde{U}$, after a possible replacement of u_i by $u_i + c_i$, $c_i \in k$, $2 \leq i \leq d$, we may suppose that $\{u_1, \ldots, u_d\}$ and $\{x_1, \ldots, x_d\}$ are regular systems of parameters in y_0 and in $\eta \circ \varphi(y_0)$, and besides, the local morphism $\eta^\sharp : \mathcal{O}_{V, \eta(y_0)} \to \mathcal{O}_{\widetilde{U}, y_0}$ is given by

$$(11)\qquad \begin{array}{ll} x_1 & \mapsto u_1^{m_1} \\ x_2 & \mapsto \sum_{1 \leq i \leq m_2} \lambda_{2,i}\ u_1^i + u_1^{m_2}\ u_2 \\ x_3 & \mapsto \sum_{1 \leq i \leq m_3} \lambda_{3,i}(u_2)\ u_1^i + u_1^{m_3}\ u_3 \\ \cdots & \cdots\cdots \\ x_\delta & \mapsto \sum_{1 \leq i \leq m_\delta} \lambda_{\delta,i}(u_2, \ldots, u_{\delta-1})\ u_1^i + u_1^{m_\delta}\ u_\delta \\ x_{\delta+1} & \mapsto u_{\delta+1} \\ \cdots & \cdots\cdots \\ x_d & \mapsto u_d \end{array}$$

where $\delta = \mathrm{codim}_{\mathbb{A}^d}\ \overline{\eta(\xi_E)}$, $m_1 \leq \mathrm{ord}_{u_1} x_j$, $2 \leq j \leq d$, $0 < m_1 \leq m_2 \leq \ldots \leq m_d$, and, for $2 \leq j \leq \delta$ and $0 \leq i \leq m_j$, $\lambda_{j,i}(u_2, \ldots, u_{j-1})$ belongs to the *henselization* $k < u_2, \ldots, u_{j-1} >$ of the local ring $k[u_2, \ldots, u_{j-1}]_{(u_2, \ldots u_{j-1})}$, and, if $i < m_{j'}$, $j' < j$, then $\lambda_{j,i}$ belongs to $k < u_2, \ldots, u_{j'-1} >$; moreover, with no loss of generality we may also suppose that $\lambda_{j,m_j}(u_2, \ldots, u_{j-1})$ is a unit for $2 \leq j \leq \delta$ ((4) in [Re4], see also [Re3], proof of prop. 4.5).

3.3. Now, we consider the following situation: Let j, $2 \leq j \leq d+1$, let $v_2, \ldots, v_{j-1}$ so that $u_1, v_2, \ldots, v_{j-1}, u_j, \ldots, u_d \in \mathcal{O}_{\widetilde{U}}$ defines a regular system of parameters of $\mathcal{O}_{\widetilde{U}, y_0}$ for all closed points y_0 in an open subset of $\widetilde{E}$ (more precisely, there exist $(c_i)_i \in k^{d-1}$ such that $(u_1, \{v_i + c_i\}_{i=2}^{\delta}, \{v_i + c_i\}_{i=\delta+1}^{d})$ is a regular system of parameters of $\mathcal{O}_{\widetilde{U}, y_0}$), and let $\theta : \widetilde{U} \to$ Spec $k[v_2, \ldots, v_{j-1}]_h[x_1, y]$ be the k-morphism given by

$$\begin{array}{lll} x_1 & \mapsto & u_1^{m_1} \\ y & \mapsto & \displaystyle\sum_{m_1 \leq i \leq m} \lambda_i(v_2, \ldots, v_{j-1})\ u_1^i + u_1^m\ \varrho \mod (u_1)^{m+1} \end{array}$$

where $h \in k[v_2, \ldots, v_{j-1}] \setminus (v_2, \ldots, v_{j-1})$, $m \geq m_1$, $\lambda_i(v_2, \ldots, v_{j-1}) \in R_{j-1} := k < v_2, \ldots, v_{j-1} >$, $\varrho \in \mathcal{O}_{Y, y_0}$ and one of the following conditions holds:

(a) ϱ is transcendental over $k(u_1, v_2, \ldots, v_{j-1})$
(b) $\varrho = 0$.

Set $\mathfrak{e} := g.c.d.(\{m_1\} \cup \{i \ / \ \lambda_i \neq 0\})$, and define $\beta_0 := e_0 := m_1$, and $\beta_{r+1} := min\,\{i \ / \ \lambda_i \neq 0 \text{ and } g.c.d.\{\beta_0, \dots, \beta_r, i\} < e_r \ \}, \quad e_{r+1} := g.c.d.\{\beta_0, \dots, \beta_{r+1}\}$ for $1 \leq r < g$, being g such that $e_g = \mathfrak{e}$, and $\beta_{g+1} := m$. Let $n_r = \frac{e_{r-1}}{e_r}$, $1 \leq r \leq g-1$. We define $\{\overline{\beta}_r\}_{r=0}^{g+1}$ from $\{\beta_r\}_{r=0}^{g+1}$ as in (6) in 2.7.

Let B be a domain which is an étale extension of $k[v_2, \dots, v_{j-1}]_h$ and contains $\lambda_i(v_2, \dots, v_{j-1})$, $m_1 \leq i \leq m$. Let $\widetilde{\nu}$ be the order function on $B[x_1, y]$ extending ν and such that $\widetilde{\nu}(\ell) = 0$ for all $\ell \in B$ (note that $\widetilde{\nu}$ is a valuation if there is no nonzero element h with $\widetilde{\nu}(h) = \infty$, for instance in case (a)). As in example 2.7, we define $\widetilde{q}_0, \dots, \widetilde{q}_g \in B[x_1, y]$ such that $\widetilde{\nu}(\widetilde{q}_r) = \overline{\beta}_r$ for $0 \leq r \leq g+1$ as follows: $\widetilde{q}_0 = x_1, \ \widetilde{q}_1 = y - \sum_{i < \overline{\beta}_1} \lambda'_i \ (\widetilde{q}_0)^{\frac{i}{\beta_0}}$ and, for $1 \leq r \leq g$,

$$\text{(12)} \qquad \widetilde{q}_{r+1} = \widetilde{q}_r^{\,n_r} - \widetilde{c}_r \ \widetilde{q}_0^{\,b_{r,0}} \dots \widetilde{q}_{r-1}^{\,b_{r,r-1}} - \sum_{\gamma = (\gamma_0, \dots, \gamma_r)} \widetilde{c}_\gamma \ \widetilde{q}_0^{\,\gamma_0} \dots \widetilde{q}_r^{\,\gamma_r}, \quad 1 \leq r < g$$

where $\{b_{r,i}\}_{i=0}^{r-1}$ are the unique nonnegative integers satisfing $b_{r,i} < n_i$, $1 \leq i \leq r-1$, and $n_r \overline{\beta}_r = \sum_{0 \leq i < r} b_{r,i} \overline{\beta}_i$, we have $\widetilde{\nu}(\widetilde{q}_0^{\,\gamma_0} \dots \widetilde{q}_r^{\,\gamma_r}) > n_r \overline{\beta}_r$ for each sequence γ of nonnegative integers in the right hand side, and $\widetilde{c}_r, \widetilde{c}_\gamma \in B$, $\ \widetilde{c}_r \neq 0$ and $\widetilde{c}_\gamma \neq 0$ only for a finite number of γ's. In case (a), we also define $\widetilde{q}_{g+1}$ as in (12); then we have that $\{\overline{\beta}_r\}_{r=0}^{g+1}$ is the minimal generating sequence for the semigroup $\widetilde{\nu}(B[x_1, y] \setminus \{0\})$ and $\widetilde{q}_0, \dots, \widetilde{q}_{g+1} \in B[x_1, y]$ is a minimal generating sequence for $\widetilde{\nu}$ ([Sp] theorem 8.6). In case (b), $\widetilde{q}_{g+1} \in B[x_1, y]$, also defined as in (12), defines the kernel of $B[x_1, y] \to \mathcal{O}_{\widetilde{U}}$.

In case (a), by induction on r, $1 \leq r \leq g+1$, we will define elements $\{q'_r\}_{r=1}^{g+1}$ in $k(v_2, \dots, v_{j-1}, x_1, y)$ more precisely,

$$q'_r \in \prod_{r'=0}^{r-1} T_{r'}^{-1} \ k[v_2, \dots, v_{j-1}, x_1, y]$$

where $T_{r'}$ is the multiplicative system generated by $q'_{r'}$, satisfying the following: $q'_0 := x_1$ and, for $1 \leq r \leq g+1$ the image of q'_r in the fraction field $K(\mathcal{O}_{Y,y_0})$ of $\mathcal{O}_{Y,y_0}$ belongs to $\mathcal{O}_{Y,y_0}$ and, if we identify q'_r with its image, then

$$\text{(13)} \qquad \begin{aligned} q'_r \quad &= \mu_r(v_2, \dots, v_{j-1}) \ u^{\overline{\beta}_r} \qquad \mod (u)^{\overline{\beta}_r + 1} \text{ for } 1 \leq r \leq g \\ q'_{g+1} \quad &= \mu_{g+1}(v_2, \dots, v_{j-1}) \ u^{\overline{\beta}_{g+1}} \ \varrho \qquad \mod (u)^{\overline{\beta}_{g+1} + 1} \end{aligned}$$

where $\mu_r(v_2, \dots, v_{j-1})$ is a unit in R_{j-1}. In fact, once defined $q'_0, \dots, q'_r$, the element q'_{r+1} is defined as follows: let

$$h_{r,1} := {q'_0}^{b_{r,0}} \cdots {q'_{r-1}}^{b_{r,r-1}} \ P_{r,1} \left(\frac{\overline{\mu}_{r,1} (q'_r)^{n_r}}{{q'_0}^{b_{r,0}} \cdots {q'_{r-1}}^{b_{r,r-1}}}, \ v_2, \dots, v_{j-1} \right)$$

where the integers $\{b_{r,r'}\}_{r'=0}^{r-1}$ are as in (12), $\overline{\mu}_{r,1} := \mu_1^{b_{r,1}} \cdots \mu_{r-1}^{b_{r,r-1}}$ is a unit, and $P_{r,1} \in k[z, v_2, \dots, v_{j-1}]$ is such that

$$\text{(14)} \qquad P_{r,1}(\mu_r^{n_r}, v_2, \dots, v_{j-1}) = 0, \quad \frac{\partial P_{r,1}}{\partial z}(\mu_r^{n_r}, v_2, \dots, v_{j-1}) \text{ is a unit in } R_{j-1}.$$

Then, we have $n_r \overline{\beta}_r < \nu(h_1) \leq \overline{\beta}_{r+1}$. If $\nu(h_1) = \overline{\beta}_{r+1}$, we set $q'_{r+1} := h_1$. If not, we define recursively

$$h_{r,s} := {q'}_0^{b_0^s} \cdots {q'}_{r-1}^{b_{r-}^s} \ P_{r,s} \left(\frac{\overline{\mu}_{r,s} h_{r,s-1}}{q_0^{b_0^s} \cdots q_{r-1}^{b_{r-1}^s}}, \ v_2, \dots, v_{j-1} \right)$$

where $\{b_{r'}^s\}_{r'=0}^{r-1}$ are the unique nonnegative integers satisfying $b_{r'}^s < n_{r'}$, $1 \leq r' \leq r-1$, and $\nu(h_{r,s-1}) = \sum_{0 \leq r' \leq r-1} b_{j,r'}^s \overline{\beta}_{j,r'}$, $\overline{\mu}_{r,s} := \mu_1^{b_1^s} \cdots \mu_{r-1}^{b_{r-1}^s}$ is a unit, and

$P_{r,s} \in k[z, v_2, \ldots, v_{j-1}]$ is such that

$$(15) \quad P_{r,s}(\lambda_{s-1}, v_2, \ldots, v_{j-1}) = 0, \qquad \frac{\partial P_{r,s}}{\partial z}(\lambda_{s-1}, v_2, \ldots, v_{j-1}) \text{ is a unit in } R_{j-1}$$

being $\lambda_{s-1} \in R_{j-1}$ the initial form of $h_{r,s-1}$. We have $\nu(h_{r,s-1}) < \nu(h_{r,s}) \leq \overline{\beta}_{r+1}$ hence, after a finite number of steps we obtain s such that $\nu(h_{r,s}) = \overline{\beta}_{r+1}$ and we set $q_{r+1} := h_{r,s}$ (for more details see [Re4], lemma 3.1).

The elements q'_r and $\widetilde{q}_r$ are related. In fact, for $0 \leq r \leq g+1$, q'_r and $\widetilde{q}_r$ define the same initial form in an étale covering of a localization of the graded algebra $\mathrm{gr}_\nu k[v_2, \ldots v_{j-1}, x_1, y]_{(x_1,y)}$. More precisely, there exist $\widetilde{\ell}, \widetilde{h} \in \prod_{0 \leq r' < r} T_{r'}^{-1} B[x_1, y]$, $\widetilde{\ell}$ being a unit and $\widetilde{\nu}(\widetilde{h}) > \overline{\beta}_r$ such that $q'_r \;=\; \widetilde{q}_r \cdot \widetilde{\ell} + \widetilde{h}$.

3.4. Recall the expression in (11). Fixed j, $2 \leq j \leq \delta$, we apply the previous study to

$$\begin{aligned} x_1 &\mapsto u_1^{m_1} \\ x_j &\mapsto \sum_{1 \leq i \leq m_j} \lambda_{j,i}(u_2, \ldots, u_{j-1})\ u_1^i + u_1^{m_j}\ u_j. \end{aligned}$$

Let B_{j-1} be a domain which is an étale extension of $k[u_2, \ldots, u_{j-1}]$ and contains $\lambda_{j,i}(u_2, \ldots, u_{j-1})$, $m_1 \leq i \leq m_j$, let $\widetilde{\nu}_j$ be the valuation on $B_{j-1}[x_1, x_j]$ extending ν and let $\{\overline{\beta}_{j,r}\}_{r=0}^{g_j+1}$ the minimal generating sequence for the semigroup $\widetilde{\nu}_j(B_{j-1}[x_1, x_j] \setminus \{0\})$. Let $\{\widetilde{q}_{j,r}\}_{r=0}^{g_j+1} \in B_{j-1}[x_1, x_j]$ be a minimal generating sequence for $\widetilde{\nu}_j$, and $\{q'_{j,r}\}_{r=0}^{g_j+1} \in k(u_2, \ldots, u_{j-1}, x_1, x_j)$ defined as in 3.3.

Consider the following sets with the lexicographic order

$$\mathcal{J}^* := \{(1,0)\} \cup \{(j,r) \,/\, 2 \leq j \leq \delta,\ 1 \leq r \leq g_j\}, \quad \mathcal{J} := \mathcal{J}^* \cup \{(j, g_j{+}1) \,/\, 2 \leq j \leq \delta\}.$$

Applying the argument in 3.3 and arguing by induction on $(j,r) \in \mathcal{J}$, we can define elements $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$,

$$(16) \qquad q_{j,r} \in \prod_{\substack{(j',r') \in \mathcal{J}^* \\ (j',r') < (j,r)}} T_{j',r'}^{-1}\ k[x_1, \ldots, x_j]$$

where $T_{j',r'}$ is the multiplicative system generated by $q_{j',r'}$, satisfying the following: $q_{1,0} := x_1$ and, for $(j,r) \in \mathcal{J}$, the image of $q_{j,r}$ in the fraction field $K(\mathcal{O}_{Y,y_0})$ of $\mathcal{O}_{Y,y_0}$ belongs to $\mathcal{O}_{Y,y_0}$ and, if we identify $q_{j,r}$ with its image, then

$$(17) \quad \begin{aligned} q_{j,r} &= \mu_{j,r}(u_2, \ldots, u_{j-1})\ u^{\overline{\beta}_{j,r}} && \bmod (u)^{\overline{\beta}_{j,r}+1} \text{ for } 1 \leq r \leq g_j \\ q_{j,g_j+1} &= \mu_{j,g_j+1}(u_2, \ldots, u_{j-1})\ u^{\overline{\beta}_{j,g_j+1}}\ u_j && \bmod (u)^{\overline{\beta}_{j,g_j+1}+1} \end{aligned}$$

where $\mu_{j,r}(u_2, \ldots, u_{j-1})$ is a unit in $k < u_2, \ldots, u_{j-1} >$. Besides, if $b_{j,0}, \ldots, b_{j,g_j}$ are the unique nonnegative integers satisfying $b_{j,r} < n_{j,r}$, $1 \leq r \leq g_j$, and $\overline{\beta}_{j,g_j+1} = \sum_{0 \leq i \leq g_j} b_{j,r} \overline{\beta}_{j,r}$, and we set $q_{j,0} := q_{1,0} = x_1$, then, identifying $q_{j,r}$ with its image in $\mathcal{O}_{Y,y_0}$, we have

$$(18) \qquad \frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}} \cdots q_{j,g_j}^{b_{j,g_j}}} \;=\; v_j \;\in\; \mathcal{O}_{Y,y_0}.$$

where $v_j = \gamma_j\ u_j \mod (u)$, being γ_j a unit in $k < u_2, \ldots, u_{j-1} >$. In particular note that $k < u_2, \ldots, u_j >= k < v_2, \ldots, v_j >$. Note also that $q_{j,r}$ is obtained from $q'_{j,r}$ by replacing $v_{j'}$ by $\frac{q_{j',g_{j'}+1}}{q_{j',0}^{b_{j',0}} \cdots q_{j',g_{j'}}^{b_{j',g_{j'}}}}$, for $1 \leq j' < j$. We will denote $\{P_{j,r,s}\}_s$ the polynomials in $k[z, v_2, \ldots, v_{j-1}]$ defined in order to obtain $q'_{j,r+1}$ from $q'_{j,r}$, hence satisfying (14) (resp. (15)) for $s = 1$ (resp. $s > 1$). The elements $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ are called a *system of transverse generators for* $\eta : Y \to \mathbb{A}^d_k$ *with respect to* E.

3.5. Finally, for every element $q \in \mathcal{O}_{Y,y_0}$ which is the image of an element in the fraction field of $k[x_1, \ldots, x_d]$, i.e. we can identify $q = l/g$ where $l, g \in k[x_1, \ldots, x_d]$, we can define $\{\overline{Q}_n\}_{n\geq 0}$ in $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ such that, in the ring $\mathcal{O}_{Y_\infty, P^Y_{eE}}$, we have

$$Q_n \equiv \overline{Q}_n \mod P^Y_{eE}. \tag{19}$$

More precisely, since P^Y_{eE} is a stable point and the image of g in $\mathcal{O}_{Y,y_0}$ is nonzero, there exists $c \in \mathbb{N}$ such that $G_0, \ldots, G_{c-1} \in P^Y_{eE}$, $G_c \notin P^Y_{eE}$. Hence we have

$$G_c Q_n + \ldots + G_{n+c} Q_e \;\equiv\; L_{n+c} \mod P^Y_{eE} \quad \text{for } n \geq 0$$

((14) in [Re3] proof of prop. 4.1) and we can define by recurrence $\overline{Q}_n \in S^{-1}\mathcal{O}_{\mathbb{A}^d_\infty}$, where S is the multiplicative system generated by G_c, satisfying (19) (see also lemma 4.1 in [Re4]).

Applying this to each $q_{j,r}$, we obtain $\overline{Q}_{j,r;n} \in \mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$, $n \geq 0$, such that $Q_{j,r;n} \equiv \overline{Q}_{j,r;n}$ modulo P^Y_{eE}. More precisely,

$$\overline{Q}_{j,r;n} \in \prod_{\substack{(j',r')\in\mathcal{J}^* \\ (j',r')<(j,r)}} \overline{T}^{-1}_{j',r'} \;\; k[x_1, \ldots, x_j]_\infty$$

where $k[x_1, \ldots, x_j]_\infty$ denotes $\mathcal{O}_{(\mathrm{Spec}\ k[x_1,\ldots,x_j])_\infty}$ and $\overline{T}_{j',r'}$ is the multiplicative system generated by $\overline{Q}_{j',r';e\overline{\beta}_{j',r'}}$. Then, let

$$\mathcal{Q} := \{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J}, en_{j,r-1}\overline{\beta}_{j,r-1} \leq n \leq e\overline{\beta}_{j,r}-1}.$$

It is clear (see (17)) that $(\mathcal{Q})\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}} \subseteq P^{\mathbb{A}^d}_{eE}\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$. Besides, note that, applying (6), (11) and, for the last equality, also (9), we have

$$\begin{aligned} \sharp\mathcal{Q} &= \textstyle\sum_{j=2}^{\delta}(e\overline{\beta}_{j,1} + e(\overline{\beta}_{j,2} - n_{j,1}\overline{\beta}_{j,1}) + \ldots + e(\overline{\beta}_{j,g_j+1} - n_{j,g_j}\overline{\beta}_{j,g_j})) = \\ &= e\ \textstyle\sum_{j=2}^{\delta} \left(\beta_{j,1} + (\ \beta_{j,2} - \beta_{j,1}) + \ldots + (\beta_{j,g_j+1} - \beta_{j,g_j})\right) = \\ &= e\ \textstyle\sum_{j=2}^{\delta} \beta_{j,g_j+1} = e\ \sum_{j=2}^{\delta} m_j = e\ (k_E(\mathbb{A}^d_k) + 1) = e\ (\widehat{k}_E(X) + 1). \end{aligned} \tag{20}$$

and recall that $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ is a regular local ring of dimension $e\ (k_E(\mathbb{A}^d_k) + 1)$ (see (xiii)). In [Re4] we have proved that $\mathcal{Q}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$; then $\mathcal{Q}$ is called *regular system of parameters of* $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{E}}$ *associated to* $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$. The proof is based in the study of the graded algebra $gr_{\nu_E} k[x_1, \ldots, x_d]$. In fact, the main idea in the proof is to show that $(\mathcal{Q})\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}}$ is a prime ideal and it follows from the following: It is proved that, modulo étale extension, $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}} / (\mathcal{Q})$ is isomorphic to a polynomial ring in countable many variables over certain localization of $gr_{\nu_E} k[x_1, \ldots, x_d]$. Since $gr_{\nu_E} k[x_1, \ldots, x_d]$ is a domain because ν_E is a valuation it follows that $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_{eE}} / (\mathcal{Q})$ is a domain (see [Re4], theorem 4.8).

More generally, let $\widetilde{q}_0, \ldots, \widetilde{q}_{g+1} \in B[x_1, y]$ be as in 3.3, and let us define $\widetilde{\mathcal{Q}} := \{\widetilde{Q}_{r;n}\}_{0\leq r\leq g+1, en_{j,r-1}\overline{\beta}_{j,r-1}\leq n\leq e\overline{\beta}_{j,r}-1}$ where $\widetilde{Q}_{r;n} \in B[x_1, y]_\infty$ and $\widetilde{L} := \prod_{r=0}^{g} \widetilde{Q}_{r;e\overline{\beta}_r}$. Then $(\widetilde{\mathcal{Q}})$ is a prime ideal of $(B[x_1, y]_\infty)_{\widetilde{L}}$ ([Re4], prop. 4.5).

In order to study the ring $\widehat{\mathcal{O}_{X_\infty, P_{eE}}}$ we may first suppose that the irreducible component X_0 of X where the valuation ν is defined is analytically irreducible. In fact, there exists an étale morphism $\widetilde{X} \to X$ such that each irreducible component of $\widetilde{X}$ is analytically irreducible. Hence, there exists an irreducible component $\widetilde{X}_0$ of $\widetilde{X}$ whose image is X_0, and there exists a divisorial valuation $\widetilde{\nu}$ on $\widetilde{X}_0$ extending

ν. Let $\widetilde{P}_e$ be the stable point on $\widetilde{X}_0$ defined by $\widetilde{\nu}$ and e, whose image is P_{eE}. Then $\widehat{\mathcal{O}_{\widetilde{X}_0,\widetilde{P}_e}} \cong \widehat{\mathcal{O}_{X_0,P_{eE}}} \otimes_{\kappa(P_{eE})} \kappa(\widetilde{P}_e)$ (see (xii)). So, assume that X_0 is analytically irreducible. We will embed X_0 in a complete intersection scheme $X' \subseteq \mathbb{A}^M_k$ of dimension $d = \dim X_0$. For any such X' we have

$$\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_{eE}} \cong \mathcal{O}_{(X'_\infty)_{\mathrm{red}},P_{eE}} \quad \text{and} \quad \widehat{\mathcal{O}_{(X_\infty),P_{eE}}} \cong \widehat{\mathcal{O}_{(X'_\infty),P_{eE}}}$$

where we also denote by P_{eE} the point induced by P^X_{eE} in X'_∞ or in $(X'_\infty)_{\mathrm{red}}$ (see (ii) and (ix) in 2.3).

Proposition 3.6. *Assume that char $k = 0$. Let X_0 be a reduced separated k-scheme of finite type. Assume that X_0 is analytically irreducible. Let $\nu = \nu_E$ be a divisorial valuation on X_0 and let $e \in \mathbb{N}$. Then, there exist a complete intersection scheme*

$$X' = Spec\ k[y_1, \ldots, y_N] \ / \ (f_{d+1}, \ldots, f_N) \quad \subseteq \quad \mathbb{A}^N_k$$

which contains X_0, and of dimension $d = \dim X_0$, and elements $\{z_{l,s}\}_{d+1 \le l \le N, 1 \le s \le g_l}$ in $k[y_1, \ldots, y_N]$, such that, if given $g \in k[y_1, \ldots, y_N]$ we denote $\nu(g)$ the ν-value of the class of g in $\mathcal{O}_{X_0}$, then the following holds:

(a) *For $d+1 \le l \le N, 1 \le s \le g_l$ let $\overline{\alpha}_{l,s} := \nu(z_{l,s})$ and let*

$$\mathcal{Z} = \cup_{l=d+1}^N \mathcal{Z}_l \quad \textit{where} \quad \mathcal{Z}_l := \{Z_{l,s;n}\}_{\substack{1 \le s \le g_l \\ 0 \le n < e\overline{\alpha}_{l,s}}}$$

being $Z_{j,r;n} \in k[y_1, \ldots, y_N]_\infty$. Then there exists $G \in \mathcal{O}_{(\mathbb{A}^N)_\infty}$ such that $(\mathcal{Q} \cup \mathcal{Z}) \left(\mathcal{O}_{(\mathbb{A}^N)_\infty}\right)_G$ is a prime ideal and

$$P^{X'}_{eE} \mathcal{O}_{X'_\infty, P^{X'}_{eE}} = (\mathcal{Q} \cup \mathcal{Z})\, \mathcal{O}_{X'_\infty, P^{X'}_{eE}}$$

(b) *For $d+1 \le l \le N$, $f_l = f_l(y_1, \ldots, y_d, y_l) \in k[y_1, \ldots, y_d, y_l]$ satisfies:*
   (i) *$\nu(Jac(f_l)) = \nu(\frac{\partial f_l}{\partial y_l})$; set $\epsilon_l := \nu(Jac(f_l))$,*
   (ii) *for all $n \ge 0$, the class of $\frac{\partial F_{l;e\epsilon_l+n}}{\partial Y_{l;n}}$ in $\mathcal{O}_{X'_\infty,P_{eE}}$ is a unit and, for $n' > n$, the class of $\frac{\partial F_{l;e\epsilon_l+n}}{\partial Y_{l;n'}}$ in $\mathcal{O}_{X'_\infty,P_{eE}}$ belongs to $P_{eE}\mathcal{O}_{X'_\infty,P_{eE}}$. Besides, if we denote $f'_{l,l} := \frac{\partial f_l}{\partial y_l}$ then the class of $\frac{\partial F_{l;e\epsilon_l+n}}{\partial Y_{l;n}} - F'_{l,l;e\epsilon_l}$ in $\mathcal{O}_{X'_\infty,P_{eE}}$ belongs to P_{eE}.*
   (iii) *there exists $L \in \mathcal{O}_{\mathbb{A}^d_\infty} = k[x_1, \ldots, x_d]_\infty$, $L \notin P^{\mathbb{A}^d}_{eE}$, such that the elements $F_{l;0}, \ldots, F_{l;e\epsilon_l-1}$ belong to $(\mathcal{Q} \cup \mathcal{Z}_l)^2 \left(\mathcal{O}_{(\mathbb{A}^N_k)_\infty}\right)_L$.*

Proof. Let $\pi : Y \to X_0$, $\rho : X_0 \to \mathbb{A}^d_k$ and $\eta = \rho \circ \pi : Y \to \mathbb{A}^d_k$ be as in the beginning of this section. Let us consider an étale morphism $\widetilde{U} \to U$ as in 3.2 and keep the notation in 3.2. From the discussion in 3.2, 3.3 and 3.4 it follows that there exist $\{u, v_2, \ldots, v_d\} \in \mathcal{O}_{\widetilde{U}}$ and $\{x_1, \ldots, x_d, x_{d+1}, \ldots, x_N\} \in \mathcal{O}_X$ such that, after replacing v_i by $v_i + c_i$ where $c_i \in k$, $2 \le i \le d$, the following property holds for the points y_0 in an open subset of $\widetilde{E}$: $\{u, v_2, \ldots, v_d\}$ (resp. $\{x_1, \ldots, x_d\}$) is a regular system of parameters of $\mathcal{O}_{\widetilde{U},y_0}$ (resp. $\mathcal{O}_{\mathbb{A}^d_k,\eta(y_0)}$) and $\{x_1, \ldots, x_d, x_{d+1}, \ldots, x_N\}$ generate the maximal ideal of $\mathcal{O}_{X_0,\pi(y_0)}$ such that:

(i) The local expression for η in (11) holds for the regular system of parameters $\{u, v_2, \ldots, v_d\}$ of $\mathcal{O}_{\widetilde{U},y_0}$ and $\{x_1, \ldots, x_d\}$ of $\mathcal{O}_{\mathbb{A}^d,\eta(y_0)}$ (i.e. in (11) replace u_1 by u, u_i by v_i for $2 \le i \le \delta$, and set $v_i = u_i$ for $\delta < i \le d$).
(ii) There exists a system of transverse generators $\{q_{j,r}\}_{(j,r) \in \mathcal{J}}$ for $\eta : Y \to \mathbb{A}^d_k$ with respect to E, hence satisfying (16), (17) and (18).

(iii) For $d+1 \leq l \leq N$, the image of x_l in $\mathcal{O}_{Y,y_0}$ is expressed as

$$x_l = \sum_{m_1 \leq i} \lambda_{l,i}(\underline{v})\ u^i \tag{21}$$

where, $\underline{v} := (v_2, \dots, v_d)$ and

$$\begin{array}{ll} \lambda_{l,i}(\underline{v}) & \in k < \underline{v} > \cap\ \mathcal{O}_{\widetilde{U},y_0} \\ \lambda_{l,i}(\underline{v}) & \in k < v_2, \dots, v_{j-1} > \cap\ \mathcal{O}_{\widetilde{U},y_0} \quad \text{if } i < m_j \quad \text{for } 2 \leq j \leq \delta \end{array} \tag{22}$$

(recall (9) for the second assertion in (22)).

Fix l, $d+1 \leq l \leq N$. Let $\overline{\beta}_{l,0}, \dots, \overline{\beta}_{l,g_l}$ be a minimal system of generators of the semigroup defined by the restriction ν_l of ν_E to $k(\underline{v})[x_1, x_l]_{(x_1,x_l)}$. Let $e_{l,r} = g.c.d.\{\overline{\beta}_{l,0}, \dots, \overline{\beta}_{l,r}\}$, $0 \leq r \leq g_l$, $n_{l,r} = \frac{e_{l,r-1}}{e_{l,r}}$, $1 \leq r \leq g_l$, and let $\beta_{l,0}, \dots, \beta_{l,g_l}$ be defined by $\overline{\beta}_{l,r} - n_{l,r-1}\overline{\beta}_{l,r-1} = \beta_{l,r} - \beta_{l,r-1}$ as in (6). Consider $h \in k[\underline{v}]$ such that $k[\underline{v}]_h$ is contained in the ring $\mathcal{O}_{\widetilde{U}}$ and consider the morphism $\theta_l : \widetilde{U} \to$ Spec $k[\underline{v}]_h[x_1, y]$ given by

$$x_1 \mapsto u^{m_1}$$

$$y \mapsto \sum_{m_1 \leq i} \lambda_{l,i}(\underline{v})\ u^i$$

Since X_0 is analytically irreducible, there exists a domain B_l such that $B_l[x_1, y]$ is an étale extension of $k[\underline{v}]_h[x_1, y]$ and there exist $x_1', y' \in B_l[x_1, y]$ with

$$x_1' = \gamma_1\ x_1, \quad y' = \gamma_l\ y \quad \text{where } \gamma_1, \gamma_l \in B_l[x_1, y] \quad \text{are units}$$

and $u' = \mu\ u$ where μ is a unit in an étale extension of $k[\underline{v}]_h[u]$, such that the induced morphism $\widetilde{\theta}_l : \widetilde{\widetilde{U}} \to$ Spec $B_l[x_1', y']$, being $\widetilde{\widetilde{U}} \to \widetilde{U}$ étale, is given by

$$x_1' \mapsto (u')^{m_1}$$

$$y' \mapsto \sum_{m_1 \leq i \leq m} \lambda_{l,i}'\ (u')^i$$

where $\lambda_{l,i}' \in B_l$ for $m_1 \leq i \leq m$ (see remark 2.8). Let $\widetilde{q}_{l,0}, \dots, \widetilde{q}_{l,g_l}, \widetilde{q}_{l,g_l+1} \in B_l[x_1', y']$ be the elements defined as in 3.3 applied to the previous expression, hence case (b) in 3.3. Hence $\widetilde{q}_{l,g_l+1}$ defines the kernel of $B_l[x_1, y] \to \mathcal{O}_{\widetilde{\widetilde{U}}}$, i.e.

$$B_l[x_1, x_l]\ \cong\ B_l[x_1, y]\ /\ (\widetilde{q}_{l,g_l+1}).$$

Thus $\widetilde{q}_{l,g_l+1}$ defines an equation of a plane curve in Spec $L_l[x_1', y']$, where L_l is a field extension of k containing $\lambda_{l,i}'$ for $m_1 \leq i \leq m$, which is analytically irreducible, and $\widetilde{q}_{l,1}, \dots, \widetilde{q}_{l,g_l}$ are its approximate roots. Let us also consider the following elements in $k[\underline{v}]_h[x_1, y]$: Let $f_0' := \widetilde{q}_0 = x_1$ and, for $1 \leq r \leq g+1$, let us define f_r' to be an irreducible polynomial in $k[\underline{v}]_h[x_1, y]$ defining the contracted ideal of $(\widetilde{q}_r)B_l[x_1, y]$ to $k[\underline{v}]_h[x_1, y]$. Set $f_l' := f_{l,g_l+1}$ and note that we have

$$f_l'(\underline{v}, x_1, y)\ =\ \widetilde{q}_{l,g_l+1} \cdot \widetilde{h} \tag{23}$$

where $\widetilde{h} \in B_l[x_1, y]$ and $\widetilde{q}_{l,g_l+1}$ does not divide $\widetilde{h}$. Let

$$C_l := \text{Spec } k[\underline{v}]_h[x_1, y]\ /\ (f_l') \qquad \widetilde{C}_l := \text{Spec } B_l[x_1, y]\ /\ (\widetilde{q}_{l,g_l+1}).$$

Note that the induced morphism $\widetilde{C}_l \to C_l$ is étale.

We consider now the spaces of arcs of C_l, $\widetilde{C}_l$. Let $\widetilde{\nu}$ be a divisorial valuation on $B_l[x_1, y]\ /\ (\widetilde{q}_{l,g_l+1})$ extending ν_l, hence $\widetilde{\nu}(\lambda) = 0$ for all $\lambda \in B_l$ (recall that $\nu_l(v_j) = 0$, $2 \leq j \leq d$) and let P_l' (resp. $\widetilde{P}_l$) be the stable point of $\mathcal{O}_{(C_l)_\infty}$ (resp. $\mathcal{O}_{(\widetilde{C}_l)_\infty}$) defined by ν_l and e (resp. $\widetilde{\nu}$ and e). Note that we have

$$\widehat{\mathcal{O}}_{(C_l)_\infty, P_l'}\ \prec\ \widehat{\mathcal{O}}_{(\widetilde{C}_l)_\infty, \widetilde{P}_l}$$

i.e. the ring on the right hand side dominates the ring on the left hand side. Following 3.5, let $\widetilde{\mathcal{Q}}_l := \{\widetilde{Q}_{l,r;n}\}_{0\le r\le g_l, en_{j,r-1}\overline{\beta}_{j,r-1}\le n\le e\overline{\beta}_{l,r}-1}$, then $(\widetilde{\mathcal{Q}}_l)$ defines a prime ideal $\widetilde{\mathbb{P}}_l$ in $(B_l[x_1,y]_\infty)_{\widetilde{L}}$, where $\widetilde{L}=\prod_{r=0}^{g}\widetilde{Q}_{r;e\overline{\beta}_r}$, and we have

$$(\widetilde{\mathcal{Q}}_l)\ \mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l} \;=\; \widetilde{P}_l\ \mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$$

(this argument has already been applied in Example 2.7, it is based on [Re4], prop. 4.5, see also 3.5). Besides $\widetilde{\mathbb{P}}_l$ is a stable point of $B_l[x_1,y]_\infty$, since $\widetilde{\mathcal{Q}}_l$ is a finite set. Let $\mathbb{P}'_l$ be the image of $\widetilde{\mathbb{P}}_l$ in $(\text{Spec } k[\underline{v}]_h[x_1,y])_\infty$. Since the morphism $k[\underline{v}]_h[x_1,y]_{(x_1,y)}\to B_l[x_1,y]_{(x_1,y)}$ is étale, $\mathbb{P}'_l$ is a stable point and we have

$$(B_l\widehat{[x_1,y]_\infty})_{\widetilde{\mathbb{P}}_l} \;\cong\; (k[\underline{v}]_h\widehat{[x_1,y]_\infty})_{\mathbb{P}'_l}\otimes_{\kappa(\mathbb{P}'_l)}\kappa(\widetilde{\mathbb{P}}_l) \tag{24}$$

([Re4] prop. 2.5, see (xii)). Let $\mathcal{F}'_l := \{F'_{l,r;n}\}_{0\le r\le g_l, 0\le n<\nu(f'_{l,r})}$ and let $L' = H_0\cdot\prod_{r=0}^{g}F'_{r;e\nu(f'_r)}$, then $(\mathcal{F}'_l)\,(k[\underline{v},x_1,y]_\infty)_{L'}$ is a prime ideal ([Re4], proof of prop. 4.5, see 3.5) and we have

$$(\mathcal{F}'_l)\ \ (k[\underline{v}]_h[x_1,y]_\infty)_{L'} = \mathbb{P}'_l\ \ (k[\underline{v}]_h[x_1,y]_\infty)_{L'} \tag{25}$$

and

$$(\mathcal{F}'_l)\ \mathcal{O}_{(C_l)_\infty,P'_l} \;=\; P'_l\ \mathcal{O}_{(C_l)_\infty,P'_l}.$$

Now, for $\widetilde{q}_{l,g_l+1}$, we have

(a.1) $\widetilde{\nu}(Jac(\widetilde{q}_{l,g_l+1})) = \widetilde{\nu}\left(\frac{\partial\widetilde{q}_{l,g_l+1}}{\partial y}\right) = \widetilde{\nu}\left(\frac{\partial\widetilde{q}_{l,g_l+1}}{\partial y'}\right) = (n_{l,g_l}-1)\overline{\beta}_{l,g_l}+\ldots+ +(n_1-1)\overline{\beta}_{l,1} = n_{l,g_l}\overline{\beta}_{l,g_l}-\beta_{l,g_l}$. Set $\widetilde{\epsilon} := n_{l,g_l}\overline{\beta}_{l,g_l}-\beta_{l,g_l}$, then:

(b.1) for all $n\ge 0$, the class of $\frac{\partial\widetilde{Q}_{l,g_l+1;e\widetilde{\epsilon}+n}}{\partial Y'_n}$ in $\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$ is equal to the class of $n_{l,g_l}\ldots n_{l,1}\ \widetilde{Q}^{n_{l,g_l}-1}_{l,g_l;e\overline{\beta}_{l,g_l}}\cdots\widetilde{Q}^{n_{l,1}-1}_{l,1;e\overline{\beta}_{l,1}}$ modulo $\widetilde{P}_l$, hence $\frac{\partial\widetilde{Q}_{l,g_l+1;e\widetilde{\epsilon}+n}}{\partial Y'_n}$ is a unit in $\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$,

(c.1) for $n'>n$, the class of $\frac{\partial\widetilde{Q}_{l,g_l+1;e\widetilde{\epsilon}+n}}{\partial Y'_{n'}}$ in $\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$ belongs to $\widetilde{P}_l\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$; . Therefore:

(b'.1) for all $n\ge 0$, the class of $\frac{\partial\widetilde{Q}_{l,g_l+1;e\widetilde{\epsilon}+n}}{\partial Y_n}$ in $\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$ is is a unit in $\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$

(c'.1) for $n'>n$, the class of $\frac{\partial\widetilde{Q}_{l,g_l+1;e\widetilde{\epsilon}+n}}{\partial Y_{n'}}$ in $\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$ belongs to $\widetilde{P}_l\mathcal{O}_{(\widetilde{C}_l)_\infty,\widetilde{P}_l}$;

(d.1) $\widetilde{Q}_{l,g_l+1;0},\ldots,\widetilde{Q}_{l,g_l+1;e\widetilde{\epsilon}-1}$ belong to $(\widetilde{\mathcal{Q}}_l)^2B_l[[x_1,y]]_\infty$.

In fact, to prove (d.1) we argue by recurrence, and prove that, for $1\le r\le g_l+1$,

$$\widetilde{Q}_{l,r;n}\in\Big(\{\widetilde{Q}_{l,r';n}\}_{0\le r'\le r-1, 0\le n\le\overline{\beta}_{l,r'}-1}\Big)^2\ B_l[x_1,y]_\infty \tag{26}$$

for $0\le n<e\ \big((n_{l,r-1}-1)\overline{\beta}_{l,r-1}+\ldots+(n_{l,1}-1)\overline{\beta}_{l,1}\big)=e(\overline{\beta}_{l,r}-\beta_{l,r})$. Now, from (24) and (25) we obtain that $F'_{l;0},\ldots,F'_{l;e\epsilon'-1}$ belong to $(\mathcal{F}'_l)^2\ \ (k[\underline{v},x_1,y]_\infty)_{\mathbb{P}'_l}$, where $\epsilon'=\widetilde{\nu}(\widetilde{h})+n_{l,g_l}\overline{\beta}_{l,g_l}-\beta_{l,g_l}$. Therefore (recall (23)), we conclude that:

(a.2) $\nu_l(Jac(f'_l))=\nu_l(\frac{\partial f'_l}{\partial y})=\widetilde{\epsilon}+\widetilde{\nu}(\widetilde{h})$. Let ϵ' denote this integer, then:

(b.2) for all $n\ge 0$, the class of $\frac{\partial F'_{l;e\epsilon'+n}}{\partial Y_n}$ in $\mathcal{O}_{(C_l)_\infty,P'_l}$ is a unit. Besides, if $h_l := \frac{\partial f'_l}{\partial y}$ then the class of $\frac{\partial F'_{l;e\epsilon'+n}}{\partial Y_n}-H_{l;e\epsilon'}$ in $\mathcal{O}_{(C_l)_\infty,P'_l}$ belongs to P'_l.

(c.2) for $n'>n$, the class of $\frac{\partial F'_{l;e\epsilon+n}}{\partial Y_{n'}}$ in $\mathcal{O}_{(C_l)_\infty,P'_l}$ belongs to $P'_l\mathcal{O}_{(C_l)_\infty,P'_l}$.

(d.2) $F'_{l;0},\ldots,F'_{l;e\epsilon-1}$ belong to $(\mathcal{F}'_l)^2\ \ (k[\underline{v},x_1,y]_\infty)_{H_0}$.

Now, let b be the smallest nonnegative integer such that $g'_l := h^b f'_l$ belongs to $k[\underline{v}, x_1, y]$ and let $\{b_{j,r}\}_{(j,r)\in\mathcal{J}^*}$ be a minimal sequence of nonnegative integers such that

$$f_l(x_1,\ldots,x_d,y_l) :=$$
$$= \prod_{(j,r)\in\mathcal{J}^*} q_{j,r}^{b_{j,r}}\; g'_l\left(\frac{q_{2,g_2+1}}{q_{1,0}^{b_{2,0}}\cdots q_{2,g_2}^{b_{2,g_2}}},\ldots,\frac{q_{\delta,g_\delta+1}}{q_{\delta,0}^{b_{\delta,0}}\cdots q_{\delta,g_\delta}^{b_{\delta,g_\delta}}}, x_{\delta+1},\ldots,x_d,x_1,y_l\right)$$

belongs to $k[x_1,\ldots,x_d,y_l]$, being y_l an indeterminacy (recall (16) and (18)). Therefore we have

$$f_l(x_1,\ldots,x_d,x_l) = 0. \tag{27}$$

From (9) and (a.2) it follows that

$$\epsilon_l := \nu(Jac(f_l)) = \nu\left(\frac{\partial f_l}{\partial y_l}\right) = \nu\left(\prod_{(j,r)\in\mathcal{J}^*} q_{j,r}^{b_{j,r}} h^b\right) + \epsilon' \tag{28}$$

i.e. (i) in the statement of the proposition holds. From (b.2) and (c.2) we obtain that (ii) also holds.

For $0 \le s \le g_l+1$, let $b(l,s)$ be the smallest nonnegative integer such that $g'_{l,s} := h^{b(l,s)} f'_{l,s}$ belongs to $k[\underline{v}, x_1, y]$ and let $\{b_{j,r}(l,s)\}_{(j,r)\in\mathcal{J}^*}$ be a minimal sequence of nonnegative integers such that

$$z_{l,s} :=$$
$$\prod\nolimits_{(j,r)\in\mathcal{J}^*} q_{j,r}^{b_{j,r}(l,s)} \cdot g'_{l,s}\left(\frac{q_{2,g_2+1}}{q_{2,0}^{b_{2,0}}\cdots q_{2,g_2}^{b_{2,g_2}}},\ldots,\frac{q_{\delta,g_\delta+1}}{q_{\delta,0}^{b_{\delta,0}}\cdots q_{\delta,g_\delta}^{b_{\delta,g_\delta}}}, x_{\delta+1},\ldots,x_d,x_1,y_l\right) \tag{29}$$

belongs to $k[x_1,\ldots,x_d,y_l]$. Set $\overline{\alpha}_{l,s} := \nu(x_{l,s})$, being $x_{l,s}$ the class of $z_{l,s}$ in $\mathcal{O}_{X'}$, and $\mathcal{Z}_l := \{Z_{l,s;n}\}_{1\le s\le g_l,\ 0\le n<e\overline{\alpha}_{l,s}}$. Then, from (d.2) and applying also the second assertion in (22), we conclude that

$$F_{l;0},\ldots,F_{l;e\epsilon_l-1} \;\in\; (\mathcal{Q}\cup\mathcal{Z}_l)^2\left(\prod_{(j,r)\in\mathcal{J}^*} \overline{T}_{j,r}^{-1}\; k[x_1,\ldots,x_d,y_l]_\infty\right)_{\overline{H}_0}$$

where, if we consider h as an element of $k(x_1,\ldots,x_d)$, i.e. replace v_j by $\frac{q_{j,g_j+1}}{q_{j,0}^{b_{j,0}}\cdots q_{j,g_j}^{b_{j,g_j}}}$ (resp. x_j), for $1 \le j < \delta$ (resp. $\delta+1 \le j \le d$), then $\overline{H}_0 \in \prod_{(j,r)\in\mathcal{J}^*} \overline{T}_{j,r}^{-1}\; k[x_1,\ldots,x_d]_\infty$ satisfies $H_0 \equiv \overline{H}_0 \mod P_{eE}^Y$, as in 3.5. In particular, if $L := \overline{H}_0 \cdot \prod_{(j,r)\in\mathcal{J}^*} \overline{Q}_{j,r;e\overline{\beta}_{j,r}}$, we obtain that $F_{l;0},\ldots,F_{l;e\epsilon_l-1} \in (\mathcal{Q}\cup\mathcal{Z}_l)^2\left(k[x_1,\ldots,x_d,y_l]_\infty\right)_L$. Setting $G_l = L\cdot\prod_{s=1}^{g_l} Z_{l,s;e\overline{\alpha}_{l,s}}$, and applying (25) and that $\mathcal{Q}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$, we have that $(\mathcal{Q}\cup\mathcal{Z}_l)(k[x_1,\ldots,x_d,y_l]_\infty)_{G_l}$ is a prime ideal.

Finally, applying (27) we conclude that

$$X' = \operatorname{Spec} k[x_1,\ldots,x_d,y_{d+1},\ldots,y_N] \;/\; (f_{d+1},\ldots,f_N)$$

is a d-dimensional complete intersection scheme in $\mathbb{A}_k^N$ containing X_0 and satisfying (i) to (iii) in (b). Besides, if we set $G = L\cdot\prod_{l=d+1}^{N}\prod_{s=1}^{g_l} z_{l,s;\overline{\alpha}_{l,s}}$ then we conclude that $(\mathcal{Q}\cup\mathcal{Z}_l)(k[x_1,\ldots,x_d,y_{d+1},\ldots,y_N]_\infty)_G$ is a prime ideal such that

$$(\mathcal{Q}\cup\mathcal{Z}_l)\mathcal{O}_{X'_\infty, P_{eE}^{X'}} = P_{eE}^{X'}\mathcal{O}_{X'_\infty, P_{eE}^{X'}}.$$

Thus, the the proposition is proved. □

Remark 3.7. Keep the notation in prop. 3.6, and fix l, $d+1 \leq l \leq N$. Denote $\underline{Y}_n^{(l)} := (Y_{1;n}, \ldots, Y_{d;n}, Y_{l;n})$, $n \geq 0$, and $f'_{l,j} := \frac{\partial f_l}{\partial y_j}$, $j \in \{1, \ldots, d, l\}$. Then, applying Taylor's formula it follows that, for $n \geq e\epsilon_l$,

$$F_{l;n+e\epsilon_l+1} = H_{l;n+e\epsilon_l+1} + \sum_{j=1}^{d} \sum_{i=0}^{e\epsilon_l} F'_{l,j;i}\; Y_{j;n+e\epsilon_l+1-i} + \sum_{i=0}^{e\epsilon_l} F'_{l,l;i}\; Y_{l;n+e\epsilon_l+1-i}, \tag{30}$$

where $H_{l;n+e+1} \in k[\underline{Y}_0^{(l)}, \ldots, \underline{Y}_n^{(l)}]$ is the coefficient in $t^{n+e\epsilon_l+1}$ of $f_l(\sum_{i=0}^{n} \underline{Y}_i^{(l)}\; t^i)$ (see [Re1] proof of lemma 3.2). In particular, since $\epsilon_l := \nu(Jac(f_l)) = \nu(\frac{\partial f_l}{\partial y_l})$, it follows that, for $n \geq e\epsilon_l$,

$$\frac{\partial F_{l;n+e\epsilon_l+1}}{Y_{l;n+1}} = F'_{l,l;e\epsilon_l} \notin P_{eE}$$

$$\frac{\partial F_{l;n+e\epsilon_l+1}}{Y_{l;n'+1}} = \begin{cases} F'_{l,l;e\epsilon_l-(n'-n)} \in P_{eE} & \text{for } n+1 \leq n' \leq n+e\epsilon_l \\ 0 & \text{for } n+e\epsilon_l < n'. \end{cases}$$

This idea, generalized to complete intersection schemes (see[Re2], proof of lemma 4.2) is a key point in the proof of [Re2], th. 4.1 (see 2.3 (vii) and (viii)). Proposition 3.6 is an improvement of the previous assertion to a similar property for $0 \leq n \leq e\epsilon_l$.

Theorem 3.8. *Assume that char $k = 0$. Let X be a reduced separated k-scheme of finite type, let $\nu = \nu_E$ be a divisorial valuation on an irreducible component X_0 of X, and let $e \in \mathbb{N}$. Then*

$$embdim\ \mathcal{O}_{(X_\infty)_{red}, P_{eE}} \;=\; embdim\ \mathcal{O}_{\widehat{(X_\infty)_{red}, P_{eE}}} \;= e\ (\widehat{k}_E + 1). \tag{31}$$

where $\widehat{k}_E$ is the Mather discrepancy of X with respect to E.

Moreover, if $\rho : X \to \mathbb{A}_k^d$, where $d = \dim X_0$, is a general projection, more precisely a projection that satisfies (9), and $\mathcal{Q} = \{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J},\ en_{j,r-1}\overline{\beta}_{j,r-1} \leq n \leq e\overline{\beta}_{j,r}-1}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}_k^d)_\infty, P_{eE}^{\mathbb{A}^d}}$, then $\mathcal{Q}$ is a minimal system of coordinates of $((X_\infty)_{red}, P_{eE}^X)$, that is, we have $\sharp\mathcal{Q} = e\ (\widehat{k}_E + 1)$ and

$$P_{eE}^X\ \mathcal{O}_{(X_\infty)_{red}, P_{eE}^X} = (\mathcal{Q})\, \mathcal{O}_{(X_\infty)_{red}, P_{eE}^X}.$$

Proof. First recall that, since $\mathcal{Q}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P_{eE}^{\mathbb{A}^d}}$ ([Re4], theorem 4.8) and $\rho : X \to \mathbb{A}_k^d$ is a dominant morphism, we have

$$P_{eE}^X\ \widehat{\mathcal{O}_{X_\infty, P_{eE}^X}} \;=\; (\mathcal{Q})\, \widehat{\mathcal{O}_{X_\infty, P_{eE}^X}}$$

([Re3], prop. 4.5, see (xi)). From this and Nakayama's lemma, the second assertion of the theorem follows (see also (20)). Therefore, we only have to prove (31), or equivalently, the independence of the elements of $\mathcal{Q}$ in $P_{eE}^X\ /\ (P_{eE}^X)^2$.

Let $\widetilde{X} \to X$ be an étale morphism such that each irreducible component of $\widetilde{X}$ is analytically irreducible. Let $\widetilde{X}_0$ be an irreducible component of $\widetilde{X}$ whose image is X_0 and let $\widetilde{\nu}$ be a divisorial valuation on $\widetilde{X}_0$ extending ν. More precisely, if $Y \to X$ is a resolution of singularities of X and E is a divisor on Y such that $\nu = \nu_E$ then $\widetilde{Y} := Y \otimes_X \widetilde{X} \to \widetilde{X}$ is a resolution of singularities of $\widetilde{X}$ and we may choose a divisor $\widetilde{E}$ on $\widetilde{Y}$ whose image on $\widetilde{Y}$ is E, and take $\widetilde{\nu} = \nu_{\widetilde{E}}$. Then $\mathcal{O}_{\widetilde{X}_\infty, P_{e\widetilde{E}}}$ is étale over $\mathcal{O}_{X_\infty, P_{eE}}$ and, since $\Omega_{\widetilde{X}/X} = 0$ we have $\widehat{k}_{\widetilde{E}}(\widetilde{X}) = \widehat{k}_E(X)$. Therefore, it suffices to prove the theorem for $\widetilde{X}$, equivalently, we may suppose that X_0 is analytically irreducible.

So, let us assume that X_0 is analytically irreducible. Then, we can apply prop. 3.6. Let X' be the d-dimensional complete intersection scheme containing X_0 defined in 3.6 and keep the notation in prop. 3.6. We have $\mathcal{O}_{(X_\infty)_{\rm red},P_{eE}} \cong \mathcal{O}_{(X'_\infty)_{\rm red},P^{X'}_{eE}}$ and $\widehat{\mathcal{O}_{(X_\infty),P_{eE}}} \cong \widehat{\mathcal{O}_{(X'_\infty),P^{X'}_{eE}}}$ (see (ii) and (ix)). Therefore, in order to prove (31) we may suppose that $X = X'$. We will next describe the ring $\widehat{\mathcal{O}_{X_\infty,P_{eE}}}$, where $X = X'$ and $P_{eE} = P^X_{eE}$. We will follow the ideas in example 2.7 (or corol. 4.6 in [Re3]), where an analogous description is made.

The residue field of $P^{\mathbb{A}^d}_{eE}$ is

$$\kappa(P^{\mathbb{A}^d}_{eE}) \cong k\left(\{X_{1;n}\}_{n>em_1} \cup \{X_{j;n}\}_{\substack{2\leq j\leq d\\ n\geq em_j}}\right)\left[\{W_{j,r}\}_{(j,r)\in\mathcal{J}^*}\right] \ / \ J$$

where we set $m_j := 0$ for $\delta+1 \leq j \leq d$ (see (11)), $W_{j,r}$ is the class of $\overline{Q}_{r,j;e\overline{\beta}_{j,r}}$, and J is the ideal generated by

$$(32) \qquad P_{j,r,1}\left(\frac{\overline{\mu}_{j,r,1}(W_{j,r})^{n_{j,r}}}{W_{1,0}^{b_{j,0}}\cdots W_{j,r-1;}^{b_{j,r-1}}}, \frac{W_{2,g_2+1}}{W_{1,0}^{b_{2,0}}\cdots W_{2,g_2}^{b_{2,g_2}}}, \ \dots \ , \frac{W_{j-1,g_{j-1}+1}}{W_{1,0}^{b_{j-1,0}}\cdots W_{j-1,g_{j-1}}^{b_{j-1,g_{j-1}}}}\right)$$

(recall 3.3 and 3.4). From the property (14) satisfied by $P_{j,r,1}$ and Hensel's lemma, it follows that we can define an embedding $\kappa(\mathbb{A}^d_k) \hookrightarrow \widehat{\mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}_{eE}}}$ sending $X_{j;n}$ to $X_{j;n} \in \widehat{\mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}_{eE}}}$, for $j = 1, n > em_1$, and $2 \leq j \leq d, n \geq em_j$, sending $W_{1,0}$ to $X_{1;em_1}$ and, recursively, for $(j,r) \in \mathcal{J}^* \setminus \{(1,0)\}$, sending $W_{j,r}$ to a root of the polynomial obtained from (32) by replacing $W_{j',r'}$, $(j',r') < (j,r)$, by its image in $\widehat{\mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}_{eE}}}$; this root exists by Hensel's lemma. Then we have

$$\widehat{\mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}_{eE}}} \ \cong \ \kappa(P^{\mathbb{A}^d}_{eE})\left[\left[\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J}\\ en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}}\right]\right]$$

where the image of $X_{j,r;n}$ in $\widehat{\mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}_{eE}}}$ is $\overline{Q}_{r,j;n}$. Besides $\widehat{\mathcal{O}_{X_\infty,P^X_{eE}}}$ is a quotient of $\kappa(P^X_{eE})\left[\left[\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J}\\ en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}}\right]\right]$ where the residue field $\kappa(P^X_{eE})$ of P^X_{eE} is a finite field extension of $\kappa(P^{\mathbb{A}^d}_{eE})$.

Now, fix l, $d+1 \leq l \leq N$. Arguing analogously we obtain that

$$\kappa_l := \kappa(P^{\mathbb{A}^d}_{eE})\left[\{W_{l,s}\}_{s=1}^{g_l}\right] \ / \ J_l \quad \hookrightarrow \quad \kappa(P^X_{eE}).$$

where $W_{l,s}$ is the class of $Z_{l,s;e\overline{\alpha}_{l,s}}$ and J_l is the ideal generated by the relations on $\{W_{l,s}\}_{s=1}^{g_l}$ induced by $G'_{l,s;\nu(f'_{l,s})-(\overline{\beta}_{l,s}-n_{l,s-1}\overline{\beta}_{l,s-1})}$, $2 \leq s \leq g_l$ (see (29)). Applying recursively Hensel's lemma to these relations we can define an embedding $\kappa_l \hookrightarrow \widehat{\mathcal{O}_{X_\infty,P^X_{eE}}}$ sending $X_{j;n}$ to $X_{j;n} \in \widehat{\mathcal{O}_{X_\infty,P^X_{eE}}}$, for $j = 1, n > em_1$, and $2 \leq j \leq d, n \geq em_j$, and sending $W_{1,0}$ to $X_{1;em_1} \in \widehat{\mathcal{O}_{X_\infty,P^X_{eE}}}$. In particular, for each $n \geq 0$ we have defined $Y^{(0)}_{l;n} \in \kappa_l$ such that $Y_{l;n} - Y^{(0)}_{l;n} \in (\mathcal{Q} \cup \mathcal{Z}_l)$. Arguing recursively on $m \geq 1$ and $n \geq 0$, with the lexicographic order on (m,n), from $\{F_{l;e\epsilon_l+n}\}_{n\geq 0}$, applying property (ii) in prop. 3.6 (b) and Hensel's lemma, and reasoning as in corol. 5.6 in [Re3] it follows that, for $m, n \geq 0$, there exists $Y^{(m)}_{l;n} \in \kappa_l[\{X_{j,r;n}\}_{(j,r)\in\mathcal{J},en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}]$ such that,

$$(33) \qquad F_{e\epsilon_l+n} \equiv L^{(m,n)}_{e\epsilon_l}\ (Y_{l;n} - Y^{(m)}_{l;n}) \quad \mod \ (\mathcal{Q} \cup \mathcal{Z}_l)^m$$

in the ring $(k[x_1,\dots,x_d,y_l]_\infty)_{(\mathcal{Q}\cup\mathcal{Z}_l)}$ where $L^{(m,n)}_{e\epsilon_l}$ is a unit. More precisely, $L^{(m,n)}_{e\epsilon_l} - F'_{l,l;e\epsilon_l} \in (\mathcal{Q} \cup \mathcal{Z}_l)$ where recall that $f'_{l,l} := \frac{\partial f_l}{\partial y_l}$.

Therefore, $Y^{(m)}_{l;n} - Y^{(m)}_{l;n} \in (\mathcal{Q} \cup \mathcal{Z}_l)^m$ by (33), hence we have defined series $\widetilde{Y}_{l;n} \in \kappa_l \left[\left[\{X_{j,r;n}\}_{(j,r)\in\mathcal{J}, en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}\right]\right]$, $\widetilde{Y}_{l;n} = \lim_m Y^{(m)}_{l;n}$ and we conclude that

$$\kappa(P^X_{eE}) = \kappa(P^Z_{eE})\left[\{W_{l,s}\}_{(l,s)\in\mathcal{L}}\right] \Big/ \sum_{l=d+1}^{N} J_l$$

and

$$\widehat{\mathcal{O}_{X_\infty,P^X_{eE}}} \cong \kappa(P^X_{eE}) \left[\left[\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J} \\ en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}}\right]\right] \Big/ \left(\{\widetilde{F}_{l;n}\}_{\substack{d+1\leq l\leq N \\ 0\leq n\leq e\epsilon_l-1}}\right) \tag{34}$$

where, for $d+1 \leq l \leq N$, $0 \leq n \leq e\epsilon_l - 1$, $\widetilde{F}_{l;n}$ is obtained from $F_{l;n}$ by substituting $Y_{l;n'}$ by $\widetilde{Y}_{l;n}$, $0 \leq n' \leq n$ (see (25) in [Re3]). In fact, we have applied the definition $\widehat{\mathcal{O}_{X_\infty,P^X_{eE}}} := \lim_{\leftarrow m} \mathcal{O}_{X_\infty,P^X_{eE}}/(P^X_{eE})^{m+1}$ and also that $P^X_{eE}\mathcal{O}_{X_\infty,P^X_{eE}} = (\mathcal{Q}\cup\mathcal{Z})\,\mathcal{O}_{X_\infty,P^X_{eE}}$ and $\mathcal{O}_{X_\infty} = k[x_1,\ldots,x_d,y_{d+1},\ldots y_N]_\infty/(\{F_{l;n}\}_{d+1\leq l\leq N, n\geq 0})$. Besides, if $\widetilde{Z}_{l,s;n}$ denotes the series obtained from $Z_{l,s;n}$ by substituting $Y_{l;n'}$ by $\widetilde{Y}_{l;n}$, $0 \leq n' \leq n$, then we have

$$\widetilde{Z}_{l,s;n} \in \left(\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J} \\ en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}}\right) \quad \text{for } d+1 \leq l \leq N, 0 \leq n \leq e\overline{\alpha}_{l,s}. \tag{35}$$

Since $F_{l;0},\ldots,F_{l;e\epsilon_l-1} \in (\mathcal{Q}\cup\mathcal{Z}_l)^2\;\kappa(P^{\mathbb{A}^d}_{eE})[[\{X_{j,r;n}\}_{(j,r)\in\mathcal{J}, en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}]]$ by (iii) in (b) in prop. 3.6, applying (35) we conclude that that

$$\widetilde{F}_{l;n} \in \left(\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J} \\ en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}}\right)^2 \quad \text{for } d+1 \leq l \leq N,\ 0 \leq n \leq; e\ \epsilon_l - 1.$$

Therefore, the images of $\{X_{j,r;n}\}_{(j,r)\in\mathcal{J}, en_{j,r-1}\overline{\beta}_{j,r-1}\leq n<e\overline{\beta}_{j,r}}$ define a basis of $P^X_{eE}\widehat{\mathcal{O}_{X_\infty,P^X_{eE}}}/(P^X_{eE}\widehat{\mathcal{O}_{X_\infty,P^X_{eE}}})^2$. Thus we obtain(31), and this finishes the proof. □

Remark 3.9. Let X be a reduced separated scheme of finite type over a field k of characteristic zero. Let P be any stable point of X_∞ and suppose that X is nonsingular at the center P_0 of P and that P_0 is not the generic point of X. There exists a birational and proper morphism $\pi : Y \to X$ such that the center of ν_P on Y is a divisor E, and $e \in \mathbb{N}$ such that $\nu_P = e\nu_E$ ([Re3], (vii) in prop. 3.7, see (v)). Let $P^Y \in Y_\infty$ whose image by π_∞ is P, let $\rho : X \to \mathbb{A}^d_k$ be a general projection and let $P^{\mathbb{A}^d}$ be the image of P in $(\mathbb{A}^d_k)_\infty$. Then $k_E(\mathbb{A}^d) = \widehat{k}_E$ where $\widehat{k}_E$ is the Mather discrepancy of X with respect to E, and we have $\dim \mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}} = e\ \widehat{k}_E + \dim \mathcal{O}_{Y_\infty,P^Y}$ (see (xiii)). Recall that $P \supseteq P^X_{eE}$, hence $P^{\mathbb{A}^d} \supseteq P^{\mathbb{A}^d}_{eE}$ and, if $\mathcal{Q}$ is a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty,P^{\mathbb{A}^d}_{eE}}$, then $\mathcal{Q} \subset P$. Note that, since $\nu_P = e\nu_E$, the proof of prop. 3.6 extends to this case, and we obtain that the complete intersection scheme X' and the set $\mathcal{Z}$ defined in proposition 3.6 for the valuation ν_E and e also satisfy the properties obtained replacing P_{eE} by P in (i) to (iii) in prop 3.6 (b). Then, from the proof of theorem 3.8 it follows that

$$\text{embdim}\,\mathcal{O}_{(X_\infty)_{\text{red}},P} \;=\; \text{embdim}\,\widehat{\mathcal{O}_{(X_\infty)_{\text{red}},P}} \;= e\,(\widehat{k}_E + \dim \mathcal{O}_{Y_\infty,P^Y}).$$

4. A lower bound for the dimension

Recall that, given a divisorial valuation $\nu = \nu_E$ on X, the Mather-Jacobian log-discrepancy of X with respect to E is defined to be

$$a_{MJ}(E;X) := \widehat{k}_E - \nu_E(Jac_X) \ + \ 1$$

where Jac_X is the Jacobian ideal of X (see [I]).

Theorem 4.1. *Assume that char $k = 0$. Let X be a reduced separated k-scheme of finite type, let $\nu = \nu_E$ be a divisorial valuation on an irreducible component X_0 of X, and let $e \in \mathbb{N}$. Then we have*

$$\dim \widehat{\mathcal{O}_{X_\infty, P^X_{eE}}} \ \geq \ e \ a_{MJ}(E;X).$$

In particular, if X is normal and complete intersection then

$$\dim \widehat{\mathcal{O}_{X_\infty, P^X_{eE}}} \ \geq \ e \ (k_E + 1).$$

Proof. It is always possible to embed X in a complete intersection scheme X' such that $\widehat{k}_E(X) = \widehat{k}_E(X')$ and $\nu_E(Jac_X) = \nu_E(Jac_{X'})$. Hence, since $\widehat{\mathcal{O}_{(X_\infty),P_{eE}}} \cong \widehat{\mathcal{O}_{(X'_\infty),P^{X'}_{eE}}}$ (see (ii) and (ix)), it suffices to prove the result for X'. That is, we may assume that X is a complete intersection, more precisely, we may suppose that

$$X = \text{Spec } k[x_1, \ldots, x_N]/(f_1, \ldots, f_{N-d}).$$

We may also suppose that (9) holds, i.e.

$$\text{ord}_E \ \pi^*(dx_1 \wedge \ldots \wedge dx_d) = \widehat{k}_E. \tag{9}$$

For simplicity in the notation we will prove the result when $e = 1$; the proof when $e > 1$ follows in the same way. Let $\rho : X \to \mathbb{A}^d_k$ be the projection on the first d coordinates, let $\eta : Y \to \mathbb{A}^d_k$ be the composition $\eta = \rho \circ \pi$, let $P^{\mathbb{A}^d}_E$ be the image of P^Y_E by η_∞ and let $\mathcal{Q} = \{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J},\ n_{j,r-1}\overline{\beta}_{j,r-1}\leq n\leq\overline{\beta}_{j,r}-1}$ be a regular system of parameters of $\mathcal{O}_{(\mathbb{A}^d)_\infty, P^{\mathbb{A}^d}_E}$ associated to $\{q_{j,r}\}_{(j,r)\in\mathcal{J}}$, as in 3.5. So we have

$$P^X_E \ \mathcal{O}_{(X_\infty)_{\text{red}}, P^X_E} = \left(\{\overline{Q}_{j,r;n}\}_{(j,r)\in\mathcal{J},\ n_{j,r-1}\overline{\beta}_{j,r-1}\leq n\leq\overline{\beta}_{j,r}-1}\right) \mathcal{O}_{(X_\infty)_{\text{red}}, P^X_E}. \tag{36}$$

(theorem 3.8).

Let us consider the following $(N-d) \times (N-d)$-matrix with coefficients in $k[x_1, \ldots, x_N]$:

$$\Delta := \left(\frac{\partial f_i}{\partial x_{d+j}}\right)_{1\leq i,j\leq N-d}$$

and let $d^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ denote the determinant of the $r \times r$- minor of Δ defined by the rows $i_1, \ldots, i_r$ and the columns $j_1, \ldots, j_r$. After reordering $\{x_{d+j}\}_{j=1}^{N-d}$ we may suppose that

$$\nu_E\left(d^{1,\ldots,i}_{1,\ldots,i}\right) = \inf\left\{\nu_E\left(d^{1,\ldots,i-1,i}_{1,\ldots,i-1,j}\right)\right\}_{j=i}^{N-d} \quad \text{for } 1 \leq i \leq N-d. \tag{37}$$

For $1 \leq i \leq N-d$ set

$$\delta_i := \nu_E\left(d^{1,\ldots,i}_{1,\ldots,i}\right) \qquad\qquad \epsilon_i := \inf\left\{\nu_E\left(\frac{\partial f_i}{\partial x_{d+j}}\right)\right\}_{j=1}^{N-d} = \inf\left\{\nu_E\left(d^i_j\right)\right\}_{j=1}^{N-d}$$

and note that $\delta_1 = \epsilon_1$ and $\delta_{N-d} := \nu_E(Jac_X)$ by (9). It can be proved by recurrence that, for $1 \leq l \leq N-d$, $l \leq i,j \leq N-d$, we have

$$d^{1,\ldots,l-1,i}_{1,\ldots,l-1,j} \cdot d^{1,\ldots,l-2}_{1,\ldots,l-2} \ = \ d^{1,\ldots,l-2,i}_{1,\ldots,l-2,j} \cdot d^{1,\ldots,l-1}_{1,\ldots,l-1} \ - \ d^{1,\ldots,l-2,i}_{1,\ldots,l-2,l-1} \cdot d^{1,\ldots,l-2,l-1}_{1,\ldots,l-2,j} \ . \tag{38}$$

Let $f'_{1,i} := \frac{\partial f_1}{\partial x_i}$, $1 \leq i \leq N$, thus $f'_{1,d+i} = d^1_i$, $1 \leq i \leq N-d$. Let $\sum_{n\geq 0} F'_{1,i;n}t^n$ (resp. $\sum_{n\geq 0} D^{i_1,\ldots,i_r}_{j_1,\ldots,j_r;n}t^n$) denote the image of $f'_{1,i}$ (resp. $d^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$) in $k[x_1,\ldots,x_N]_\infty$. Given $a_1 > \epsilon_1$ and $n > (a_1-\epsilon_1)$, applying Taylor's formula to $f_1(w_0+t^{n-(a_1-\epsilon_1)}w_1)$, where $w_0 = \sum_{i=0}^{n-(a_1-\epsilon_1)-1} \underline{x}_i\, t^i$ and $w_1 = \sum_{i\geq n-(a_1-\epsilon_1)} \underline{x}_i\, t^{i-(n-(a_1-\epsilon_1))}$, we obtain that for $n > n_1 := 2a_1 - \epsilon_1$ (i.e. $2(n-(a_1-\epsilon_1)) > n+\epsilon_1$) we have

$$F_{1;\epsilon_1+n} \;=\; H'_{1;n}(\underline{X}_0,\ldots,\underline{X}_{n-(a_1-\epsilon_1)-1}) \;+\; \sum_{i=1}^{N}\sum_{r=0}^{a_1} F'_{1,i;r}X_{i;n+\epsilon_1-r}$$

where $H'_{1,n} \in k[\underline{X}_0,\ldots,\underline{X}_{n-(a_1-\epsilon_1)-1}]$ (see [Re2], proof of theorem 4.1, or equality (30) in remark 3.7, where the same argument is applied). Hence, there exists a polynomial $H_{1;n} \in k\left[\underline{X}_0,\ldots,\underline{X}_{n-(a_1-\epsilon_1)-1}, \{X_{j;n'}\}_{\substack{1\leq j\leq d\\ n-(a_1-\epsilon_1)\leq n'\leq n+\epsilon_1}}\right]$ such that

$$(39)\quad \begin{aligned} F_{1;\epsilon_1+n} \;=\; & H_{1;n}(\underline{X}_0,\ldots,\underline{X}_{n-(a_1-\epsilon_1)-1}, \{X_{j;n'}\}_{\substack{1\leq j\leq d\\ n'\leq n+\epsilon_1}}) \;+ \\ & + \sum_{i=1}^{N-d}\sum_{r=\epsilon_1}^{a_1} D^1_{i;r}X_{d+i;n+\epsilon_1-r} \quad \mathrm{mod}\ \left(\{D^1_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_1}}\right). \end{aligned}$$

It follows that, for $n > n_1$ there exists

$$X^{(1)}_{d+1;n} \in k\left[\{X_{j;n'}\}_{\substack{1\leq j\leq d\\ 0\leq n'\leq n+\epsilon_1}} \cup \{X_{d+1;n'}\}_{0\leq n'\leq n_1} \cup \{X_{d+i;n'}\}_{\substack{2\leq i\leq N-d\\ 0\leq n'\leq n}}\right]_{D^1_{1;\epsilon_1}}$$

such that

$$F_{1;\epsilon_1+n} = D^1_{1;\epsilon_1}\ (X_{d+1;n} - X^{(1)}_{d+1;n}) \quad \mathrm{mod}\ \left(\{D^1_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_1}} \cup \{F_{1;\epsilon_1+n'}\}_{n_1<n'<n}\right)$$

in the ring $(k[x_1,\ldots,x_N]_\infty)_{D^1_{1;\epsilon_1}}$. Besides, it can be proved by recurrence that, for $n > n_1 + a_1 - \epsilon_1$, $2 \leq i \leq N-d$ and $0 \leq r \leq a_1 - \epsilon_1$ we have

$$(40)\qquad \frac{\partial X^{(1)}_{d+1;n}}{\partial X_{d+i;n-r}} \;=\; -\sum_{s=0}^{r} \frac{D^1_{i;\epsilon_1+s}}{D^1_{1;\epsilon_1}}\ B^1_{r-s} \quad \mathrm{mod}\ \left(\{D^1_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_1}}\right).$$

where

$$B^1_{r-s} := \sum_{k_1,\ldots,k_m,b_1,\ldots,b_m} (-1)^b \frac{b!}{b_1!\cdots b_m!}\ \frac{(D^1_{1;\epsilon_1+k_1})^{b_1}\cdots(D^1_{1;\epsilon_1+k_m})^{b_m}}{(D^1_{1;\epsilon_1})^b}.$$

with $k_1,\ldots,k_m,b_1,\ldots,b_m$ runnig over all positive integers satisfying $k_1 < k_2 < \ldots < k_m$ and $\sum_{i=1}^m b_ik_i = r-s$, and $b := \sum_{i=1}^m b_i$.

Analogously, taking $a_2 > \epsilon_2$, applying Taylor's formula to f_2, and then replacing $X_{d+1;n'}$ by $X^{(1)}_{d+1;n'}$ for $n' > n_1$, i.e. considering the image $F^{(1)}_{2;\epsilon_2+n}$ of $F_{2;\epsilon_2+n}$ in $k\left[\{X_{j;n'}\}_{\substack{1\leq j\leq d\\ 0\leq n'\leq \epsilon_2+n}} \cup \{X_{d+1;n'}\}_{0\leq n'\leq n_1} \cup \{X_{d+i;n'}\}_{\substack{2\leq i\leq N-d\\ 0\leq n'\leq n}}\right]_{D^1_{1;\epsilon_1}}$, we obtain that for $n >> 0$, $2 \leq i \leq N-d$, $0 \leq r \leq \inf\{(a_1-\epsilon_1),(a_2-\epsilon_2)\}$, we have

$$(41)\quad \frac{\partial F^{(1)}_{2;\epsilon_2+n}}{\partial X_{d+i;n-r}} \;=\; \sum_{s=0}^{r} \frac{D^{1,2}_{1,i;\epsilon_1+\epsilon_2+s}}{D^1_{1;\epsilon_1}}\ B^1_{r-s} \quad \mathrm{mod}\ (\{D^1_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_1}} \cup \{D^2_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_2}}).$$

In fact, to conclude (41) we have to apply Taylor's development as in (39) and also the identities (40). Hence, if $(a_1-\epsilon_1)$ and $(a_2-\epsilon_2)$ are bigger than $(\delta_2-\delta_1-\epsilon_2)$, for $n >> 0$, $0 \leq r \leq \inf\{(a_1-\epsilon_1)-(\delta_2-\delta_1-\epsilon_2),(a_2-\epsilon_2)-(\delta_2-\delta_1-\epsilon_2)\}$ and $2 \leq i \leq N-d$, we have

$$\frac{\partial F^{(1)}_{2;\delta_2-\delta_1+n}}{\partial X_{d+i,n-r}} = \sum_{s=0}^{r} \frac{D^{1,2}_{1,i;\delta_2+s}}{D^1_{1;\epsilon_1}}\ B^1_{r-s}$$
$$\mod \left(\{D^1_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_1}} \cup \{D^2_{i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\epsilon_2}} \cup \{D^{1,2}_{1,i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\delta_2}}\right).$$

In particular

$$\frac{\partial F^{(1)}_{2;\delta_2-\delta_1+n}}{\partial X_{d+i,n}} \equiv \frac{D^{1,2}_{1,i;\delta_2}}{D^1_{1;\epsilon_1}} \quad \text{and} \quad \frac{\partial F^{(1)}_{2;\delta_2-\delta_1+n}}{\partial X_{d+i,n'}} \equiv 0 \quad \text{for } n' > n.$$

This implies that there exists n_2 such that for $n > n_2$ there exists

$$X^{(1)}_{d+2;n} \in k\left[\{X_{j;n'}\}_{\substack{1\leq j\leq d\\ n'\leq n+\delta_2-\delta_1}} \cup \{X_{d+i;n'}\}_{\substack{1\leq i\leq 2\\ n'\leq n_i}} \cup \{X_{d+i;n'}\}_{\substack{3\leq i\leq N-d\\ n'\leq n}}\right]_{D^1_{1;\epsilon_1}\cdot D^{1,2}_{1,2;\delta_2}}$$

such that

$$F_{2,\delta_2-\delta_1+n} = \frac{D^{1,2}_{1,2;\delta_2}}{D^1_{1;\epsilon_1}}\ (X_{d+2;n} - X^{(1)}_{d+2;n})$$

$$\mod \left(\{D^j_{i;s_1}, D^{1,2}_{1,i;s_2}\}_{\substack{1\leq i\leq N-d\\ 1\leq j\leq 2\\ s_1<\delta_1, s_2<\delta_2}} \cup \{F_{1;\epsilon_1+n'}\}^{n+(\delta_2-\delta_1-\epsilon_2)}_{n'=n_1+1} \cup \{F_{2;\delta_2-\epsilon_1+n'}\}_{n_2<n'<n}\right)$$

in the ring $(k[x_1,\ldots,x_N]_\infty)_{D^1_{1;\epsilon_1}\cdot D^{1,2}_{1,2;\delta_2}}$ and

$$\frac{\partial X^{(1)}_{d+2;n}}{\partial X_{d+i;n-r}} = -\sum_{s=0}^{r} \frac{D^{1,2}_{1,i;\delta_2+s}}{D^{1,2}_{1,2;\delta_2}}\ B^2_{r-s} \mod \left(\{D^j_{i;s}\}_{\substack{1\leq i\leq N-d\\ 1\leq j\leq 2\\ 0\leq s<\epsilon_1}} \cup \{D^{1,2}_{1,i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\delta_2}}\right)$$

for $2 \leq i \leq N-d$ and $0 \leq r \leq \inf\{(a_l-\epsilon_l)-(\delta_l-\delta_{l-1}-\epsilon_l)-\ldots-(\delta_2-\delta_1-\epsilon_2)\}_{1\leq l\leq 2}$, where we set $\delta_0 := 0$.

Now let

$$\mathcal{D} := \{D^j_{i;s}\}_{\substack{1\leq i,j\leq N-d\\ 0\leq s<\epsilon_j}} \cup \{D^{1,2}_{1,i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\delta_2}} \cup \ldots \cup \{D^{1,2,\ldots,N-d-1,N-d}_{1,2,\ldots,N-d-1,i;s}\}_{\substack{1\leq i\leq N-d\\ 0\leq s<\delta_{N-d}}}$$

and $D_0 := D^1_{1;\epsilon_1} \cdot D^{1,2}_{1,2;\delta_2} \cdots D^{1,2,\ldots,N-d}_{1,2,\ldots,N-d;\delta_{N-d}}$. Recall that, by (37) and since $\delta_i = \nu_E(d^{1,\ldots,i}_{1,\ldots,i})$, we have that, for each element in $\mathcal{D}$, its class in $\mathcal{O}_{X_\infty,P^X_E}$ is in P^X_E and also that the class of D_0 is a unit in $\mathcal{O}_{X_\infty,P^X_E}$. Following as before, we obtain that, for $1 \leq i \leq N-d$, given $a_i > \epsilon_i$, there exists n_i such that for $n > n_i$ there exists

$$X^{(1)}_{d+i;n} \in k\left[\{X_{j;n'}\}_{\substack{1\leq j\leq d\\ 0\leq n'\leq n+\delta_i-\delta_{i-1}}} \cup \{X_{d+j;n'}\}_{\substack{1\leq j\leq i\\ 0\leq n'\leq n_j}} \cup \{X_{d+j;n'}\}_{\substack{i+1\leq j\leq N-d\\ 0\leq n'\leq n}}\right]_{D_0}$$

satisfying

(42)
$$F_{i;\delta_i-\delta_{i-1}+n} = \frac{D^{1,\ldots,i}_{1,\ldots,i;\delta_i}}{D^{1,\ldots,i-1}_{1,\ldots,i-1;\delta_{i-1}}}\ (X_{d+i;n} - X^{(1)}_{d+i;n})$$
$$\mod (\mathcal{D} \cup \{F_{j;\delta_j-\delta_{j-1}+n'}\}_{\substack{1\leq j<i\\ n_j<n'<+n+(\delta_i-\delta_{i-1}-\epsilon_i)}} \cup \{F_{i;\delta_i-\delta_{i-1}+n'}\}_{n_i<n'<n})$$

in the ring $(k[x_1,\ldots,x_N]_\infty)_{D_0}$, and

(43)
$$\frac{\partial X^{(1)}_{d+i;n}}{\partial X_{d+j;n-r}} = -\sum_{s=0}^{r} \frac{D^{1,\ldots,i-1,i}_{1,\ldots,i-1,j;\delta_i+s}}{D^{1,\ldots,i}_{1,\ldots,i;\delta_i}}\ B^i_{r-s} \qquad \mod (\mathcal{D})$$

for $i \leq j \leq N-d$ and $r \leq \inf\{(a_l-\epsilon_l)-(\delta_l-\delta_{l-1}-\epsilon_l)-\ldots-(\delta_i-\delta_{i-1}-\epsilon_i)\}_{1\leq l\leq i}$ where

$$B^i_{r-s} := \sum_{k_1,\ldots,k_m,b_1,\ldots,b_m} (-1)^b \frac{b!}{b_1!\cdots b_m!} \frac{(D^{1,\ldots,i}_{1,\ldots,i;\delta_i+k_1})^{b_1}\cdots(D^{1,\ldots,i}_{1,\ldots,i;\delta_i+k_m})^{b_m}}{(D^{1,\ldots,i}_{1,\ldots,i;\delta_i})^b}.$$

$k_1,\ldots,k_m,b_1,\ldots,b_m$ running over all positive integers such that $k_1<\ldots<k_m$ and $\sum_{i=1}^m b_ik_i = r-s$, and $b := \sum_{i=1}^m b_i$. Note that from (43) and applying the equalities (38) it follows that for $n>>0$, the image $F^{(1)}_{i+1;\delta_{i+1}-\delta_i+n}$ of $F_{i+1;\delta_{i+1}-\delta_i+n}$ in $k\left[\{X_{j;n'}\}_{\substack{1\leq j\leq d\\ 0\leq n'\leq\epsilon_{i+1}+n}} \cup \{X_{d+j;n'}\}_{\substack{1\leq j\leq i\\ 0\leq n'\leq n_j}} \cup \{X_{d+j;n'}\}_{\substack{i+1\leq j\leq N-d\\ 0\leq n'\leq n}}\right]_{D_0}$ satisfies

$$\frac{\partial F^{(1)}_{i+1;\delta_{i+1}-\delta_i+n}}{\partial X_{d+j,n-r}} = \sum_{s=0}^{r} \frac{D^{1,\ldots,i,i+1}_{1,\ldots,i,j;\delta_i+\delta_{i+1}+s}}{D^{1,\ldots,i}_{1,\ldots,i;\delta_i}} B^i_{r-s} \mod(\mathcal{D}).$$

for $i+1\leq j\leq N-d$ and $r\leq \inf\{(a_l-\epsilon_l)-(\delta_l-\delta_{l-1}-\epsilon_l)-\ldots-(\delta_{i+1}-\delta_i-\epsilon_i)\}_{1\leq l\leq i+1}$. This is used in the recurrence reasoning. Therefore, taking $a_l>\epsilon_l+(\delta_l-\delta_{l-1}-\epsilon_l)+\ldots+(\delta_{N-d}-\delta_{N-d-1}-\epsilon_{N-d})$ for $1\leq l\leq N-d$, we conclude the existence of n_i, $1\leq i\leq N-d$, and $X^{(1)}_{d+i;n}$, $1\leq i\leq N-d$, $n>n_i$, satisfying (42) and (43).

From the previous discussion and arguing by recurrence on (m,i,n), $m\geq 1, 1\leq i\leq N-d, n\geq n_i+1$, with the lexicographic order, we obtain

$$X^{(m)}_{d+i;n} \in k\left[\{X_{j;n'}\}_{\substack{1\leq j\leq d\\ n'\geq 0}} \cup \{X_{d+j;n'}\}_{\substack{1\leq j\leq N-d\\ 0\leq n'\leq n_j}}\right]_{D_0}$$

satisfying

$$F_{i;\delta_i-\delta_{i-1}+n} = \frac{D^{1,\ldots,i}_{1,\ldots,i;\delta_i}}{D^{1,\ldots,i-1}_{1,\ldots,i-1;\delta_{i-1}}} (X_{d+i;n}-X^{(m)}_{d+i;n})$$

$$\mod (\mathcal{D})^m + \left(\{F_{j;\delta_j-\delta_{j-1}+n'}\}_{\substack{1\leq j\leq N-d\\ n_j<n'}}\right)$$

in $(k[x_1,\ldots,x_N]_\infty)_{D_0}$. Thus we have

$$X^{(m+1)}_{d+i;n} - X^{(m)}_{d+i;n} \in (\mathcal{D})^m + \left(\{F_{j;\delta_j-\delta_{j-1}+n'}\}_{\substack{1\leq j\leq N-d\\ n_j<n'}}\right).$$

Recall (36) and that the image of $\mathcal{D}$ in $\mathcal{O}_{X_\infty,P^X_E}$ is in P^X_E. Fix an embedding $\kappa(P^X_E)\hookrightarrow \widehat{\mathcal{O}_{X_\infty,P^X_E}}$ sending $X_{j;n}$ to $X_{j;n}\in\widehat{\mathcal{O}_{X_\infty,P^X_E}}$, for $1\leq j\leq d$, $n\geq m_j$ (see the proof of theorem 3.8). Then, for $1\leq i\leq N-d$ and $n>n_i$, the polynomials $\{X^{(m)}_{d+i;n}\}_{m\geq 1}$ define a series

$$\widetilde{X}_{d+i;n} \in \kappa(P)\left[\left[\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J}\\ n_{j,r-1}\overline{\beta}_{j,r-1}\leq n<\overline{\beta}_{j,r}}} \cup \{X_{d+j;n'}-\overline{X}_{d+j;n'}\}_{\substack{1\leq j\leq N-d\\ 0\leq n'\leq n_j}}\right]\right]$$

where we identify $X_{j,r;n}$ with $\overline{Q}_{j,r;n}$, as in the proof of th. 3.8, and where $\overline{X}_{d+j;n'}\in\widehat{\mathcal{O}_{X_\infty,P^X_E}}$ is the image of the class of $X_{d+j;n'}$ in $\kappa(P^X_E)$, for $1\leq j\leq N-d$, $0\leq n'\leq n_j$. Setting $Y_{d+j;n'} := X_{d+j;n'}-\overline{X}_{d+j;n'}$, $1\leq j\leq N-d$, $0\leq n'\leq n_j$, we conclude that $\widehat{\mathcal{O}_{X_\infty,P^X_E}}$ is isomorphic to

$$\kappa(P^X_{eE})\left[\left[\{X_{j,r;n}\}_{\substack{(j,r)\in\mathcal{J}\\ n_{j,r-1}\overline{\beta}_{j,r-1}\leq n<\overline{\beta}_{j,r}}} \cup \{Y_{d+j;n'}\}_{\substack{1\leq j\leq N-d\\ n'\leq n_j}}\right]\right] \Big/ \left(\{\widetilde{F}_{j;n}\}_{\substack{1\leq j\leq N-d\\ n\leq\delta_j-\delta_{j-1}+n_j}}\right)$$

where for $1 \leq j \leq N-d$, $0 \leq n \leq \delta_j - \delta_{j-1} + n_j$, $\widetilde{F}_{j;n}$ is obtained from $F_{j;n}$ by substituting $X_{d+i;n'}$ by $\widetilde{X}_{d+i;n'}$, for $1 \leq i \leq N-d$ and $0 \leq n' \leq n$, and $X_{d+j;n'}$ by $\overline{X}_{d+j;n'} + Y_{d+j;n'}$ for $1 \leq j \leq N-d$, $0 \leq n' \leq n_j$. Applying Krull's theorem we obtain that

$$\dim \widehat{\mathcal{O}_{X_\infty, P^X_{eE}}} \geq \widehat{k}_E + 1 + \sum_{i=1}^{N-d} (n_i + 1) - \sum_{i=1}^{N-d} (\delta_i - \delta_{i-1} + n_i + 1) = \widehat{k}_E + 1 - \delta_{N-d} = a_{MJ}(E).$$

Finally, if X is normal and complete intersection, we have $a_{MJ}(E) = k_E + 1$ ([EM] appendix). Hence we conclude the result. □

4.2. Recall that, given an extension of fields $k \subseteq K$, a K-*wedge* on X is a k-morphism Spec $K[[\xi, t]] \to X$; equivalently it is a K-arc on X_∞ (see (3)). Given a birational and proper k-morphism $p : Y \to X$ and a stable point P of X_∞, we say that *p satisfies the property of lifting wedges centered at P* if, for any field extension K of the residue field $\kappa(P)$ of P in X_∞, and for any K-wedge ϕ : Spec $K[[\xi, t]] \to X$ on X whose special arc is P (i.e. P is the image in X_∞ of the closed point of Spec $K[[\xi]]$), there exists a K-wedge $\widetilde{\phi}$: Spec $K[[\xi, t]] \to Y$ on Y such that $p \circ \widetilde{\phi} = \phi$.

In [Re3], corol. 5.12, it is proved that, if $\nu = \nu_E$ is an essential divisorial valuation on X, then, the following are equivalent:

(i) $\dim \widehat{\mathcal{O}_{X_\infty, P^X_E}} = 1$ and Spec $\widehat{\mathcal{O}_{X_\infty, P^X_E}}$ is irreducible.
(ii) $\dim \mathcal{O}_{X_\infty, P^X_E} = 1$.
(iii) For every resolution of singularities $p : Y \to X$, p satisfies the property of lifting wedges centered at P_E.
(iii') There exists a resolution of singularities $p : Y \to X$ that satisfies the condition in (iii), and such that the center of ν on Y has codimension 1.

T. de Fernex and R. Docampo [dFD] have proved that, if ν_E is a terminal valuation then condition (iii) above holds. In fact, this follows from the proof of th.1.1 in [dFD], note that their statement in th.1.1 is weaker to condition (iii) (see [Re2], th.5.1 or [Re3] section 5). Terminal valuations are the divisorial valuations defined by the exceptional divisors of a minimal model of X, hence they are essential (see [dFD]).

From this and theorem 4.1 above, corollaries 4.3 and 4.4 below follow:

Corollary 4.3. *Let X be a reduced separated scheme of finite type over a field k of char $k = 0$. Let $\nu = \nu_E$ be an essential divisorial valuation on an irreducible component X_0 of X. Consider the following conditions:*

(1) *ν_E is a terminal valuation.*
(2) $\dim \widehat{\mathcal{O}_{X_\infty, P^X_E}} = 1$.
(3) *$a_{MJ}(E; X) \leq 1$, in particular $k_E(X) \leq 0$ if X is normal and complete intersection.*

We have that (1) implies (2) and (2) implies (3).

The following example shows that (2) does not imply (1). It has been pointed out to us by M. Mustata.

Remark 4.4. In [dFD], example 6.3, the toric variety X defined by the cone σ in $\mathbb{R}^3$ spanned by the vectors $(1,0,0), (0,1,0)$ and $(1,1,2)$ is considered, and the divisorial valuation ν_E defined by $(1,1,1)$, which is not a terminal valuation. It can be proved that $\dim \widehat{\mathcal{O}_{X_\infty, P^X_E}} = 1$. In this case we have $\widehat{k}_E(X) = 2$ and $\nu_E(Jac_X) = 3$, hence $a_{MJ}(E; X) = 0$.

Corollary 4.5. *Let X be a reduced separated scheme of finite type over a field k of char $k = 0$. Suppose that X is normal and complete intersection. Let $\nu = \nu_E$ be an essential divisorial valuation on an irreducible component X_0 of X and suppose that $k_E \geq 1$. Then, for every resolution of singularities $p : Y \to X$ such that the center of ν on Y has codimension 1, p does not satisfy the property of lifting wedges centered at P_E, i.e. there exist a field extension K of $\kappa(P_E)$ and a K-wedge $\phi : Spec\ K[[\xi, t]] \to X$ on X whose special arc is P_E and which does not lift to Y.*

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